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Positivity

A category analogue of the density topology non-homeomorphic with the $\mathcal{I}$-density topology

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Abstract
The paper deals with the category analogue of a density point and a density topology (with respect to a Lebesgue measure) on the real line which is different from the $\mathcal{I}$-density topology considered in Poreda et al. (Fundam Math 125:167–173, 1985; Comment Math Univ Carol 26:553–563, 1985). This topology called the intensity topology, manifests several properties analogous to that of $\mathcal{I}$-density topology, but there are also differences. The class of function which are continuous as functions from $\mathbb{R}$ equipped with an intensity topology to $\mathbb{R}$ equipped with the natural topology is included in the first class of Baire Darboux functions.

Keywords Density point · Density topology · $\mathcal{I}$-density point · $\mathcal{I}$-density topology · Intensity topology · Intensely continuous functions · Baire class

Mathematics Subject Classification 54 A 10 · 26 A 21 · 28 A 05

1 Introduction

In [8] one can find a characterization of a Lebesgue density point which uses only the σ-ideal of nullsets. This was a starting point for the definition of category density point, called the $\mathcal{I}$-density point and for the construction of $\mathcal{I}$-density topology, which was studied later in numerous papers. In this paper we shall formulate another characterization of a Lebesgue density point, again not using a Lebesgue measure but only the σ-ideal of nullsets. This characterization leads again to a definition of category density point, which is, however, not equivalent to $\mathcal{I}$-density and to a category density topology, which is different from the $\mathcal{I}$-density topology.

In the sequel $\mathcal{L}$ will denote the σ-algebra of Lebesgue measurable sets on the real line, $\mathcal{N}$—the σ-ideal of nullsets, $\mathcal{B}$—the σ-algebra of sets having the Baire property,

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$\mathcal{I}$—the σ-ideal of sets of the first category, λ will stand for the Lebesgue measure. We shall use also the following denotation:

$$A + x = \{t + x : t \in A\} \text{ and } xA = \{xt : t \in A\}.$$

Recall that $x_0 \in \mathbb{R}$ is a density point of $A \in \mathcal{L}$ if and only if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1,$$

x_0 is a right-hand (left-hand) density point if and only if $\lim_{h\to 0^+} \frac{\lambda(A\cap[x_0,x_0+h])}{h} = 1$ ($\lim_{h\to 0^+} \frac{\lambda(A\cap[x_0-h,x_0])}{h} = 1$). A point x_0 is a dispersion (right-hand dispersion, left-hand dispersion) point of $A \in \mathcal{L}$ if and only if it is a density (right-hand density, left-hand density) point of A', or, equivalently, if the above limits are equal to 0. Observe also that x_0 is a density point of A if and only if 0 is a density point of $A - x_0$. The set of all density points of A is denoted by $\Phi(A)$. The operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ has the following properties (see [7], Ch. 22):

1. $\forall_{A\in\mathcal{L}} \quad \lambda(A \triangle \Phi(A)) = 0,$ (the Lebesgue density theorem)
2. $\forall_{A,B\in\mathcal{L}} \quad (\lambda(A \triangle B) = 0 \implies \Phi(A) = \Phi(B)),$
3. $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,
4. $\forall_{A,B\in\mathcal{L}} \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B).$

From (1) it follows immediately that $\Phi : \mathcal{L} \to \mathcal{L}$ (in fact $\Phi : \mathcal{L} \to G_{\delta\sigma}$).

The family $\mathcal{T}_d = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ is a topology on $\mathbb{R}$ (called the density topology), which is essentially stronger than a natural topology.

Since x_0 is a density (resp. dispersion) point of $A \in \mathcal{L}$ if and only if it is simultaneously a left-hand and a right-hand density (resp. dispersion) point of A, we shall deal in the sequel with the right-hand case, the remaining being obvious. The expression "in measure" will always mean with respect to the Lebesgue measure restricted to $[0, 1]$, $[-1, 0]$ or $[-1, 1]$.

In [8] it was observed that x_0 is a right-hand density (resp. dispersion) point of $A \in \mathcal{L}$ if and only if the sequence $\{\chi_{(n\cdot(A-x_0)\cap[0,1])}\}_{n\in\mathbb{N}}$ of characteristic functions converges in measure to $\chi_{[0,1]}$ (resp. to $\chi_\emptyset$). By virtue of theorem of Riesz, x_0 is a right-hand density (resp. dispersion) point of $A \in \mathcal{L}$ if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\{\chi_{n_{m_p}} \cdot (A - x_0) \cap [0, 1]\}_{p\in\mathbb{N}}$ converges to $\chi_{[0,1]}$ (resp. to $\chi_\emptyset$) almost everywhere. The last condition indeed does not use the Lebesgue measure, only the σ-ideal $\mathcal{N}$ is important.

Definition 1 [8] A point $x_0 \in \mathbb{R}$ is an $\mathcal{I}$-density point of $A \in \mathcal{B}$ if and only if $\{\chi_{n\cdot(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ converges in category to $\chi_{[-1,1]}$.

The convergence in category means that for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\{\chi_{n_{m_p}\cdot(A-x_0)\cap[-1,1]}\}_{p\in\mathbb{N}}$ converges to $\chi_{[-1,1]}$ $\mathcal{I}$-almost everywhere (i.e. except on a set belonging to $\mathcal{I}$) (compare [13]).

$\mathcal{I}$-dispersion points, right-hand $\mathcal{I}$-density or $\mathcal{I}$-dispersion points and left-hand $\mathcal{I}$-density or $\mathcal{I}$-dispersion points are defined in an obvious way. If $\Phi_{\mathcal{I}}(A) = \{x \in \mathbb{R} : x$ is an $\mathcal{I}$-density point of $A\}$ for $A \in \mathcal{B}$, then the operator $\Phi_{\mathcal{I}}$ has all properties analogous to properties (1)–(4) of Φ, also $\Phi_{\mathcal{I}} : \mathcal{B} \to \mathcal{B}$. The family $\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}(A)\}$ is a topology on $\mathbb{R}$ (called the $\mathcal{I}$-density topology), which is essentially stronger than a natural topology (see [8] or [3]).

2 New characterization

Suppose that A is a Lebesgue measurable set. Put $a_n = \lambda(A \cap [\frac{1}{n+1}, \frac{1}{n}])$ for $n \in \mathbb{N}$ and let $b_n = n(n+1)a_n$ be an average density of the set A on the interval $[\frac{1}{n+1}, \frac{1}{n}]$.

Definition 2 We shall say that 0 is the right-hand C-density point of A (C-dispersion point of A) if and only if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} b_i = 1$ ($\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} b_i = 0$).

The definition of the left-hand C-density point(C-dispersion point) is formulated in an obvious way. 0 is the C-density (C-dispersion) point of A if and only if it is simultaneously the right-hand and the left-hand C-density (C-dispersion) point of A. We say that x_0 is the right-hand (left-hand) C-density (C-dispersion) point of A if and only if 0 is the right-hand (left-hand) C-density (C-dispersion) point of $A - x_0$.

Observe that 0 is the right-hand C-density point of A if and only if it is the right-hand C-dispersion point of A'.

Theorem 3 *0 is the right-hand C-dispersion point of A if and only if 0 is the right-hand dispersion point of A.*

Proof "⇒" Take $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have $\frac{1}{2n-1} \cdot \sum_{i=1}^{2n-1} b_i < \frac{\varepsilon}{4}$. Hence also $\frac{1}{2n-1} \cdot \sum_{i=n}^{2n-1} b_i < \frac{\varepsilon}{4}$. Since $2n \leq 4 \cdot \frac{n(n+1)}{2n-1} \leq 4 \cdot \frac{i(i+1)}{2n-1}$ for $i \geq n$, then $\frac{\lambda(A\cap[\frac{1}{2n},\frac{1}{n}])}{(\frac{1}{n}-\frac{1}{2n})} = 2n \cdot \sum_{i=n}^{2n-1} a_i \leq \frac{4}{2n-1} \cdot \sum_{i=n}^{2n-1} i(i+1)a_i = \frac{4}{2n-1} \cdot \sum_{i=n}^{2n-1} b_i < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$ for $n \geq n_0$. So we have shown that $\lim_{n\to\infty} \frac{\lambda(A\cap[\frac{1}{2n},\frac{1}{n}])}{(\frac{1}{n}-\frac{1}{2n})} = 0$. This implies $\lim_{n\to\infty} \frac{\lambda(A\cap[0,\frac{1}{n}])}{\frac{1}{n}} = 0$ and 0 is the right-hand dispersion point of A.

