Strict K -monotonicity and K -order continuity in symmetric spaces

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Abstract This paper is devoted to strict K -monotonicity and K -order continuity in symmetric spaces. Using a local approach to the geometric structure in a symmetric space E we investigate a connection between strict K -monotonicity and global convergence in measure of a sequence of the maximal functions. Next, we solve an essential problem whether an existence of a point of K -order continuity in a symmetric space E on $[0, \infty)$ implies that the embedding $E \hookrightarrow L^1[0, \infty)$ does not hold. We present a complete characterization of an equivalent condition to K -order continuity in a symmetric space E using a notion of order continuity and the fundamental function of E . We also investigate a relationship between strict K -monotonicity and K -order continuity in symmetric spaces and show some examples of Lorentz spaces and Marcinkiewicz spaces having these properties or not. Finally, we discuss a local version of a crucial correspondence between order continuity and the Kadec–Klee property for global convergence in measure in a symmetric space E .

Keywords Symmetric space · Marcinkiewicz space · K -order continuity · Strict K -monotonicity · The Kadec–Klee property for global convergence in measure

Mathematics Subject Classification 46E30 · 46B20 · 46B42

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1 Introduction

In 1992, Kurc [17] presented a relation between the best dominated approximation and order continuity and monotonicity properties. In the paper [8], authors answered the question about the full characterization of the local monotonicity structure and order continuity of Banach lattices and applications in the best dominated approximation. The next motivating research was published in [9], where there has been established among others a connection between the best dominated approximation and the Kadec–Klee property for global convergence in measure in Banach function spaces. Recently, in view of the previous investigation there appeared many results [5–7, 10, 14] devoted to exploration of the global and local monotonicity and rotundity structure of Banach spaces applicable in the best approximation problems. The main inspiration for this article appeared in paper [4], where there has been introduced a new type of the best dominated approximation with respect to the Hardy–Littlewood–Pólya relation $\prec$. It is worth mentioning that author has investigated an application of strict K -monotonicity and K -order continuity to the best dominated approximation problems in the sense of the Hardy–Littlewood–Pólya relation. In the spirit of the previous articles, the essential purpose of this paper is to focus on the complete criteria for K -order continuity in symmetric spaces.

It is necessary to recall the significant paper [2], in which there has been shown a correlation between strict K -monotonicity and global convergence in measure of a sequence of the decreasing rearrangements in symmetric spaces. The next intention of this paper is to find a local version of a correspondence between strict K -monotonicity and global convergence in measure of a sequence of the maximal functions in symmetric spaces.

The present article is organized as follows. Preliminaries contain all needed terminologies, which will be used later. Here we also recall an earlier result, which play a crucial role in our further exploration.

Section 3 is dedicated to strict K -monotonicity in a symmetric space E . We start our investigation with the auxiliary result proving an existence of two elements less than $x \in E$ with respect to the Hardy–Littlewood–Pólya relation $\prec$ which are upper bounds of two disjoint complementary subsets of $\mathcal{M}(x, \tau, \epsilon)$ the family of the decreasing rearrangements bounded by x with respect to $\prec$. Then, in view of the above result, we present an application of a point of lower K -monotonicity to global convergence in measure of a sequence of decreasing rearrangements and a sequence of maximal functions to a decreasing rearrangement and a maximal function of a function with a finite distribution, respectively.

In the last Sect. 4 we answer the key question whether an existence of a point of K -order continuity in a symmetric space E on $[0, \infty)$ guarantees that E is not embedded in $L^1[0, \infty)$. Next, we discuss a full criteria for a point of K -order continuity in a symmetric space E . Namely, we show a complete characterization of an equivalent condition for a point of K -order continuity in a symmetric space E given in terms of a point of order continuity and the fundamental function ϕ of E in case when $I = [0, \infty)$. Next, we investigate an interesting relationship between strict K -monotonicity and K -order continuity in symmetric spaces. We also present several auxiliary examples of symmetric spaces having strict K -monotonicity and K -order continuity or not.

It is worth pointing out that strict K -monotonicity and K -order continuity coincide in certain symmetric spaces. Nevertheless, we show some symmetric spaces which possess one property but not another. Finally, using the local approach we describe a crucial relationship between order continuity and the Kadec–Klee property for global convergence in measure in a symmetric space E .

2 Preliminaries

Assume that $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ are the sets of reals, nonnegative reals and positive integers, respectively. For a Banach space $(X, \|\cdot\|_X)$ we denote by $S(X)$ (resp. $B(X)$) the unit sphere (resp. the closed unit ball). A nonnegative function ϕ defined on $\mathbb{R}^+$ is said to be *quasiconcave* if $\phi(t)$ is increasing and $\phi(t)/t$ is decreasing on $\mathbb{R}^+$ and also $\phi(t) = 0 \Leftrightarrow t = 0$. Let us denote as usual by μ the Lebesgue measure on $I = [0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$, and by L^0 the set of all (equivalence classes of) extended real valued Lebesgue measurable functions on I . We employ the notation $A^c = I \setminus A$ for any measurable set A . Given a Banach lattice $(E, \|\cdot\|_E)$ is said to be a *Banach function space* (or a *Köthe space*) if it is a sublattice of L^0 satisfying the following conditions

- (1) If $x \in L^0$, $y \in E$ and $|x| \leq |y|$ a.e., then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
- (2) There exists a strictly positive $x \in E$.

Additionally, to simplify the notation in the whole paper it is used the symbol $E^+ = \{x \in E : x \geq 0\}$.

An element $x \in E$ is called a *point of order continuity* if for any sequence $(x_n) \subset E^+$ such that $x_n \leq |x|$ and $x_n \rightarrow 0$ a.e. we have $\|x_n\|_E \rightarrow 0$. A Banach function space E is said to be *order continuous* (shortly $E \in (OC)$) if any element $x \in E$ is a point of order continuity (see [18]). We say that a Banach function space E has the *Fatou property* if for any $(x_n) \subset E^+$, $\sup_{n \in \mathbb{N}} \|x_n\|_E < \infty$ and $x_n \uparrow x \in L^0$, then we get $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$. Unless it is said otherwise, in the whole article it is considered that E has the Fatou property.

