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Problems of existence of order copies of ℓ^∞ and $L_p(\nu)$ in some non-Banach Köthe spaces

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Abstract

For a monotone Orlicz function Φ taking only values 0 and ∞, it is showed that in both cases, the s-homogeneous norm $\|\cdot\|_{\Phi,s}$, if Φ is s-convex $(0 < s \leq 1)$ and the Mazur–Orlicz F-norm $\|\cdot\|_{\Phi}$, if Φ is non-decreasing on $\mathbb{R}_+$, we have that $L^{\Phi}(\mu) = L^{\infty}(\mu)$ and both these norms are proportional to $\|\cdot\|_{\infty}$. The problems of existence of order linearly isometric copy of $\ell^\infty \backslash B_{\ell^\infty}(0,\varepsilon)$ for any $\varepsilon > 0$ as well as an order linearly isometric copy of the whole ℓ^∞ in Orlicz F-normed function and sequence spaces are considered. In the last section the problem of the existence of order linearly isometric copies of $L_p(\nu)$ with $0 < p \leq 1$ in F-normed Orlicz spaces are considered.

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1 Introduction

In 1976, Turett [35] proved that Orlicz spaces over a non-atomic finite measure space, generated by convex Orlicz functions and equipped with the Luxemburg norm, contain an order isometric copy of ℓ^∞ if and only if the generating convex Orlicz functions do not satisfy the Δ_2-condition at infinity. This result was later extended to other measure space cases by other mathematicians (see [7] and its references, and [17]).

The aim of Sect. 3 is to extend those results to the case of Orlicz spaces generated by non-decreasing (non-convex, in general) Orlicz functions and equipped with the Mazur–Orlicz F-norm.

First, in Theorem 1, it is proved that if a monotone Orlicz function Φ takes only values 0 and ∞, then in both cases, the s-homogeneous F-norm $\|\cdot\|_{\Phi,s}$, if Φ is s-convex ($0 < s \leq 1$), and the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$, if Φ is non-decreasing on $\mathbb{R}_+$, the equality $L^\Phi(\mu) = L^\infty(\mu)$ holds and the respective norms are proportional to the norm $\|\cdot\|_\infty$.

Let us recall that if the F-norm on a σ-Dedekind complete F-lattice X is *not* order continuous, then X contains an order-isomorphic copy of the space ℓ^∞, see [3, Theorem 10.8]. However, in a concrete case we may have even a more unusual situation. Namely, in Theorems 2 and 3 it is proved that if Φ is a non-decreasing Orlicz function, vanishing only at zero and not fulfilling the suitable Δ_2-condition, then for every $\varepsilon > 0$, there is a linear order isomorphism $P_\varepsilon\colon \ell^\infty \to L^\Phi(\mu)$ such that $\|P_\varepsilon x\|_\Phi = \|x\|_\infty$ for every $x \in L^\Phi(\mu)$ with $\|x\|_\infty \geq \varepsilon$. For a non-atomic infinite measure space, as well as for the counting measure space, it is proved that if Φ vanishes outside 0, then the Orlicz space $(L^\Phi(\mu), \|\cdot\|_\Phi)$ contains an order linearly isometric copy of the whole ℓ^∞.

Next, are given two remarks concerning the characterization of the open balls $B_{\|\cdot\|_\Phi}(0,\varepsilon)$ for the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$ in Orlicz function and sequence spaces, for any $\varepsilon > 0$, in terms of the modular I_Φ. Moreover, the problem of the openness of the modular balls $B_{\Phi,\varepsilon} := \{x \in L^\Phi(\mu)\colon I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon\}$ in the metric topology, in both cases: the s-homogeneous norm $\|\cdot\|_{\Phi,s}$ ($0 < s \leq 1$) and the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$ is solved. For the s-homogeneous norm this is done for $\varepsilon = 1$ and for the F-norm this is done for any $\varepsilon > 0$.

In the last section of this paper, we study the problem when X embeds order-isometrically into Y whenever X embeds isometrically into Y. In Proposition 2 we show that this is the case for $X = L_p(\nu)$, for some $p \in (0, 1]$, extending a similar result proved in 2003 by the fourth present author [36, Theorem 1′] for a class of Banach lattices. As an application, we obtain that, for a (non-convex) strictly monotone Orlicz function Φ, for any measure space (Γ, Θ, ν) and any $p \in (0, 1]$, the order continuous part $E^\Phi(\mu)$ of the Orlicz space $L^\Phi(\mu)$, endowed with the Mazur–Orlicz F-norm, does not contain an isometric copy of $L^p(\nu)$. This is in contrast with the Banach lattice case where a strictly monotone Banach lattice E contains an order-isometric copy of $L^1(\nu)$ iff E contains an isometric copy of $L^1(\nu)$; see [36, Corollary 2].

2 Preliminaries

In the whole paper (Ω, Σ, μ) denotes a σ-finite and complete measure space with a non-atomic measure μ on Σ or the counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with $\mu(A) = \mathrm{Card}(A)$ for any $A \subseteq \mathbb{N}$, and it is assumed that all operators acting between two given spaces considered in the paper are linear. Moreover, the word *isomorphism* will mean a linear topological isomorphism and the expression *an order isomorphism* will mean a topological isomorphism preserving the order.

A function $x \mapsto \|x\|$ defined on a linear (real) space X and with values in $\mathbb{R}_+ := [0, \infty)$ is said to be an F-norm if it satisfies the following conditions:

1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$,
3. $\|x\| = \|-x\|$ for any $x \in X$,
4. if $a_k \to a$ and $\|x_k - x\| \to 0$ as $k \to \infty$, where $a_k, a \in \mathbb{R}$ and $x_k, x \in X$, then $\|a_k x_k - ax\| \to 0$ as $k \to \infty$.

If an F-normed space X is $\|\cdot\|$-complete, then $(X, \|\cdot\|)$ is said to be an F-space. If a lattice E is endowed with a monotone F-norm $\|\cdot\|$ (i.e., the condition $|x| \le |y|$ implies that $\|x\| \le \|y\|$ for any $x, y \in E$), under which E is complete, then $E = (E, \|\cdot\|)$ is said to be an F-lattice.

We will consider two kinds of Orlicz functions. All of them are denoted by Φ and they are functions from $\mathbb{R}_+ := [0, \infty)$ into $\mathbb{R}_+^e = [0, \infty]$ vanishing at zero with $b(\Phi) \in (0, \infty]$, where $b(\Phi) := \sup\{u \ge 0 \colon \Phi(u) < \infty\}$. The first class of Orlicz functions is the class of s-convex $(0 < s \le 1)$ continuous Orlicz functions, that is, such functions Φ that $\Phi(\alpha u + \beta v) \le \alpha^s \Phi(u) + \beta^s \Phi(v)$ for all $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$ and all $u, v \ge 0$. The second class of Orlicz functions is the class of non-decreasing functions $\Phi \colon \mathbb{R}_+ \to \mathbb{R}_+^e$, which are continuous on $[0, b(\Phi))$ and $\Phi(b(\Phi)) := \lim_{x \to b(\Phi)^-} \Phi(x)$, $\Phi(u) \to 0$ as $u \to 0^+$ and with $\lim_{u \to \infty} \Phi(u) > 0$, called shortly non-decreasing Orlicz functions.

For any Orlicz function Φ let us define $a(\Phi) = \sup\{u \ge 0 \colon \Phi(u) = 0\}$. We say that Φ satisfies the $\Delta_2(\infty)$-condition ($\Phi \in \Delta_2(\infty)$ for short) if $b(\Phi) = \infty$ and $\limsup_{u \to \infty} \frac{\Phi(2u)}{\Phi(u)} < \infty$. We say that Φ satisfies the $\Delta_2(0)$-condition ($\Phi \in \Delta_2(0)$ for short) if $a(\Phi) = 0$ and $\limsup_{u \to 0} \frac{\Phi(2u)}{\Phi(u)} < \infty$, and we will write $\Phi \in \Delta_2(\mathbb{R}_+)$ if $\Phi \in \Delta_2(\infty)$ and $\Phi \in \Delta_2(0)$. By the suitable Δ_2-condition we will mean the $\Delta_2(\infty)$-condition in the case of a non-atomic finite measure space, the $\Delta_2(\mathbb{R}_+)$-condition in the case of a non-atomic infinite measure space and the $\Delta_2(0)$-condition in the case of the counting measure space.