"⇐" Take $\varepsilon > 0$. Fix a positive integer $k > \frac{3}{\varepsilon} + 1$. Then we have $\frac{n}{nk-1} < \frac{\varepsilon}{3}$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\lambda(A\cap[0,\frac{1}{n}])}{\frac{1}{n}} = 0$, then also $\lim_{n\to\infty} \frac{\lambda(A\cap[\frac{1}{nk},\frac{1}{n}])}{\frac{1}{n}-\frac{1}{nk}} = \lim_{n\to\infty} \frac{nk}{k-1} \cdot \lambda(A \cap [\frac{1}{nk}, \frac{1}{k}]) = 0$. There exists $n_1 \in \mathbb{N}$ such that $\frac{nk}{k-1}\lambda(A \cap [\frac{1}{nk}, \frac{1}{k}]) < \frac{\varepsilon}{3k}$ for $n \geq n_1$. Since $\frac{i(i+1)}{nk-1} \leq \frac{(nk-1)nk}{nk-1} = nk$ for $i \leq nk-1$, then $\frac{1}{nk-1}\sum_{i=n}^{nk-1} b_i = \frac{1}{nk-1}\sum_{i=n}^{nk-1} i(i+1)a_i \leq nk\sum_{i=n}^{nk-1} a_i = nk\lambda(A \cap [\frac{1}{nk}, \frac{1}{n}]) < \frac{\varepsilon}{3k} \cdot (k-1) < \frac{\varepsilon}{3}$.

Let $n_2 \in \mathbb{N}$ be such that $\frac{k}{nk-1} < \frac{\varepsilon}{3}$ for $n \geq n_2$. Then for $n \geq n_0 = \max(n_1, n_2)$ and $nk-1 < m \leq (n+1)k-1$ we have $\frac{1}{m}\sum_{i=n}^{m} b_i = \frac{1}{m}\sum_{i=n}^{nk-1} b_i + \frac{1}{m}\sum_{i=nk}^{m} b_i \leq \frac{1}{nk-1}\sum_{i=n}^{nk-1} b_i + \frac{1}{nk-1}\sum_{i=nk}^{m} b_i < \frac{\varepsilon}{3} + \frac{1}{nk-1}\sum_{i=nk}^{m} 1 \leq \frac{\varepsilon}{3} + \frac{k}{nk-1} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 2\frac{\varepsilon}{3}$. At last for $n \geq n_0$ and m between $nk-1$ and $(n+1)k-1$ we have $\frac{1}{m}\sum_{i=1}^{m} b_i = \frac{1}{m}\sum_{i=1}^{n-1} b_i + \frac{1}{m}\sum_{i=n}^{m} b_i < \frac{n-1}{nk-1} + \frac{2}{3}\varepsilon < \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon$. Hence $\frac{1}{m}\sum_{i=1}^{m} b_i < \varepsilon$ for $m \geq n_0k-1$, which means that 0 is the right-hand C-dispersion point of A. □

Remark 4 Obviously similar theorem holds also for the left-hand C-dispersion point of A. Also 0 is the right-hand (or left-hand) C-density point of A if and only if it is the right-hand (or left-hand) density point of A.

3 Intensity topology

Now we shall formulate the definition of C-dispersion (and C-density) point without using the Lebesgue measure. By virtue of Theorem 3 it will be another characterization of the dispersion (and density) point. This will enable us to introduce the category analogue of the density and dispersion point.

Put $A_n = [n(n+1) \cdot (A - \frac{1}{n+1})] \cap [0, 1]$. Then $b_n = \lambda(A_n) = \int_0^1 \chi_{A_n}(t)dt$, so $\frac{1}{n}\sum_{i=1}^n b_i = \frac{1}{n}\int_0^1 \sum_{i=1}^n \chi_{A_i}(t)dt$. If we denote $f_n = \frac{1}{n}\sum_{i=1}^n \chi_{A_i}$ for $n \in \mathbb{N}$, then we have $0 \leq f_n(t) \leq 1$ for each $n \in \mathbb{N}$ and $t \in [0, 1]$.

So 0 is a right hand C-dispersion point (dispersion point) of A if and only if $\int_0^1 f_n \underset{n\to\infty}{\to} 0$, which means $f_n \underset{n\to\infty}{\to} 0$ in L^1 norm. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is commonly bounded this is equivalent to $f_n \underset{n\to\infty}{\to} 0$ in measure (compare [2], p. 245, Th. 5.36 and p. 246 Th. 5.37).

Definition 5 We say that 0 is the right-hand rarefaction (intensity) point of $A \in \mathcal{B}$ if and only if the sequence $\{f_n\}_{n\in\mathbb{N}}$ defined above converges to $0 = \chi_\emptyset$ (converges to $\chi_{[0,1]}$) in category.

The left-hand rarefaction (intensity) point is defined in the obvious way. 0 is rarefaction (intensity) point of $A \in \mathcal{B}$ if and only if it is simultaneously the right-hand and the left-hand rarefaction (intensity) point of A. A point x_0 is the right-hand rarefaction (intensity) point of A if and only if 0 is the right-hand rarefaction (intensity) point of $A - x_0$.

In the sequel we shall use the following denotation:

$$A_n(x) = \left[n(n+1)(A - x) - \frac{1}{n+1}\right] \cap [0, 1].$$

If $x = 0$ we shall write A_n rather than $A_n(0)$.

Let $\Phi_i(A) = \{x \in \mathbb{R} : x \text{ is the intensity point of } A\}$ for $A \in \mathcal{B}$.

Theorem 6 *Φ_i is a lower density operator, i.e.*

1. $\forall_{A\in\mathcal{B}} \quad \Phi_i(A) \triangle A \in \mathcal{I}$,
2. $\forall_{A,B\in\mathcal{B}} \quad (A \triangle B \in \mathcal{I} \implies \Phi_i(A) = \Phi_i(B))$,
3. $\Phi_i(\emptyset) = \emptyset$, $\Phi_i(\mathbb{R}) = \mathbb{R}$,
4. $\forall_{A,B\in\mathcal{B}} \quad \Phi_i(A \cap B) = \Phi_i(A) \cap \Phi_i(B)$.

Proof Ad 1. Observe that if $A = G \triangle P$, where G is open and $P \in \mathcal{I}$, then $G \subset \Phi_i(A) \subset \bar{G}$ and $\bar{G} \setminus G$ is nowhere dense.

Proofs of 2, 3 and 4 are standard. □

Theorem 7 *The family $\mathcal{T}_i = \{A \in \mathcal{B} : A \subset \Phi_i(A)\}$ is a topology essentially stronger than a natural topology on the real line.*

Proof Exactly the same as in [8] for $\mathcal{T}_{\mathcal{I}}$. □

We shall show that the notion of intensity point is not equivalent to that of $\mathcal{I}$-density point. However, the topology $\mathcal{T}_i$ shares most of the properties (but not all) with the topology $\mathcal{T}_{\mathcal{I}}$.