An element $x \in E$ is called an H_g *point* in E if for any sequence $(x_n) \subset E$ such that $x_n \rightarrow x$ globally in measure and $\|x_n\|_E \rightarrow \|x\|_E$, then $\|x_n - x\|_E \rightarrow 0$. Let us recall that the space E has the *Kadec–Klee property for global convergence in measure* if each element $x \in E$ is an H_g point in E .

The *distribution function* for any function $x \in L^0$ is given by

$$d_x(\lambda) = \mu \{s \in [0, \alpha) : |x(s)| > \lambda\}, \quad \lambda \geq 0.$$

For each element $x \in L^0$ we define its *decreasing rearrangement* by

$$x^*(t) = \inf \{\lambda > 0 : d_x(\lambda) \leq t\}, \quad t \geq 0.$$

We use the convention $x^*(\infty) = \lim_{t \rightarrow \infty} x^*(t)$ if $\alpha = \infty$ and $x^*(\infty) = 0$ if $\alpha = 1$. For any function $x \in L^0$ we define the *maximal function* of x^* by

$$x^{**}(t) = \frac{1}{t} \int_0^t x^*.$$

Given $x \in L^0$, it is commonly known that $x^* \leq x^{**}$, x^{**} is decreasing, continuous and subadditive. For more details of d_x , x^* and x^{**} the reader is referred to see [1, 16].

Let us recall that two functions $x, y \in L^0$ are called *equimeasurable* (shortly $x \sim y$) if $d_x = d_y$. A Banach function space $(E, \|\cdot\|_E)$ is said to be *symmetric* or *rearrangement invariant* (r.i. for short) if whenever $x \in L^0$ and $y \in E$ such that $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. For a symmetric space E we define ϕ_E its *fundamental function* given by $\phi_E(t) = \|\chi_{(0,t)}\|_E$ for any $t \in [0, \alpha)$ (see [1]). Given $x, y \in L^1 + L^\infty$ we define the *Hardy-Littlewood-Pólya relation* $\prec$ by

$$x \prec y \Leftrightarrow x^{**}(t) \leq y^{**}(t) \text{ for all } t > 0.$$

A symmetric space E is said to be *K-monotone* (shortly $E \in (KM)$) if for any $x \in L^1 + L^\infty$ and $y \in E$ with $x \prec y$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$. It is well known that a symmetric space is *K-monotone* if and only if E is exact interpolation space between L^1 and L^∞ . Let us also recall that a symmetric space E equipped with an order continuous norm or with the Fatou property is *K-monotone* (see [16]).

Given $x \in E$ is said to be a *point of lower K-monotonicity* of E (shortly an *LKM point*) if for any $y \in E$, $x^* \neq y^*$ with $y \prec x$, then $\|y\|_E < \|x\|_E$. Let us mention that given a symmetric space E is called *strictly K-monotone* (shortly $E \in (SKM)$) if any element of E is an *LKM point*. Recall that a symmetric space E has the *Kadec-Klee property with respect to $L^1 \cap L^\infty$* if for any $(x_n) \subset E$, $x \in E$ and any functional $f \in L^1 \cap L^\infty$ we have

$$f(x_n) \rightarrow f(x) \quad \text{and} \quad \|x_n\|_E \rightarrow \|x\|_E \quad \Rightarrow \quad \|x_n - x\|_E \rightarrow 0.$$

An element $x \in E$ is called a *point of K-order continuity* of E if for any sequence $(x_n) \subset E$ with $x_n \prec x$ and $x_n^* \rightarrow 0$ a.e. we have $\|x_n\|_E \rightarrow 0$. A symmetric space E is said to be *K-order continuous* (shortly $E \in (KOC)$) if any element x of E is a point of *K-order continuity*. We refer the reader for more information to see [2, 4, 5, 10].

For any quasiconcave function ϕ on I the Marcinkiewicz function space $M_\phi^{(*)}$ (resp. M_ϕ) is a subspace of L^0 such that

$$\begin{aligned} \|x\|_{M_\phi^{(*)}} &= \sup_{t>0} \{x^*(t)\phi(t)\} < \infty \\ \left(\text{resp. } \|x\|_{M_\phi} &= \sup_{t>0} \{x^{**}(t)\phi(t)\} < \infty \right). \end{aligned}$$

Clearly, $M_\phi \xhookrightarrow{1} M_\phi^{(*)}$. It is well known that the Marcinkiewicz space $M_\phi^{(*)}$ (resp. M_ϕ) is a r.i. quasi-Banach function space (r.i. Banach function space) with the fundamental

function ϕ on I . It is worth reminding that for any symmetric space E with the fundamental function ϕ we have $E \xhookrightarrow{1} M_\phi$ (for more details see [1, 16]).

Let us recall auxiliary result proved in [4] which will be useful in our further investigation.

Lemma 1 *Let $x \in L^1 + L^\infty$ and $x^*(\infty) = 0$, then $x^{**}(\infty) = 0$.*

3 Local approach to strict K -monotonicity in symmetric spaces

Now, we introduce the special set which is similar to a notion that was presented in [16]. Let E be a symmetric space and $x \in E$, $x = x^*$. Define for any $\epsilon > 0$ and $\tau > 0$,

$$\mathcal{M}(x, \tau, \epsilon) = \left\{ y \in E : y = y^*, y < x, \int_0^\tau y + \epsilon \leq \int_0^\tau x \right\}.$$

The first result shows that for any collection $\mathcal{M}(x, \tau, \epsilon)$ there exist two elements less than x with respect to the relation $<$ which are upper bounds of two disjoint complementary subsets of $\mathcal{M}(x, \tau, \epsilon)$. We wish to recall that the analogous problem in the opposite case, when the relations $<, \leq$ are replaced by $>, \geq$ respectively, has been considered in Corollary 2.2 in paper [2]. However we investigate the similar issue, the proof of Proposition 1 is quite long and requires new techniques. Although the case 1 in the proof follows from the case 2, we present the whole details for the sake of the reader's convenience and because of the construction that might arouse the reader's interest.