If Φ is an Orlicz function from one of the above two classes, then the modular generated by Φ on the space $L^0(\Omega, \Sigma, \mu)$ of all abstract classes of Σ-measurable real functions defined on Ω, where the functions which are equal μ-a.e. in Ω are identified, is a functional defined by the formula

$$I_\Phi(x) = \int_\Omega \Phi(|x(t)|)d\mu(t) \qquad \left(\forall\, x \in L^0(\Omega, \Sigma, \mu)\right).$$

Obviously, if (Ω, Σ, μ) is the counting measure space with $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$, then $I_\Phi(x) = \sum_{n=1}^{\infty} \Phi(|x(n)|)$ for all $x = (x(n))_{n=1}^{\infty} \in \ell^0$, where ℓ^0 is the space of all real sequences (since in this case $L^0(\mu) = \ell^0$).

In any of the above two kinds of Orlicz function Φ, the Orlicz space $L^\Phi(\mu) := L^\Phi(\Omega, \Sigma, \mu)$ is defined by the formula

$$L^\Phi(\mu) = \left\{x \in L^0(\Omega, \Sigma, \mu)\colon I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\right\}$$

and in the case when (Ω, Σ, μ) is the counting measure space, the Orlicz space $\ell^\Phi(\mu)$ is denoted simply by ℓ^Φ. In the case of an s-convex $(0 < s \le 1)$ Orlicz function Φ we define in the Orlicz space $L^\Phi(\mu)$ or ℓ^Φ the s-norm

$$\|x\|_{\Phi,s} = \inf\left\{\lambda > 0\colon I_\Phi\left(\frac{x}{\lambda^{1/s}}\right) \le 1\right\}.$$

Note that for $s = 1$, Φ is a convex Orlicz function and $\|x\|_{\Phi,1}$ is the Luxemburg norm. Finally, when Φ is a non-decreasing Orlicz function, we define in the Orlicz space $L^\Phi(\mu)$ or ℓ^Φ the Mazur–Orlicz F-norm (see [29])

$$\|x\|_\Phi = \inf\left\{\lambda > 0\colon I_\Phi\left(\frac{x}{\lambda}\right) \le \lambda\right\}.$$

It is known that in the case of both Orlicz functions the convergence in $L^\Phi(\mu)$ or ℓ^Φ of a sequence $(x_n)_{n=1}^{\infty}$ in the Orlicz space to its element x is equivalent to the condition $I_\Phi(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for any $\lambda > 0$ (see [19] or [38]). For more information about Orlicz normed (also s-normed, $0 < s < 1$) spaces we refer, for instance, to [7,24,27,28,30,34,38].

If $(x_n)_{n=1}^{\infty}$ is a pairwise disjoint sequence in $L^0(\mu)$ and $(a_n)_{n=1}^{\infty}$ is a sequence of scalars, then the symbol $\sum_{n=1}^{\infty} a_n x_n$ denotes the formal pointwise sum of the functions $a_n x_n$ in $L^0(\mu)$.

An F-space $(E, \|\cdot\|_E)$ is called an F-normed Köthe space if it is a linear subspace of L^0 satisfying the following conditions:

(i) If $x \in L^0$, $y \in E$ and $|x| \le |y|$ μ-a.e., then $x \in E$ and $\|x\|_E \le \|y\|_E$.
(ii) There exists a strictly positive $x \in E$ (called a weak unit).

For a Köthe space E, let $E_+ := \{x \in E\colon x \ge 0\}$ and $S_+(E) := S(E) \cap E_+$.

An element x of a Köthe space E over (Ω, Σ, μ) is said to be order continuous if for any sequence $(x_n)_{n=1}^{\infty}$ in E such that $0 \le x_n \le |x|$ for any $n \in \mathbb{N}$ and $x_n \to 0$ μ-a.e., we have $\|x_n\|_E \to 0$. The subspace of E that consists of all order continuous elements from E is denoted by E_a and it is called the subspace of order continuous elements of E.

It is known that for the Orlicz spaces $L^\Phi(\mu)$ over a non-atomic measure space, we have that $(L^\Phi(\mu))_a \neq \{0\}$ if and only if $b(\Phi) = \infty$, and

$$(L^\Phi(\mu))_a = E^\Phi(\mu) := \left\{x \in L^0(\Omega, \Sigma, \mu) \colon I_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\right\}$$

(see [19], cf. [37,38]). Moreover, in the case of the counting measure space,

$$(\ell^\Phi)_a = h^\Phi := \left\{x = (x(n))_{n=1}^\infty \in \ell^0 : \underset{\lambda>0}{\forall}\ \underset{n_\lambda \in \mathbb{N}}{\exists} \sum_{n=n_\lambda}^\infty \Phi(\lambda|x(n)|) < \infty\right\}$$

(see again [19], cf. [37,38]). In case when the distinguishing of the function and sequence Orlicz spaces would not be necessary, we will write just $L^\Phi(\mu)$ in both kinds of measure spaces.

If A is a non-empty subspace of ℓ^∞, then a linear operator $P \colon A \to L^\Phi(\mu)$ (resp. $P \colon A \to \ell^\Phi(\mu)$) is said to be a linear order isometry (or shortly an order isometry) if P keeps the order (i.e. $Px \geq 0$ for any $x \in A$, $x \geq 0$) and $\|Px - Py\|_\Phi = \|P(x-y)\|_\Phi = \|x-y\|_\infty$ for any $x, y \in A$.

A Köthe function space E is said to contain an order-isometric copy of ℓ^∞ if there exists in E_+ a sequence $(x_n)_{n=1}^\infty$ of pairwise orthogonal elements (i.e., $\mu(\operatorname{supp} x_n \cap \operatorname{supp} x_m) = 0$ for any $m, n \in \mathbb{N}$, $m \neq n$) such that

$$\left\|\sum_{n=1}^\infty c_n x_n\right\|_E = \|c\|_\infty \quad (\forall\, c = (c_n)_{n=1}^\infty \in \ell^\infty).$$

An s-normed Köthe space E $(0 < s \leq 1)$ is said to contain an order-isometric copy of $(\ell^\infty)^s$ if there exists in E_+ a sequence $(x_n)_{n=1}^\infty$ with $\mu(\operatorname{supp} x_n \cap \operatorname{supp} x_m) = 0$ for any $m, n \in \mathbb{N}$, $m \neq n$, such that

$$\left\|\sum_{n=1}^\infty c_n x_n\right\|_E = \||c|^s\|_\infty \quad (\forall\, c = (c_n)_{n=1}^\infty \in \ell^\infty),$$

where $|c|^s := (|c_1|^s, \ldots, |c_n|^s, \ldots)$.

If E, G are two linear lattices then the symbol $\mathcal{L}_r(E, G)$ denotes the linear space of all regular linear operators $T \colon E \to G$, i.e. operators of the form $T = U - V$, where $U, V \in \mathcal{L}^+(E, G)$, the cone of positive (i.e. non-negative) linear operators; hence we also have that the space $\mathcal{L}_r(E, G)$ is partially ordered by $\mathcal{L}^+(E, G)$. We say that a linear operator $T \colon E \to G$ preserves disjointness if the condition $|x_1| \wedge |x_2| = 0$ in E implies that $|Tx_1| \wedge |Tx_2| = 0$ (notice that such an operator T need not to be regular, in general: see, e.g., [1, Examples 1, 2]). If T is injective, it is said to be an order isomorphism if both T and T^{-1} are positive; equivalently, $|Tx| = T(|x|)$ for every $x \in E$. An endomorphism J on a linear space X is said to be an involution if its square J^2 is the identity on X.

Let X and Y be two real F-lattices, and let the F-norm $\|\cdot\|$ on Y be strictly monotone, i.e. $\|y_1\| < \|y_2\|$ whenever $y_1, y_2 \in Y$, $|y_1| \leq |y_2|$ and $y_1 \neq y_2$. The classical F-space $L_p(\nu)$, $0 < p < 1$, endowed with the F-norm $\|x\|_p := \int_\Gamma |x|^p \mathrm{d}\nu$ and the standard (ν-a.e.) partial ordering, is a simple nontrivial example of a strictly monotone non-Banach F-lattice. Other examples, within the class of non-Banach Orlicz spaces, are given in the recent paper [19].

