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A sharp estimate for Muckenhoupt class A_∞ and BMO

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Abstract
A classical fact in the weighted theory asserts that a weight w belongs to the Muckenhoupt class A_∞ if and only if its logarithm $\log w$ is a function of bounded mean oscillation. We prove a sharp quantitative version of this fact in dimension one: for a weight w defined on some interval $J \subset \mathbb{R}$, we provide best lower and upper bounds for the BMO norm of $\log w$ in terms of A_∞ characteristics of w. The proof rests on the precise evaluation of associated Bellman functions.

Keywords Weight · BMO · Best constant · Bellman function

Mathematics Subject Classification Primary 46B20; Secondary 46A35

1 Introduction

A real-valued locally integrable function φ defined on $\mathbb{R}^n$ is said to be in BMO, the space of functions of bounded mean oscillation, if

$$\sup_Q \left\langle |\varphi - \langle\varphi\rangle_Q| \right\rangle_Q < \infty. \tag{1.1}$$

Here the supremum is taken over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes and

$$\langle\varphi\rangle_Q = \frac{1}{|Q|}\int_Q \varphi(x)\mathrm{d}x$$

denotes the average of φ over Q. The BMO class, introduced by John and Nirenberg in [7], plays an important role in analysis and probability, since many classical operators

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(maximal, singular integral, etc.) map L^∞ into BMO. Another remarkable result, due to Fefferman [4], asserts that BMO is a dual to the Hardy space H^1. It is well-known that the functions of bounded mean oscillation have very strong integrability properties (see e.g. [7]). In particular, the p-oscillation

$$||\varphi||_{BMO^p} := \sup_Q \langle |\varphi - \langle\varphi\rangle_Q|^p\rangle_Q^{1/p}, \quad 1 < p < \infty,$$

is finite for any $\varphi \in BMO$. It turns out that $||\cdot||_{BMO^p}$ forms an equivalent quasinorm on $BMO(\mathbb{R}^n)$. In what follows, we will work with $||\cdot||_{BMO^2}$ and denote it simply by $||\cdot||_{BMO}$. One of the reasons we choose this particular norm is that we have the identity

$$||\varphi||_{BMO^2} = \sup_Q \left\{\langle\varphi^2\rangle_Q - \langle\varphi\rangle_Q^2\right\}^{1/2}, \tag{1.2}$$

which makes the norm very convenient to handle. We will also restrict ourselves to the localized setting. That is, if $\mathcal{Q} \subset \mathbb{R}^n$ is a fixed cube with edges parallel to the coordinate axes, we define the class $BMO(\mathcal{Q})$ as the collection of all integrable $\varphi : \mathcal{Q} \to \mathbb{R}$ for which the expression (1.2) is finite, this time the supremum being taken over all cubes $Q \subseteq \mathcal{Q}$ with edges parallel to the coordinate axes.

There is a well-known connection between BMO and weights satisfying the so-called A_∞ condition of Muckenhoupt [9]. In what follows, the word 'weight' refers to a nonnegative, locally integrable function on some base space $\mathcal{B}$ (which is typically $\mathbb{R}^n$ or some cube $\mathcal{Q} \subset \mathbb{R}^n$). Following [8], we say that a weight w satisfies the condition A_∞ (or belongs to the class $A_\infty(\mathcal{B})$), if

$$[w]_{A_\infty(\mathcal{B})} = \sup_Q \langle w\rangle_Q \exp(-\langle\log w\rangle_Q) < \infty,$$

where, as above, the supremum is taken over all cubes Q contained in the base space $\mathcal{B}$, having edges parallel to the coordinate axes. The aforementioned connection between the space BMO and the class A_∞ can be (a little informally) stated as

$$BMO = \log A_\infty.$$

More precisely, a weight w satisfies the condition A_∞ if and only if $\log w$ belongs to the class BMO. The principal goal of this paper is to provide a sharp quantitative version of this result in the case $n = 1$. In this particular setting, the cubes become intervals; to stress that we work in the one-dimensional case, we will use the letters I, J instead of Q.

Here is our main result. For a given $c \geq 1$, let $d_\pm = d_\pm(c)$ be the constants introduced in Lemma 3.1 below.

Theorem 1.1 *Let $J \subset \mathbb{R}$ be a given interval. If w is a weight on J, then we have*

$$-d_-([w]_{A_\infty(J)}) \leq ||\log w||_{BMO(J)} \leq d_+([w]_{A_\infty(J)}). \tag{1.3}$$

Both estimates are sharp: for any $c \geq 1$ *there are weights* $w_\pm$ *satisfying* $[w_\pm]_{A_\infty} = c$ *such that* $||\log w_\pm||_{BMO} = \pm d_\pm(c)$.

Actually, we will prove more. Our approach will rest on the so-called Bellman function method, a powerful technique which is now used widely in various contexts of analysis and probability theory. Roughly speaking, this technique enables to extract the optimal constants in a given estimate from the existence of a certain special function enjoying an appropriate size condition and concavity. The method originates from the theory of optimal control (the theory of dynamic programming): see [1]. The connection of this approach to the estimates for martingale transforms was observed in the eighties by Burkholder [3] and then it was exploited in the study of other semimartingale inequalities (consult [12] for an overview of the results in this direction). A decisive step towards the application of the Bellman function method to general problems of harmonic analysis was made by Nazarov et al. [11] (see also [10]). Since then, the technique has been successfully applied in numerous settings: see e.g. [2,5,6,13–17] and consult the references therein.

We will identify the explicit formula for the Bellman functions associated with the double inequality (1.3). As usual in this type of problems, the argumentation splits naturally into two parts. The first part, which contains an *informal reasoning* leading to the discovery (or the *guess*) of the Bellman functions, is presented in the next section. The formal verification that these guessed objects are indeed the desired Bellman functions, is the contents of Sect. 3.

2 Associated Bellman functions

The purpose of this section is to rephrase both estimates in (1.3) in the language of the associated Bellman functions and to describe the main steps which lead to the discovery of these crucial objects. For the reader's convenience, we split the reasoning into several separate parts.

