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Positivity

A result on the ideal structure of $\mathcal{L}^r(X)$ for uniformly convex X

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Abstract
We consider the question of when is every positive compact operator, between two given Banach lattices, approximable regular. An immediate consequence of our main result is that, within the class of uniformly convex Banach lattices, the purely atomic ones are completely characterized by the fact that every positive compact operator on them is approximable regular.

Keywords Banach lattice · Compact operator · Ideal · Regular operator

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1 Introduction

Given a Banach lattice X, let $\mathcal{A}^r(X)$ be the Banach algebra of approximable regular operators on X (see below for definition), and let $\mathcal{K}^r(X)$ be the linear span of the positive compact operators on X. One readily sees that $\mathcal{A}^r(X) \subseteq \mathcal{K}^r(X)$, and it is known that if X and X' are order continuous, then $\mathcal{A}^r(X)$ and $\mathcal{K}^r(X)$ are both closed order and algebra ideals of the Banach algebra $\mathcal{L}^r(X)$ of regular operators on X.

It was first discovered by Fremlin [4], that $\mathcal{A}^r(L^2[0,1]) \subsetneq \mathcal{K}^r(L^2[0,1])$. This was later extended by Wickstead to arbitrary non-atomic L^p-spaces, for $1 < p < \infty$ [7, Theorem 3.4]. (For $p = 1$ or ∞, the corresponding algebra $\mathcal{L}^r(X)$ coincides with the Banach algebra of all bounded operators on X, so the result no longer holds).

Since $\mathcal{A}^r(X) = \mathcal{K}^r(X)$ whenever X is order continuous and atomic, one easily obtains, combining this latter fact with the above result, that within the class of L^p-spaces with $1 < p < \infty$, the equality $\mathcal{A}^r(X) = \mathcal{K}^r(X)$ completely characterizes the atomic ones [7, Corollary 3.5]. It is one of the main purposes of this short note

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to further extend this last result to the class of uniformly convex Banach lattices. We shall consider, though, the more general question of when is every positive compact operator, between two Banach lattices, approximable regular.

2 Some preliminaries

Throughout, we write X' for the topological dual of a Banach space X, T' for the topological adjoint of a linear operator T between Banach spaces and $\|T\|$ for its operator norm.

Recall a Banach lattice X is said to satisfy an **upper** (resp. a **lower**) p**-estimate** for some $1 < p < \infty$ if for some constant C and every finite disjoint sequence $x_1, \ldots, x_n \in X$,

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \quad \left(\text{resp.} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \leq C \left\| \sum_{i=1}^{n} x_i \right\| \right).$$

The **lower** (resp. **upper**) **index of** X, denoted $s(X)$ (resp. $\sigma(X)$), is then defined by $s(X) := \sup \{ p \in [1, \infty] : X \text{ satisfies an upper } p\text{-estimate} \}$ (resp. $\sigma(X) := \inf \{ p \in [1, \infty] : X \text{ satisfies a lower } p\text{-estimate} \}$). For any Banach lattice X, $s(X) \leq \sigma(X)$ and $\sigma(X)^{-1} + s(X')^{-1} = \sigma(X')^{-1} + s(X)^{-1} = 1$ (with the convention that $\infty^{-1} = 0$). Furthermore, if $\sigma(X) < \infty$ then X is order continuous. It is also known that a Banach lattice X is uniformly convex with respect to some equivalent lattice norm if and only if $1 < s(X) \leq \sigma(X) < \infty$ (see for instance [5, Theorems 1.f.1 & 1.f.7]).

Recall a linear operator T from a Banach lattice X to a Banach lattice Y is said to be p**-convex** (resp. p**-concave**) for some $1 \leq p < \infty$ if for some constant C and every finite sequence $x_1, \ldots, x_n \in X$,

$$\left\| \left(\sum_{i=1}^{n} |Tx_i|^p \right)^{\frac{1}{p}} \right\| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \quad \left(\text{resp.} \left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} \leq C \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \right).$$

A Banach lattice X is said to be p-convex (resp. p-concave) for some $1 \leq p < \infty$ if id_X (:= the identity operator on X) is p-convex (resp. p-concave).

