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Isoperimetric problem for exponential measure on the plane with ℓ_1-metric

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Abstract We give a solution to the isoperimetric problem for the exponential measure on the plane with the ℓ_1-metric. As it turns out, among all sets of a given measure, the simplex or its complement (i.e. the ball in the ℓ_1-metric or its complement) has the smallest boundary measure. The proof is based on a symmetrisation (along the sections of equal ℓ_1-distance from the origin).

Keywords Isoperimetric inequality · Exponential measure · Symmetrisation

Mathematics Subject Classification 52A40 · 60E15

1 Introduction and main result

For a metric space (X, d) equipped with a Borel measure μ we define the boundary measure μ^+ of a Borel set A as

$$\mu^+(A) := \liminf_{h\to 0+} \frac{\mu(A^h) - \mu(A)}{h},$$

where $A^h := \{x \in X : \exists y \in A \ \ d(x, y) < h\}$ is an h-neighbourhood of A with respect to d. It is interesting to study the isoperimetric problem: among all sets of a given measure find a set with the smallest boundary measure. In other words, we want

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to find a set which measure grows the slowest among all sets of a given measure. Such a set is said to be *extremal*.

This problem seems to be difficult in general and the solution to it is known only in a few cases. If μ is the Lebesgue measure in the n-dimensional Euclidean space, then balls are extremal sets. This follows for example by the Brunn–Minkowski inequality and can be proven in many other ways (see for example [10, Sect. 2]). Lévy [9] and Schmidt [11] proved that the extremal sets with respect to the Haar measure on the n-dimensional sphere equipped with the geodesic metric are balls in the geodesic metric, i.e. the intersections of half-spaces in $\mathbb{R}^{n+1}$ with the sphere.

Another example of the full solution to the isoperimetric problem is the Gaussian measure in the Euclidean space $\mathbb{R}^n$, i.e. the product measure with the density $(2\pi)^{-n/2}e^{-|x|^2/2}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Borell [5] and Sudakov with Tsirelson [12] proved that in this case half-spaces $\{x : \langle x, u\rangle \geq \lambda\}$ are extremal. As Bobkov and Houdré proved in [4], on the real line this result can be generalized into the case of an arbitrary symmetric log-concave measure. Bobkov [3] also studied the isoperimetric problem in the product metric space $(X^n, d_{\sup})$ equipped with a product probability measure, where $d_{\sup}(x, y) := \sup_{i\leq n} d(x_i, y_i)$. In this case, if the extremal sets in X^2 are of the form $A \times X$ and A are extremal in X, then $A \times X^{n-1}$ are extremal in X^n.

The discrete version of the isoperimetric problem on the cube $\{-1, 1\}^n$ (with the uniform measure and the Hamming distance) was considered by Harper in [7]. Roughly speaking, he showed that balls in the Hamming distance are extremal. This was generalized to sets $\{0, 1, \ldots, d-1\}^n$ (instead of $\{-1, 1\}^n$) by Wang and Wang in [13].

The isoperimetric problem for the product exponential measure was investigated by Bobkov, who showed in [1] that for $\mathbb{R}^n_+ = [0, \infty)^n$ equipped with the ℓ_∞-distance, the cubes (i.e. the balls in ℓ_∞) have the smallest boundary measure among all monotone Borel sets of a given measure (for an analogue of this result in $\mathbb{R}^2_+$ with ℓ_1-metric see Remark 2 below). Recall that a set $A \subset \mathbb{R}^n_+$ is called monotone, if for every $x \in A$ all the points $y \in \mathbb{R}^n_+$ satisfying $y_1 \leq x_1, \ldots, y_n \leq x_n$ belong to A.

It should be noted, that once we know the solution to the isoperimetric problem, we can obtain concentration properties for the measure μ (see for example Chapter 2.1 of [8]). However, it is probably the most difficult way to derive concentration inequalities, since it relies on finding the exact value of the isoperimetric function (and the sets which achieve the smallest boundary measure), not only a reasonable estimate on it.

In this note we will find the extremal sets in the case of the exponential measure on the plane with ℓ_1-metric. Let ν be the product exponential measure on $\mathbb{R}^n_+ = [0, \infty)^n$, i.e. the measure with the density $e^{-\sum_{i=1}^n x_i}\mathbf{1}_{x\in\mathbb{R}^n_+}$, and let B_1^n be the unit ball in the ℓ_1-distance (centred at the origin).