Step 1. Geometric interpretation of A_∞ *weights* We follow the work [17] by Vasyunin; the reader can also find in that paper the related interpretation for A_p weights in the range $1 \leq p < \infty$. For a given $c \geq 1$, introduce the domain

$$\Omega_c = \left\{(x, y) \in (0, \infty) \times \mathbb{R} \,:\, y \leq \log x \leq y + \log c\right\}.$$

Then a weight $w : J \to \mathbb{R}_+$ satisfies $[w]_{A_\infty(J)} \leq c$ if and only if for any interval $I \subseteq J$ we have $(\langle w\rangle_I, \langle \log w\rangle_I) \in \Omega_c$. Indeed, the inequality $\log\langle w\rangle_I \leq \langle \log w\rangle_I + \log c$ is guaranteed directly by the A_∞ condition, while the left estimate $\langle \log w\rangle_I \leq \log\langle w\rangle_I$ is a mere consequence of Jensen's inequality.

We will require the following well-known fact (we include an easy proof for the sake of completeness).

Lemma 2.1 *For any* $(x, y) \in \Omega_c$ *and any interval* J*, there is a weight* $w : J \to \mathbb{R}_+$ *satisfying* $[w]_{A_\infty(J)} \leq c$ *such that* $\langle w\rangle_J = x$ *and* $\langle \log w\rangle_J = y$.

Proof If $y = \log x$, then the constant weight $w \equiv x$ does the job. Suppose then, that $y < \log x$. Let PR be a line segment, passing through (x, y), tangent to the lower boundary of Ω_c, with P, R lying on the upper boundary of this set (i.e., such that $P_y = \log P_x$, $R_y = \log R_x$). (If (x, y) does not belong to the lower boundary, then there are two such line segments; take any of them). Then PR is entirely contained in Ω_c and there is a number $\alpha \in (0, 1)$ such that $\alpha P + (1-\alpha)R = (x, y)$. Split the interval J into two subintervals $J_\pm$ such that $|J_-| = \alpha|J|$ and $|J_+| = (1-\alpha)|J|$, and define $w = P_x\chi_{J_-} + R_x\chi_{J_+}$. Then

$$\langle w\rangle_J = \alpha P_x + (1-\alpha)R_x = x,$$
$$\langle \log w\rangle_J = \alpha \log P_x + (1-\alpha)\log R_x = \alpha P_y + (1-\alpha)R_y = y$$

and hence it is enough to check that $[w]_{A_\infty(J)} \leq c$. To this end, pick an arbitrary subinterval I of J and note that

$$\langle w\rangle_I = \frac{|I \cap J_-|}{|I|}P_x + \frac{|I \cap J_+|}{|I|}R_x$$

and

$$\langle \log w\rangle_I = \frac{|I \cap J_-|}{|I|}\log P_x + \frac{|I \cap J_+|}{|I|}\log R_x = \frac{|I \cap J_-|}{|I|}P_y + \frac{|I \cap J_+|}{|I|}R_y.$$

This means that $(\langle w\rangle_I, \langle \log w\rangle_I)$ lies on the line segment PR, and hence is contained in Ω_c. This is exactly what we need.

Let us briefly note that if (x, y) does not lie on the upper boundary of Ω_c, then the weight w constructed above actually satisfies the *equality* $[w]_{A_\infty(J)} = c$. Indeed, it follows from Darboux property that if I is chosen appropriately, then the average $(\langle w\rangle_I, \langle \log w\rangle_I)$ is the point of tangency of PR to the lower boundary of Ω_c. This amounts to saying that $\langle w\rangle_I \exp(-\langle \log w\rangle_I) = c$, which gives the desired reverse estimate $[w]_{A_\infty(J)} \geq c$. □

Step 2. Abstract Bellman functions Fix $c \geq 1$, an interval $J \subset \mathbb{R}$ and consider the functions $\mathbb{B}^c_\pm : \Omega_c \to \mathbb{R}$ given by the abstract formulas

$$\mathbb{B}^c_-(x, y) = \inf\left\{\langle \log^2 w\rangle_J\right\}, \qquad \mathbb{B}^c_+(x, y) = \sup\left\{\langle \log^2 w\rangle_J\right\}, \tag{2.1}$$

where the infimum (supremum) is taken over all weights $w \in A_\infty(J)$ such that $[w]_{A_\infty(J)} \leq c$, $\langle w\rangle_J = x$ and $\langle \log w\rangle_J = y$. By the previous lemma, $\mathbb{B}^c_\pm$ are well-defined objects on the whole Ω_c. It is clear that the above functions do not depend on the base interval J. Indeed, for any two intervals J_1 and J_2, there is an affine mapping putting J_1 onto J_2; such a mapping preserves the averages and puts the classes $A_\infty(J_1)$ and $A_\infty(J_2)$ in one-to-one correspondence. To describe the relation between $\mathbb{B}^c_\pm$ and (1.3), observe that

$$\mathbb{B}^c_-(x,y) - y^2 = \inf \left\{ \langle \log^2 w \rangle_J - \langle \log w \rangle_J^2 \right\},$$

$$\mathbb{B}^c_+(x,y) - y^2 = \sup \left\{ \langle \log^2 w \rangle_J - \langle \log w \rangle_J^2 \right\},$$

where the infimum and supremum are taken over the same class as previously. In particular, this shows that $d_+^2(c) = \sup_{(x,y)\in\Omega_c} \big(\mathbb{B}^c_+(x,y) - y^2\big)$. Indeed, given an arbitrary weight w with $[w]_{A_\infty(J)} = c$, we see that $[w]_{A_\infty(I)} \le c$ for any subinterval $I \subseteq J$ and hence

$$\langle \log^2 w \rangle_I - \langle \log w \rangle_I^2 \le \mathbb{B}^c_+(\langle w \rangle_I, \langle \log w \rangle_I) - \langle \log w \rangle_I^2 \le \sup_{(x,y)\in\Omega_c} \big(\mathbb{B}^c_+(x,y) - y^2\big).$$