Theorem 8 *There exists a set $A \subset [0, 1]$ such that 0 is a right-hand rarefaction point of A but not the right-hand $\mathcal{I}$-dispersion point of A.*

Proof Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a standard example of a sequence of functions defined on $[0, 1]$ which converges in measure and diverges everywhere on $(0, 1]$, i.e. $\phi_1 = \chi_{(0,1]}$, $\phi_2 = \chi_{(0,\frac{1}{2}]}$, $\phi_3 = \chi_{(\frac{1}{2},1]}, \ldots, \phi_{2^k} = \chi_{(0,\frac{1}{2^k}]}, \ldots, \phi_{2^{k+1}-1} = \chi_{(1-\frac{1}{2^k},1]}$. Put $E_n = \{x \in (0, 1] : \phi_n(x) = 1\}$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \frac{1}{n(n+1)} \cdot E_n + \frac{1}{n+1}$. Observe that for each $k \in \mathbb{N}$ we have $\sum_{n=2^k}^{2^{k+1}-1} \chi_{(n(n+1)(A-\frac{1}{n+1})\cap[0,1])}(x) = \sum_{n=2^k}^{2^{k+1}-1} \chi_{E_n}(x) = 1$ for each $x \in (0, 1]$. Hence $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{(n(n+1)(A-\frac{1}{n+1})\cap[0,1])}(x) = 0$ for each $x \in (0, 1]$ and 0 is a right-hand rarefaction point of A.

At the same time the set $A \cap [\frac{1}{n+1}, \frac{1}{n}]$ includes a non-empty open set for each $n \in \mathbb{N}$, from which it follows easily that 0 is not a right-hand $\mathcal{I}$-dispersion point of A. □

Theorem 9 *There exists a set $A \subset [0, 1]$ such that 0 is the right-hand $\mathcal{I}$-dispersion point of A but not the right-hand rarefaction point of A.*

Proof Put $C_1 = [\frac{1}{6}, 1] \setminus (\frac{1}{4}, \frac{1}{3})$. Let I_1^1 and I_1^2 be closed components of C_1 numbered from the left to the right. Suppose that we have defined $C_1, C_2, \ldots, C_{n-1}$ such that $C_{n-1} \subset [\frac{1}{6^{n-1}}, \frac{1}{6^{n-2}}]$ and $I_{n-1}^1, \ldots, I_{n-1}^{2^{n-1}}$ is the sequence of closed components of C_{n-1} numbered from the left to right. We shall define $C_n \subset [\frac{1}{6^n}, \frac{1}{6^{n-1}}]$ in the following way: let J_{n-1}^i for $i = \{1, \ldots, 2^{n-1}\}$ be an open interval of the form $(\frac{1}{m+1}, \frac{1}{m})$, where m is a positive integer such that the center of $\frac{1}{6} \cdot I_{n-1}^i$ belongs to $[\frac{1}{m+1}, \frac{1}{m})$. Put $C_n = \frac{1}{6} C_{n-1} \setminus \bigcup_{i=1}^{2^{n-1}} J_{n-1}^i$. Then C_n is the union of 2^n closed intervals $I_n^1, \ldots, I_n^{2^n}$. At last put $A = \bigcup_{n=1}^{\infty} C_n$. Observe that $B = \bigcap_{n=1}^{\infty} 6^{n-1} A \cap [\frac{1}{6}, 1]$ is a perfect nowhere dense subset of $[\frac{1}{6}, 1]$. Observe also that the interval $[\frac{1}{6^n}, \frac{1}{6^{n-1}}]$ is the union of $5 \cdot 6^{n-1}$ intervals of the form $[\frac{1}{m+1}, \frac{1}{m}]$, namely $[\frac{1}{6^n}, \frac{1}{6^{n-1}}] = \bigcup_{m=6^{n-1}}^{6^n-1} [\frac{1}{m+1}, \frac{1}{m}]$ while the set C_n is the union of $5 \cdot 6^{n-1} - (6^{n-1} + 2 \cdot 6^{n-2} + \cdots + 2^{n-2} \cdot 6 + 2^{n-1})$ among these intervals. Since $A_k = [k(k+1)(A - \frac{1}{k+1})] \cap [0, 1] = [k(k+1)(C_n - \frac{1}{k+1})] \cap [0, 1]$ for $k \in \{6^{n-1}, \ldots, 6^n - 1\}$, we have $\frac{1}{5\cdot 6^{n-1}} \cdot \sum_{k=6^{n-1}}^{6^n-1} \chi_{A_k}(t) > \frac{7}{10}$ for each $n \in \mathbb{N}$ and $t \in [0, 1]$ and 0 is not a right-hand rarefaction point of A. Indeed observe that $\frac{1}{5\cdot 6^{n-1}}(6^{n-1} + 2 \cdot 6^{n-2} + \cdots + 2^{n-2} \cdot 6 + 2^{n-1}) = \frac{1}{5}(1 + \frac{2}{6} + \frac{2^2}{6^2} + \cdots + \frac{2^{n-1}}{6^{n-1}}) < \frac{1}{5} \cdot \frac{1}{1-\frac{1}{3}} = \frac{3}{10}$.

To show that 0 is a right-hand $\mathcal{I}$-dispersion point of A take an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers. For each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$

such that $\frac{1}{n_m} \in [\frac{1}{6^{k_m}}, \frac{1}{6^{k_m-1}})$. Since the sequence $\frac{6^{k_m}}{n_m}$ is bounded ($1 \le \frac{6^{k_m}}{n_m} < 6$), there exists a convergent subsequence $\left\{\frac{6^{k_{m_p}}}{n_{m_p}}\right\}$. Put $c = \lim_{p\to\infty} \frac{6^{k_{m_p}}}{n_{m_p}}$. We have $\limsup_{p\to\infty}(n_{m_p} \cdot A) \cap [0, 1] = (c \cdot B) \cap [0, 1]$. Since this is a perfect nowhere dense set, the proof is finished. □

Now we shall present some basic properties of the topology $\mathcal{T}_i$. The proofs go exactly in the same way as for the topology $\mathcal{T}_{\mathcal{I}}$, so we shall give only references at the end of the following theorem.

Theorem 10 *The topology $\mathcal{T}_i$ on $\mathbb{R}$ has the following properties:*

1. *a subset A of $\mathbb{R}$ is closed and discrete with respect to $\mathcal{T}_i$ if and only if $A \in \mathcal{I}$,*
2. *a subset A of $\mathbb{R}$ is $\mathcal{T}_i$-nowhere dense if and only if it is $\mathcal{T}_i$—of the first category if and only if $A \in \mathcal{I}$,*
3. *$\mathcal{T}_i$ is neither first countable, nor separable, nor has the Lindelöf property,*
4. *$\mathcal{T}_i$ is not regular,*
5. *a subset A of $\mathbb{R}$ is compact with respect to $\mathcal{T}_i$ if and only if A is finite,*
6. *the family of $\mathcal{T}_i$—Borel sets coincides with the family of sets having the Baire property (in the natural topology), each set which is a $\mathcal{T}_i$—Borel set is the union of $\mathcal{T}_i$-open and $\mathcal{T}_i$-closed set,*
7. *if $A \subset \mathbb{R}$, then $\text{Int}_i(A) = A \cap \Phi_i(B)$, where $B \subset A$ is a Baire kernel of A.*

For the proofs of 1, 3, 5 see [3], p. 37, proof of 2, 6 see [9], proof of [4] see [14] and of 7 see [4].

The following property of $\mathcal{T}_i$ needs a separate proof.

Theorem 11 *Each interval $[a, b] \subset \mathbb{R}$ is connected in $\mathcal{T}_i$.*

Proof Suppose that $[a, b] = A \cup B$, where $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A, B \in \mathcal{T}_i$ (it means that A, B are open in the subspace topology $\mathcal{T}_i|_{[a,b]}$). We have $A \subset \Phi_i(A)$ and $B \subset \Phi_i(B)$, where at the endpoints of $[a, b]$ one sided intensity is considered. At the same time $A = G_1 \triangle P_1$, $B = G_2 \triangle P_2$, where G_1, G_2 are regular open and P_1, P_2 are of the first category (all in the natural topology). Let $(c, d) \subset [a, b]$ be a component of G_1 (in the natural topology). We shall prove that $[c, d] \subset A$. First observe that $P_1 \cap (c, d) = \emptyset$. Indeed, if there exists $x_0 \in P_1 \cap (c, d)$, then $x_0 \in B$, so $x_0 \in \Phi_i(B)$. It is impossible because A is residual in some neighbourhood of x_0 (in the natural topology) and $x_0 \in \Phi_i(A)$. A point d is a left-hand point of intensity of A, so it cannot belong to B and similarly for a point c. Obviously if (c, d) is a component of G_2, then $[c, d] \subset B$.