Definition 1 For a Borel set $A \subset \mathbb{R}$ we define the set B_A by a formula

$$B_A := \begin{cases} tB_1^n & \text{if} \quad \nu(A) \geq \frac{1}{2} \\ \mathbb{R}^n_+ \setminus tB_1^n & \text{if} \quad \nu(A) < \frac{1}{2} \end{cases},$$

where t is the unique positive number for which $\nu(B_A) = \nu(A)$. We call such a number t the radius of B_A.

In other words B_A is a simplex or a complement of a simplex, and has the same measure as A. As will be clear from Lemma 2 below, out of these two sets we pick the one of smaller boundary measure.

Our main result is the following theorem, which states that among all Borel sets of a given measure, a simplex or its complement has the smallest boundary measure. Unfortunately, we are able to give the complete proof only in the case $n = 2$, but a part of our reasoning works also for a general n.

Theorem 1 *If A is a Borel set in $\mathbb{R}^2_+$, then*

$$\nu^+(A) \geq \nu^+(B_A). \tag{1}$$

We call a Borel subset $A \subset \mathbb{R}^n$ 1-unconditional if $x \in A$ implies that $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in A$ for every choice of signs $(\varepsilon_i)_{i=1}^n \in \{-1, 1\}^n$. Note that if A is such a set and $x \in A^h \cap \mathbb{R}^n_+$, then there exists $y \in A \cap \mathbb{R}^n_+$ such that $\|x - y\|_1 < h$. This together with the previous theorem implies the following isoperimetric inequality.

Corollary 1 *Let μ be the symmetric exponential measure on the plane, i.e. the measure with density $\frac{1}{4}e^{-|x|-|y|}$. Then, among all 1-unconditional Borel sets A, a ball or its complement has the smallest boundary measure.*

However, the balls are not extremal sets for the symmetric exponential measure on the plane. An example is the set $A := \{x + y \leq 3\}$, which boundary measure is smaller than the boundary measure of the simplex of the same measure.

We believe that Theorem 1 holds also in higher dimensions.

Conjecture 1 If A is a Borel set in $\mathbb{R}^n_+$, then

$$\nu^+(A) \geq \nu^+(B_A).$$

Remark 1 In the next section we prove in fact the isoperimetric inequality (1) for the exponential measure not only in $\mathbb{R}^2_+$, but also in $\mathbb{R}_+$. Indeed, Lemmas 2 and 3 are valid for any n, and in the case $n = 1$ they imply (1) directly, since the only connected compact subsets of $\mathbb{R}$ are the closed intervals, i.e. one-dimensional *trapezoids*. This result is a special case of [2, Proposition 2.1].

Note that in $\mathbb{R}$ the ℓ_1-metric and the ℓ_2-metric coincide. Therefore Corollary 1 remains true also for the symmetric exponential measure on $\mathbb{R}$.

Corollary 2 *Let μ be the symmetric exponential measure (i.e. the measure with density $\frac{1}{2}e^{-|x|}$) on the one-dimensional Euclidean space. Then, among all 1-unconditional Borel sets A, the symmetric interval or its complement has the smallest boundary measure.*

Remark 2 In the proof of Theorem 1 we justify inequality (13), which states in particular that for every $p \in (0, 1)$ among all connected Borel sets $A \subset \mathbb{R}^2_+$ containing the origin, and with $\nu(A) = p$ the smallest boundary measure is attained by $tB_1^2 \cap \mathbb{R}_+^2$ with t such that $\nu(tB_1^2 \cap \mathbb{R}_+^2) = p$. Thus, as a by-product of Theorem 1 we obtain a two-dimensional analogue of Bobkov's result for monotone sets, cf. [1].

The organization of this paper is the following. First in Sect. 2.1 we prove that among all *trapezoids* (i.e. the sets of the form $RB_1^n \backslash rB_1^n$ for $0 \le r < R \le \infty$) of a given measure the simplex tB_1^n or its complement has the smallest boundary measure (see Lemma 2). Then we show that in order to prove the isomerimetric inequality (1), it suffices to consider connected compact sets only. After that in Sect. 2.2 we restrict our attention to the case $n = 2$. We do symmetrisations, which lead us from a given connected compact set A to a *trapezoid* of the same measure (see Lemma 5 and proof of Theorem 1).