Since I was arbitrary, this gives $\|\log w\|^2_{BMO} \le \sup_{(x,y)\in\Omega_c} \big(\mathbb{B}^c_+(x,y) - y^2\big)$ and hence also $d_+^2(c) \le \sup_{(x,y)\in\Omega_c} \big(B^c_+(x,y) - y^2\big)$, by taking the supremum over all w. To get the reverse bound, take $\varepsilon > 0$ and a point $(x_0, y_0) \in \Omega_c$ such that $\mathbb{B}^c_+(x_0, y_0) - y_0^2 \ge \sup_{(x,y)\in\Omega_c} \big(B^c_+(x,y) - y^2\big) - \varepsilon$. There is a weight w on J satisfying $[w]_{A_\infty} \le c$, $\langle w \rangle_J = x_0$, $\langle \log w \rangle_J = y_0$ and $\mathbb{B}^c_+(x_0, y_0) \le \langle \log^2 w \rangle_J + \varepsilon$. Putting all these properties together, we see that w satisfies

$$\|\log w\|^2_{BMO} \ge \langle \log^2 w \rangle_J - \langle \log w \rangle_J^2 \ge \sup_{(x,y)\in\Omega_c} \big(\mathbb{B}^c_+(x,y) - y^2\big) - 2\varepsilon.$$

This proves $d_+^2(c) \ge \sup_{(x,y)\in\Omega_c} \big(\mathbb{B}^c_+(x,y) - y^2\big)$, since ε was arbitrary. A similar reasoning establishes an analogous relation between $\mathbb{B}^c_-$ and $d_-(c)$. Thus, having found the explicit formulas for $\mathbb{B}^c_\pm$, we immediately extract the solution to our main problem. As we will see now, the identification of these formulas is possible due to certain structural properties of Bellman functions.

Step 3. Properties of Bellman functions Suppose that (x, y) belongs to the upper boundary of Ω_c. Then for any interval J, by Jensen's inequality, there is only one weight $w : J \to \mathbb{R}_+$ satisfying $\langle w \rangle_J = x$ and $\langle \log w \rangle_J = y$: the constant $w \equiv x$. Consequently, directly from the definition of $\mathbb{B}^c_\pm$, we have

$$\mathbb{B}^c_\pm(x,y) = y^2$$

on the upper boundary of Ω_c. The next observation is a certain homogeneity-type condition, which allows to express $\mathbb{B}^c_\pm$ in terms of some functions of one variable. Namely, fix a weight $w : J \to \mathbb{R}_+$ satisfying $[w]_{A_\infty(J)} \le c$, $\langle w \rangle_J = x$ and $\langle \log w \rangle_J = y$. Then for any $\lambda > 0$ we have $[\lambda w]_{A_\infty(J)} \le c$, $\langle \lambda w \rangle_J = \lambda x$ and $\langle \log(\lambda w) \rangle_J = y + \log \lambda$, so

$$\langle \log^2 w \rangle_J = \langle \log^2(\lambda w) \rangle_J - 2y \log \lambda - \log^2 \lambda \le \mathbb{B}^c_+(\lambda x, y + \log \lambda) - 2y \log \lambda - \log^2 \lambda.$$

Since w was arbitrary, this yields

$$\mathbb{B}^c_+(x,y) \le \mathbb{B}^c_+(\lambda x, y + \log \lambda) - 2y \log \lambda - \log^2 \lambda$$

and replacing x, y, λ by λx, $y+\log\lambda$ and λ^{-1}, respectively, gives the reverse bound. Consequently, setting $\lambda = x^{-1}$, we see that

$$\mathbb{B}_+^c(x,y) - y^2 = \mathbb{B}_+^c(1, y-\log x) - (y-\log x)^2 = \Phi_+^c(y-\log x), \tag{2.2}$$

for some function $\Phi_+^c : [-\log c, 0] \to \mathbb{R}$ to be found. The same reasoning shows that $\mathbb{B}_-^c(x,y) - y^2 = \Phi_-^c(y-\log x)$, for some unknown $\Phi_-^c : [-\log c, 0] \to \mathbb{R}$.

To find $\Phi_\pm^c$, we go back to the structure of the "whole" function $\mathbb{B}_\pm^c$. In a typical situation, Bellman functions are locally convex/concave on their domains (here the phrase "local convexity/concavity" means the convexity/concavity along any line segment contained in the domain of the function). Let us describe *informally* how the idea works for the function $\mathbb{B}_+^c$. Suppose that the line segment PR is entirely contained in Ω_c, $\alpha \in (0,1)$ is a fixed parameter and let $S = \alpha P + (1-\alpha)R$. Let w_- be a weight on the interval $[0,\alpha)$, satisfying $[w]_{A_\infty([0,\alpha))} \leq c$, $\langle w\rangle_{[0,\alpha)} = P_x$, $\langle \log w\rangle_{[0,\alpha)} = P_y$. Let w_+ be a weight on $[\alpha,1]$, satisfying $[w]_{A_\infty([\alpha,1])} \leq c$, $\langle w\rangle_{[\alpha,1]} = R_x$, $\langle w\rangle_{[\alpha,1]} = R_y$. Splicing these two weights into one weight w on $[0,1]$, we see that

$$\langle \log^2 w\rangle_{[0,1]} = \alpha\langle \log^2 w_-\rangle_{[0,\alpha)} + (1-\alpha)\langle \log^2 w_+\rangle_{[\alpha,1]}.$$

Suppose that the weight w satisfies $[w]_{A_\infty(0,1)} \leq c$ (unfortunately, this cannot be formally proved; however, let us proceed under this assumption). This would yield

$$\mathbb{B}_+^c(S) \geq \alpha\langle \log^2 w_-\rangle_{[0,\alpha)} + (1-\alpha)\langle \log^2 w_+\rangle_{[\alpha,1]}$$

and taking the supremum over all $w_\pm$ as above, we would obtain

$$\mathbb{B}_+^c(S) \geq \alpha\mathbb{B}_+^c(P) + (1-\alpha)\mathbb{B}_+^c(R),$$

i.e., the desired local concavity of $\mathbb{B}_+^c$. A similar heuristic argument indicates that $\mathbb{B}_-^c$ should be locally convex.