Take a closed component $[c_1, d_1] \subset [a, b]$ of A. Obviously $c_1 > a$ or $d_1 < b$. Suppose that the second case holds. Consider the interval $I_1 = [c_1, d_1 + (d_1 - c_1)] = [c_1, 2d_1 - c_1]$. Observe that there exists a point $x_1 \in B \cap (d_1, 2d_1 - c_1)$. Since $B \subset \Phi_i(B)$, in each neighbourhood of x_1 one can find a closed component of B. In particular there exists a closed component $[c_2, d_2]$ of B such that $c_2 \in (d_1, \min(b, 2d_1 - c_1))$. Similarly as above we can find a point $x_2 \in A \cap (2c_2 - d_2, c_2)$ and a closed component $[c_3, d_3]$ of A such that $d_3 \in (2c_2 - d_2, c_2)$. Continuing in this way we obtain a sequence of disjoint intervals $\{[c_n, d_n]\}_{n\in\mathbb{N}}$ such that $\bigcup_{n=1}^{\infty}[c_{2n-1}, d_{2n-1}] \subset A$,

$\bigcup_{n=1}^{\infty}[c_{2n}, d_{2n}] \subset B$, $d_n - c_n \underset{n\to\infty}{\to} 0$ and there exists $x_0 \in (a, b)$ such that $c_n \underset{n\to\infty}{\to} x_0$ and $d_n \underset{n\to\infty}{\to} x_0$. Observe also that the sequence $\{c_{2n-1}\}_{n\in\mathbb{N}}$ is increasing and $\{c_{2n}\}_{n\in\mathbb{N}}$ is decreasing. From the construction it follows that $d_n - c_n \geq \operatorname{dist}(x_0, [c_n, d_n])$. Consider the sequence $\{[c_{2n}, d_{2n}]\}_{n\in\mathbb{N}}$. Let k_n and m_n for $n \in \mathbb{N}$ be positive integers such that

$$\bigcup_{i=k_n}^{m_n}\left[x_0 + \frac{1}{i+1}, x_0 + \frac{1}{i}\right] \subset [c_{2n}, d_{2n}] \subset B.$$

Both sequences $\{k_n\}_{n\in\mathbb{N}}$, $\{m_n\}_{n\in\mathbb{N}}$ are increasing and tend to infinity. Moreover $\liminf_{n\to\infty}\frac{m_n}{k_n} \geq 2$. Then $\limsup_{n\to\infty}\frac{1}{m_n}\sum_{i=1}^{m_n}\chi_{i(i+1)((B-x_0)-\frac{1}{i+1})}(x) \geq \frac{1}{2}$ for each $x \in [0, 1]$ and the same holds for each subsequence $\{m_{n_p}\}_{p\in\mathbb{N}}$ of $\{m_n\}_{n\in\mathbb{N}}$. Hence x_0 is not a right-hand rarefaction point of B. Similarly we see that x_0 is not a left-hand rarefaction point of A. Then $x_0 \in (a, b)$ is neither intensity point of A, nor of B—a contradiction. □

Corollary 12 *The family of all $\mathcal{T}_i$—connected subsets of $\mathbb{R}$ coincides with the family of intervals of all kinds (open, closed, half-open, bounded and unbounded).*

4 Intense continuity

To introduce the notion of intense continuity (right- or left intense continuity we shall need the notion of interior intensity point of an arbitrary set $A \subset \mathbb{R}$. Recall that the Baire kernel of $A \subset \mathbb{R}$ is a set $B \subset A$ having the Baire property such that for each set C having the Baire property if $C \subset A$, then $C\setminus B \in \mathcal{I}$. For each set $A \subset \mathbb{R}$ there exists the Baire kernel of A (compare [11] or [6], §11.IV, Cor. 1, p. 90).

Definition 13 We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ is intensely continuous at a point $x_0 \in \mathbb{R}$ if and only if x_0 is an intensity point of a Baire kernel of $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$, in another words x_0 is an interior intensity point of $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ for each $\varepsilon > 0$. The right- or left intense continuity of f at x_0 is defined in an obvious way.

Definition 14 We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ is intensely continuous if for each set G open in the natural topology ($\mathcal{T}_{nat}$) $f^{-1}(G) \in \mathcal{T}_i$, i.e. f is continuous as a function from $(\mathbb{R}, \mathcal{T}_i)$ to $(\mathbb{R}, \mathcal{T}_{nat})$. The right- or left intense continuity as well as intense continuity $\mathcal{I}$-almost everywhere is defined in an obvious way.

Theorem 15 *A function $f : \mathbb{R} \to \mathbb{R}$ is intensely continuous (right intensely continuous) if and only if is intensely continuous (right intensely continuous) at each point $x \in \mathbb{R}$.*

Proof We shall prove first that if for each point $x \in A$ there exists a set $B_x \subset A$ having the Baire property such that $x \in \Phi_i(B_x)$, then A has the Baire property. Suppose that it is not the case. Then there exists an interval $(a, b) \subset \mathbb{R}$ such that for each $(c, d) \subset (a, b)$ the set $A \cap (c, d)$ is of the second category but not residual in (c, d). If B is an arbitrary Baire kernel of $A \cap (a, b)$, then B is of the first category, so for each $x \in A \cap (a, b)$ a point x is not an intensity point of B, a contradiction. The rest of the proof is routine. The proof in the case of right intense continuity is analogue. □

To prove basic properties of intensely continuous functions we shall need some lemmas. In the sequel $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ will denote the basis of the natural topology on $(0, 1)$.

Lemma 16 *If* 0 *is a right-hand intensity point of a regular open set* A*, then for each* $m \in \mathbb{N}$ *there exists a positive integer* n_m *such that for each* $i \in \{1, 2, \ldots, m\}$ *there exists an open interval* $(c_i^m, d_i^m) \subset (a_i, b_i)$ *such that*

$$\frac{1}{n_m}\sum_{j=1}^{n_m} \chi_{A_j}(t) \geq \frac{m-1}{m} \quad \text{for each } t \in \bigcup_{i=1}^{m}(c_i^m, d_i^m).$$

The sequence $\{n_m\}_{m\in\mathbb{N}}$ *can be chosen to be increasing. Here* $A_j = j(j+1)(A - \frac{1}{j+1}) \cap (0, 1)$*, as before.*

Proof This is obvious for $m = 1$. Suppose that the thesis holds for some $m \in \mathbb{N}$.

Let $\{r_p\}_{p\in\mathbb{N}}$ be an arbitrary increasing sequence of natural numbers such that $r_1 > n_m \geq m$. There exists a subsequence $\{r_{p_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \frac{1}{r_{p_k}} \sum_{j=1}^{r_{p_k}} \chi_{A_j}(t) = 1$ $\mathcal{I}$-almost everywhere on $(0, 1)$. Hence for each $i \in \{1, 2, \ldots, m, m+1\}$ there exists $t_i \in (a_i, b_i)$ and p_{k_i} such that for each $p \geq p_{k_i}$ we have $\frac{1}{r_p} \sum_{j=1}^{r_p} \chi_{A_j}(t_i) \geq \frac{m}{m+1}$. Let $p(m+1) = \max\{p_{k_1}, p_{k_2}, \ldots, p_{k_m}, p_{k_{m+1}}\}$ and $n_{m+1} = r_{p(m+1)}$. Then for each $i \in \{1, 2, \ldots, m, m+1\}$ we have $\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} \chi_{A_j}(t_i) \geq \frac{m}{m+1}$ and obviously $n_{m+1} > n_m \geq m$.