2 Proof of Theorem 1

2.1 Initial simplifications

Define the function $I : [0, 1] \to \mathbb{R}$ by the formula $I(p) = \nu^+(B_A)$, where A is any Borel set with $\nu(A) = p$. Note that I is continuous on $[0, 1]$ and smooth on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Indeed, one can calculate that

$$\nu\left(tB_1^n\right) = \nu\left(\mathbb{R}_+^n \cap tB_1^n\right) = C_n \int_0^t e^{-x} x^{n-1} dx$$

and therefore

$$\nu^+\left(tB_1^n\right) = \nu^+\left(\mathbb{R}_+^n \cap tB_1^n\right) = C_n e^{-t} t^{n-1} = \nu^+\left(\mathbb{R}_+^n \setminus tB_1^n\right) = \nu^+\left(\left(tB_1^n\right)^c\right),$$

where

$$C_n = \left(\int_0^\infty e^{-x} x^{n-1} dx\right)^{-1} = \frac{1}{(n-1)!}.$$

Moreover, if A is a finite sum of ℓ_1-balls, we can write the true limit in the definition of $\mu^+(A)$. We use these facts to deduce the following technical lemma.

Lemma 1 (i) *Inequality (1) holds for every Borel set A of measure at least $\frac{1}{2}$ if and only if for every finite union B of ℓ_1-balls, such that $\nu(B) \ge \frac{1}{2}$, we have*

$$\varphi^{-1}\big(\nu(B^h)\big) \ge \varphi^{-1}\big(\nu(B)\big) + h, \qquad \textit{for all } h > 0, \tag{2}$$

where $\varphi(t) := \nu(tB_1^n) = C_n \int_0^t e^{-x} x^{n-1} dx$. Moreover, inequalities (1) for finite unions of ℓ_1-balls of measure at least $\frac{1}{2}$ and (2) for finite unions of ℓ_1-balls of measure at least $\frac{1}{2}$ are equivalent.

(ii) *Inequality* (1) *holds for every Borel set A of measure less than $\frac{1}{2}$ if and only if for every $h > 0$ and every finite union B of ℓ_1-balls, such that $\nu(B^h) < \frac{1}{2}$, we have*

$$\psi^{-1}\big(\nu(B^h)\big) \leq \psi^{-1}\big(\nu(B)\big) - h, \tag{3}$$

where $\psi(t) := \nu(\mathbb{R}^n \setminus tB_1^n) = C_n \int_t^\infty e^{-x}x^{n-1}dx$. Moreover, inequalities (1) *for finite unions of ℓ_1-balls of measure less than $\frac{1}{2}$ and* (2) *for finite unions of ℓ_1-balls of measure less than $\frac{1}{2}$ are equivalent.*

Proof We will only show (i), since the proof of (ii) is similar.

Assume first that (1) holds for Borel sets of measure at least $\frac{1}{2}$. To prove inequality (2) let us introduce the function $h \mapsto \varphi^{-1}(\nu(B^h))$ and note that (1) for B^h (which is also a finite union of balls) implies that the derivative of this function is bounded from below by 1. Note that we use (1) only for finite unions of balls, so we also proved the second part of (i).

Now suppose that (2) holds for finite unions of balls with measure at least $\frac{1}{2}$. It is obvious that (2) for a set A implies (1) for this A, so we only have to show (2) for the set A instead of B. Note that for $r > 0$ the set A^r is open and therefore it can be represented as a countable union of balls $\bigcup \mathcal{U}_r$. Let $\mathcal{U}_{r,m}$ be a subfamily of the family $\mathcal{U}_r$ containing the first m balls (so that $\mathcal{U}_{r,m+1} \setminus \mathcal{U}_{r,m}$ contains a single ball). Then, by the continuity and the monotonicity of φ, and the inequality (2) for $\bigcup \mathcal{U}_{r,m}$, we have