Before we proceed, let us make a crucial comment. We expect the Bellman functions $\mathbb{B}_\pm^c$ to yield the best constants in (1.3). In such a situation, one typically assumes that for each point (x,y) lying in the interior of the domain, the convexity/concavity assumption degenerates in some direction; that is, there is a (short) line segment passing through (x,y) along which the Bellman function is *linear*. As we shall see, this assumption leads to the key second-order differential equation for $\Phi_\pm^c$ which can be solved explicitly, and hence it identifies the formulas for $\mathbb{B}_\pm^c$.

Step 4. On the search of $\mathbb{B}_\pm^c$ We put all the above facts together. Let us *assume* that $\mathbb{B}_\pm^c$ are of class C^2. Then the local convexity/concavity can be reformulated in terms of the corresponding Hessian matrices. Furthermore, the aforementioned degeneration condition implies that the determinant of the Hessian must vanish at each point belonging to the interior of the domain. We compute that

$$D^2\mathbb{B}_\pm^c(x,y) = \begin{bmatrix} ((\Phi_\pm^c)''(t) + (\Phi_\pm^c)'(t))x^{-2} & -(\Phi_\pm^c)''(t)x^{-1} \\ -(\Phi_\pm^c)''(t)x^{-1} & (\Phi_\pm^c)''(t)+2 \end{bmatrix},$$

where, for brevity, we denoted $y - \log x$ by t. Then the requirement $\det D^2\mathbb{B}^c_\pm = 0$ yields the following ODE for $\Phi^c_\pm$:

$$(\Phi^c_\pm)''(t)((\Phi^c_\pm)'(t) + 2) + 2(\Phi^c_\pm)'(t) = 0.$$

This equation can be easily solved, taking into account the initial condition obtained from the behavior of $\mathbb{B}^c_\pm$ at the upper boundary of Ω_c. This gives us the *candidates* for $\Phi^c_\pm$, which in turn yield the candidates for the Bellman functions $\mathbb{B}^c_\pm$. These candidates, denoted from now on by $\Psi^c_\pm$ and $B^c_\pm$, will be explicitly introduced and studied in the next section.

3 Formal verification

Now we will present the formal proof of Theorem 1.1 and the rigorous identification of the explicit formulas for the Bellman functions $\mathbb{B}^c_\pm$. Throughout, $c \geq 1$ is a fixed parameter.

3.1 Special functions and their properties

Introduce the auxiliary function $G : (-1, \infty) \to \mathbb{R}$ by $G(x) = x - \log\big(c(1 + x)\big)$. One easily computes that $G'(x) = x/(1 + x)$, so G is strictly decreasing on $(-1, 0)$ and strictly increasing on $(0, \infty)$. Furthermore, one checks immediately that $\lim_{x\downarrow -1} G(x) = \lim_{x\uparrow\infty} G(x) = \infty$ and $G(0) = -\log c \leq 0$, so in particular we have the following fact, which we formulate as a separate statement.

Lemma 3.1 *There are unique $d_- = d_-(c) \in (-1, 0]$ and $d_+ = d_+(c) \in [0, \infty)$ such that*

$$d_\pm = \log\big(c(1 + d_\pm)\big). \tag{3.1}$$

Furthermore, the function G maps each of the intervals $[d_-, 0]$, $[0, d_+]$ monotonically onto $[-\log c, 0]$.

We are ready to introduce explicitly the functions $\Psi^c_\pm : [-\log c, 0] \to [0, \infty)$ we have constructed in the preceding section. These functions are defined by

$$\Psi^c_\pm(x) = d^2_\pm - (G^{-1}(x))^2,$$

where G^{-1} is the inverse to G (considered as a function from $[-\log c, 0]$ to $[d_-, 0]$ in the case of Ψ^c_-, and from $[-\log c, 0]$ to $[0, d_+]$ in the case of Ψ^c_+). In other words, the functions $\Psi^c_\pm$ satisfy

$$\Psi^c_-\big(x - \log\big(c(1 + x)\big)\big) = d^2_- - x^2, \qquad \text{for } x \in [d_-, 0], \tag{3.2}$$

$$\Psi^c_+\big(x - \log\big(c(1 + x)\big)\big) = d^2_+ - x^2, \qquad \text{for } x \in [0, d_+]. \tag{3.3}$$

The (candidates for) Bellman functions are given explicitly by

$$B_{\pm}^{c}(x,y)=\Psi_{\pm}^{c}(y-\log x)+y^{2}. \tag{3.4}$$

Let us verify formally the local convexity/concavity of these objects.

Lemma 3.2 *The Hessian matrix of B_{+}^{c} is nonpositive-definite. The Hessian matrix of B_{-}^{c} is nonnegative-definite.*

Proof Actually, we have already performed all the necessary calculations in the preceding section, but we rewrite the computations for the sake of completeness. Setting $t:=y-\log x$, the Hessian matrix of $B_{\pm}^{c}$ is given by

$$D^{2}B_{\pm}^{c}(x,y)=\begin{bmatrix}((\Psi_{\pm}^{c})''(t)+(\Psi_{\pm}^{c})'(t))x^{-2} & -(\Psi_{\pm}^{c})''(t)x^{-1}\\ -(\Psi_{\pm}^{c})''(t)x^{-1} & (\Psi_{\pm}^{c})''(t)+2\end{bmatrix}.$$

Observe that

$$\det D^{2}B_{\pm}^{c}(x,y)=x^{-2}\left[(\Psi_{\pm}^{c})''(t)((\Psi_{\pm}^{c})'(t)+2)+2(\Psi_{\pm}^{c})'(t)\right]=0,$$

because by (3.2) and (3.3),

$$(\Psi_{\pm}^{c})'\big(x-\log\big(c(1+x)\big)\big)=-2(1+x),\quad (\Psi_{\pm}^{c})''\big(x-\log\big(c(1+x)\big)\big)=-2(1+x)/x.$$

(Here the identities are valid for all $x\in[0,d_{+}]$ if Ψ_{+}^{c} is concerned; and for all $x\in[d_{-},0]$ if Ψ_{-}^{c} is considered). So, to establish the assertion of the lemma, it is enough to prove that $(\Psi_{+}^{c})''+(\Psi_{+}^{c})'$ is nonpositive and $(\Psi_{-}^{c})''+(\Psi_{-}^{c})'$ is nonnegative. But this follows immediately from the identity

$$(\Psi_{\pm}^{c})''\big(x-\log\big(c(1+x)\big)\big)+(\Psi_{\pm}^{c})'\big(x-\log\big(c(1+x)\big)\big)=-2(1+x)^{2}/x$$

(with the same restrictions on the range of x as above, depending on whether Ψ_{+}^{c} or Ψ_{-}^{c} is studied). □

We will also exploit the following size estimate for $B_{\pm}^{c}$.