Since A is regular open and all A_n's are regular open too, for each $i \in \{1, 2, \ldots, m, m+1\}$ there exists a neighbourhood $(c_i^{m+1}, d_i^{m+1}) \subset (a_i, b_i)$ of t_i such that $\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} \chi_{A_j}(t) \geq \frac{m}{m+1}$ for each $t \in \bigcup_{i=1}^{m+1}(c_i^{m+1}, d_i^{m+1)})$.

Indeed, let $C_{m+1,i} = \{j : 1 \leq j \leq n_{m+1} \wedge t_i \in A_j\}$ for $i \in \{1, 2, \ldots, m+1\}$. If $j \in C_{m+1,i}$, then there exists $\varepsilon_{i,j}^{m+1} > 0$ such that $(t_i - \varepsilon_{i,j}^{m+1}, t_i + \varepsilon_{i,j}^{m+1}) \subset (a_i, b_i) \cap A_j$. Put $\varepsilon_i^{m+1} = \min\{\varepsilon_{i,j} : j \in C_{m+1,i}\}$. We have $(c_i^{m+1}, d_i^{m+1}) = (t_i - \varepsilon_i^{m+1}, t_i + \varepsilon_i^{m+1}) \subset (a_i, b_i) \cap A_j$ for each $i \in \{1, 2, \ldots, m, m+1\}$ and $j \in C_{m+1,i}$, so $\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} \chi_{A_j}(t) \geq \frac{m}{m+1}$ for each $t \in \bigcup_{i=1}^{m+1}(c_i^{m+1}, d_i^{m+1})$.

Hence there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers fulfilling all requirements. □

Lemma 17 *If* 0 *is a right-hand intensity point of a regular open set* A *if* $\{n_m\}_{m\in\mathbb{N}}$ *is a sequence of positive integers and* $\{(c_i^m, d_i^m)\}_{m\in\mathbb{N}, i\in\{1,2,\ldots,m\}}$*—a sequence of intervals constructed in the previous lemma, then for each* $m \in \mathbb{N}$ *there exist intervals* $(e_i^m, f_i^m) \subset (c_i^m, d_i^m)$ *for* $i = \{1, 2, \ldots, m\}$ *and a neighbourhood* O_m *of* 0 *such that*

$$\frac{1}{n_m}\sum_{j=1}^{n_m}\chi_{A_j(x)}(t) \geq \frac{m-1}{m} \quad \textit{for each } x \in O_m \textit{ and each } t \in \bigcup_{i=1}^{m}(e_i^m, f_i^m).$$

Proof Fix $m \in \mathbb{N}$. Put $C_{m,i} = \{j : 1 \leq j \leq n_m \wedge (c_i^m, d_i^m) \subset A_i\}$ for $i \in \{1, 2, \ldots, m\}$. Observe that $A_j(x) = (j(j+1)(A - \frac{1}{j+1}) - j(j+1)x) \cap (0, 1)$, so if $\varepsilon_{i,j}^m < \frac{1}{3j(j+1)}(d_i^m - c_i^m)$ for $i = \{1, 2, \ldots, m\}$ and $j \in C_{m,i}$, then $(e_i^m, f_i^m) = (\frac{2}{3}c_i^m + \frac{1}{3}d_i^m, \frac{1}{3}c_i^m + \frac{2}{3}d_i^m) \subset A_j(x)$ for $x \in (-\varepsilon_{i,j}^m, \varepsilon_{i,j}^m)$. Put now $\varepsilon_i^m = \min\{\varepsilon_{i,j}^m : j \in C_{m,i}\}$ and $\varepsilon^m = \min\{\varepsilon_i^m : i \in \{1, 2, \ldots, m\}\}$. Then for $x \in (-\varepsilon_m, \varepsilon_m) = O_m$ we have $\frac{1}{n_m}\sum_{j=1}^{n_m}\chi_{A_j(x)}(t) \geq \frac{m-1}{m}$ for $t \in \bigcup_{i=1}^{m}(e_i^m, f_i^m)$. □

Since the intensity of a set at points different from zero is defined with the use of the translation, we obtain immediately:

Corollary 18 *If $x_0 \in \mathbb{R}$ is a right-hand intensity point of a regular open set A, then for each $m \in \mathbb{N}$ there exists a natural number $n_m(x_0) \geq m$ such that there exists a neighbourhood $O_m(x_0)$ of x_0 and for each $i = \{1, 2, \ldots, m\}$ there exists an open interval $(e_i^m, f_i^m) \subset (a_i, b_i)$ such that*

$$\frac{1}{n_m(x_0)}\sum_{j=1}^{n_m(x_0)}\chi_{A_j(x)}(t) \geq \frac{m-1}{m}$$

$$\textit{for each } m \in \mathbb{N}, \ \textit{each } x \in O_m(x_0) \textit{ and each } t \in \bigcup_{i=1}^{m}(e_i^m, f_i^m).$$

Theorem 19 *If a function $f : \mathbb{R} \to \mathbb{R}$ is right intensely continuous at each point, then f is of the first Baire class.*

Proof Suppose that this is not the case. Then there exists a perfect set $P \subset \mathbb{R}$ and two real numbers a, b $(a < b)$ such that the sets $T_1 = \{x \in P : f(x) < a\}$ and $T_2 = \{x \in P : f(x) > b\}$ are both dense in P in the natural topology (see for example [10] or [6], p. 395). Let $C_i < T_i$ be a countable set, dense in T_i (thus also dense in P) for $i = 1, 2$. Consider first the set $C_1 = \{x_1, x_2, \ldots, x_k, \ldots\}$. For each $m, k \in \mathbb{N}$ let $O_{m,k}$ be the neighbourhood of x_k such that

$$\frac{1}{n_m(x_k)}\sum_{j=1}^{n_m(x_k)}\chi_{A_j(x)}(t) \geq \frac{m-1}{m}$$

$$\text{for each } x \in O_{m,k} \text{ and each } t \in \bigcup_{i=1}^{m}\left(e_{i,k}^m, f_{i,k}^m\right),$$

where $\{n_m(x_k)\}_{m\in\mathbb{N}}$, a sequence associated with x_k is described in the above corollary and $\{(e_{i,k}^m, f_{i,k}^m)\}_{m\in\mathbb{N}, i\in\{1,2,\ldots,m\}}$ also depends on x_k.

Put $O_m = P \cap \bigcup_{k=1}^{\infty} O_{m,k}$ and $O_a = P \cap \bigcap_{m=1}^{\infty} O_m$. The set O_a is residual in P, since each O_m is open and dense in P.

From the assumption it follows that each point of T_1 is a right intensity point of T_1. We shall show that if $x \in O_a \cap P$ and G_a is a regular open part of T_1, then x is not a rarefaction point of G_a. Indeed, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that $x \in \bigcap_{m=1}^{\infty} O_{m,k_m}$.

Let $n_m = n_m(x_{k_m})$ be a number associated with $m \in \mathbb{N}$ and $x_{k_m} \in C_1$. We can (and, choosing a subsequence if necessary, shall) suppose that $\{n_m\}_{m\in\mathbb{N}}$ is an increasing sequence. For each $m \in \mathbb{N}$ we have

$$\frac{1}{n_m}\sum_{j=1}^{n_m} \chi_{G_{a,j(x)}}(t) \geq \frac{m-1}{m} \quad \text{for each } t \in \bigcup_{i=1}^{m}\left(e_{i,k_m}^m, f_{i,k_m}^m\right),$$

where

$$G_{a,j}(x) = j \cdot (j+1)\left((G_a - x) - \frac{1}{j+1}\right) \cap (0,1),$$

as usual. So

$$\limsup_{m\to\infty} \frac{1}{n_m}\sum_{j=1}^{n_m} \chi_{G_{a,j(x)}}(t) = 1 \quad \text{for } t \in \limsup_{m\to\infty}\bigcup_{i=1}^{m}\left(e_{i,k_m}^m, f_{i,k_m}^m\right)$$

and the last set is residual in $(0, 1)$. The same argument works also for each subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ of $\{n_m\}_{m\in\mathbb{N}}$. That means that x is not a right-hand rarefaction point of G_a. Since $G_a \triangle T_1$ is the set of the first category, x is not a right-hand rarefaction point of T_1. Similarly, considering the set $C_2 \subset T_2$ one can prove the existence of the set O_b residual in P such that if $x \in O_b \cap T$, then x is not a right-hand rarefaction point of T_2.