Proposition 1 *Let $x \in L^0$, $x = x^*$, $x^*(\infty) = 0$ and $\tau > 0$, $\epsilon > 0$. Then, there exist $\epsilon_1 > 0$, $\tau_1 \in (0, \tau)$, $z \in \mathcal{M}(x, \tau - \tau_1, \epsilon_1)$ and $w \in \mathcal{M}(x, \tau + \tau_1, \epsilon_1)$ such that for any $y \in \mathcal{M}(x, \tau, \epsilon)$ we have $y < z$ or $y < w$.*

Proof Let $y \in \mathcal{M}(x, \tau, \epsilon)$. Define for any decreasing function $u \in L^0$ and $t > 0$,

$$\phi_u(t) = \int_0^t u.$$

Since $x^*(\infty) = 0$, by monotonicity of functions x and y it is clear that ϕ_x and ϕ_y are concave, increasing and not affine on I . By assumption we have $\phi_y \leq \phi_x$ on I and also $\phi_y(\tau) \leq \phi_x(\tau) - \epsilon$. Let γ be an intersection of ϕ_x and $p(t) = \phi_x(\tau) - \epsilon$ and let β be an intersection of ϕ_x and $q(t) = t(\phi_x(\tau) - \epsilon)/\tau$. Denote

$$\xi = \frac{\phi_x(\beta) - \phi_x(\gamma)}{\beta - \gamma} = \frac{1}{\beta - \gamma} \int_\gamma^\beta x^* \quad \text{and} \quad \phi(t) = \xi(t - \gamma) + \phi_x(\gamma).$$

Now, we show the proof in cases.

Case 1 Assume that there exists $t \in (\gamma, \beta)$ such that $\phi(t) < \phi_x(t)$. Then, by concavity of ϕ_x for any $t \in (\gamma, \beta)$ we have $\phi(t) < \phi_x(t)$. Define

$$z = x^* \chi_{[0, \gamma) \cup [\beta, \infty)} + \xi \chi_{[\gamma, \beta]}.$$

Since ξ is an average value of x^* on (γ, β) , by monotonicity of x^* we observe $z = z^*$, $z < x$ and $z \neq x$. Moreover, by concavity of ϕ_y we notice that a tangent line of ϕ_y at τ has a slope $\eta \in [0, (\phi(\tau) - \epsilon)/\tau]$ and for any $t \in I$,

$$\phi_y(t) \leq \eta(t - \tau) + \phi_x(\tau) - \epsilon.$$

Hence, for any $t \in [\gamma, \beta]$ we have

$$\phi_y(t) \leq \eta(t - \tau) + \phi_x(\tau) - \epsilon \leq \xi(t - \tau) + \phi_x(\gamma) = \phi(t) = \phi_z(t).$$

Consequently, since $\phi_x = \phi_z$ on $[0, \gamma] \cup [\beta, \infty)$, by assumption $y < x$ we get $y < z$.

Case 2 Assume that for any $t \in [\gamma, \beta]$, $\phi_x(t) = \phi(t)$. Denote

$$\gamma_0 = \inf\{t > 0 : \phi_x(t) = \phi(t)\}.$$

Clearly, $\gamma_0 \leq \gamma$. We claim that $\gamma_0 > 0$. Indeed, if we suppose that $\gamma_0 = 0$, then $\phi_x = \phi$ on $[0, \beta]$ and so $\phi_x(0) = -\xi\gamma + \phi_x(\gamma) = 0$. Hence, by definition of ξ and by assumption that β is the intersection of ϕ_x and q we have

$$\xi = \frac{\phi_x(\gamma)}{\gamma} = \frac{\phi_x(\beta)}{\beta} = \frac{\phi_x(\tau) - \epsilon}{\tau}.$$

On the other hand, since ϕ_x is affine on $[\gamma_0, \beta]$ it follows that $\xi = \phi_x(\tau)/\tau$. Therefore, we obtain a contradiction which proves our claim. Now, without loss of generality we may suppose that $\epsilon > 0$ is small enough such that there exist $\gamma_1 > 0$ and $\beta_1 > 0$ the intersections of the functions ϕ_x and $\phi - \epsilon$. Observe $0 < \gamma_1 < \gamma_0 < \gamma < \beta < \beta_1$. Consider temporary the slope η of the tangent line of ϕ_y at τ is $\eta \in [\xi, (\phi_x(\tau) - \epsilon)/\tau]$. Define

$$z = x^* \chi_{[0, \gamma_1] \cup [\tau, \infty)} + \frac{\phi_x(\tau) - \phi_x(\gamma_1)}{\tau - \gamma_1} \chi_{[\gamma_1, \tau)}.$$

Then, since $\frac{\phi_x(\tau) - \phi_x(\gamma_1)}{\tau - \gamma_1}$ is the average value of x^* on (γ_1, τ) , by definition of γ_0 and by assumption ϕ_x is affine on the interval (γ_0, β) and by the concavity of ϕ_x it follows that $z = z^*$, $z < x$ and $z \neq x$. Furthermore, we have for every $t \in [\gamma_1, \tau]$,

$$\phi_y(t) \leq \eta(t - \tau) + \phi_x(\tau) - \epsilon \leq \phi_z(t).$$

Thus, since $\phi_x = \phi_z$ on $[0, \gamma_1] \cup [\tau, \infty)$, by assumption $y < x$ we conclude $y < z$, whenever $\eta \in [\xi, (\phi_x(\tau) - \epsilon)/\tau]$. Now, assume that $\eta \in [0, \xi]$. Define

$$w = x^* \chi_{[0, \gamma) \cup [\beta_1, \infty)} + \frac{\phi_x(\beta_1) - \phi_x(\gamma)}{\beta_1 - \gamma} \chi_{[\gamma, \beta_1)}.$$

Since $\frac{\phi_x(\beta_1) - \phi_x(\gamma)}{\beta_1 - \gamma}$ is the average value of x^* on $[\gamma, \beta_1]$, by definition of β_1 and by concavity of ϕ_x we get $w = w^*$, $w \prec x$ and $w \neq x$. Notice that for any $t \in [\gamma, \beta_1]$,

$$\phi_y(t) \leq \eta(t - \tau) + \phi_x(\tau) - \epsilon \leq \phi_w(t).$$

Moreover, $\phi_x = \phi_w$ on $[0, \gamma] \cup [\beta_1, \infty)$ and by assumption $y \prec x$ we obtain $y \prec w$ for any $\eta \in [0, \xi)$ which completes case 2.