Lemma 3.3 *We have $y^{2}\leq B_{\pm}^{c}(x,y)\leq d_{\pm}^{2}+y^{2}$ for all $(x,y)\in\Omega_{c}$.*

Proof As shown in the proof of the previous lemma, the functions Ψ_{+}^{c} and Ψ_{-}^{c} are decreasing on $[-\log c,0]$. Consequently,

$$B_{\pm}^{c}(x,y)\geq\Psi_{\pm}^{c}(0)+y^{2}=y^{2}\quad\text{and}\quad B_{\pm}^{c}(x,y)\leq\Psi_{\pm}^{c}(-\log c)+y^{2}=y^{2}$$

and we are done. □

3.2 Proof of the inequalities $\mathbb{B}_+^c \leq B_+^c$, $\mathbb{B}_-^c \geq B_-^c$ and (1.3)

The argument rests on the inductive use of the local convexity/concavity of the Bellman functions. We will need the following technical fact (see Lemma 4_∞ in [17]).

Lemma 3.4 *For any $\varepsilon > c$ and an arbitrary weight on J with $[w]_{A_\infty(J)} \leq c$ there exists a splitting $J = J^- \cup J^+$, $|J^\pm| = \alpha_\pm |J|$, such that the entire interval with the endpoints $p^\pm = (\langle w\rangle_{J^\pm}, \langle \log w\rangle_{J^\pm})$ is in Ω_ε. Moreover, the splitting parameters $\alpha_\pm$ can be chosen bounded away from* 0 *and* 1 *uniformly with respect to w and, therefore, with respect to J as well.*

Proof of $\mathbb{B}_+^c \leq B_+^c$. Fix an A_∞ weight w on J, satisfying $[w]_{A_\infty(J)} \leq c$, $\langle w\rangle_J = x$ and $\langle \log w\rangle_J = y$. Furthermore, pick an arbitrary $\varepsilon > c$.

Step 1. Consider the following family $\{\mathcal{J}^n\}_{n\geq 0}$ of partitions of J, generated by the inductive use of Lemma 3.4. We start with $\mathcal{J}^0 = \{J\}$; then, given $\mathcal{J}^n = \{J^{n,1}, J^{n,2}, \dots, J^{n,2^n}\}$, we split each $J^{n,k}$ according to Lemma 3.4, applied to the function w and the parameter ε. Finally, put

$$\mathcal{J}^{n+1} = \left\{J_-^{n,1}, J_+^{n,1}, J_-^{n,2}, J_+^{n,2}, \dots, J_-^{n,2^n}, J_+^{n,2^n}\right\}.$$

Next, we define the sequences $(f_n)_{n\geq 0}$, $(g_n)_{n\geq 0}$ of functions on J by

$$f_n(t) = \frac{1}{|J^n(t)|}\int_{J^n(t)} w(s)\mathrm{d}s, \qquad g_n(t) = \frac{1}{|J^n(t)|}\int_{J^n(t)} \log w(s)\mathrm{d}s.$$

Here $J^n(t) \in \mathcal{J}^n$ is an interval containing t; if there are two such intervals, we pick the one which has t as its right endpoint. Since $[w]_{A_\infty(J)} \leq c$, we have $(f_n(t), g_n(t)) \in \Omega_\varepsilon$ for each n and almost all $t \in J$.

Step 2. Let B_+^ε be the Bellman function studied in the previous section, corresponding to the parameter ε. We will prove that for any nonnegative integer n and any $J^{n,k} \in \mathcal{J}^n$ we have

$$\int_{J^{n,k}} B_+^\varepsilon(f_{n+1}(t), g_{n+1}(t))\,\mathrm{d}t \leq \int_{J^{n,k}} B_+^\varepsilon(f_n(t), g_n(t))\,\mathrm{d}t. \tag{3.5}$$

To this end, observe that the pair (f_n, g_n) is constant on $J^{n,k}$ (equal to $p = (\langle w\rangle_{J^{n,k}}, \langle \log w\rangle_{J^{n,k}})$ there), while (f_{n+1}, g_{n+1}) takes two values on this interval: $p_\pm = (\langle w\rangle_{J_\pm^{n,k}}, \langle \log w\rangle_{J_\pm^{n,k}})$). By Lemma 3.4, the entire interval with the endpoints $p_\pm$ is contained within Ω_ε, and hence B_+^ε is concave along this interval. It remains to note that (3.5) follows immediately from this concavity.

Step 3. Summing (3.5) and (3.7) over k, we get

$$\int_J B_+^\varepsilon(f_{n+1}(t), g_{n+1}(t))\,\mathrm{d}t \leq \int_J B_+^\varepsilon(f_n(t), g_n(t))\,\mathrm{d}t.$$

Consequently, for any nonnegative integer n we have

$$\begin{aligned}\frac{1}{|J|}\int_J B_+^\varepsilon(f_n(t),g_n(t))\,\mathrm{d}t &\le \frac{1}{|J|}\int_J B_+^\varepsilon(f_0(t),g_0(t))\,\mathrm{d}t\\ &= B_+^\varepsilon\left(\langle w\rangle_J,\langle \log w\rangle_J\right)=B_+^\varepsilon(x,y).\end{aligned}$$

Combining this estimate with Lemma 3.3, we get

$$\frac{1}{|J|}\int_J g_n^2(t)\mathrm{d}t \le B_+^\varepsilon\left(x,y\right). \tag{3.6}$$