Take now a point $x_0 \in O_a \cap O_b$ and ε such that $0 < \varepsilon < \frac{b-a}{3}$. Put $T = f^{-1}(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. A point x_0 is a right-hand intensity point of T and is a right-hand rarefaction point neither of T_1 nor of T_2. Hence $T \cap T_1 \neq \emptyset$ and $T \cap T_2 \neq \emptyset$. Let $x' \in T \cap T_1$ and $x'' \in T \cap T_2$. We have $|f(x') - f(x_0)| < \varepsilon$ and $|f(x'') - f(x_0)| < \varepsilon$, Recall that $f(x') < a$ and $f(x'') > b$, so $f(x'') - f(x') > b - a > 3\varepsilon$, but simultaneously $|f(x'') - f(x')| < 2\varepsilon$, a contradiction. □

Theorem 20 *If $f : \mathbb{R} \to \mathbb{R}$ is intensely continuous at each point, then f is of the first Baire class and has the Darboux property.*

Proof The first part follows from the previous theorem. The second is an immediate consequence of the following result of Z. Zahorski: if $f : \mathbb{R} \to \mathbb{R}$ is Baire one and for each $a \in \mathbb{R}$ the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are bilaterally dense in itself, then f has the Darboux property ([16], see also [1], Th. 1.1 (8)). □

Theorem 21 *A function $f : \mathbb{R} \to \mathbb{R}$ has the Baire property if and only if it is intensely continuous $\mathcal{I}$-almost everywhere.*

Proof Suppose that f has the Baire property. Then there exists a residual set $E \subset \mathbb{R}$ such that the restriction $f|_E$ is continuous. Then f is intensely continuous at each point of E, so it is intensely continuous $\mathcal{I}$-almost everywhere.

Suppose now that f is intensely continuous $\mathcal{I}$-almost everywhere. Consider the set $A = \{x \in \mathbb{R} : f(x) < a\}$ for $a \in \mathbb{R}$. Let E be the set of points where f is intensely continuos. Since $A \setminus E$ is of the first category, it is enough to show that $E \cap A$ has the Baire property. If $x \in E \cap A$, then there exists a $\mathcal{T}_i$-neighbourhood $A(x)$ of x such that $A(x) \subset A$. Then $A(x) \cap E$ is also a $\mathcal{T}_i$-neighbourhood of x and $E \cap A = \bigcup_{x \in E \cap A}(A(x) \cap E)$, so $E \cap A \in \mathcal{T}_i$ and obviously it has the Baire property. □

5 Restrictional intense continuity

Until now we have considered "topological" intense continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$. One can also consider a kind of "restrictional" or "path" intense continuity defined in the following way (compare [9] or [12]).

Definition 22 We say that a function $f : \mathbb{R} \to \mathbb{R}$ is restrictively intensely continuous at a point x_0 if and only if there exists a $\mathcal{T}_i$-neighbourhood U of x_0 such that $f(x_0) = \lim\limits_{\substack{x \to x_0 \\ x \in U}} f(x)$.

According to [5] if a topology $\mathcal{T}$ on the real line is invariant with respect to translations, then the topological and restrictional continuities (both with respect to $\mathcal{T}$) are equivalent if and only if the following condition (called (J_2) in [12]), pp. 28–30) holds:

(J_2) For each descending sequence $\{U_n\}_{n \in \mathbb{N}}$ of right-hand $\mathcal{T}$-neighbourhoods of 0 there exists a decreasing sequence $\{h_n\}_{n \in \mathbb{N}}$ tending to 0 such that $\{0\} \cup \bigcup_{n=1}^{\infty}(U_n \cap [h_{n+1}, h_n))$ is also a right-hand $\mathcal{T}$-neighbourhood of 0 and the analogous condition for the left-hand case (with necessary changes) holds.

For the topology $\mathcal{T}_i$ the above condition for the right-hand case can be formulated in the following more suitable way:

For each ascending sequence $\{A_n\}_{n \in \mathbb{N}}$ of sets having the Baire property such that 0 is a right-hand rarefaction point of each A_n there exists a decreasing sequence $\{h_n\}_{n \in \mathbb{N}}$ of positive numbers tending to 0 such that 0 is a right-hand rarefaction point of $\bigcup_{n=1}^{\infty}(A_n \cap [h_{n+1}, h_n))$. The formulation for the left-hand case is analogous.

Observe that restrictional continuity for the above mentioned topologies always implies the topological continuity.

It is well known ([1], Th. 5.6, p. 23 or [12], Ex. 14.1, p. 27) that a function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at a point x_0 if and only if it is restrictively approximately continuous at that point. In [9] it was proved that for $\mathcal{I}$-approximately continuous the situation is different, namely the condition (J_2) does not hold. Now we shall prove that for intensely continuous functions restrictional and topological continuities at a point also are different because (J_2) fails for $\mathcal{T}_i$.

Lemma 23 *There exists a double sequence* $\{E_{n,m}\}_{n,m \in \mathbb{N}}$ *of subsets of the interval* $(0, 1)$ *having the Baire property such that* $E_{n,m} \subset E_{n,m+1}$ *for each* $n, m \in \mathbb{N}$, $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{E_{n,m}}(x) = 0$ *for each* $x \in (0, 1)$ *and each* $m \in \mathbb{N}$ *and for each increasing sequence* $\{k_m\}_{m \in \mathbb{N} \cup \{0\}}$ *of positive integers (where* $k_0 = 0$*) if* $F_n = E_{n,m}$ *for* $k_{m-1} \le n < k_m$, *then* $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{F_i}(x) = 1$ *on a set residual in* $(0, 1)$.

Proof For $(a, b) \subset (0, 1)$ let $\{B_n(a, b)\}_{n\in\mathbb{N}}$ be a descending sequence of intervals defined in the following way: $B_n(a, b) = (a, a + \frac{1}{n}(b - a))$ for $n \in \mathbb{N}$. It is not difficult to see that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{B_i(a,b)}(x) = 0$ everywhere on $(0, 1)$. Put $E_{n,1} = B_n(0, 1)$, $E_{n,2} = B_n(0, 1) \cup B_n(0, \frac{1}{2}) \cup B_n(\frac{1}{2}, 1) = E_{n,1} \cup B_n(0, \frac{1}{2}) \cup B_n(\frac{1}{2}, 1)$ for $n \in \mathbb{N}$ and generally $E_{n,m+1} = E_{n,m} \cup \bigcup_{i=1}^{m+1} B_n(\frac{i-1}{m+1}, \frac{i}{m+1})$ for $n \in \mathbb{N}$. Immediately, we have $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{E_{i,m}}(x) = 0$ for each $x \in (0, 1)$ and each $m \in \mathbb{N}$.

If $\{k_m\}_{m\in\mathbb{N}\cup\{0\}}$ ($k_0 = 0$) is an increasing sequence of positive integers, we choose a subsequence $\{k_{m_p}\}_{p\in\mathbb{N}\cup\{0\}}$ (where $k_{m_0} = k_0 = 0$) such that $k_{m_p} \geq p \cdot k_{m_{p-1}}$ for $p \in \mathbb{N}$.