$$\begin{aligned}\varphi^{-1}\big(\nu(A^{r+h})\big) &\geq \varphi^{-1}\Big(\nu\Big(\Big(\bigcup \mathcal{U}_{r,m}\Big)^h\Big)\Big) \geq \varphi^{-1}\Big(\nu\Big(\bigcup \mathcal{U}_{r,m}\Big)\Big) + h \\ &\xrightarrow{m\to\infty} \varphi^{-1}\big(\nu(A^r)\big) + h \geq \varphi^{-1}\big(\nu(A)\big) + h\end{aligned}$$

for sufficiently large n (depending on $r > 0$) . We take $r \to 0$ on the left-hand side of this estimate to get (2) for $\overline{A^h}$. In particular, for any $h > \varepsilon > 0$ we have (2) for $\overline{A^{h-\varepsilon}}$, so $\varphi^{-1}\big(\nu(A^h)\big) \geq \varphi^{-1}\big(\nu(A)\big) + h - \varepsilon$. We take $\varepsilon \to 0$ to get (2) for A. This finishes the proof. □

Corollary 3 *It suffices to prove the isoperimetric inequality* (1) *for finite unions of ℓ_1-balls.*

We start the main part of the proof of the isoperimetric inequality by showing that the simplex or its complement is the set growing most slowly among all *trapezoids* of a given measure.

Lemma 2 *The isoperimetric inequality* (1) *holds for sets A of the form $\{x \in \mathbb{R}_+^n : a < \|x\|_1 < b\}$, where $0 \leq a < b \leq \infty$.*

Proof Let c be the radius of the set B_A (see Definition 1). We consider three cases.

Case 1 Assume $a = 0$ or $b = \infty$. To prove the isoperimetric inequality in this case we need only to prove that if $\int_0^x e^{-t}t^{n-1}dt = \int_y^\infty e^{-t}t^{n-1}dt < \frac{1}{2}\int_0^\infty e^{-t}t^{n-1}dt$, then $e^{-x}x^{n-1} \geq e^{-y}y^{n-1}$ (in the other case we can consider the complements of these sets). Note that this means that in the definition of B_A, among the simplex and the complement of the simplex, we always pick the set of smaller boundary measure.

Let us first show that the condition $\int_y^\infty e^{-t}t^{n-1}dt < \frac{1}{2}\int_0^\infty e^{-t}t^{n-1}dt$ implies that $y \geq n-1$. To this end we only have to show that $\int_{n-1}^\infty e^{-t}t^{n-1}dt \geq \frac{1}{2}\int_0^\infty e^{-t}t^{n-1}dt$. Integration by parts yields

$$\frac{\int_{n-1}^\infty e^{-t}t^{n-1}dt}{\int_0^\infty e^{-t}t^{n-1}dt} = e^{-(n-1)}\left(\frac{(n-1)^{n-1}}{(n-1)!} + \frac{(n-1)^{n-2}}{(n-2)!} + \ldots + \frac{n-1}{1!} + 1\right),$$

so we only have to show that $\mathbb{P}(\text{Poiss}(k) \leq k) \geq \frac{1}{2}$, where $\text{Poiss}(\lambda)$ is the random variable of Poisson distriubution with parameter λ. Due to [6, Theorem 1], the smallest integer l for which $\mathbb{P}(\text{Poiss}(\lambda) \leq l) \geq \frac{1}{2}$ satisfies $\lambda - \log 2 \leq l < \lambda + \frac{1}{3}$. This implies that $\mathbb{P}(\text{Poiss}(k) \leq k) \geq \frac{1}{2}$ and therefore finishes the proof of the inequality $y \geq n-1$.

One can easily check that the function $t^{n-1}e^{-t}$ is decreasing on the half-line $[n-1, \infty)$, so if $n-1 \leq x \leq y$, then $e^{-x}x^{n-1} \geq e^{-y}y^{n-1}$ and we are done. Otherwise $x \leq n-1 \leq y$ (since $x \leq y$ and $n-1 \leq y$). Let us consider this case now. Note that the equation $\int_0^x e^{-t}t^{n-1}dt = \int_y^\infty e^{-t}t^{n-1}dt$ determines y as a function of x and $e^{-x}x^{n-1} = -y'e^{-y}y^{n-1}$. For $x = 0$ the inequality $e^{-x}x^{n-1} \geq e^{-y}y^{n-1}$ holds (and is in fact an equality, since $y(0) = \infty$). For $x = n-1$ the function $e^{-x}x^{n-1}$ attains its maximum on $[0, \infty]$, so $e^{-x}x^{n-1} \geq e^{-y}y^{n-1}$ if $x = n-1$. Therefore it suffices to check whether $e^{\frac{x-y}{n-1}}\frac{y}{x} \leq 1$ for every $x \leq n-1$ at which the derivative of $e^{-x}x^{n-1} - e^{-y}y^{n-1}$ vanishes. This derivative is equal to