However, recall that the splitting ratios $\alpha_\pm$ of Lemma 3.4 were bounded away from 0 and 1. Therefore, the diameter of $\mathcal{J}^n$ tends to 0 as $n\to\infty$, i.e., we have $\lim_{n\to\infty}\sup_{1\le k\le 2^n}|J^{n,k}|=0$. Consequently, by Lebesgue's differentiation theorem, we have $g_n(t)\to\log w(t)$ for almost all $t\in J$. By Fatou's lemma, (3.6) yields

$$\langle \log^2 w\rangle_J \le B_+^\varepsilon\left(x,y\right)$$

and since w was arbitrary, we get $\mathbb{B}_+^c(x,y)\le B_+^\varepsilon(x,y)$. It remains to let $\varepsilon\to c$ to get the desired bound. □

Proof of (1.3), *the right estimate.* Let w be an A_∞ weight on J with $[w]_{A_\infty(J)}\le c$. Then for any subinterval $I\subseteq J$, we have $[w]_{A_\infty(I)}\le c$ and hence, by Lemma 3.3 and the inequality $\mathbb{B}_+^c\le B_+^c$,

$$\langle \log^2 w\rangle_I \le \mathbb{B}_+^c(\langle w\rangle_I,\langle\log w\rangle_I)\le B_+^c(\langle w\rangle_I,\langle\log w\rangle_I)\le d_+^2(c)+\langle\log w\rangle_I^2.$$

Since I was arbitrary, this gives the desired upper bound for $\|\log w\|_{BMO(J)}$. □

Proof of the inequality $\mathbb{B}_-^c\ge B_-^c$. The argument is very similar to that used above, however, there are some small differences, so we have decided to write the proof separately. Let w be an A_∞ weight with $[w]_{A_\infty(J)}\le c$. Set $x=\langle w\rangle_J$, $y=\langle\log w\rangle_J$ and let $\varepsilon>c$. Construct the partitions $\{\mathcal{J}^n\}_{n\ge 0}$ of J and the functional sequences $(f_n)_{n\ge0}$, $(g_n)_{n\ge0}$ using the same formulas as previously. Let B_-^ε be the Bellman function corresponding to the parameter ε. Then Lemma 3.2 shows that for any n and k,

$$\int_{J^{n,k}} B_-^\varepsilon(f_{n+1}(t),g_{n+1}(t))\,\mathrm{d}t \ge \int_{J^{n,k}} B_-^\varepsilon(f_n(t),g_n(t))\,\mathrm{d}t \tag{3.7}$$

and hence, summing over k,

$$\int_J B_-^\varepsilon(f_{n+1}(t),g_{n+1}(t))\,\mathrm{d}t \ge \int_J B_-^\varepsilon(f_n(t),g_n(t))\,\mathrm{d}t.$$

This proves that for any $n\ge 0$ we have

$$\int_J B_-^\varepsilon(f_n(t),g_n(t))\,\mathrm{d}t \ge \int_J B_-^\varepsilon(f_0(t),g_0(t))\,\mathrm{d}t=|J|B_-^\varepsilon(x,y). \tag{3.8}$$

Now we use Lebesgue's differentiation theorem and Lebesgue's dominated convergence theorem. The function B_-^ε is continuous on Ω_ε, so we have

$$B_-^\varepsilon(f_n(t), g_n(t)) \xrightarrow{n\to\infty} B_-^\varepsilon(w(t),\ \log w(t)) = \log^2 w(t)$$

for almost all $t \in J$. In addition, by Lemma 3.3,

$$0 \le B_-^\varepsilon(f_n(t), g_n(t)) \le d_-^2(\varepsilon) + g_n^2(t) \le d_-^2(\varepsilon) + \big(\mathcal{M}\log w(t)\big)^2,$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator. The expression on the right is integrable on J: indeed, $\log w$ is square-integrable as a function from the class BMO and $\mathcal{M}$ is L^2-bounded. Therefore, letting $n \to \infty$ in (3.8) yields, by Lebesgue's dominated convergence theorem,

$$\langle \log^2 w\rangle_J \ge B_-^\varepsilon(x, y).$$

Since w was arbitrary, this implies $\mathbb{B}_-^c(x, y) \ge B_-^\varepsilon(x, y)$. It remains to let $\varepsilon \to c$ to get the claim. □

Proof of (1.3), *the left estimate.* Pick a weight w on J such that $\log w$ belongs to the class BMO. Then w is an A_∞ weight. Set $c = [w]_{A_\infty}$; we may assume that $c > 1$, since otherwise there is nothing to prove (indeed, if $c = 1$, then w, and hence also $\log w$, are constant). Pick $c' \in (1, c)$ and choose $I \subset J$ such that $\langle w\rangle_I \exp(-\langle \log w\rangle_I) \ge c'$; such a choice is possible by the very definition of the A_∞ condition. Since $\mathbb{B}_-^c \ge B_-^c$, we get, by the very definition of B_-^c,

$$\begin{aligned}\langle \log^2 w\rangle_I &\ge \mathbb{B}_-^c(\langle w\rangle_I, \langle \log w\rangle_I)\\ &\ge B_-^c(\langle w\rangle_I, \langle \log w\rangle_I)\\ &= \Psi_-^c(\langle \log w\rangle_I - \log\langle w\rangle_I) + \langle \log w\rangle_I^2\\ &\ge \Psi_-^c(-\log c') + \langle \log w\rangle_I^2,\end{aligned}$$

where in the last passage we have exploited the fact that Ψ_-^c is a decreasing function. This shows that $\|\log w\|_{BMO(J)} \ge \Psi_-^c(-\log c')$ and it remains to let $c' \to c$ and note that $\Psi_-^c(-\log c) = -d_-(c)$ (see (3.2)). □

3.3 Sharpness of (1.3) and proofs of the inequalities $\mathbb{B}_+^c \ge B_+^c$, $\mathbb{B}_-^c \le B_-^c$

Let us split this subsection into two parts.