As customary, a linear map T between Banach lattices X and Y shall be said to be **positive** if $T(X_+) \subseteq Y_+$ and **regular** if it can be written as a linear combination of positive maps. We shall write $\mathcal{L}^r(X, Y)$ for the Banach space of all regular maps from X to Y, endowed with the regular norm $\| \cdot \|_r$.

We shall write $\mathcal{A}^r(X, Y)$ for the **approximable regular operators** from X to Y (i.e., the norm-closure of the finite-rank operators in $\mathcal{L}^r(X, Y)$), and $\mathcal{K}^r(X, Y)$ for the linear span of the positive compact operators from X to Y. Clearly, $\mathcal{A}^r(X, Y) \subseteq \mathcal{K}^r(X, Y)$. Furthermore, if X' and Y are order continuous, then $\mathcal{A}^r(X, Y)$ and $\mathcal{K}^r(X, Y)$ are both order ideals of $\mathcal{L}^r(X, Y)$ and $\mathcal{K}^r(X, Y)$ is norm closed. As customary, if $X = Y$, we write $\mathcal{A}^r(X)$, $\mathcal{K}^r(X)$ and $\mathcal{L}^r(X)$ for $\mathcal{A}^r(X, Y)$, $\mathcal{K}^r(X, Y)$ and $\mathcal{L}^r(X, Y)$, respectively.

Lastly, we assume all our Banach lattices to be real.

3 When is $\mathcal{A}^r(X, Y) \subsetneq \mathcal{K}^r(X, Y)$?

Our main result, Theorem 1 below, shall provide conditions on a pair X, Y of Banach lattices under which the inclusion $\mathcal{A}^r(X, Y) \subseteq \mathcal{K}^r(X, Y)$ is strict. The extension announced in the introduction will follow readily from this.

It is easy to see that if either one of X or Y is order continuous and atomic, or if $\mathcal{L}^r(X, Y) = \mathcal{L}(X, Y)$ (e.g., if X is an AL-space and Y is Levi, or if Y is order complete with a strong order unit [1, Theorem 9]) and Y has the (Grothendieck) approximation property, then $\mathcal{A}^r(X, Y) = \mathcal{K}^r(X, Y)$. In the opposite direction, we now prove the following:

Theorem 1 *Let X and Y be non-atomic Banach lattices such that $s(X) > 1$, $\sigma(Y) < \infty$ and either X or Y' is order continuous. Then $\mathcal{A}^r(X, Y) \subsetneq \mathcal{K}^r(X, Y)$.*

Proof Suppose first X is order continuous. Then, since X and Y are non-atomic, they both contain non-trivial bands without atoms and with weak order units (see for instance [5, Proposition 1.a.9]). We shall continue to denote these bands by X and Y, respectively. Furthermore, we shall assume (as we can, by [5, Theorem 1.b.14]) that there are probability spaces $(\Lambda, \Sigma_\Lambda, \lambda)$ and $(\Omega, \Sigma_\Omega, \mu)$ such that X and Y are norm-dense order ideals of $L^1(\lambda)$ and $L^1(\mu)$, respectively, and also that $\|x\|_1 \leq \|x\| \leq 2\|x\|_\infty$ $(x \in L^\infty(\lambda))$ and $\|y\|_1 \leq \|y\| \leq 2\|y\|_\infty$ $(y \in L^\infty(\mu))$.