$$\begin{aligned} &e^{-x}x^{n-2}(-x + (n-1)) - y'e^{-y}y^{n-2}(-y + (n-1)) \\ &= e^{-x}x^{n-2}\left(-x + (n-1) + \frac{x}{y}(n-1) - x\right), \end{aligned}$$

so we should check values of x satisfying $y(x) = \frac{x(n-1)}{2x-(n-1)}$. In particular, these values are greater than $\frac{n-1}{2}$. Note that if $y(x) = \frac{x(n-1)}{2x-(n-1)}$, then $e^{\frac{x-y}{n-1}} \cdot \frac{y}{x} = e^{-2\lambda\frac{1-\lambda}{2\lambda-1}}/(2\lambda-1)$, where $\lambda := \frac{x}{n-1} \in (\frac{1}{2}, 1]$. The derivative of $e^{-2\lambda\frac{1-\lambda}{2\lambda-1}}/(2\lambda-1)$ is equal to $4(1-\lambda)^2e^{-2\lambda\frac{1-\lambda}{2\lambda-1}}/(2\lambda-1)^3 \geq 0$, so this function is non-decreasing and therefore less than its value in 1 (for $\lambda \in (\frac{1}{2}, 1]$). Hence the inequality $e^{-x}x^{n-1} \geq e^{-y}y^{n-1}$ holds for x such that $y(x) = \frac{x(n-1)}{2x-(n-1)}$ and the claim is proved.

Case 2 Assume $\nu(A) \leq \frac{1}{2}$. Then we have

$$\int_a^b e^{-t}t^{n-1}dt = \int_c^\infty e^{-t}t^{n-1}dt. \tag{4}$$

For a fixed $b > 0$ this equality determines a as a function of c and $a'e^{-a}a^{n-1} = e^{-c}c^{n-1}$. If c is such that $a(c) = 0$, then the inequality we want to prove, $e^{-a}a^{n-1} + e^{-b}b^{n-1} \geq e^{-c}c^{n-1}$, holds as we proved in Case 1. Therefore it suffices to show that the derivative of $e^{-a}a^{n-1} - e^{-c}c^{n-1}$ as a function of c is non-negative. Integration by parts of (4) yields

$$(n-1)\int_a^b e^{-t}t^{n-2}dt - (n-1)\int_c^\infty e^{-t}t^{n-2}dt = -e^{-a}a^{n-1} + e^{-b}b^{n-1} + e^{-c}c^{n-1},$$

so the derivative of $e^{-a}a^{n-1} - e^{-c}c^{n-1}$ is equal to

$$(n-1)(a'e^{-a}a^{n-2} - e^{-c}c^{n-2}) = (n-1)e^{-c}c^{n-2}\left(\frac{c}{a} - 1\right).$$

Since $c \geq a$, the derivative we consider is indeed non-negative.

Case 3 Assume $\nu(A) \geq \frac{1}{2}$. We proceed similarly as in Case 2. Since $\nu(A) \geq \frac{1}{2}$, we have

$$\int_a^b e^{-t}t^{n-1}dt = \int_0^c e^{-t}t^{n-1}dt.$$

For a fixed $a > 0$ this equality determines b as a function of c and $b'e^{-b}b^{n-1} = e^{-c}c^{n-1}$. For c such that $b(c) = \infty$ the inequality $e^{-a}a^{n-1} + e^{-b}b^{n-1} \geq e^{-c}c^{n-1}$, holds, what we proved in Case 1. Therefore it suffices to prove that the derivative of $(e^{-b}b^{n-1} - e^{-c}c^{n-1})$ is negative. Calculations similar to those carried out in Case 2 show that this derivative is equal to $(n-1)e^{-c}c^{n-2}(\frac{c}{b} - 1)$, which is negative, since $b > c$. □

The next lemma allows us to restrict our attention to connected compact sets.