The constants $d_\pm(c)$ in (1.3) *cannot be improved.* Fix $c \ge 1$ and consider the weight w on $(0, 1)$ given by $w(s) = s^d$, where $d \in \{d_-(c), d_+(c)\}$. We will prove that

$$[w]_{A_\infty(0,1)} = c \qquad \text{and} \qquad \|\log w\|_{BMO(0,1)} = d, \tag{3.9}$$

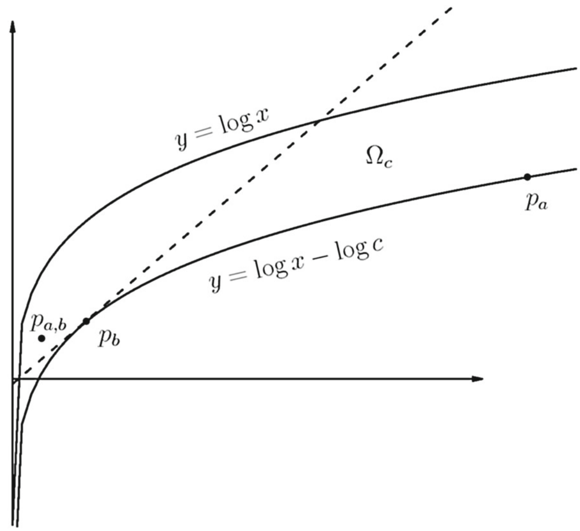

Fig. 1 The point $p_{a,b}$ belongs to Ω_c. Furthermore, it lies above the line tangent to the lower boundary of Ω_c at the point p_b

which clearly gives the announced sharpness of (1.3). To show the first part of (3.9), fix $a > 0$ and compute that

$$p_a = (x_a, y_a) := \left(\frac{1}{a}\int_0^a w(s)\mathrm{d}s, \frac{1}{a}\int_0^a \log w(s)\mathrm{d}s\right) = \left(\frac{a^d}{d+1}, d\log a - d\right). \tag{3.10}$$

We easily check that p_a lies at the lower boundary of Ω_c: $\log x_a = y_a + \log c$, or $\langle w\rangle_{(0,a)} \exp\big(-\langle \log w\rangle_{(0,a)}\big) = c$, directly by (3.1). Thus, the A_∞ constant of w is at least c and all we need is the estimate $[w]_{A_\infty(0,1)} \leq c$. We will check that

$$\langle w\rangle_{[a,b]} \exp(-\langle \log w\rangle_{[a,b]}) \leq c. \tag{3.11}$$

for any $a < b$. Consider the point

$$p_{a,b} = (x_{a,b}, y_{a,b}) := \left(\frac{1}{b-a}\int_a^b w(s)\mathrm{d}s, \frac{1}{b-a}\int_a^b \log w(s)\mathrm{d}s\right).$$

The estimate (3.11) is equivalent to showing that $p_{a,b} \in \Omega_c$. By Jensen's inequality, $p_{a,b}$ cannot lie above the upper boundary of Ω_c, i.e., we have $y_{a,b} \leq \log x_{a,b}$. To show that $y_{a,b} \geq \log x_{a,b} - \log c$, it suffices to note that the lower boundary of Ω_c is a graph of a concave function $y = \log x - c$ and

$$p_b = \frac{a}{b}p_a + \frac{b-a}{b}p_{a,b}.$$

This proves the first equality in (3.9); actually, it proves a little more (which will be useful later): the point $p_{a,b}$ lies above the line which is tangent to the lower boundary of Ω_c at the point p_b. See Fig. 1.

To show the second equality in (3.9), let $0 \leq a < b \leq 1$. We have

$$\langle \log w\rangle_{[a,b]} = \frac{1}{b-a}\int_a^b \log w(s)\mathrm{d}s = \frac{d}{b-a}(b\log(b/e) - a\log(a/e))$$

and similarly

$$\langle \log^2 w\rangle_{[a,b]} = \frac{d^2}{b-a}\big[b\log^2 b - a\log^2 a - 2(b\log(b/e) - a\log(a/e))\big]. \quad (3.12)$$

After some lengthy but rather straightforward computations, one obtains

$$\langle \log^2 w\rangle_{[a,b]} - \langle \log w\rangle_{[a,b]}^2 - d^2 = -\frac{d^2 ab\log^2(b/a)}{(b-a)^2} \leq 0.$$

Since a and b were arbitrary, we obtain $||\log w||_{BMO} \leq d$; furthermore, taking $a = 0$ we see that actually equality holds here. This is precisely the desired assertion. □

Proof of $\mathbb{B}_+^c \geq B_+^c$, $\mathbb{B}_-^c \leq B_-^c$. Fix $c \geq 1$, $d \in \{d_-(c), d_+(c)\}$, $\kappa \geq 1$ and consider a modification of the above weight, given by $w(s) = s^d\chi_{(0,1]}(s) + \chi_{[1,\kappa)}(s)$. This weight satisfies $[w]_{A_\infty(0,\kappa)} = c$. Indeed, we have $[w]_{A_\infty(0,\kappa)} \geq [w]_{A_\infty(0,1)} = c$ and, as we will prove now,

$$p_{a,b} = (x_{a,b}, y_{a,b}) := \left(\frac{1}{b-a}\int_a^b w(s)\mathrm{d}s, \frac{1}{b-a}\int_a^b \log w(s)\mathrm{d}s\right) \in \Omega_c \quad (3.13)$$

for any $0 \leq a < b \leq \kappa$. This is clear if $a \geq \kappa$ (then $p_{a,b} = (1,0)$), we have also checked the inclusion if $b \leq 1$. Therefore, it remains to check (3.13) for $a < 1 < b$. We have proved above that $p_{a,1}$ lies above the line which is tangent to the lower boundary of Ω_c at the point p_1 (where p_1 is given by (3.10)): see Fig. 1 above. The tangent line has the equation

$$y = (d+1)x - 1 - \log(c(d+1)),$$

and hence, by (3.1), it contains the point $(1,0) = (\frac{1}{b-1}\int_1^b w, \frac{1}{b-1}\int_1^b \log w)$. Therefore, the point

$$p_{a,b} = \frac{1-a}{b-a}p_{a,1} + \frac{b-1}{b-a}(1,0)$$

must lie above this line, and hence (3.13) holds.