Since $s(X) > 1$, X is p-convex for some $1 < p \leq 2$ (see [5, Theorem 1.f.7]), and therefore, the embedding $X \hookrightarrow L^1(\lambda)$ is p-convex. It follows easily from the proof of [5, Theorem 1.d.11] (recall $L^1(\lambda)$ is r-concave for every $r \geq 1$) that there is a p-additive norm $|||\cdot|||_p$ on X such that both inclusion maps, $\imath_1 : X \to (X, |||\cdot|||_p)$ and $\imath_2 : (X, |||\cdot|||_p) \to L^1(\lambda)$, are continuous. It is not hard to see that the map $\nu : \Sigma_\Lambda \to [0, +\infty)$, $A \mapsto |||\chi_A|||_p^p$, defines a measure on Λ (its σ-additivity follows easily from the order continuity of $|||\cdot|||_p$), which is absolutely continuous with respect to λ (for $\|\chi_A\|_1 = 0 \Rightarrow \|\chi_A\| = 0$) and satisfies $|||x|||_p^p = \int_\Lambda |x|^p d\nu$ $(x \in X)$. Then a standard application of [5, Theorem 2.c.9] allows one to construct a collection $\{\Lambda_{i,n} : 1 \leq i \leq 2^n,\ n \in \mathbb{N} \cup \{0\}\} \subset \Sigma_\Lambda$ such that $\Lambda_{1,0} = \Lambda$ and for every $n \in \mathbb{N}$: (i) $\Lambda_{i,n} \cap \Lambda_{j,n} = \emptyset$ whenever $i \neq j$; (ii) $\bigcup_{i=1}^{2^n} \Lambda_{i,n} = \Lambda$; (iii) $\Lambda_{2i-1,n} \cup \Lambda_{2i,n} = \Lambda_{i,n-1}$ $(1 \leq i \leq 2^{n-1})$; and (iv) $\left(\int_{\Lambda_{i,n}} \imath_2'(\chi_\Lambda)^q \, d\nu\right)^{1/q} = 2^{-n/q}\, |||\, \imath_2'(\chi_\Lambda)|||_p'$ $(1 \leq i \leq 2^n)$, where we have written $|||\cdot|||_p'$ for the norm on $(X, |||\cdot|||_p)'$ and χ_Λ for the linear functional $x \mapsto \int_\Lambda x \, d\lambda$ $(x \in L^1(\lambda))$.

Also, since $\sigma(Y) < \infty$, Y is r-concave for some $r < \infty$ (see [5, Theorem 1.f.7]), and in turn, the embedding $L^\infty(\mu) \hookrightarrow Y$ is r-concave. Once again, the proof of [5, Theorem 1.d.11] (this time, taking into account that $L^\infty(\mu)$ is s-convex for every $s \geq 1$) yields the existence of an r-additive norm $|||\cdot|||_r$ on $L^\infty(\mu)$ such that the inclusion maps $\jmath_1 : L^\infty(\mu) \to (L^\infty(\mu), |||\cdot|||_r)$ and $\jmath_2 : (L^\infty(\mu), |||\cdot|||_r) \to Y$ are both continuous. In turn, $A \mapsto |||\chi_A|||_r^r$ $(A \in \Sigma_\Omega)$ defines a μ-absolutely continuous measure, and therefore, for some $g \in L^1(\mu)_+$, we have that

$$\|y\| \leq \|\jmath_2\| \left(\int_\Omega |y|^r g \, d\mu \right)^{1/r} \quad (y \in L^\infty(\mu)). \tag{1}$$

Note that we can assume $\int_\Omega g\,d\mu = 1$, for $|||\chi_\Omega|||_r \le \|\jmath_1\| \le$ the convexity constant of $L^\infty(\mu)$ (see [5, Theorem 1.d.11]), which is 1. As in the previous paragraph, choose a family $\{\Omega_{i,n} : 1 \le i \le 2^n,\ n \in \mathbb{N}\cup\{0\}\} \subset \Sigma_\Omega$ satisfying $\Omega_{1,0} = \Omega$, the same conditions (i)–(iii) as $\{\Lambda_{i,n}\}$, and also $\int_{\Omega_{i,n}} g\,d\mu = 2^{-n}$ $(1 \le i \le 2^n,\ n \in \mathbb{N})$.