Finally, combining both cases, by construction of z and w it is easy to see that there exist $\tau_1 \in (0, \tau)$ and $\epsilon_1 > 0$ such that $z \in \mathcal{M}(x, \tau - \tau_1, \epsilon_1)$ and $w \in \mathcal{M}(x, \tau + \tau_1, \epsilon_1)$. $\square$

Using the local approach we investigate a relationship between strict K -monotonicity and a convergence in measure of a sequence of decreasing rearrangements (resp. a convergence in measure of a sequence of maximal functions) to a decreasing rearrangement (resp. a maximal function) of a function with a finite distribution. Although the proof of the above theorem is similar to the proof of Proposition 2.4 in [2] we show the details for the sake of the reader's convenience.

Theorem 1 *Let E be a symmetric space. Assume that $x \in E$ is an LKM point, $x^*(\infty) = 0$ and $(x_n) \subset E$. If $x_n \prec x$ and $\|x_n\|_E \rightarrow \|x\|_E$, then $x_n^{**} \rightarrow x^{**}$ and $x_n^* \rightarrow x^*$ globally in measure.*

Proof Let ϕ be the fundamental function of E . Since $x_n \prec x$ for any $n \in \mathbb{N}$, by symmetry and by Theorem 4.6 in [1] it follows that

$$x_n^*(t)\phi(t) \leq \|x_n\|_E \leq \|x\|_E$$

for any $t \in I$ and $n \in \mathbb{N}$. Hence, by Helly's Selection Theorem in [20] passing to subsequence and relabelling if necessary there is $z \in E$ such that $z = z^*$ and x_n^* converges to z^* a.e. on I . It is easy to notice that for all $0 < t < s < \alpha$,

$$\int_t^s x_n^* \rightarrow \int_t^s z^*. \quad (1)$$

We claim that for any $t > 0$,

$$\int_0^t x_n^* \rightarrow \int_0^t x^*. \quad (2)$$

Indeed, if it is not true then there exist $\tau > 0$ and $\epsilon > 0$ and a sequence $(n_k) \subset \mathbb{N}$ such that for all $k \in \mathbb{N}$,

$$\int_0^\tau x_{n_k}^* \leq \int_0^\tau x^* - \epsilon.$$

Consequently, by Proposition 1 there are $\epsilon_1 > 0$, $\tau_1 > 0$ and $y \in \mathcal{M}(x, \tau - \tau_1, \epsilon_1)$ and $w \in \mathcal{M}(x, \tau + \tau_1, \epsilon_1)$ such that for any $k \in \mathbb{N}$,

$$x_{n_k} \prec y \quad \text{or} \quad x_{n_k} \prec w.$$

Since $y^* \neq x^*$, $w^* \neq x^*$ and $y \prec x$, $w \prec x$, by assumption that x is an LKM point we get for every $k \in \mathbb{N}$,

$$\|x_{n_k}\|_E \leq \max\{\|y\|_E, \|w\|_E\} < \|x\|_E.$$

Thus, by assumption $\|x_{n_k}\|_E \rightarrow \|x\|_E$ we obtain a contradiction which provides our claim (2). Now, combining (1) and (2) we have $x^* = z^*$ a.e. on I . Therefore, x_n^* converges to x^* a.e. on I . Since $x^*(\infty) = 0$, it is easy to prove that x_n^* converges to x^* globally in measure. Indeed, there exists $t_0 > 0$ such that $x^*(t) < \epsilon/2$ for any $t \geq t_0$. Without loss of generality we may assume that

$$|x_n^*(t_0) - x^*(t_0)| < \epsilon/2$$

for large enough $n \in \mathbb{N}$. In consequence, by monotonicity of a decreasing rearrangement for all $t \geq t_0$ and large enough $n \in \mathbb{N}$ we obtain

$$x_n^*(t) \leq x_n^*(t_0) \leq |x_n^*(t_0) - x^*(t_0)| + x^*(t_0) < \epsilon.$$

Hence, for any $t \geq t_0$ and for large enough $n \in \mathbb{N}$ we get

$$|x_n^*(t) - x^*(t)| < \epsilon$$

and since x_n^* converges to x^* locally in measure we conclude global convergence in measure. Moreover, by Lemma 1 we have $x^{**}(\infty) = 0$. Hence, there is $t_0 > 0$ such that for all $t \geq t_0$ we have $x^{**}(t) < \epsilon$. Since $x_n \prec x$ for any $n \in \mathbb{N}$ and by (2) it follows that x_n^{**} converges to x^{**} globally in measure. $\square$

4 K -order continuity in symmetric spaces

In this section we study a full criteria for K -order continuity in symmetric spaces. This notion was discussed for the first time in papers [12] (see Theorem 4.2) for a separable symmetric space with the Fatou property. Next, the problem was improved in paper [11], where authors have established K -order continuity in the more general setting of symmetric spaces of measurable operators under additional assumption. We start our investigation with an equivalent condition for K -order continuity in a symmetric space E . Moreover, we show a necessary and sufficient condition for K -order continuity in symmetric spaces using a simple direct proof. We also present a simple example of the order continuous rearrangement invariant space with the Fatou property that is not K -order continuous.

Lemma 2 *Let E be a symmetric space and $x_n, x \in E$ for all $n \in \mathbb{N}$. Consider the following assertions:*

- (i) *If $x_n \prec x$ and $x_n^* \rightarrow 0$ a.e., then $\|x_n\|_E \rightarrow 0$.*
- (ii) *If $x_n \prec x$ and $x_n \rightarrow 0$ globally in measure, then $\|x_n\|_E \rightarrow 0$.*

We have, (i) $\Rightarrow$ (ii); and if $x^(\infty) = 0$ then (ii) $\Rightarrow$ (i).*

Proof (i) $\Rightarrow$ (ii). Let assumption be satisfied. Then, by property 2.11 in [16] it follows that $x_n^* \rightarrow 0$ a.e. Consequently, since $x_n \prec x$ for any $n \in \mathbb{N}$, by symmetry of E and by (i) we get $\|x_n\|_E \rightarrow 0$.