Next, we derive directly that

$$\langle w\rangle_{(0,\kappa)} = 1 - \frac{d}{\kappa(d+1)}, \qquad \langle \log w\rangle_{(0,\kappa)} = -\frac{d}{\kappa}$$

and, by (3.12), $\langle \log w\rangle_{(0,\kappa)} = \frac{1}{\kappa}\langle \log^2 w\rangle_{(0,1)} = \frac{2d^2}{\kappa}$. This, directly from the definition of $\mathbb{B}^c_\pm$, yields the estimates

$$\mathbb{B}^c_+\left(1-\frac{d_+(c)}{\kappa(d_+(c)+1)},-\frac{d_+(c)}{\kappa}\right)\geq\frac{2d_+(c)^2}{\kappa}$$

and

$$\mathbb{B}^c_-\left(1-\frac{d_-(c)}{\kappa(d_-(c)+1)},-\frac{d_-(c)}{\kappa}\right)\leq\frac{2d_-(c)^2}{\kappa}.$$

On the other hand, by (3.3),

$$\begin{aligned}
B^c_+&\left(1-\frac{d_+(c)}{\kappa(d_+(c)+1)},-\frac{d_+(c)}{\kappa}\right)\\
&=\Psi_+\left(-\frac{d_+(c)}{\kappa}-\log\left(1-\frac{d_+(c)}{\kappa(d_+(c)+1)}\right)\right)+\left(\frac{d_+(c)}{\kappa}\right)^2\\
&=\Psi_+\left(d_+(c)-\frac{d_+(c)}{\kappa}-\log\left(c\left(1+d_+(c)-\frac{d_+(c)}{\kappa}\right)\right)\right)+\left(\frac{d_+(c)}{\kappa}\right)^2\\
&=d_+(c)^2-\left(d_+(c)-\frac{d_+(c)}{\kappa}\right)^2+\left(\frac{d_+(c)}{\kappa}\right)^2\\
&=\frac{2d_+(c)^2}{\kappa}
\end{aligned}$$

and similarly

$$B^c_-\left(1-\frac{d_-(c)}{\kappa(d_-(c)+1)},-\frac{d_-(c)}{\kappa}\right)=\frac{2d_-(c)^2}{\kappa}.$$

In other words, we have proved the estimate $\mathbb{B}^c_+\geq B^c_+$ at the set of all points of the form $\left(1-\frac{d_+(c)}{\kappa(d_+(c)+1)},-\frac{d_+(c)}{\kappa}\right)$ and the inequality $\mathbb{B}^c_-\leq B^c_-$ at the set of all points of the form $\left(1-\frac{d_-(c)}{\kappa(d_-(c)+1)},-\frac{d_-(c)}{\kappa}\right)$, where κ ranges from 1 to infinity. In addition, these inequalities hold also for the limit case $\kappa=\infty$ (and become $\mathbb{B}^c_+(1,0)\geq B^c_+(1,0)$, $\mathbb{B}^c_-(1,0)\leq B^c_-(1,0)$): this follows at once from the formula for $B^c_\pm$ and from Step 3 of the previous section. Now, as κ goes from 1 to infinity, the points $\left(1-\frac{d_+(c)}{\kappa(d_+(c)+1)},-\frac{d_+(c)}{\kappa}\right)$ form a continuous curve which starts at the lower boundary of Ω_c and terminates at the upper boundary of Ω_c. Since both $\mathbb{B}^c_+$ and B^c_+ satisfy the homogeneity-type conditions (see (2.2) and (3.4)), the estimate $\mathbb{B}^c_+\geq B^c_+$ propagates from the curve to the whole domain Ω_c. An analogous argument proves the second inequality $\mathbb{B}^c_-\leq B^c_-$ on Ω_c. This completes the proof. □

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References

1. Bellman, R.: Dynamic Programming. Reprint of the 1957 Edition. Princeton Landmarks in Mathematics. Princeton University Press, Princeton (2010)
2. Borichev, A., Janakiraman, P., Volberg, A.: On Burkholder function for orthogonal martingales and zeros of Legendre polynomials. Am. J. Math. **135**, 207–236 (2012)
3. Burkholder, D.L.: Boundary value problems and sharp inequalities for martingale transforms. Ann. Probab. **12**, 647–702 (1984)
4. Fefferman, C.: Characterizations of bounded mean oscillation. Bull. Am. Math. Soc. **77**, 587–588 (1971)
5. Ivanisvili, P., Stolyarov, D., Zatitskii, P.: Bellman vs Beurling: sharp estimates of uniform convexity for L_p spaces (Russian). Algebra i Analiz **27**(2), 218–231 (2015); translation in St. Petersburg Math. J. **27**(2), 333–343 (2016)
6. Ivanisvili, P., Osipov, N., Stolyarov, D., Vasyunin, V., Zatitskiy, P.: Bellman function for extremal problems in BMO. Trans. Am. Math. Soc. **368**, 3415–3468 (2016)
7. John, F., Nirenberg, L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. **14**, 415–426 (1961)
8. Khrushchev, S.V.: A description of weights satisfying the A_∞ condition of Muckenhoupt. Proc. Am. Math. Soc. **90**, 253–257 (1984)
9. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972)
10. Nazarov, F.L., Treil, S.R.: The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis. St. Petersb. Math. J. **8**, 721–824 (1997)
11. Nazarov, F.L., Treil, S.R., Volberg, A.: The Bellman functions and two-weight inequalities for Haar multipliers. J. Am. Math. Soc. **12**, 909–928 (1999)
12. Osękowski, A.: Sharp martingale and semimartingale inequalities. Monografie Matematyczne, vol. 72. Birkhäuser, Boston (2012)
13. Reznikov, A.: Sharp weak type estimates for weights in the class A_{p_1,p_2}. Rev. Mat. Iberoam. **29**, 433–478 (2013)
14. Slavin, L., Stokolos, A., Vasyunin, V.: Monge–Ampère equations and Bellman functions: the dyadic maximal operator. C. R. Acad. Sci. Paris Ser. I **346**, 585–588 (2008)
15. Slavin, L., Vasyunin, V.: Sharp results in the integral-form John–Nirenberg inequality. Trans. Am. Math. Soc. **363**, 4135–4169 (2011)
16. Slavin, L., Vasyunin, V.: Sharp L^p estimates on BMO. Indiana Math. J. **61**, 1051–1110 (2012)
17. Vasyunin, V.: The exact constant in the inverse Hölder inequalityfor Muckenhoupt weights (Russian). Algebra i Analiz **15**, 73–117 (2003); translation in St. Petersburg Math. J. **15**, 49–79 (2004)