Now, for every $n \in \mathbb{N}\cup\{0\}$, let Σ_n be the σ-algebra generated by $\{\Lambda_{i,n} \times \Omega_{j,n} : 1 \le i, j \le 2^n\}$; let $\mathcal{E}_n$ be the conditional expectation with respect to Σ_n; let $\mathfrak{M}_n$ be the space of Σ_n-measurable functions on $\Lambda \times \Omega$, and given $k \in \mathfrak{M}_n$, let $\mathrm{Int}(k) : X \to Y$, $x \mapsto \int_\Lambda k(\omega, t)x(t)\,d\lambda(t)$. For each pair $m, n \in \mathbb{N}\cup\{0\}$, with $m < n$, set

$$\mathfrak{M}_{m,n} := \left\{k \in \mathfrak{M}_n : |k| \equiv 1 \text{ and } \mathcal{E}_m(k) = 0\right\}.$$

The key fact needed in the proof of Theorem 1 (established in [4] for $L^2[0,1]$ and then in [7] for any L^p-space with $1 < p < \infty$) can now be stated as follows:

Lemma 1 *Let $\{\Lambda_{i,n} : 1 \le i \le 2^n,\ n \in \mathbb{N}\cup\{0\}\}$ and $\{\Omega_{i,n} : 1 \le i \le 2^n,\ n \in \mathbb{N}\cup\{0\}\}$ be as above. Then for every $m \in \mathbb{N}\cup\{0\}$ and $h \in \mathfrak{M}_m$,*

$$\inf_{n\,:\,n>m}\ \min_{k\in\mathfrak{M}_{m,n}} \|Int(hk)\| = 0.$$

Proof Let $r_n := \sum_{i=1}^{2^n}(-1)^i \chi_{\Omega_{i,n}}$ $(n \in \mathbb{N})$. Fix n and define $k : \Lambda \times \Omega \to \mathbb{R}$ by

$$k(\omega, t) := \sum_{i=1}^{2^n} \chi_{\Lambda_{i,n}}(t) r_{n+i}(\omega) \quad (t \in \Lambda,\ \omega \in \Omega).$$

Note that $k \in \mathfrak{M}_{m,N}$ $(0 \le m \le 2^n,\ N \ge 2^{n+2^n})$. For each $1 \le i \le 2^n$ let $\lambda_{i,n}$ be the linear functional on X, defined by $\lambda_{i,n}(x) := \int_{\Lambda_{i,n}} x\,d\lambda$ $(x \in X)$. Then, for every $x \in X$,

$$\begin{aligned}\left\|\mathrm{Int}(k)(x)\right\| &= \left\|\sum_{i=1}^{2^n} \lambda_{i,n}(x) r_{n+i}\right\| \le \|\jmath_2\| \left(\int_\Omega \left|\sum_{i=1}^{2^n} \lambda_{i,n}(x) r_{n+i}\right|^r g\,d\mu\right)^{1/r} \\ &\le C\left(\sum_{i=1}^{2^n} \left|\lambda_{i,n}(x)\right|^2\right)^{1/2}, \end{aligned} \tag{2}$$

for some constant C independent of n [by (1) above and Khintchine's inequalities]. Next, set $x_i := \chi_{\Lambda_{i,n}} x$ $(1 \le i \le 2^n)$, let $\widehat{\lambda_{i,n}}$ be the norm continuous extension of $\lambda_{i,n}$ to $L^1(\lambda)$ $(1 \le i \le 2^n)$, and let $\imath : X \to L^1(\lambda)$ be the inclusion map, so $\imath = \imath_2 \circ \imath_1$. Then

$$\begin{aligned}|\lambda_{i,n}(x)| = |\lambda_{i,n}(x_i)| &= \left|\widehat{\lambda_{i,n}}(\imath(x_i))\right| = \left|\left(\imath_2'(\widehat{\lambda_{i,n}})\right)(\imath_1(x_i))\right| \\ &\le |||\imath_2'(\widehat{\lambda_{i,n}})\,|||_p'\ |||\imath_1(x_i)|||_p \le \frac{|||\imath_2'(\chi_\Lambda)|||_p'}{2^{n/q}}\ |||\,\imath_1(x_i)|||_p,\end{aligned}$$