Observe that if $x \in D_p = \bigcup_{i=1}^{m_p} (\frac{i-1}{m_p}, \frac{i-1}{m_p} + \frac{1}{k_{m_p}-1} \cdot \frac{1}{m_p})$, then $x \in F_n = E_{n,m_p}$ for each n such that $k_{m_{p-1}} \leq n < k_{m_p}$. Hence $\frac{1}{k_{m_p}-1} \sum_{n=1}^{k_{m_p}-1} \chi_{F_n}(x) \geq \frac{1}{k_{m_p}-1} \sum_{n=k_{m_{p-1}}}^{k_{m_p}-1} \chi_{F_n}(x) = (k_{m_p} - k_{m_{p-1}}) \cdot \frac{1}{k_{m_p}-1} \geq \frac{k_{m_p}-k_{m_{p-1}}}{k_{m_p}} \geq (1 - \frac{1}{p})$ for each $x \in D_p$.

So, if $x \in \limsup_{p\to\infty} D_p$, then $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{F_n}(x) = 1$. Observe that $\limsup_{p\to\infty} D_p$ is residual in $(0, 1)$. In fact $\limsup_{p\to\infty} \frac{1}{k_{m_p}-1} \sum_{n=1}^{k_{m_{p-1}}} \chi_{F_n}(x) = 1$ for $x \in \limsup_{p\to\infty} D_p$ and the same result holds for each subsequence $\{k_{m_p}^{(1)}\}_{p\in\mathbb{N}}$ of $\{k_{m_p}\}_{p\in\mathbb{N}}$. □

Theorem 24 *There exists an increasing sequence $\{A_m\}_{m\in\mathbb{N}}$ of subsets of $(0, 1)$ having the Baire property such that 0 is a right-hand rarefaction point of A_m for each $m \in \mathbb{N}$ and for each increasing sequence $\{k_m\}_{m\in\mathbb{N}}$ of positive integers 0 is not a right-hand rerefaction point of the set $\bigcup_{m=1}^{\infty}(A_m \cap (\frac{1}{k_m}, \frac{1}{k_{m-1}}))$.*

Proof Put $A_m = \bigcup_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \cdot E_{n,m} + \frac{1}{n+1}\right)$ for each $m \in \mathbb{N}$, where $\{E_{n,m}\}_{n,m\in\mathbb{N}}$ is a double sequence from Lemma 23.

Then $E_{n,m} = n(n + 1)(A_m - \frac{1}{n+1}) \cap (0, 1)$ for $n, m \in \mathbb{N}$ and 0 is a right-hand rarefaction point of each A_m, so the result follows immediately from the above lemma. □

Corollary 25 *The topology $\mathcal{T}_i$ does not fulfil the condition (J_2).*

Proof Let $\{A_m\}_{m\in\mathbb{N}}$ be a sequence of sets from Theorem 11 and let $\{h_m\}_{m\in\mathbb{N}\cup\{0\}}$ be an arbitrary decreasing sequence convergent to 0. If $\{k_m\}_{m\in\mathbb{N}\cup\{0\}}$ is an increasing sequence of positive integers such that $\frac{1}{k_m} \leq h_m$ for each $m \in \mathbb{N}$, then

$$\bigcup_{m=1}^{\infty} \left(A_m \cap \left(\frac{1}{k_m}, \frac{1}{k_{m-1}}\right)\right) \setminus \{h_m : m \in \mathbb{N}\} \subset \bigcup (A_m \cap (h_m, h_{m-1}))$$

and the conclusion follows immediately from the above theorem. □

6 More properties of the intense topology

It is well known that if $A \subset [0, 1]$ is a regular open set such that 0 is a right-hand $\mathcal{I}$-density point of A, then there exists a right interval set $E \subset A$ such that 0 is also a right-hand $\mathcal{I}$-density point of E (see, for example [3], Lemma 2.2.4, p. 25). We shall show that for intensity the situation is different.

Recall that a right interval set (open or closed) at $a \in \mathbb{R}$ is a set of the form $\bigcup_{n=1}^{\infty}(a_n, b_n)$ (or $\bigcup_{n=1}^{\infty}[a_n, b_n]$) such that $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $a = \lim_{n\to\infty} a_n$. A left interval set at a is defined in the same way. The set E is an interval set if it is the union of a right interval set and a left interval set at the same point.

Theorem 26 *There exists a sequence* $\{A_n\}_{n\in\mathbb{N}}$ *of regular open subsets of* $[0, 1]$ *such that* $\lim_{j\to\infty} \frac{1}{n}\sum_{n=1}^{n} \chi_{a_j}(x) = 1$ *$\mathcal{I}$-a.e. and for each sequence* $\{B_n\}_{n\in\mathbb{N}}$ *of open sets such that* B_n *consists of finite number of components of* A_n *there exists an increasing sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of positive integers such that for each subsequence* $\{n_{k_p}\}_{p\in\mathbb{N}}$

$$\liminf_{p\to\infty} \frac{1}{n_{k_p}} \sum_{j=1}^{n_{k_p}} \chi_{B_j}(x) \le \frac{1}{2}$$

on a residual G_δ *subset of* $[0, 1]$.

Proof Each positive integer n can be represented uniquely in a form $n = 2^k + l$, where $k \in \mathbb{N} \cup \{0\}$ and $l \in \{0, 1, \ldots, 2^k - 1\}$. Let $q \in (0, 1)$.

Put $E_n = (0, 1)\setminus \bigcup_{m=0}^{\infty}[q^{2^k\cdot m+l+1}, q^{2^k\cdot m+l}]$ for $n = 2^k + l$ and $A_n = \bigcup_{i=0}^{k}(E_n + \frac{i}{k+1}) \cap (\frac{i}{k+1}, \frac{i+1}{k+1})$ for $n \in 2^k + l$.

Observe that for each $k \in \mathbb{N}$ and $l \in \{0, 1, \ldots, 2^{k-1}\}$ we have $\sum_{n=2^k}^{2^k+l} \chi_{A_n}(x) \ge l$ for all $x \in (0, 1)$ except on the set $\left\{\frac{i}{k+1} + q^m : i \in \{0, 1, \ldots, k\}, m \in \mathbb{N}\right\}$. Hence it is not difficult to see that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{A_j}(x) = 1$ except on a set $\mathbb{Q} \cap (0, 1)$, which means that the convergence is $\mathcal{I}$-a.e. which is stronger than the convergence in category. We shall show that the subsequence $n_k = 2^k$ is a required subsequence. Indeed, let $B_n \subset A_n$ for each $n \in \mathbb{N}$ be a set consisting of a finite number of components of A_n. Observe that for each $k \in \mathbb{N}$ there exists a positive number ε_k such that

$$B_n \cap \bigcup_{i=0}^{k}\left(\frac{i}{k+1}, \frac{i}{k+1} + \varepsilon_k\right) = \emptyset \quad \text{for each } n \in \{2^k, \ldots, 2^{k+1} - 1\}.$$

Hence $\frac{1}{2^k}\sum_{n=2^k}^{2^{k+1}-1} \chi_{B_n}(x) = 0$ for $x \in \bigcup_{i=0}^{k}\left(\frac{i}{k+1}, \frac{i}{k+1} + \varepsilon_k\right) = D_k$. From this it follows immediately that if $D = \limsup_{k\to\infty} D_k$, then $\liminf_{k\to\infty} \frac{1}{2^k-1}\sum_{n=1}^{2^{k+1}-1} \chi_{B_n}(x) \le \frac{1}{2}$. Obviously D is dense G_δ set in $(0, 1)$, so it is residual in $(0, 1)$.

The same holds for each subsequence of $\{n_k\}_{k\in\mathbb{N}}$, which ends the proof. □

Theorem 27 *There exists a set* $A \subset [0, 1]$ *such that* 0 *is a right-hand intensity point of* A *but it is not a right-hand intensity point of any interval set included in* A.