3 Results

Let us start with the following

Theorem 1 *Assume that $u_0 \in (0, \infty)$ and $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+^e$ is defined by the formula $\Phi(u) = 0$ if $0 \le u \le u_0$ and $\Phi(u) = \infty$ if $u > u_0$. Then $L^\Phi(\mu) = L^\infty(\mu)$, $|||x|||_\Phi = \frac{1}{u_0}\|x\|_\infty$ and $\|x\|_{\Phi,s} = \frac{1}{(u_0)^s}\|x\|_\infty^s$ for any $x \in L^\Phi(\mu)$, any $0 < s \le 1$ and any measure space (Ω, Σ, μ).*

Proof First, we will easily prove that $L^\Phi(\mu) = L^\infty(\mu)$. Let us assume that $x \in L^\Phi(\mu)$. There exists $\lambda > 0$ such that $I_\Phi(\lambda x) < \infty$, whence $\lambda|x(t)| \le u_0$ μ-a.e., that is, $\|x\|_\infty \le \frac{u_0}{\lambda}$ μ-a.e., which means that $x \in L^\infty(\mu)$. Now, let us assume that $x \in L^\infty(\mu)$. Then $\frac{u_0}{\|x\|_\infty}|x(t)| \le u_0$ μ-a.e., whence $I_\Phi\left(\frac{u_0}{\|x\|_\infty}x\right) < \infty$, that is, $x \in L^\Phi(\mu)$. In such a way we have proved two inclusions $L^\Phi(\mu) \subseteq L^\infty(\mu) \subseteq L^\Phi(\mu)$, which gives the equality $L^\Phi(\mu) = L^\infty(\mu)$.

Let us assume that $0 < s \le 1$. If $x = 0$, then $\|x\|_{\Phi,s} = \|x\|_\infty = 0$, whence in this case the desired equality holds. So, let us now assume that $x \in L^\Phi(\mu)\backslash\{0\}$. Then $\|x\|_\infty > 0$. It is obvious by the formula for Φ that $I_\Phi\left(\frac{x}{\|x\|_\infty/u_0}\right) = 0$, whence $\|x\|_{\Phi,s} \le \left(\frac{1}{u_0}\|x\|_\infty\right)^s$ and $|||x|||_\Phi \le \frac{1}{u_0}\|x\|_\infty$. For any $\lambda \in (0, 1)$ we have that $\frac{|x(t)|}{\lambda\|x\|_\infty/u_0} > u_0$ on a set of positive measure, whence

$$I_\Phi\left(\frac{x}{\lambda\|x\|_\infty/u_0}\right) = \infty \qquad (\forall\, \lambda \in (0, 1)).$$

Consequently, $\|x\|_{\Phi,s} \ge \left(\frac{\lambda}{u_0}\right)^s \|x\|_\infty^s$ and $|||x|||_\Phi \ge \frac{\lambda}{u_0}\|x\|_\infty$. By the arbitrariness of $\lambda \in (0, 1)$, we obtain

$$\|x\|_{\Phi,s} \ge \frac{1}{(u_0)^s}\|x\|_\infty^s \text{ and } |||x|||_\Phi \ge \frac{1}{u_0}\|x\|_\infty,$$

which finishes the proof. □

In two theorems below we will deal with isometric copies of ℓ^∞ in non-Banach Orlicz spaces. It is well known (see [3]) that a σ-Dedekind complete F-lattice E is not order continuous if and only if E contains an order-isomorphic copy of ℓ^∞. In the Banach lattice case, there are function spaces E with a much stronger property: E is not order continuous if and only if E contains an order-isometric copy of ℓ^∞ (for Orlicz spaces see, e.g., [7,8,21,34,38], for Marcinkiewicz spaces see [22], for Orlicz–Lorentz spaces see [6] and for some class of general Banach lattices see [17]). The problem of existence in Banach spaces of almost isometric, asymptotically isometric or even isometric copies was considered in various papers (see for example [8,10,11,20,21]). In Theorems 2 and 3 we will show that non-Banach Orlicz spaces $L^\Phi(\mu)$, endowed with the Mazur–Orlicz F-norm, have a nearing property.

Theorem 2 *Let $\ell^\Phi := \ell^\Phi(\mu)$ be the Orlicz sequence space over the counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ generated by a non-decreasing Orlicz function $\Phi: \mathbb{R}_+ \to \mathbb{R}_+^e$ and equipped with the Mazur–Orlicz F-norm $\|\cdot\|_\Phi$. Then the following holds:*

(i) *If $a(\Phi) = 0$ (i.e., Φ vanishes only at 0) and Φ does not satisfy condition $\Delta_2(0)$, then for any $\varepsilon > 0$ there is an order-isomorphism $P_\varepsilon: \ell^\infty \to \ell^\Phi$ such that*

  (a) *$\|P_\varepsilon c\|_\Phi \geq \|c\|_\infty$ for every $c \in \ell^\infty$, and*

  (b) *$\|P_\varepsilon c\|_\Phi = \|c\|_\infty$ for every $c \in \ell^\infty$ with $\|c\|_\infty \geq \varepsilon$, i.e., if $\|x - y\|_\infty \geq \varepsilon$ then $\|P_\varepsilon x - P_\varepsilon y\|_\Phi = \|P_\varepsilon(x - y)\|_\Phi = \|x - y\|_\infty$, for any $x, y \in \ell^\Phi$.*

(ii) *If $a(\Phi) > 0$, then $(\ell^\Phi, \|\cdot\|_\Phi)$ contains an order-isometric copy of ℓ^∞.*

Proof Statement (i). Let us assume first that $a(\Phi) = 0$ and that Φ does not satisfy condition $\Delta_2(0)$. Then for any $K > 0$ and any $a \in \left(0, \frac{b(\Phi)}{2}\right)$ there exists a sequence $(u_n)_{n=1}^\infty$ of positive numbers such that

$$\Phi(u_n) \leq \Phi(a) \text{ and } \Phi\left(\left(1 + \frac{1}{n}\right) u_n\right) > K\Phi(u_n) \qquad (\forall\, n \in \mathbb{N}). \tag{1}$$

Consequently, given any $\varepsilon \in (0, 1)$, there exists a sequence $(u_n)_{n=1}^\infty$ of positive numbers such that

$$\Phi(u_n) \leq \frac{\varepsilon}{2^{n+1}} \text{ and } \Phi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^{n+1}\Phi(u_n) \qquad (\forall\, n \in \mathbb{N}). \tag{2}$$

Let, for any $n \in \mathbb{N}$, k_n be the biggest natural number such that $k_n\Phi(u_n) \leq \varepsilon/2^n$. Then $(k_n + 1)\Phi(u_n) > \frac{\varepsilon}{2^n}$, whence $k_n\Phi(u_n) > \frac{\varepsilon}{2^n} - \Phi(u_n) \geq \frac{\varepsilon}{2^{n+1}}$. Consequently, the couple (u_n, k_n) satisfies the inequalities

$$\frac{\varepsilon}{2^{n+1}} < k_n\Phi(u_n) \leq \frac{\varepsilon}{2^n} \qquad (\forall\, n \in \mathbb{N}). \tag{3}$$

Defining

$$x = \left(\overbrace{u_1, \ldots, u_1}^{k_1 \text{ times}}, \ldots, \overbrace{u_n, \ldots, u_n}^{k_n \text{ times}}, \ldots\right),$$

by the second inequality in (3), we obtain that $I_\Phi(x) \le \varepsilon$. Moreover, taking any $\lambda \in (0,1)$, one can find $n_\lambda \in \mathbb{N}$ such that $\frac{1}{\lambda} \ge 1 + \frac{1}{n}$ for any $n \ge n_\lambda$. Consequently,

$$\begin{aligned} I_\Phi\left(\frac{x}{\lambda}\right) &\ge \sum_{n=n_\lambda}^{\infty} k_n \Phi\left(\frac{u_n}{\lambda}\right) \ge \sum_{n=n_\lambda}^{\infty} k_n \Phi\left(\left(1+\frac{1}{n}\right)u_n\right) \\ &\ge \sum_{n=n_\lambda}^{\infty} k_n 2^{n+1}\Phi(u_n) \ge \sum_{n=n_\lambda}^{\infty} 2^{n+1}\frac{\varepsilon}{2^{n+1}} = \varepsilon\sum_{n=n_\lambda}^{\infty} 1 = \infty. \end{aligned}$$

This inequality together with $I_\Phi(x) \le \varepsilon < 1$, gives the equality $\|x\|_\Phi = 1$.