where we have used that $\iota_2'(\widehat{\lambda_{i,n}}) = \chi_{\Lambda_{i,n}}\iota_2'(\chi_\Lambda)$, which is easy to verify. In turn, letting $M := |||\iota_2'(\chi_\Lambda)|||_p'$, we obtain that

$$\sum_{i=1}^{2^n} |\lambda_{i,n}(x)|^2 \leq \frac{M^2}{2^{2n/q}} \sum_{i=1}^{2^n} |||\iota_1(x_i)|||_p^2$$
$$\leq \frac{M^2}{2^{2n/q}} \left(\sum_{i=1}^{2^n} |||\iota_1(x_i)\;|||_p^p \right)^{2/p} = \frac{M^2}{2^{2n/q}}\; |||\;\iota_1(x)|||_p^2,$$

and combining this last estimate with (2), we arrive at

$$\left\|\mathrm{Int}(k)\right\| \leq C\frac{M}{2^{n/q}}\|\iota_1\|.$$

To finish, simply note that if $h = \sum_{i,j} h_{ij}\chi_{\Lambda_{i,m}\times\Omega_{j,m}} \in \mathfrak{M}_m$ and $k \in \mathfrak{M}_N$, with $m < N$, then

$$\left\|\mathrm{Int}(hk)\right\| \leq \sum_{i,j} |h_{ij}| \left\|\mathrm{Int}(\chi_{\Lambda_{i,m}\times\Omega_{j,m}}k)\right\| \leq \left(\sum_{i,j}|h_{ij}|\right)\left\|\mathrm{Int}(k)\right\|.$$

□

We now resume the first part of the proof by constructing a positive compact operator $S : X \to Y$ (essentially as in [4]) which is not approximable. First, fix $\varepsilon \in (0, 1/2)$ and set $h_0 := \chi_{\Lambda\times\Omega}$. Choose $n_1 \in \mathbb{N}$ big enough so that $\min_{k\in\mathfrak{M}_{0,n_1}} \|\mathrm{Int}(h_0k)\| \leq \varepsilon$ (which exists by the lemma), then choose $k_1 \in \mathfrak{M}_{0,n_1}$ (so $\mathcal{E}_0(h_0k_1) = 0$) such that $\|\mathrm{Int}(h_0k_1)\| \leq \varepsilon$ and set $h_1 := h_0 + k_1$. In general, if h_i and n_i have been chosen for some $i \geq 1$, choose $n_{i+1} > n_i$ so that $\min_{k\in\mathfrak{M}_{n_i,n_{i+1}}} \|\mathrm{Int}(h_ik)\| \leq \varepsilon/2^i$ (again possible by the lemma), choose $k_{i+1} \in \mathfrak{M}_{n_i,n_{i+1}}$ such that $\|\mathrm{Int}(h_ik_{i+1})\| \leq \varepsilon/2^i$ and set $h_{i+1} := h_i(h_0 + k_{i+1})$. (Note that $\mathcal{E}_{n_i}(h_ik_{i+1}) = 0$.) For every $i \in \mathbb{N}\cup\{0\}$, let $T_i := \mathrm{Int}(h_i)$ and define $S := \lim_i T_i = \mathrm{Int}(h_0) + \sum_{i=0}^{\infty} \mathrm{Int}(h_ik_{i+1})$. It is clear from its definition that S is a non-zero positive compact operator. Furthermore, for every $i \in \mathbb{N}$, $\|T_{i+1} - T_i\|_r = \|T_i\| \geq \|T_i(\chi_\Lambda)\| = \|\chi_\Omega\| \geq 1$ (the first equality because $|h_{i+1} - h_i| = |h_ik_{i+1}| = h_i$ $(i \in \mathbb{N})$).