(ii) $\Rightarrow$ (i). Let $x_n \prec x$ for all $n \in \mathbb{N}$ and $x_n^* \rightarrow 0$ a.e. and let $x^*(\infty) = 0$. Then, by Lemma 1 we get $x^{**}(\infty) = 0$. Hence, since

$$x_n^*(t) \leq x_n^{**}(t) \leq x^{**}(t)$$

for any $t > 0$ and $n \in \mathbb{N}$, we easily observe that $x_n^* \rightarrow 0$ globally in measure. Therefore, since $x_n \prec x$ for any $n \in \mathbb{N}$, by symmetry of E and by (ii) we complete the proof. $\square$

Immediately, by definition of K -order continuity and by the above lemma as well as by Lemma 2.5 in [8] in case when E is order continuous we conclude the following corollary.

Corollary 1 *Let E be a symmetric space and $x_n, x \in E$ for all $n \in \mathbb{N}$. If for any $x \in E$ we have $x^*(\infty) = 0$ or E is order continuous then the following conditions are equivalent.*

- (i) E is K -order continuous.
- (ii) If $x_n \prec x$ and $x_n \rightarrow 0$ globally in measure, then $\|x_n\|_E \rightarrow 0$.

Now we answer the question about a correspondence between a point of K -order continuity in a symmetric space E and an embedding $E \hookrightarrow L^1[0, \infty)$. For the sake of reader's convenience we present details of the proof of the following lemma.

Lemma 3 *Let E be a symmetric space on $I = [0, \infty)$ and let ϕ be the fundamental function of E . If $x \in E \setminus \{0\}$ is a point of K -order continuity, then $\lim_{t \rightarrow \infty} \phi(t)/t = 0$ and E is not embedded in $L^1[0, \infty)$. Additionally, if $x^*(\infty) = 0$ we have $\lim_{t \rightarrow \infty} \phi(t)x^{**}(t) = 0$.*

Proof Let $t_x > 0$ be such that $x^*(t_x) > 0$. Define

$$y_n = \frac{x^*(t_x)}{n} \chi_{[0, nt_x]}$$

for any $n \in \mathbb{N}$. Clearly, $y_n = y_n^* \prec y_1^* \leq x^*$ and $\int_0^\infty y_n^* = x^*(t_x)t_x < \infty$ for any $n \in \mathbb{N}$. Thus, since $y_n^* \rightarrow 0$ a.e. and by assumption that x is a point of K -order continuity we obtain

$$x^*(t_x)t_x \frac{\phi(nt_x)}{nt_x} = x^*(t_x) \frac{\phi(nt_x)}{n} = \|y_n\|_E \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $\lim_{t \rightarrow \infty} \phi(t)/t = 0$. Let us recall the well known fact that $\lim_{t \rightarrow \infty} \phi(t)/t > 0$ if and only if $\phi(t) \approx t$ for large enough $t > 0$. Indeed, by Corollary 5.3 in [1] it follows that the fundamental function ϕ of E is quasiconcave which concludes $\phi(t)/t$ is decreasing and continuous and provides the above fact. Now, let us show the claim

that $\phi(t) \approx t$ for large enough $t > 0$ if and only if $E \xhookrightarrow{C} L^1[0, \infty)$. In fact, if $\phi(t) \approx t$ for sufficiently large $t > 0$, then there exist $A, B > 0$ and $t_0 > 0$ such that for all $t \geq t_0$ we have $At \leq \phi(t) \leq Bt$. Thus, by Proposition 5.9 in [1] we get

$$A \|z\|_{L^1} = A \sup_{t \geq t_0} \left\{ \int_0^t z^*(s) ds \right\} \leq \sup_{0 < t < \infty} \{z^{**}(t)\phi(t)\} = \|z\|_{M_\phi} \leq \|z\|_E$$

for any $z \in E$, which implies the embedding of E in $L^1[0, \infty)$ with $C = 1/A$. Assuming conversely that $E \xhookrightarrow{C} L^1[0, \infty)$ with $C > 0$ we obtain immediately $t \leq C\phi(t)$ for all $t > 0$. On the other hand, since $\phi(t)/t$ is decreasing on I , there exists $B > 0$ such that $\phi(t) \leq Bt$ for large enough $t > 0$. Consequently, $\phi(t) \approx t$ for sufficiently large $t > 0$ and this proves the claim. Hence, since $\lim_{t \rightarrow \infty} \phi(t)/t = 0$ we get E is not embedded in $L^1[0, \infty)$. Now, assume that $x^*(\infty) = 0$. Denote for any $n \in \mathbb{N}$,

$$x_n = x^{**}(n)\chi_{[0,n]}.$$

Then, we have $x_n = x_n^* < x$ for all $n \in \mathbb{N}$. Since $x^*(\infty) = 0$, by Lemma 1 it follows that $x_n^* \rightarrow 0$ a.e. on I . Finally, according to assumption that x is a point of K -order continuity we obtain $x^{**}(n)\phi(n) = \|x_n\|_E \rightarrow 0$ as $n \rightarrow \infty$. $\square$

In the next result we investigate a complete characterization of K -order continuity in symmetric spaces. Namely, we show a connection between order continuity and K -order continuity in symmetric spaces.

Theorem 2 *Let E be a symmetric space with the fundamental function ϕ and let $x \in E$. Then, the following conditions are equivalent.*

(i) *x is a point of order continuity and*

$$\lim_{s \rightarrow \infty} \phi(s)x^{**}(s) = 0.$$

(ii) *x is a point of K -order continuity and $x^*(\infty) = 0$.*

Proof (ii) $\Rightarrow$ (i). The proof in this case follows immediately from Lemma 3 and it is omitted because of its triviality.

(i) $\Rightarrow$ (ii). We prove only the theorem in case when $\alpha = \infty$, because the proof in the case when $\alpha = 1$ is similar. Let $(x_n) \subset E$ and $x_n^* \rightarrow 0$ a.e., $x_n < x$. Clearly, we may assume $x \neq 0$. First, we claim that x_n^* converges to zero globally in measure. Indeed, since for any $t > 0$ and $n \in \mathbb{N}$,

$$x_n^*(t) \leq x_n^{**}(t) \leq x^{**}(t), \quad (3)$$

by Lemma 2.5 in [8] and Lemma 1 for any $\epsilon > 0$ there is $t_\epsilon > 0$ such that for all $t \geq t_\epsilon$ and $n \in \mathbb{N}$ we have

$$\max\{x_n^*(t), x^*(t)\} \leq \frac{\epsilon}{2}. \quad (4)$$

Hence, since $x_n^* \rightarrow 0$ locally in measure we get

$$\mu \left(t : x_n^*(t) \geq \epsilon \right) = \mu \left(t \in [0, t_\epsilon] : x_n^*(t) \geq \epsilon \right) \rightarrow 0,$$