Let us note that the technique presented above gives us the possibility of building for any $\varepsilon > 0$ a sequence $(x_{n,\varepsilon})_{n=1}^{\infty}$ of positive elements in ℓ^Φ with pairwise disjoint supports such that $I_\Phi(x_{n,\varepsilon}) \le 2^{-n}\varepsilon$ and $I_\Phi\left(\frac{x_{n,\varepsilon}}{\lambda}\right) = \infty$ for any $\lambda \in (0,1)$. In order to do this, it is enough to divide the set $\mathbb{N}$ into a countable family $(\mathbb{N}_n)_{n=1}^{\infty}$ of infinite and pairwise disjoint subsets of $\mathbb{N}$ and consider the sequence of Orlicz sequence spaces $(\ell^\Phi(\mathbb{N}_n, 2^{\mathbb{N}_n}, \mu/2^{\mathbb{N}_n}))$. Applying the technique presented above for building the element x, one can build for any $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$ an element $x_{n,\varepsilon}$ such that $\operatorname{supp} x_{n,\varepsilon} \subseteq \mathbb{N}_n$, $I_\Phi(x_{n,\varepsilon}) \le \varepsilon 2^{-n}$ and $I_\Phi(x_{n,\varepsilon}/\lambda) = \infty$ for any $\lambda \in (0,1)$. Consequently, $\|x_{n,\varepsilon}\|_\Phi = 1$ for any $n \in \mathbb{N}$. Moreover, if $x_\varepsilon := \sup_{n\ge 1} x_{n,\varepsilon} = \sum_{n=1}^{\infty} x_{n,\varepsilon}$, then $I_\Phi(x_\varepsilon) = \sum_{n=1}^{\infty} I_\Phi(x_{n,\varepsilon}) \le \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon$ and $I_\Phi\left(\frac{x_\varepsilon}{\lambda}\right) \ge I_\Phi\left(\frac{x_{n,\varepsilon}}{\lambda}\right) = \infty$ for any $\lambda > 1$ and any $n \in \mathbb{N}$, whence $\|x_\varepsilon\|_\Phi = 1$.

Let us now define an operator $P_\varepsilon \colon \ell^\infty \to \ell^\Phi$ by

$$P_\varepsilon c = \sum_{n=1}^{\infty} c_n x_{n,\varepsilon} \qquad (\forall\, c = (c_n)_{n=1}^{\infty} \in \ell^\infty).$$

Since the elements $x_{n,\varepsilon}$ are positive and pairwise disjoint, the operator is non-negative, so continuous as well (see [3, Theorem 16.6]). Moreover, for any $c = (c_n)_{n=1}^{\infty} \in \ell^\infty$, $c \ne 0$, we have

$$I_\Phi\left(\frac{P_\varepsilon c}{\|c\|_\infty}\right) = \sum_{n=1}^{\infty} I_\Phi\left(\frac{|c_n|}{\|c\|_\infty} x_{n,\varepsilon}\right) \le \sum_{n=1}^{\infty} I_\Phi(x_{n,\varepsilon}) \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon, \tag{4}$$

which shows that $P_\varepsilon c \in \ell^\Phi(\mu)$ for any $c \in \ell^\infty$ as well as that if $c \in \ell^\infty$ and $\|c\|_\infty \ge \varepsilon$, then $I_\Phi\left(\frac{P_\varepsilon c}{\|c\|_\infty}\right) \le \|c\|_\infty$, whence $\|P_\varepsilon c\|_\Phi \le \|c\|_\infty$ for any $c \in \ell^\infty$ with $\|c\|_\infty \ge \varepsilon$. Moreover, for any $c \in \ell^\infty \setminus \{0\}$ and any $\lambda \in (0,1)$, taking n_λ such that $\frac{|c_{n_\lambda}|}{\lambda\|c\|_\infty} > 1$, we get

$$I_\Phi\left(\frac{P_\varepsilon c}{\lambda\|c\|_\infty}\right) \ge I_\Phi\left(\frac{|c_{n_\lambda}| x_{n_\lambda}}{\lambda\|c\|_\infty}\right) = \infty,$$

whence $\|P_\varepsilon c\|_\Phi \ge \lambda\|c\|_\infty$ and, by the arbitrariness of $\lambda \in (0,1)$, we get that $\|P_\varepsilon c\|_\Phi \ge \|c\|_\infty$ for any $c \in \ell^\infty \setminus \{0\}$, so also for any $c \in \ell^\infty$ because $\|P_\varepsilon 0\|_\Phi =$

$\|0\|_\Phi = 0 = \|0\|_\infty$. Since the operators P_ε are linear and, by the fact that the elements $x_{n,\varepsilon}$ that are used in the definition of P_ε are non-negative, P_ε are also non-negative, so they preserve the order, whence we conclude that the obtained copies of $\ell^\infty \backslash B_{\ell^\infty}(0, \varepsilon)$ are order copies. Therefore, we have proved that

$$\|P_\varepsilon c\|_\Phi \geq \|c\|_\infty \text{ for any } c \in \ell^\infty. \tag{5}$$

In particular, the operator P_ε^{-1} is continuous at 0, thus P_ε is an order-isomorphism. From inequality (4) we also obtain that if $c \in \ell^\infty$ and $\|c\|_\infty \geq \varepsilon$, then $I_\Phi\left(\frac{P_\varepsilon c}{\|c\|_\infty}\right) \leq \|c\|_\infty$, whence

$$\|P_\varepsilon c\|_\Phi \leq \|c\|_\infty \text{ for any } c \in \ell^\infty \text{ with } \|c\|_\infty \geq \varepsilon. \tag{6}$$

Thus, by (5) and (6), the proof of statement (i) of our theorem is complete.

(ii) Since the proof can be proceeded analogously as in the case when $a(\Phi) = 0$ and $\Phi \notin \Delta_2(0)$ (cf. also [8] in the case of normed Orlicz spaces), it is omitted. □

Theorem 3 *Let (Ω, Σ, μ) be a complete σ-finite non-atomic measure space and let Φ be a non-decreasing Orlicz function from $\mathbb{R}_+$ into $\mathbb{R}_+^e$. Then*:

(i) *If $\mu(\Omega) < \infty$ or $\mu(\Omega) = \infty$ and $a(\Phi) = 0$, then for any $\varepsilon > 0$ satisfying inequality $\varepsilon < \lim_{u\to\infty} \Phi(u)\mu(\Omega)$ whenever $\mu(\Omega) < \infty$ and for arbitrary $\varepsilon > 0$ if $\mu(\Omega) = \infty$, there is an order-isomorphism $P_\varepsilon: \ell^\infty \to L^\Phi(\mu)$ such that $\|P_\varepsilon c\|_\Phi \geq \|c\|_\infty$ for any $c \in \ell^\infty$ and $\|P_\varepsilon c\|_\Phi \leq \|c\|_\infty$ for any $c \in \ell^\infty$ with $\|c\|_\infty \geq \varepsilon$ (i.e., $\|P_\varepsilon x - P_\varepsilon y\|_\Phi = \|P_\varepsilon(x-y)\|_\Phi = \|x-y\|_\infty$ for all $x, y \in \ell^\infty$ with $\|x-y\|_\infty \geq \varepsilon$) if and only if Φ does not satisfy the suitable Δ_2-condition.*

(ii) *If $\mu(\Omega) = \infty$ and $a(\Phi) > 0$, then there is an order isometry $P: \ell^\infty \to L^\Phi(\mu)$.*

Proof Statement (i). Necessity. If Φ satisfies the suitable Δ_2-condition then the Orlicz space $(L^\Phi(\mu), \|\cdot\|_\Phi)$ is order continuous (i.e., the space $L^\Phi(\mu)$ has the Lebesgue property), so $L^\Phi(\mu)$ does not contain an order-isomorphic copy of ℓ^∞ (see [3, Theorem 10.8]).

Sufficiency. Following the idea from the proof of Theorem 2, it is enough to prove the existence, for any $\varepsilon > 0$, of a sequence $(x_n)_{n=1}^\infty$ of non-negative functions with pairwise disjoint supports and with $\|x_k\|_\Phi = \left\|\sum_{n=1}^\infty x_n\right\|_\Phi = 1$ for any $k \in \mathbb{N}$ and $I_\Phi\left(\sum_{n=1}^\infty x_n\right) \leq \varepsilon$.

Assume that $\Phi \notin \Delta_2(\infty)$. We can restrict our proof to the case of finite measure because if $\mu(\Omega) = \infty$, then we can work on a subset $A \subseteq \Omega$ with $0 < \mu(A) < \infty$ instead of Ω.