It remains to be shown that $S \notin \mathcal{A}^r(X, Y)$. To this end, suppose towards a contradiction $S \in \mathcal{A}^r(X, Y)$. At this point one can appeal to the integral representation of operators in $\mathcal{A}^r(X, Y)$ (as in [4,7]) to derive a contradiction. Instead, however, we shall appeal to the fact that if X' and Y are order continuous then so is $\mathcal{A}^r(X, Y)$ [3, Theorem 2.8]. First note $S\wedge(T_0-2^{-i}T_i) = 0$ $(i \in \mathbb{N})$, for if $k_i = \sum_{k,l} t_{kl}\chi_{\Lambda_{k,n_i}\times\Omega_{l,n_i}}$, and P_k and Q_l $(1 \leq k, l \leq 2^{n_i})$ are the band projections onto the bands generated by $\chi_{\Lambda_{k,n_i}}$ and $\chi_{\Omega_{l,n_i}}$, then

$$S = \lim_j T_j = \lim_j \sum_{(k,l)\,:\,t_{kl}\neq 0} Q_lT_jP_k = \sum_{(k,l)\,:\,t_{kl}\neq 0} Q_lSP_k.$$

Since $S \wedge 2^i T_0 \leq S \wedge 2^i (T_0 - 2^{-i} T_i) + S \wedge T_i$, it follows that $S \wedge T_i = S \wedge 2^i T_0$ ($i \in \mathbb{N}$), and hence that $S = \sup_i S \wedge T_i$, so $\lim_i \|S - S \wedge T_i\|_r = 0$ (by the order continuity of $\mathcal{A}^r(X, Y)$). But

$$\|T_i - S\|_r \leq \|T_i - T_i \wedge S\| + \|S - T_i \wedge S\| \leq \|T_i - S\| + 2\|S - S \wedge T_i\|_r,$$

and since $\lim_i \|S - S \wedge T_i\|_r = 0$, we would have that $\lim_i \|T_i - S\|_r = 0$, which is clearly impossible since (T_i) is not Cauchy. This concludes the proof of the theorem in the case where X is order continuous.

Now suppose Y' is order continuous, so Y is reflexive (see for instance [2, Theorems 4.69 and 4.71]). By the previous part of the proof, $\mathcal{A}^r(Y', X') \subsetneq \mathcal{K}^r(Y', X')$. Let $T \in \mathcal{K}^r(Y', X') \backslash \mathcal{A}^r(Y', X')$, let $\kappa : X \to X''$ be the canonical embedding, and let $S := T' \circ \kappa$. Clearly, $S \in \mathcal{K}^r(X, Y)$, so it will suffice to show $S \notin \mathcal{A}^r(X, Y)$. Suppose towards a contradiction that there is a sequence $(T_n) \subset \mathcal{F}(X, Y)$ (:= the finite-rank operators in $\mathcal{L}^r(X, Y)$) such that $\lim_n \|T_n - S\|_r = 0$. We would have then that $\|T_n' - S'\|_r \leq \|T_n - S\|_r$ (see [6, Corollary on page 231]), and therefore, that $\lim_n \|T_n' - S'\|_r = 0$, which is impossible since $S' = \kappa' \circ T'' = T$. $\square$

As an immediate consequence of Theorem 1, we now have the announced extension of [7, Corollary 3.5].

Corollary 1 *Let X be a Banach lattice such that $1 < s(X) \leq \sigma(X) < \infty$ (or equivalently, isomorphic to a uniformly convex Banach lattice). Then $\mathcal{A}^r(X) = \mathcal{K}^r(X)$ if and only if X is atomic.*

Little seems to be known about the ideal structure of $\mathcal{K}^r(X)$, where by ideal we mean order and algebra ideal. We bring this note to a close with what seems to us a natural question regarding the ideal structure of $\mathcal{K}^r(X)$, for which we do not have yet an answer:

Let $1 < p < \infty$. Is $\mathcal{A}^r(L^p[0, 1])$ the only closed non-trivial proper order and (two-sided) algebra ideal of $\mathcal{K}^r(L^p[0, 1])$?

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