which proves our claim. Since $x^* \neq 0$, there exists $t_0 \in (0, t_\epsilon)$ such that $x^*(t_0) > 0$. Define $\delta_0 = x^*(t_0)/2$ and for any $n \in \mathbb{N}$,

$$M_n = \left\{ t \in [0, t_0] : x^*(t) \leq x_n^*(t) \right\}.$$

Then, by monotonicity of x^* it is easy to see that for any $t \in [0, t_0]$,

$$x^*(t) \geq x^*(t_0) > \delta_0.$$

Therefore, for any $n \in \mathbb{N}$ we obtain

$$M_n \subset \left\{ t : x_n^*(t) > \delta_0 \right\},$$

whence, by the claim $x_n^* \rightarrow 0$ globally in measure we get $\mu(M_n) \rightarrow 0$. Moreover, by assumption that $x_n^* \rightarrow 0$ a.e., we may suppose without loss of generality that $x_n^*(t_0) \rightarrow 0$. Clearly, for any $n \in \mathbb{N}$ we observe

$$(x_n^* - x^*)^+ \chi_{[t_0, t_\epsilon]} \leq x_n^* \chi_{[t_0, t_\epsilon]} \leq x_n^*(t_0) \chi_{[t_0, t_\epsilon]}.$$

Hence, by symmetry of E it is easy to notice that

$$\left\| (x_n^* - x^*)^+ \chi_{[t_0, t_\epsilon]} \right\| \leq x_n^*(t_0) \left\| \chi_{[t_0, t_\epsilon]} \right\|$$

for each $n \in \mathbb{N}$. Consequently, since $x_n^*(t_0)$ converges to zero, this implies

$$\left\| (x_n^* - x^*)^+ \chi_{[t_0, t_\epsilon]} \right\| \rightarrow 0. \quad (5)$$

Now, according to (3) we have

$$(x_n^* - x^*)^+ \leq x_n^* \prec x^* \quad \text{and} \quad (x^* - x_n^*)^+ \leq x^*$$

for every $n \in \mathbb{N}$. In consequence, by Hardy's lemma [1] it follows that

$$\int_0^t ((x_n^* - x^*)^+)^* \chi_{[0, \mu(M_n)]} \leq \int_0^t x^* \chi_{[0, \mu(M_n)]}$$

for all $n \in \mathbb{N}$. Thus, by Proposition 1.1 [3] for any $t > 0$ and $n \in \mathbb{N}$ we conclude

$$\begin{aligned} ((x_n^* - x^*)^+ \chi_{M_n})^{**}(t) &= \frac{1}{t} \int_0^t ((x_n^* - x^*)^+ \chi_{M_n})^* \\ &\leq \frac{1}{t} \int_0^t ((x_n^* - x^*)^+)^* \chi_{[0, \mu(M_n)]} \\ &\leq (x^* \chi_{[0, \mu(M_n)]})^{**}(t) \leq x^{**}(t). \end{aligned}$$

Hence, since E is a symmetric space and

$$(x_n^* - x^*)^+ \chi_{M_n} = (x_n^* - x^*)^+ \chi_{[0, t_0]}$$

for a.e. on I and for any $n \in \mathbb{N}$, we have

$$\|(x_n^* - x^*)^+ \chi_{[0, t_0]}\|_E = \|(x_n^* - x^*)^+ \chi_{M_n}\|_E \leq \|x^* \chi_{[0, \mu(M_n)]}\|_E$$

for each $n \in \mathbb{N}$. In consequence, since $x^* \chi_{[0, \mu(M_n)]} \leq x^*$ and $\mu(M_n) \rightarrow 0$, by assumption that x is a point of order continuity and by Lemma 2.6 [8] it follows

$$\|(x_n^* - x^*)^+ \chi_{[0, t_0]}\|_E \rightarrow 0,$$

whence, by (5) and by the triangle inequality of the norm in E we obtain

$$\|(x_n^* - x^*)^+ \chi_{[0, t_\epsilon]}\|_E \rightarrow 0. \quad (6)$$

Furthermore, since $\min\{x_n^*, x^*\} \leq x^*$ and $\min\{x_n^*, x^*\} \rightarrow 0$ a.e., using analogous argument as previously we conclude

$$\|\min\{x_n^*, x^*\}\|_E \rightarrow 0.$$

Therefore, since $x_n^* = \min\{x_n^*, x^*\} + (x_n^* - x^*)^+$ for any $n \in \mathbb{N}$, by (6) we get

$$\|x_n^* \chi_{[0, t_\epsilon]}\|_E \rightarrow 0. \quad (7)$$

Now, we need to only show

$$\|x_n^* \chi_{[t_\epsilon, \infty)}\|_E \rightarrow 0. \quad (8)$$

Define for any $n \in \mathbb{N}$,

$$y_n = \left(\frac{1}{n} \int_0^n x^* \right) \chi_{[0, n)} + x^* \chi_{[n, \infty)}.$$

It is obvious that $y_n = y_n^*$ for all $n \in \mathbb{N}$. We claim that $y_n \prec x$ for any $n \in \mathbb{N}$. In fact, by monotonicity of the maximal function of x^* , for any $n \in \mathbb{N}$ and $t \leq n$ we have

$$y_n^{**}(t) = \frac{1}{t} \int_0^t \left(\frac{1}{n} \int_0^n x^* \right) \chi_{[0, n)} = \frac{1}{n} \int_0^n x^* \leq x^{**}(t).$$