Case I Assume additionally that $b(\Phi) = \infty$. By $\Phi \notin \Delta_2(\infty)$, given any $\varepsilon \in (0, 1)$ there exists a sequence $(u_n)_{n=1}^\infty$ of positive numbers such that $u_n \leq u_{n+1}$ for any $n \in \mathbb{N}$, $\Phi(u_n) \to \infty$ as $n \to \infty$, $\Phi(u_1)\mu(\Omega) \geq \varepsilon$ and

$$\Phi\left(\left(1 + \frac{1}{n}\right)u_n\right) > 2^n\Phi(u_n) \qquad (\forall\, n \in \mathbb{N}). \tag{7}$$

Since the measure space (Ω, Σ, μ) is non-atomic and $\Phi(u_n)\mu(\Omega) \geq \Phi(u_1)\mu(\Omega) \geq \varepsilon$ for any $n \in \mathbb{N}$, we can find in Σ a sequence $(A_n)_{n=1}^{\infty}$ with pairwise disjoint supports such that $\Phi(u_n)\mu(A_n) = \frac{\varepsilon}{2^n}$. Define

$$x = \sum_{n=1}^{\infty} u_n \chi_{A_n}. \tag{8}$$

Then, $I_\Phi(x) = \sum_{n=1}^{\infty} \Phi(u_n)\mu(A_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon < 1$. Given any $\lambda \in (0, 1)$, there exists n_λ such that $\frac{1}{\lambda} > 1 + \frac{1}{n}$ for any $n \geq n_\lambda$. Consequently, by (7),

$$\begin{aligned} I_\Phi\left(\frac{x}{\lambda}\right) &= \sum_{n=1}^{\infty} \Phi\left(\frac{u_n}{\lambda}\right)\mu(A_n) \geq \sum_{n=n_\lambda}^{\infty} \Phi\left(\left(1 + \frac{1}{n}\right)u_n\right)\mu(A_n) \\ &\geq \sum_{n=n_\lambda}^{\infty} 2^n \Phi(u_n)\mu(A_n) = \sum_{n=n_\lambda}^{\infty} 2^n \cdot 2^{-n} = \sum_{n=n_\lambda}^{\infty} 1 = \infty, \end{aligned}$$

whence $\|x\|_\Phi = 1$. Now, we will divide x into a sum $\sum_{n=1}^{\infty} x_n$ of the elements of a sequence $(x_n)_{n=1}^{\infty}$, where x_n are non-negative functions with pairwise disjoint supports and with $\|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. We will made this division by the following induction

$$\begin{aligned} x_1 &= u_1\chi_{A_1} + u_3\chi_{A_3} + u_5\chi_{A_5} + u_7\chi_{A_7} + u_9\chi_{A_9} + \cdots \\ x_2 &= u_2\chi_{A_2} + u_6\chi_{A_6} + u_{10}\chi_{A_{10}} + u_{14}\chi_{A_{14}} + u_{18}\chi_{A_{18}} + \cdots \\ x_3 &= u_4\chi_{A_4} + u_{12}\chi_{A_{12}} + u_{20}\chi_{A_{20}} + u_{28}\chi_{A_{28}} + \cdots \\ &\vdots \end{aligned}$$

In general, x_{n+1} is the sum of every second term $u_i\chi_{A_i}$ of the function $x - \sum_{k=1}^{n} x_k$. Then it is obvious that for any $n \in \mathbb{N}$, $I_\Phi(x_n) \leq I_\Phi(x) = \varepsilon < 1$. Repeating the method from x into x_n $(n \in \mathbb{N})$, we can prove that $I_\Phi\left(\frac{x_n}{\lambda}\right) = \infty$ for any $\lambda \in (0, 1)$, whence $\|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. We can prove in the same way as for the sequence spaces ℓ^Φ in Theorem 2 that the operator P_ε defined by the formula

$$P_\varepsilon c = \sum_{n=1}^{\infty} c_n x_n \qquad (\forall\ c = (c_n)_{n=1}^{\infty} \in \ell^\infty)$$

is the desired operator.

Case II Assume now that $b(\Phi) < \infty$. Let us take any $\varepsilon \in (0, 1)$ and any sequence $(u_n)_{n=1}^{\infty}$ such that $0 < u_n < b(\Phi)$ for any $n \in \mathbb{N}$ and $u_n \to b(\Phi)$ as $n \to \infty$. Let $(A_n)_{n=1}^{\infty}$ be any sequence in Σ such that $\mu(A_n) > 0$ and $\Phi(u_n)\mu(A_n) \leq \frac{\varepsilon}{2^n}$. Defining $x = \sum_{n=1}^{\infty} u_n\chi_{A_n}$, we have $I_\Phi(x) \leq \varepsilon$ and $I_\Phi\left(\frac{x}{\lambda}\right) = \infty$ for any $\lambda \in (0, 1)$. Dividing the set $\mathbb{N}$ into an infinite family $(\mathbb{N}_k)_{k=1}^{\infty}$ of pairwise disjoint and infinite subsets of $\mathbb{N}$ and defining $x_k = \sum_{n \in \mathbb{N}_k} u_n\chi_{A_n}$, we get $I_\Phi(x_k) \leq I_\Phi(x) \leq \varepsilon$. Since given any $\lambda \in (0, 1)$, by $u_n < b(\Phi)$ and $u_n \to b(\Phi)$, we obtain that $\frac{u_n}{\lambda} > b(\Phi)$ for n large enough, so $I_\Phi\left(\frac{x_k}{\lambda}\right) = \infty$ for any $\lambda \in (0,1)$, whence $\|x_k\|_\Phi = \|x\|_\Phi = 1$ for any $k \in \mathbb{N}$. We can easily show that the operator $P_\varepsilon \colon \ell^\infty \to L^\Phi(\mu)$ defined as

$$P_\varepsilon c = \sum_{k=1}^{\infty} c_k x_k$$

has the required properties.

Assume now that $\mu(\Omega) = \infty$, $a(\Phi) = 0$ and $\Phi \notin \Delta_2(0)$. Since μ is non-atomic, there exists a sequence $(A_n)_{n=1}^\infty$ in Σ such that $A_m \cap A_n = \emptyset$ for any $m, n \in \mathbb{N}, m \neq n$ and $\mu(A_n) = 1$ for any $n \in \mathbb{N}$. Now, we can repeat the proof of Theorem 2 to build for any $\varepsilon > 0$ the desired operator $P_\varepsilon \colon \ell^\infty \to L^\Phi(\mu)$.

Statement (ii). If $\mu(\Omega) = \infty$, we can divide the set Ω into a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint sets such that $\mu(A_n) = \infty$ for any $n \in \mathbb{N}$. Let us define $x_n = a(\Phi)\chi_{A_n}$. Then $\sum_{n=1}^\infty x_n = a(\Phi)\chi_\Omega =: x$ and $I_\Phi(x) = I_\Phi(x_n) = 0$ as well as $I_\Phi\left(\frac{x}{\lambda}\right) = I_\Phi\left(\frac{x_n}{\lambda}\right) = \infty$ for any $n \in \mathbb{N}$ and any $\lambda \in (0,1)$, whence it follows that $x, x_n \in L^\Phi(\mu)$ and $\|x\|_\Phi = \|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. Now, we can repeat the respective part of the proof of Theorem 2 to show that the operator $P \colon \ell^\infty \to L^\Phi(\mu)$ is an order isometry. Namely, defining the operator $P \colon \ell^\infty \to L^\Phi(\mu)$ by

$$Pc = \sum_{n=1}^{\infty} c_n x_n, \quad (\forall\, c = (c_n)_{n=1}^\infty \in \ell^\infty),$$

we get that the operator P is positive, so continuous as well. The inequality $\|Pc\|_\Phi \leq \|c\|_\infty$ follows from that for any $c \in \ell^\infty$ and any $\varepsilon > 0$, we have

$$\begin{aligned} I_\Phi\left(\frac{Pc}{\|c\|_\infty + \varepsilon}\right) &= I_\Phi\left(\sum_{n=1}^{\infty} \frac{|c_n| a(\Phi)\chi_{A_n}}{\|c\|_\infty + \varepsilon}\right) \\ &\leq I_\Phi\left(\sum_{n=1}^{\infty} a(\Phi)\chi_{A_n}\right) = 0 \leq \|c\|_\infty + \varepsilon. \end{aligned}$$

Moreover, the inequality $\|Pc\|_\Phi \geq \|c\|_\infty$ for any $c \in \ell^\infty$ can be proved in the same way as for the operator P_ε above. □

Remark 1 Note that if E is a Köthe space E and E^s is its s-concavification with $0 < s < 1$, then $(E^s, \|\cdot\|_{E^s})$ contains an order-isometric copy of $(\ell^\infty)^s$ if and only if E contains an order-isometric copy of ℓ^∞.