Lemma 3 *If for every connected compact set A the inequality*

$$\nu(A^h) \geq \nu(B_A^h) - Lh^2 \tag{5}$$

holds for every $h \leq h_0$ with some $L \geq 0$ and $h_0 > 0$ depending on A only, then the isoperimetric inequality (1) *holds for every Borel set A.*

Proof Let A be a Borel set of positive measure. Assuming (5) for connected bounded Borel sets, we are going to prove (1) for A.

By Corollary 3 it suffices to prove (1) for finite unions of balls. Moreover, if we show (1) for $\overline{A}$, the inequality for A will follow (since $\nu(\overline{A}^h) = \nu(A^h)$ and $\nu(\overline{A}) \geq \nu(A)$). Since (5) implies (1), it suffices to prove (5) for a compact set A with finitely many connected components $A_1, \ldots, A_N$, each of non-empty interior. Then for sufficiently small $h_0 > 0$ the sets $A_j^{h_0}$ are pairwise disjoint, so for any $h \in (0, h_0)$ we have $\nu(A^h) = \sum_{j=1}^N \nu(A_j^h) \geq \sum_{j=1}^N \nu(B_{A_j}^h) - NLh^2$, because (5) holds for A_i. To finish the proof we use Lemma 4 (see below), and an obvious induction. □

Lemma 4 *If the sets C and D are disjoint, then $\nu^+(B_C) + \nu^+(B_D) \geq \nu^+(B_{C\cup D})$.*

Proof By x, y, z we denote the radii of B_C, B_D and $B_{D\cup C}$ respectively. Note that if $\nu(C) \geq \frac{1}{2}$, then $\nu(D) < \frac{1}{2}$ and $\nu(C \cup D) > \frac{1}{2}$. Therefore it suffices to consider the following three cases.

Case 1 Assume $\nu(C), \nu(D), \nu(C \cup D) < \frac{1}{2}$. Without loss of generality we may assume $x \geq y$. By the definition of the sets B_C, B_D and $B_{C\cup D}$ we have

$$\int_x^\infty e^{-t} t^{n-1} dt + \int_y^\infty e^{-t} t^{n-1} dt = \int_z^\infty e^{-t} t^{n-1} dt. \tag{6}$$

Integration by parts implies that the inequality $e^{-x}x^{n-1} + e^{-y}y^{n-1} = \nu^+(B_C) + \nu^+(B_D) \geq \nu^+(B_{C\cup D}) = e^{-z}z^{n-1}$ is equivalent to

$$\int_x^\infty e^{-t} t^{n-2} dt + \int_y^\infty e^{-t} t^{n-2} dt \leq \int_z^\infty e^{-t} t^{n-2} dt. \tag{7}$$

We will prove that (6) implies (7) for $x, y, z \geq 0$ such that $x \geq y$. Fix $z \geq 0$. Then equality (6), determines x as a smooth function of y and $x'e^{-x}x^{n-1} + e^{-y}y^{n-1} = 0$. Note that $x(z) = 0$ and thus for $y = z$ inequality (7) holds (and is in fact an equality). To end the proof in Case 1 we will show that the left-hand side of (7) is a decreasing function of y for $y > z$ such that $x(y) > y$. The derivative of the left-hand side of (7) is equal to $-(x'e^{-x}x^{n-2} + e^{-y}y^{n-2}) = e^{-y}y^{n-2}(\frac{y}{x} - 1)$ which is negative for $x > y$. Note that in this case we have used the fact that $\nu(B_C), \nu(B_D), \nu(B_{C\cup D}) \leq \frac{1}{2}$ only to obtain (6).

Case 2 Assume $\nu(C), \nu(D) < \frac{1}{2},\ \nu(C \cup D) \geq \frac{1}{2}$. Let v be such that $\int_0^z e^{-t}t^{n-1}dt = \int_v^\infty e^{-t}t^{n-1}dt$. Then (6), and consequently (7), holds with v in place of z and thus $e^{-x}x^{n-1} + e^{-y}y^{n-1} \geq e^{-v}v^{n-1} \geq e^{-z}z^{n-1}$ – the last inequality holds by Lemma 2 applied to $a = v$ and $b = \infty$.