Next, taking $n \in \mathbb{N}$ and $t > n$ by definition of y_n we get immediately $y_n^{**}(t) = x^{**}(t)$ which proves our claim. Since $x^*(\infty) = 0$, by Lemma 1 it follows that $y_n^* \rightarrow 0$ a.e. on I . Moreover, for each $n \in \mathbb{N}$ we have

$$\|y_n\|_E \leq \phi(n)x^{**}(n) + \|x^*\chi_{[n,\infty)}\|_E.$$

In consequence, since x is a point of order continuity, by Lemma 2.6 in [8] and by assumption that $\phi(n)x^{**}(n) \rightarrow 0$ as $n \rightarrow \infty$ it follows that

$$\|y_n\|_E \rightarrow 0. \quad (9)$$

Since $x_n^* \rightarrow 0$ a.e., we may assume without loss of generality that $x_n^*(t_\epsilon) \rightarrow 0$. Moreover, we may find a subsequence $(n_k) \subset \mathbb{N}$ such that for any $k \in \mathbb{N}$,

$$x_{n_k}^*(t_\epsilon) \leq \frac{1}{k} \int_0^k x^*.$$

Fix $k \in \mathbb{N}$. Then, for any $t \in [0, k)$ we get

$$(x_{n_k}^*\chi_{[t_\epsilon,\infty)})^*(t) \leq x_{n_k}^*(t_\epsilon) \leq y_k^*(t).$$

Furthermore, since $y_k^{**}(t) = x^{**}(t)$ for all $t \geq k$ and by assumption $x_n \prec x$ for any $n \in \mathbb{N}$, considering $t \geq k$ we observe

$$(x_{n_k}^*\chi_{[t_\epsilon,\infty)})^{**}(t) \leq x_{n_k}^{**}(t) \leq x^{**}(t) = y_k^{**}(t).$$

Hence, we obtain $x_{n_k}^*\chi_{[t_\epsilon,\infty)} \prec y_k$ for each $k \in \mathbb{N}$. In consequence, by symmetry of E this yields that

$$\|x_{n_k}^*\chi_{[t_\epsilon,\infty)}\|_E \leq \|y_k\|_E$$

for any $k \in \mathbb{N}$. Thus, by (9) and by Lemma 2.3 in [2] we prove (8). Finally, according to (7) and (8) we get the end of the proof. $\square$

We show that the conclusion (i) $\Rightarrow$ (ii) in Theorem 2 without the assumption that $\lim_{s \rightarrow \infty} \phi(s)x^{**}(s) = 0$ is not true in general. Namely, we prove that any element in $L^1[0, \infty)$ is not point of K -order continuity.

Remark 1 It is well known that each element x of $L^1[0, \infty)$ is a point of order continuity (see [18]). Consider $x = \chi_{[0,1]}$ and $x_n = \frac{1}{n}\chi_{[0,n]}$ for any $n \in \mathbb{N}$. It is easy to see that $x_n \rightarrow 0$ and $x_n \prec x$ for every $n \in \mathbb{N}$. On the other hand, we can easily show $\|x_n\|_{L^1} = 1$ for any $n \in \mathbb{N}$, which concludes that x is no point of K -order continuity. Hence, it is easy to observe that any $y \in L^1 \setminus \{0\}$ is no point of K -order continuity. Indeed, finding $\beta, \gamma \in \mathbb{R}^+$ such that $\beta\chi_{(0,\gamma)} \prec y^*$ and proceeding analogously as above with x we finish our investigation. $\square$

In the next example, we show that the assumption $x^*(\infty) = 0$ in the proof (ii) $\Rightarrow$ (i) of Theorem 2 is necessary and cannot be omitted.

Example 1 We consider the Marcinkiewicz function space $M_\phi^{(*)}$ with $\phi(t) = 1 - \frac{1}{1+t}$ on $I = [0, \infty)$. Define $x = \chi_{[0, \infty)}$. We claim that x is a point of K -order continuity and it is no point of order continuity in $M_\phi^{(*)}$. It is easy to see that $x^*(\infty) = 1$ and $\|x\|_{M_\phi^{(*)}} = 1$. By Lemma 2.5 [8] it follows that x is no point of order continuity in $M_\phi^{(*)}$. Let $(x_n) \subset M_\phi^{(*)}$ be such that $x_n \prec x$ and $x_n^* \rightarrow 0$ a.e. In view of the fact that $\|\cdot\|_{M_\phi^{(*)}}$ satisfies the triangle inequality with a constant $C > 0$, we observe for all $n \in \mathbb{N}$,

$$\begin{aligned} \|x_n\|_{M_\phi^{(*)}} &\leq C \|x_n^* \chi_{[0,1]}\|_{M_\phi^{(*)}} + C \|x_n^* \chi_{(1,\infty)}\|_{M_\phi^{(*)}} \\ &\leq C \|x_n^* \chi_{[0,1]}\|_{M_\phi^{(*)}} + C x_n^*(1) \|x\|_{M_\phi^{(*)}}. \end{aligned} \quad (10)$$

By assumption $x_n \prec x$ and by definition of x and right-continuity of x^* we get $x_n^*(t) \leq 1$ for all $n \in \mathbb{N}$ and $t \in [0, \infty)$. Thus, by Bolzano-Weierstrass theorem (see [19]), passing to subsequence and relabelling if necessary we may assume that for each $n \in \mathbb{N}$ there exists $t_n \in [0, 1]$ such that $\lim_{n \rightarrow \infty} t_n = t_0$ and

$$\|x_n^* \chi_{[0,1]}\|_{M_\phi^{(*)}} = \sup_{0 < t < 1} \{x_n^*(t) \phi(t)\} \leq x_n^*(t_n) \left(1 - \frac{1}{1+t_n}\right) + \frac{1}{n} \quad (11)$$

for all $n \in \mathbb{N}$. Since x_n^* is decreasing for any $n \in \mathbb{N}$, if $t_0 > 0$, then it is easy to notice that for large enough $n \in \mathbb{N}$,

$$x_n^*(t_n) \left(1 - \frac{1}{1+t_n}\right) \leq x_n^*\left(\frac{t_0}{2}\right) \left(1 - \frac{1}{1+t_n}\right) \leq x_n^*\left(\frac{t_0}{2}\right),$$

whence, by assumption $x_n^* \rightarrow 0$ a.e. and by (10) and (11), this yields that $\|x_n\|_{M_\phi^{(*)}} \rightarrow 0$. Now, suppose $t_0 = 0$. Then, $t_n \rightarrow 0$ and since $x_n^* \leq 1$ for all $n \in \mathbb{N}$ we have

$$x_n^*(t_n) \left(1 - \frac{1}{1+t_n}\right) \leq \left(1 - \frac{1}{1+t_n}\right) \rightarrow 0.$$

Hence, proceeding analogously as in previous case we complete the proof.

The immediate consequence of Theorem 2 is the following result.