The easy proof is omitted.

Remark 2 Let Φ be a non-decreasing Orlicz function and the Orlicz space $L^\Phi(\mu)$ over a non-atomic σ-finite measure space or over the counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be equipped with the Mazur–Orlicz F-norm and let $\Phi(b(\Phi)) = \infty$ (note that in the case of a non-atomic and σ-finite measure this condition is automatically implied by $\Phi \in \Delta_2$). If Φ satisfies the suitable Δ_2-condition, then for any $x \in L^\Phi(\mu)$ and any $\varepsilon > 0$, we have that $\|x\|_\Phi < \varepsilon$ if and only if $I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon$.

Proof Under the assumptions on Φ we know that $I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi$ for any $x \in L^\Phi(\mu)\backslash\{0\}$ (see [9, Corollary 2.17 and Lemma 2.18]). Assume that $x \in L^\Phi(\mu)\backslash\{0\}$ and $\|x\|_\Phi < \varepsilon$. Then $I_\Phi\left(\frac{x}{\varepsilon}\right) \le I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi < \varepsilon$. For $x = 0$ the equivalence is obvious.

Assume now that $I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon$ and $\|x\|_\Phi = \varepsilon$. Then, we obtain that $x \neq 0$ and $\varepsilon > I_\Phi\left(\frac{x}{\varepsilon}\right) = I_\Phi\left(\frac{x}{\|x\|_\Phi}\right) = \|x\|_\Phi = \varepsilon$, a contradiction, which proves that if $I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon$, then $\|x\|_\Phi < \varepsilon$. □

Remark 3 In Remark 2 the assumptions that Φ satisfies the suitable Δ_2-condition and that $\Phi(b(\Phi)) = \infty$ are necessary for the equivalence of the conditions $I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon$ and $\|x\|_\Phi < \varepsilon$ for any $\varepsilon > 0$ and any $x \in L^\Phi(\mu)\backslash\{0\}$ (equivalently, for any $x \in L^\Phi(\mu)$).

Proof Assume that $\Phi \notin \Delta_2$. Then, there exists $x \in L^\Phi(\mu)\backslash\{0\}$ such that $I_\Phi(x) < 1$ and $I_\Phi\left(\frac{x}{\lambda}\right) = \infty$ for any $\lambda \in (0,1)$, whence $\|x\|_\Phi = 1$, which proves the necessity of $\Phi \in \Delta_2$ for the equivalence mentioned in the remark. If the measure space is non-atomic, complete and σ-finite, then $b(\Phi) = \infty$ (so also $\Phi(b(\Phi)) = \infty$) is necessary for $\Phi \in \Delta_2$, so also for the equivalence. If μ is the counting measure on $2^{\mathbb{N}}$ and $\varepsilon := \Phi(b(\Phi)) < \infty$, then defining $x = b(\Phi)e_1$, we have

$$I_\Phi\left(\frac{2\varepsilon x}{2\varepsilon}\right) = I_\Phi(x) = \Phi(b(\Phi)) = \varepsilon < 2\varepsilon$$

and for any $\lambda \in (0,1)$, we get

$$I_\Phi\left(\frac{2\varepsilon x}{2\lambda\varepsilon}\right) = I_\Phi\left(\frac{x}{\lambda}\right) = \infty > 2\lambda\varepsilon,$$

whence $I_\Phi\left(\frac{2\varepsilon x}{2\varepsilon}\right) < 2\varepsilon$ and $\|2\varepsilon x\|_\Phi = 2\varepsilon$, so the necessity of the condition $\Phi(b(\Phi)) = \infty$ for the equivalence is also proved. □

Remark 4 Let μ be an arbitrary complete and σ-finite measure space. Then

(i) if Φ is a non-decreasing Orlicz function, then all the sets

$$B_{\Phi,\varepsilon} := \left\{x \in L^\Phi(\mu)\colon\ I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon\right\},$$

corresponding to all $\varepsilon > 0$, are open in the metric $\|\cdot\|_\Phi$-topology, where $\|\cdot\|_\Phi$ is the Mazur–Orlicz F-norm, if and only if Φ satisfies the suitable Δ_2-condition and $\Phi(b(\Phi)) = \infty$.

(ii) if Φ is an s-convex Orlicz function and the Orlicz space $L^\Phi(\mu)$ is considered with the s-homogeneous norm $\|\cdot\|_{\Phi,s}$ $(0 < s \le 1)$, then the modular unit ball $B_\Phi := \{x \in L^\Phi(\mu)\colon I_\Phi(x) < 1\}$ is open in the metric $\|\cdot\|_{\Phi,s}$-topology if and only if Φ satisfies the suitable Δ_2-condition and $\Phi(b(\Phi)) \ge 1$.

Proof We will present a proof only for case (i) because the proof for case (ii) is similar.

Sufficiency. Under our assumptions about Φ, by virtue of Remark 2, we know that $\|x\|_\Phi < \varepsilon$ if and only if $I_\Phi\left(\frac{x}{\varepsilon}\right) < \varepsilon$ for any $x \in L^\Phi(\mu)$ and any $\varepsilon > 0$. Since the ball $B_{\|\cdot\|_\Phi}(0, \varepsilon) = \{x \in L^\Phi(\mu) \colon \|x\|_\Phi < \varepsilon\}$ is an open set under the metric topology in $L^\Phi(\mu)$ generated by the F-norm $\|\cdot\|_\Phi$ and under our assumptions we have $B_{\Phi,\varepsilon} = B_{\|\cdot\|_\Phi}(0, \varepsilon)$, the desired result follows.

Necessity. Assume that Φ does not satisfy the suitable Δ_2-condition or $\Phi(b(\Phi)) < \infty$. Then, by the proofs of Theorems 2 and 3, there exist $\varepsilon > 0$ and $x \in L^\Phi(\mu)\backslash\{0\}$ such that $I_\Phi(x) < \varepsilon$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$. Assume that $B_{\Phi,\varepsilon}$ is an open set in the F-norm topology. Since $\varepsilon x \in B_{\Phi,\varepsilon}$, there exists $\delta > 0$ such that $B_{\|\cdot\|_\Phi}(x, \delta) \subset B_{\Phi,\varepsilon}$. Since $\|\lambda_n \varepsilon x - \varepsilon x\|_\Phi \to 0$ for any sequence $(\lambda_n)_{n=1}^\infty$ in $\mathbb{R}_+$ such that as $\lambda_n > 1$ for any $n \in \mathbb{N}$ and $\lambda_n \to 1$ as $n \to \infty$, there exists $\sigma > 1$ such that $\sigma\varepsilon x \in B_{\|\cdot\|_\Phi}(x, \delta)$, whence $\sigma\varepsilon x \in B_{\Phi,\varepsilon}$. Consequently, $I_\Phi(\sigma x) = I_\Phi\left(\frac{\sigma\varepsilon x}{\varepsilon}\right) < \varepsilon$, which contradicts the fact that $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$.

Note Let us note that from Remarks 2, 3 and 4 as well as from their proofs, it follows that if $b(\Phi) = \infty$, then given any $\varepsilon > 0$, the openness of the set $B_{\Phi,\varepsilon}$ in the F-norm $\|\cdot\|_\Phi$ topology is equivalent to the fact that Φ satisfies the suitable Δ_2-condition.

Remark 5 Consider the same Orlicz spaces as in Theorems 2 and 3 and define, for any $x \in L^\Phi(\mu)$, the function $f_{\Phi,x}(\lambda) := I_\Phi(\lambda x)$ for all $\lambda \in \mathbb{R}_+$. Then, for all cases with respect to the kinds of the generating Orlicz functions as well as for both kinds of the measure spaces, the function $f_{\Phi,x}(\cdot)$ is continuous on $\mathbb{R}_+$ if and only if $b(\Phi) = \infty$ and Φ satisfies suitable Δ_2 condition.

Proof We can restrict ourselves to $x \in L^\Phi(\mu)\backslash\{0\}$ only. The sufficiency follows from Lemma 4.1 in [9].