Proof Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets constructed in the above lemma. The set $A = \bigcup_{n=1}^{\infty}(\frac{1}{n(n+1)}A_n + \frac{1}{n+1})$ fulfills all requirements. □

Theorem 28 *Topological spaces* $(\mathbb{R}, \mathcal{T}_{\mathcal{I}})$ *and* $(\mathbb{R}, \mathcal{T}_i)$ *are not homeomorphic.*

Proof The families of $\mathcal{T}_{\mathcal{I}}$- and $\mathcal{T}_i$- connected sets are equal. Both consist of all intervals. Suppose that $h : (\mathbb{R}, \mathcal{T}_i) \to (\mathbb{R}, \mathcal{T}_{\mathcal{I}})$ is a homeomorphism. Then it is not difficult to see that if $[a, b] \subset \mathbb{R}$ is an arbitrary interval, then $h([a, b])$ is also a closed interval and since h is one-to-one, it must be a homeomorphism $h : (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{nat})$. Suppose that h is increasing. In the remaining case the proof is similar.

Let $A \subset \mathbb{R}$ be a $\mathcal{T}_i$-open set such that 0 is a right-hand intensity point of A, but it is not a right-hand intensity point of any interval set included in the regular part G of A. Such a set exists by virtue of Theorem 27. Then $h(A) \in \mathcal{T}_{\mathcal{I}}$ and $h(0)$ is a right-hand $\mathcal{I}$-density point of $h(G)$. Let $B \subset h(G)$ be an interval set such that $h(0)$ is a right-hand $\mathcal{I}$-density point of B. Then 0 should be a right-hand intensity point of $h^{-1}(B)$. But $h^{-1}(B)$ is an interval set included in G—a contradiction. □

Lemma 29 *There exists a sequence* $\{E_n\}_{n\in\mathbb{N}}$ *of subsets of* $(0, \frac{1}{2})$ *having the Baire property such that* $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\chi_{E_i}(x) = 0$ *except on a set of the first category but a sequence* $\{\frac{1}{n}\sum_{i=1}^{n}\chi_{F_i}\}_{n\in\mathbb{N}}$, *where* $F_n = \frac{2n+1}{2n}E_n$ *for* $n \in \mathbb{N}$, *does not converge to* 0 *in category.*

Proof (Compare the proof of Theorem 4 in [15]) Let $D_k = \bigcup_{i=1}^{2^{k-1}}(\frac{i}{2^k} - \varepsilon_k, \frac{i}{2^k})$ for $k \in \mathbb{N}$, where ε_k is a positive number small enough to assure that if $E_n = \frac{2n}{2n+1}D_k$ for $2^{k-1} \le n < 2^k - 1$, then $\{E_n\}_{2^{k-1}\le n<2^k-1}$ is a family of pairwise disjoint sets. To prove the existence of positive number ε_k fulfilling the above requirement it suffices to show that if $i, j \in \{1, 2, \ldots, 2^{k-1}\}$ and $m, n \in \{2^{k-1}, \ldots, 2^k - 1\}$, then the equality $\frac{i}{2^k}\cdot\frac{2n}{2n+1} = \frac{j}{2^k}\cdot\frac{2m}{2m+1}$ is possible only when $i = j$ and $m = n$. Indeed, suppose that $\frac{i\cdot 2n}{2n+1} = \frac{j\cdot 2m}{2m+1}$ and $i \ne j$. Then $i\cdot 2n\cdot(2m+1) = j\cdot 2m\cdot(2n+1)$, so $2nm(i-j) = mj - ni$. Observe that $|2nm(i-j)| \ge 2\cdot 2^{k-1}\cdot 2^{k-1}\cdot|i-j| \ge 2^{2k-1}$ and $|mj - ni| \le (2^k - 1)\cdot 2^{k-1} - 2^{k-1} = 2^{2k-1} - 2^k$ (since $m \le 2^k - 1$, $j \le 2^{k-1}$, $n \ge 2^{k-1}$ and $i \ge 1$) for each i, j, m and n from their domains, a contradiction. Hence $i = j$ and also $m = n$. So in the set $B_k = \{\frac{i}{2^k}\cdot\frac{2n}{2n+1} : i \in \{1, 2, \ldots, 2^{k-1}\}; n \in \{2^{k-1}, \ldots, 2^k - 1\}\} \cup \{0, \frac{1}{2}\}$ all points are different and each positive ε_k less than the smallest distance between points of B_k does the job.

Observe now that for each $x \in (0, 1)$ we have $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\chi_{E_i}(x) = 0$. Simultaneously $F_n = \frac{2n+1}{2n}\cdot E_n = D_k$ for $2^{k-1} \le n < 2^k - 1$. The set $D = \limsup_{k\to\infty} D_k$ is a G_δ set dense in $(0, \frac{1}{2})$.

If $x \in D$, then for infinitely many k's $x \in \bigcap_{n=2^{k-1}}^{2^k-1} F_n = D_k$ and then $\limsup_{k\to\infty}\frac{1}{2^k-1}\sum_{i=1}^{2^k-1}\chi_{F_i}(x) \ge \frac{1}{2}$. For each subsequence $\{2^{k_p-1}\}_{p\in\mathbb{N}}$ of the sequence $\{2^k - 1\}_{k\in\mathbb{N}}$ we have also $\limsup_{p\to\infty}\frac{1}{2^{k_p}-1}\sum_{i=1}^{2^{k_p}-1}\chi_{F_i}(x) \ge \frac{1}{2}$ for $x \in \hat{D} = \limsup_{p\to\infty} D_{k_p}$ and the set $\hat{D}$ is also a G_δ set dense in $(0, \frac{1}{2})$. □

Theorem 30 *There exists a set* $A \subset (0, 1)$ *such that* 0 *is a right-hand rarefaction point of* A *but not a right-hand rarefaction point of* $\frac{1}{2}A$.

Proof Put $A_n = \frac{1}{n(n+1)} \cdot E_n + \frac{1}{n+1}$ for $n \in \mathbb{N}$, where $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of sets from the lemma 5. Then $A_n \subset (\frac{1}{n+1}, \frac{1}{n+1} + \frac{1}{2}\frac{1}{n(n+1)}) = (\frac{1}{n+1}, \frac{2n+1}{2n(n+1)})$, which means that A_n is included in the left half of the interval $(\frac{1}{n+1}, \frac{1}{n})$. Let $A = \bigcup_{n=1}^{\infty} A_n$. We have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{i\cdot(i+1)\cdot\left(A-\frac{1}{i+1}\right)\cap(0,1)}(x) = 0$$

$\mathcal{I}$-a.e., because $i(i+1)\left(A - \frac{1}{i+1}\right) \cap (0, 1) = E_i$ for $i \in \mathbb{N}$. This means that 0 is a right-hand rarefaction point of A.

Consider now $\frac{1}{2}A = \frac{1}{2}\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}(\frac{1}{2}A_n)$. Observe that $\frac{1}{2}A_n$ is included in $\left(\frac{1}{2(n+1)}, \frac{2n+1}{4n(n+1)}\right)$ (and not necessarily in $\left(\frac{1}{2(n+1)}, \frac{1}{2n+1}\right)$, because $\frac{1}{2n+1} < \frac{2n+1}{4n(n+1)}$). Further $2(n+1)(2n+1)\left(\frac{1}{2}A_n - \frac{1}{2(n+1)}\right) = \frac{1}{2} \cdot 2 \cdot (n+1)(2n+1)(A_n - \frac{1}{n+1}) = \frac{(n+1)(2n+1)}{n(n+1)} \cdot E_n = \frac{2n+1}{n} E_n = 2F_n \subset (0, \frac{2n+1}{2n})$. Simultaneously $2(n+1)(2n+1)\left(\frac{1}{2}A - \frac{1}{2(n+1)}\right) \cap \left(0, \frac{2n+1}{2n}\right) = 2(n+1)(2n+1)\left(\frac{1}{2}A - \frac{1}{2(n+1)}\right) = 2F_n$. Since $\left\{\frac{1}{2n}\sum_{i=1}^{n} \chi_{2F_i}\right\}_{n\in\mathbb{N}}$ does not converge to 0 in category, the sequence $\left\{\frac{1}{n}\sum_{i=1}^{n} \chi_{2F_i\cap[0,1]}\right\}_{n\in\mathbb{N}}$ has the same property, so 0 is not a right-hand rarefaction point of $\frac{1}{2}A$. □

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