Corollary 2 *Let E be a symmetric space with the fundamental function ϕ . The following conditions are equivalent.*

(i) *E is order continuous and for any $x \in E$,*

$$\lim_{s \rightarrow \infty} \phi(s) x^{**}(s) = 0.$$

(ii) *E is K -order continuous and $x^*(\infty) = 0$ for any $x \in E$.*

Now, we present a nontrivial symmetric space that is K -order continuous.

Example 2 Assuming that ϕ is a quasiconcave function on I with $\inf_{t>0} \phi(t) = 0$ and $\sup_{t>0} \frac{t}{\phi(t)} = \infty$, by Theorem 1.3 in [13] it is well known that the subspace of all points of order continuity in the Marcinkiewicz space M_ϕ coincides with

$$(M_\phi)_a = \left\{ y \in M_\phi : \lim_{t \rightarrow 0^+, \infty} \phi(t)y^{**}(t) = 0 \right\} \neq \{0\}.$$

Finally, according to Corollary 2 we observe that $(M_\phi)_a$ is K -order continuous.

In the following remark we discuss a connection between strict K -monotonicity and K -order continuity in symmetric spaces.

Remark 2 First, let us notice that by Theorem 2.7 in [2] and by Corollary 2, in any K -order continuous symmetric space E such that $x^*(\infty) = 0$ for all $x \in E$, the Kadec–Klee property with respect to $L^1 \cap L^\infty$ is equivalent to strict K -monotonicity and the Kadec–Klee property for global convergence in measure. On the other hand, it is easy to see that strict K -monotonicity does not imply K -order continuity in general. Indeed, considering the Lorentz space $\Lambda_{1,\phi'}$ with a strictly concave fundamental function ϕ such that $\phi(0^+) = 0$, $\phi(\infty) = \infty$ and $\sup_{t>0} \frac{t}{\phi(t)} < \infty$, by Proposition 1.4 in [15] and by Theorem 2.11 in [2] we obtain $\Lambda_{1,\phi'}$ is strictly K -monotone. Furthermore, since $\lim_{t \rightarrow \infty} \phi(t)(\chi_{[0,1]})^{**}(t) > 0$, by Corollary 2 it follows that the Lorentz space $\Lambda_{1,\phi'}$ is not K -order continuous. However, by Corollary 1.3 in [2] and by Theorem 1, we may observe additionally that for any $(x_n) \subset \Lambda_{1,\phi'}$, $x \in \Lambda_{1,\phi'}$,

$$x_n \prec x \quad \text{and} \quad \|x_n\|_{\Lambda_{1,\phi'}} \rightarrow \|x\|_{\Lambda_{1,\phi'}} \quad \Rightarrow \quad \|x_n^* - x^*\|_{\Lambda_{1,\phi'}} \rightarrow 0.$$

□

In the next example we show that K -order continuity does not imply strict K -monotonicity in general. Namely, we present the Marcinkiewicz space which is K -order continuous and it is not strictly K -monotone.

Example 3 Consider the Marcinkiewicz space M_ϕ where ϕ is given

$$\phi(t) = t\chi_{[0,1)}(t) + \chi_{[1,2)}(t) + (\sqrt{t} - \sqrt{2} + 1)\chi_{[2,\infty)}(t)$$

for any $t \in [0, \infty)$. It is easy to see that ϕ is quasiconcave and $\inf_{t>0} \phi(t) = 0$ and $\sup_{t>0} \frac{t}{\phi(t)} = \infty$. Define

$$x(t) = \chi_{[0, \frac{3}{2})}(t) \quad \text{and} \quad y(t) = \chi_{[0,1)}(t) + \frac{1}{2}\chi_{[1,2)}(t)$$

for any $t \in [0, \infty)$. Then, we easily observe that $x = x^* \neq y^* = y$ and $y \prec x$ and

$$\lim_{t \rightarrow 0^+, \infty} \phi(t)x^{**}(t) = \lim_{t \rightarrow 0^+, \infty} \phi(t)y^{**}(t) = 0.$$

Hence, by Theorem 1.3 in [13] we get $x, y \in (M_\phi)_a$. Moreover, we have $\|x\|_{M_\phi} = \|y\|_{M_\phi} = 1$. Thus, x is not an LKM point in $(M_\phi)_a$ and so the Marcinkiewicz space $(M_\phi)_a$ is not strictly K -monotone. On the other hand, by Example 2 we obtain that $(M_\phi)_a$ is K -order continuous.

In Proposition 1.1 [2] authors have proved that in a symmetric space E the Kadec–Klee property or Kadec–Klee property for local convergence in measure implies order continuity of the norm in E . On the other hand, they have shown that the Lorentz space, order continuous or not, has the Kadec–Klee property for global convergence in measure. In view of the previous investigation, we discuss a local version of a relationship between the Kadec–Klee property for global convergence in measure and order continuity in symmetric spaces. Namely, we establish an essential correspondence between an H_g point and a point of order continuity in symmetric spaces under an additional assumption.

Proposition 2 *Let E be a symmetric space and let $x \in E$, $x^*(\infty) = 0$. If x is an H_g point, then x is a point of order continuity in E .*

Proof Let $(x_n) \subset E^+$, $x_n \leq |x|$ and $x_n \rightarrow 0$ a.e. Since $x^*(\infty) = 0$, by property 2.12 in [16] it follows that $x_n^*(t) \rightarrow 0$ for all $t > 0$. Moreover, by Proposition 1.7 [1] we have $x_n^* \leq x^*$ a.e. and consequently it is easy to show that $x_n^* \rightarrow 0$ in measure. Hence, we get $0 \leq x^* - x_n^* \leq x^*$ a.e. for any $n \in \mathbb{N}$ and $x^* - x_n^* \rightarrow x^*$ in measure. Therefore, it is obvious that $\|x^* - x_n^*\|_E \leq \|x^*\|_E$ for any $n \in \mathbb{N}$. Furthermore, by assumption that x is an H_g point, in view of Theorem 3.3 [9] it follows that x^* is an H_g point. Then, by Theorem 3.8 [9] and by the Fatou property we conclude

$$\|x^* - x_n^*\|_E \rightarrow \|x^*\|_E.$$

Thus, since x^* is an H_g point, by symmetry of E we get $\|x_n\|_E \rightarrow 0$. $\square$

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