Necessity. Assuming that either $b(\Phi) < \infty$ or $b(\Phi) = \infty$ and Φ does not satisfy the suitable condition Δ_2, one can find $x \in L^\Phi(\mu)\backslash\{0\}$ such that $I_\Phi(x) < \infty$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$ (see the proofs of Theorems 2 and 3). This means that the function $f_{\Phi,x}(\cdot)$ is not then continuous at the point $\lambda_0 = 1$. $\square$

4 Problems of order copies of $L_p(\nu)$ in F-normed Orlicz spaces

Throughout this section, (Ω, Σ, μ) and (Γ, Θ, ν) denote σ-finite and complete measure spaces with μ, ν non-atomic on Σ and Θ, respectively, or the counting measure on $2^{\mathbb{N}}$. The lattices considered here are *real* (in particular, this assumption deals with the classical space $L^p(\nu)$, $0 < p \le 1$).

Our first result is a more detailed version than the known polar decomposition theorem, published in 1983 by Abramovich [1, Propositions A, B, C], and extended subsequently to more general settings by Abramovich et al. [2, Section 3], Grobler and Huijsmans [15], and Boulabiar and Buskes [4]. The result is included implicitly in the proof of [36, Theorem 1] for the case when X, Y are Banach lattices. For the convenience of the reader, we outline its proof focusing on an argument showing that the involution S, decomposing an operator T, is an isometry. Here we consider a general case and we work on the Dedekind completion of an F-lattice Y (instead of the bidual Y^{**} [36, pp. 5–6], which may be trivial for Y non-Banach), but in the proof for Köthe function F-lattices, which are just Dedekind complete, such assumption is superfluous.

Proposition 1 *Let X, Y be two F-lattices, and let $T: X \to Y$ be an operator preserving disjointness. Then its modulus $|T|(= T \vee -T)$ exists in $\mathcal{L}_r(X, Y)$, it is an order homomorphism $X \to Y$, and there is an isometric involution S on the closed sublattice of Y generated by $T(X)$ such that $T = S|T|$ and $|T| = ST$ (showing that S acts on Y with $|S| =$ identity on Y when T is surjective).*

Moreover, if T is a surjective isometry then $|T|$ is an order surjective isometry, too.

Proof Similarly as in the proof of [36, Theorem 1], by [1, Proposition B] and [12, Theorem], T has the decomposition $T = T^+ - T^-$, where T^+, T^- are continuous lattice homomorphisms $X \to Y$, and $|T| = T^+ + T^-$ is a lattice homomorphism. We have

$$T^{\pm}x = (Tx)^{\pm} \text{ for all } x \geq 0 \text{ and } |T|(|x|) = |Tx| \text{ for all } x \in X. \tag{9}$$

Since the sublattices $B_{\pm} := T^{\pm}(X)$ of Y are disjoint, the bands $(B_+)^{dd}$ and $(B_-)^{dd}$ in the Dedekind completion $\widehat{Y}$ of Y are also disjoint. Let Q denote the order projection from $\widehat{Y}$ onto $(B_+)^{dd}$, and let I be the identity operator on $\widehat{Y}$. Since the projections Q and $I - Q$ are evidently disjoint, the operator J on $\widehat{Y}$ of the form $J := 2Q - I$ is an involution. Further, from the form of Q, we obtain

$$QT^+x = T^+x \text{ and } QT^-x = 0 \text{ for all } x \in X. \tag{10}$$

Since for every $y \in \widehat{Y}$ the elements Qy and $(I - Q)y$ are disjoint, we obtain

$$|Jy| = |Qy - (I - Q)y| = |Qy + (I - Q)y| = |y|, \; y \in \widehat{Y}. \tag{11}$$

Now, from (10) and the equality $J^2 = I$, we get the formulas $JT = |T|$ and $J|T| = T$. Let S denote the restriction of J to the sublattice H of Y generated by $T(X)$. From (11) we obtain that S is an isometric involution on H with $|S| =$ identity on H, and from the latter formulas, we obtain the required identities $T = S|T|$ and $|T| = ST$.

Moreover, by (9), the modulus $|T|$ is an isometry whenever T is, and the above identities imply that T and $|T|$ are surjective simultaneously. □

Proposition 2 *Let $Y = (Y, \| \|)$ be a strictly monotone F-lattice, let (Γ, Θ, ν) be an arbitrary complete measure space, and let $p \in (0, 1]$ be fixed. We consider $X := L^p(\nu)$ endowed with the standard p-homogeneous F-norm $\| \|_p$. If $T: L^p(\nu) \to Y$ is an isometry then:*

(i) *T preserves disjointness,*
(ii) *Y contains a lattice-isometric copy U_p of $L^p(\nu)$ (more exactly, $U_p = |T|(L^p(\nu))$, where the operator $|T|: L^p(\nu) \to Y$ fulfills the second identity in (9), and $|T|$ is a lattice isometry), and*
(iii) *The spaces $V_p := T(L^p(\nu))$ and U_p are isometric via a mapping $S: V_p \to U_p$ of the form $STx = |T|x$, where S is an isometric involution defined in the proof of Proposition 1.*

Proof The proof of part (i) goes the same lines as in the proof of part (a′) in [36, Theorem 1′], where the lattice $L^1(\nu)$ should be replaced by $3L^p(\nu)$. For the convenience of the reader, we present its F-lattice modification. Let x, y be two disjoint elements of $L^p(\nu)$, i.e. $|x \pm y| = |x| + |y|$. Then

$$\|Tx \pm Ty\| = \|x \pm y\|_p = \|\,|x| + |y|\,\|_p = \|x\|_p + \|y\|_p = \|Tx\| + \|Ty\|. \tag{12}$$

From (12) and the inequality $|Tx \pm Ty| \le |Tx| + |Ty|$ we obtain the identity $\|\,|Tx \pm Ty|\,\| = \|\,|Tx| + |Ty|\,\|$. Hence, since Y is strictly monotone, we obtain that $|Tx + Ty| = |Tx - Ty| (= |Tx| + |Ty|)$, which implies that the elements Tx and Ty are disjoint. Thus T preserves disjointness.

Now parts (ii) and (iii) follow easily from Proposition 1 (cf. the Banach lattice case [36, p. 6]). □

From Proposition 2 we immediately obtain the following, more general version of Corollary 2 from [36], proved for $p = 1$ in the Banach lattice-case.

Corollary 1 *Let Y be a strictly monotone F-lattice, and let $p \in (0, 1]$ be fixed. Then the following two conditions are equivalent.*

(i) *Y contains an isometric copy of $L^p(\nu)$.*
(ii) *Y contains a lattice-isometric copy of $L^p(\nu)$.*

Motivated by the above corollary, we shall present our main results. In the first theorem, we study the possibility of the existence of order-isometric copies of $L^p(\nu)$ in the general case for $L^\Phi(\mu)$, and next, we apply Corollary 1 to the order continuous part $E^\Phi(\mu)$ of $L^\Phi(\mu)$.

Theorem 4 *Let Φ be a non-decreasing Orlicz function and $p \in (0, 1]$. Then $L^\Phi(\mu)$, endowed with the Mazur–Orlicz F-norm, does not contain an order-isometric copy of $\ell^p_{(2)}$. Consequently, $L^\Phi(\mu)$ does not contain an order-isometric copy of $L^p(\nu)$ for every measure space (Γ, Θ, ν), either.*

Proof Fix $p \in (0, 1]$, and let e_1, e_2 denote the standard unit vectors in $\ell^p_{(2)}$. Let us note that every lattice isometry $T : \ell^p_{(2)} \to L^\Phi(\mu)$ is of the form $T(e_i) = x_i, i = 1, 2$, where the elements x_1, x_2 are disjoint and of F-norm 1, with $\|ax_1 + bx_2\|_\Phi = |a|^p + |b|^p$ for all real numbers a, b. We will show that there are no such x_1, x_2 in $L^\Phi(\mu)$.

Let x, y be two arbitrary disjoint and positive elements in $L^\Phi(\mu)$ with $\|x\|_\Phi = \|y\|_\Phi = 1$. We will consider two cases separately.
Case $p = 1$. For two real numbers $a, b > 2$, choose $\varepsilon \in (0, 1)$ such that, for $\lambda := (1 - \varepsilon)(a + b)$, we have

$$a/\lambda < 1 \quad \text{and} \quad b/\lambda < 1. \tag{13}$$

Then, since $\lambda/a > 1$, $\lambda/b > 1$ and $a, b > 2$, from the definition of the Mazur–Orlicz F-norm, we obtain

$$I_\Phi\left(\frac{ax + by}{\lambda}\right) = I_\Phi(ax/\lambda) + I_\Phi(bx/\lambda) \le \frac{\lambda}{a} + \frac{\lambda}{b} < \lambda.$$

Hence

$$\|ax + by\|_\Phi \leq \lambda = (1-\varepsilon)(a+b) < (a+b). \tag{14}$$

Notice also that if $a = b > 2$ then, setting $\varepsilon = 0.5 + \delta$ with $\delta \in (0, 0.5)$ arbitrary fixed, we obtain that $\|ax + by\|_\Phi \leq (0.5 + \delta)2a$, whence $\|a(x + y)\|_\Phi \leq a$.

Case $0 < p < 1$. Now, let $a, b \in \mathbb{R}$ with $2 < a < b < 2^{1+p}$. Notice first that $\frac{b}{a^p+b^p} < \frac{2^{1+p}}{2a^p} < 1$. Hence, we can choose $\varepsilon \in (0, 1)$ such that for $\lambda := (1-\varepsilon)(a^p + b^p)$, we have

$$a/\lambda < b/\lambda < 1. \tag{15}$$

Then, similarly as in Case $p = 1$, we obtain that $I_\Phi(\frac{ax+by}{\lambda}) < \lambda$, whence $\|ax + by\|_\Phi \leq \lambda = (1-\varepsilon)(a^p + b^p) < (a^p + b^p)$.

Thus, we have obtained that, in both cases, inequality (14) holds for the suitable $a, b > 2$ and $\varepsilon \in (0, 1)$. Consequently, the lattice isomorphism $\ell^p_{(2)} \ni (a, b) \mapsto ax + by \in E_\Phi(\mu)$, with x, y pairwise disjoint and of F-norm 1, is not an isometry. This proves the first part of the theorem, and the second part is now obvious. □

The corollary below is an immediate consequence of Theorem 4, Corollary 1 and the fact that $E^\Phi(\mu)$ is strictly monotone for Φ strictly increasing [19, Corollary 6.7].

Corollary 2 *Let an Orlicz function Φ be strictly increasing, and suppose that the (strictly monotone) order continuous part $E^\Phi(\mu)$ of $L^\Phi(\mu)$ is nontrivial. Then $E^\Phi(\mu)$ does not contain an isometric copy of $L^p(\nu)$ for every $p \in (0, 1]$.*

Remark 6 There exist Orlicz function spaces $L^\Phi(\mu)$ containing latices-isomorphic copies of L^p for a fixed $0 < p \leq 1$ (see Theorems A and A′ in [16]).

4.1 Remarks on isometries preserving disjointness

In the previous section, we studied the form of an isometry T acting between two strictly monotone F-lattices by showing that if the domain of T is equal to $L^p(\nu)$ for some $p \in (0, 1]$, then T preserves disjointness. Let us consider the following problem:

(P) *Let X and Y be two F-lattices with X strictly monotone, and suppose that X and Y are linearly isometric. Does it follow that Y is strictly monotone, too? In other words, does every surjective isometry preserve strict monotonicity?*

Problem (P) is suggested by many results showing that, if X is a real rearrangement invariant (r.i.) Banach function space on [0, 1] [a r.i. sequence space, resp.],[1] not isometrically equal to $L^2[0, 1]$, then every surjective isometry $T: X \to X$ preserves disjointness (see [5], [23, Theorem 1], [31,33] and the references therein; cf. [13]). In either of these cases, applying our Proposition 1, problem (P) has a positive answer.

However, in 1998 Randrianantoanina [32, Theorem 4 and Remark on p. 324] obtained a somewhat surprising result: *if X and Y are real Banach spaces with 1-unconditional bases $(e_i)_{i=1}^d$ and $(f_i)_{i=1}^d$, respectively*, $2 \leq d \leq \infty$, *and either X or Y is r.i.* (in this case all autoisometries of the underlined space preserve disjointness), *then an isometry $T: X \to Y$ need not be disjointness-preserving: there is a subset A*

[1] Such an X may be regarded as a Banach lattice [26, pp. 29–30].

of $\mathbf{N}$ *such that the restricted operator* $T_{|[e_i]_{i\in A}}$ *preserves disjointness and* $T_{|[e_i]_{i\in A^c}}$ *does not* (here A may be empty or $A = \mathbf{N}$, and $A^c := \mathbf{N}\backslash A$). *More exactly, for a pair* $i, j \in A^c, i \neq j$, *there are* $\varepsilon_i, \delta_i = \pm 1$ *and distinct* $k, l \in \mathbf{N}$ *such that*

$$T(e_i) = \frac{\delta_i}{\|f_k + f_l\|}(f_k + \varepsilon_i f_l) \text{ and } T(e_j) = \frac{\delta_i}{\|f_k + f_l\|}(f_k - \varepsilon_i f_l). \tag{16}$$

The following two-dimensional example of X, Y and an isometry $T_0\colon X \to Y$ illustrates well the above-mentioned result of B. Randrianantoanina, and extends easily to the infinite-dimensional case (the isometry T_0 was defined in 1976 by Lacey and Wojtaszczyk [25], cf. [14, Example 9.5.9]).

Example 1 Let us consider two real spaces $U = \ell^1_{(2)}$ and $V = \ell^\infty_{(2)}$ endowed with their natural p-norms with $p = 1$ and $p = \infty$, respectively, and let e_i and f_i, $i = 1, 2$ denote the natural unit vectors in the respective spaces. Notice that both spaces are also Banach lattices, U is strictly monotone while V is not. Let T_0 denote the operator acting from U onto V of the form $T_0(a, b) = (a+b, a-b)$. Since $\max\{|a+b|, |a-b|\} = |a|+|b|$, T_0 is an isometry. Moreover, $T_0(e_1) = f_1 + f_2$ and $T_0(e_2) = f_1 - f_2$, and thus equations (16) hold with $\varepsilon_1 = \delta_1 = 1$.

Now let X and Y denote the ℓ^1-sums of U and V, respectively: $X = (\oplus_{s=1}^\infty U_s)_1$, $Y = (\oplus_{s=1}^\infty V_s)_1$, where $U_s = U$ and $V_s = V$ for all indices s. Then the natural norms on X and Y are monotone, whence X and Y are Banach lattices, X is strictly monotone and order-isometric to ℓ^1. It is now plain that the operator $T\colon X \to Y$ of the form

$$T((x_s)_{s=1}^\infty) := \big(T_0(x_s)_{s=1}^\infty\big)$$

is a surjective isometry that does not preserve disjointness. It also has the form (16) for $i = k = 2t-1$ and $j = l = 2t$, $t = 1, 2, \ldots$, where e_i and f_k denote the respective unit vectors in X and Y.

Thus, we have constructed in Y, a highly non-strictly monotone, an isometric copy Y of (the strictly monotone) ℓ^1.

The above construction can be modified slightly to obtain a similar example within the class of non-Banach F-lattices as follows. For $p \in (0, 1)$, let X_p and Y_p denote the ℓ^p sums of U and V, respectively: $X_p = (\oplus_{s=1}^\infty U_s)_p$, $Y_p = (\oplus_{s=1}^\infty V_s)_p$, endowed with their natural F-norms $\|(w_s)\|_p = \sum_{s=1}^\infty \|w_s\|^p$, $w_s \in U_s$ for all $s \in \mathbb{N}$, or $w_s \in V_s$ for all $s \in \mathbb{N}$. Then X_p and Y_p are isometric, X_p is strictly monotone and Y_p is not.

Remark 7 The above example shows us that if the targeted space Y *is not* strictly monotone (and X is), then a surjective isometry $T\colon X \to Y$ may not be disjointness preserving. However, if X is an AL^p-space (i.e., an order isometric copy of some $L^p(\nu)$), $0 < p \leq 1$, and Y is strictly monotone then, by Proposition 2, T preserves disjointness (whence, by Proposition 1, the modulus $|T|\colon X \to Y$ is a surjective isometry).

Remark 8 There is a class of Banach lattices where problem (P) has a positive answer. It is known that if a Banach lattice X is uniformly rotund (= uniformly convex), then it is strictly monotone; hence, since isometries preserve uniform rotundity, they preserve (in this case) additionally strict monotonicity (see [18] for details).

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