Case 3 Assume $\nu(C) < \frac{1}{2},\ \nu(D), \nu(C \cup D) \geq \frac{1}{2}$. Then

$$\int_x^\infty e^{-t} t^{n-1} dt + \int_0^y e^{-t} t^{n-1} dt = \int_0^z e^{-t} t^{n-1} dt. \tag{8}$$

Fix any $x > 0$ satisfying $\int_x^\infty e^{-t}t^{n-1}dt \leq \frac{1}{2}$. Equality (8) determines z as a function of y and $e^{-y}y^{n-1} = z'e^{-z}z^{n-1}$. We want to show that $e^{-x}x^{n-1} + e^{-y}y^{n-1} \geq e^{-z}z^{n-1}$. Note that for y such that $\nu(yB_1^n) = \frac{1}{2}$ we have $\nu(z(y)B_1^n) \geq \frac{1}{2}$ and Case 2 implies that then $e^{-x}x^{n-1} + e^{-y}y^{n-1} \geq e^{-z}z^{n-1}$. Thus it suffices to show that $e^{-y}y^{n-1} - e^{-z}z^{n-1}$ is increasing in y. We integrate (8) by parts and get

$$\begin{aligned}(n-1)\int_x^\infty e^{-t} t^{n-2} dt &+ e^{-x}x^{n-1} - \left(e^{-y}y^{n-1} - e^{-z}z^{n-1}\right) \\ &= (n-1)\int_0^z e^{-t} t^{n-2} dt - (n-1)\int_0^y e^{-t} t^{n-2} dt. \end{aligned}\tag{9}$$

Therefore we only need to show that the derivative of the right-hand-side of (9) is negative. This derivative is equal to

$$(n-1)\left(z'e^{-z}z^{n-2}-e^{-y}y^{n-2}\right)=(n-1)e^{-y}y^{n-2}\left(\frac{y}{z}-1\right),$$

what is negative since $y<z$. □

2.2 Symmetrisation

From now on we assume $n=2$. Otherwise the symmetrisation described below, as well as the final argument, does not work. Let

$$T:=\overline{B_1^2\cap\left([0,\infty)^2\cup(-\infty,0]^2\right)}=\left\{(x,y)\in\mathbb{R}:|x|+|y|\le 1,\ \operatorname{sgn}x=\operatorname{sgn}y\right\}.$$

Remark 3 Note that for sets A of the form $rB_1^2\cap\mathbb{R}_+^2$ or $\mathbb{R}_+^2\setminus rB_1^2$ we have $\nu(A+hT)=\nu(A^h)$, and for any compact set A the inequality $\nu(A+h_kT)\le\nu(A^{h_k})$ holds for some sequence $(h_k)_{k\ge 0}$ (depending on A) which tends to 0. (We pick a sequence $(h_k)_{k\ge 0}$, because it may happen that

$$\nu(A+hT)\ne\nu\Big(A+h\big(B_1^2\cap([0,\infty)^2\cup(-\infty,0]^2)\big)\Big),$$

but only for finitely many $h>0$.) Therefore in order to prove (5) it suffices to show that every connected compact set A satisfies

$$\nu(A+hT)\ge\nu(B_A+hT)-Lh^2\quad\text{for } h\le h_0, \tag{10}$$

where L is an absolute constant and h_0 depends on A only.

Definition 2 For a Borel set $A\subset\mathbb{R}_+^2$ and $t>0$ we define

$$f_A(t):=\mathcal{H}_1(A\cap S_t),$$

where $\mathcal{H}_1$ is the one-dimensional Hausdorff measure and $S_t:=\{(x,y)\in\mathbb{R}_+^2:x+y=t\}$.

Clearly, f_A is a measurable function. Moreover, for any Borell set A of $\mathbb{R}_+^2$ we have

$$\mu(A)=\int_{(x,y)\in A}e^{-(x+y)}dxdy=\int_0^\infty\int_{y:(t-y,y)\in A}e^{-t}dydt=\int_0^\infty\frac{1}{\sqrt{2}}f_A(t)e^{-t}dt.$$

The next lemma introduces a symmetrisation which preserves the function f_A. Moreover, the lemma states that this symmetrisation does dot increase the boundary measure of the symmetrised set. This symmetrisation is illustrated on next two figures (Fig. 1).

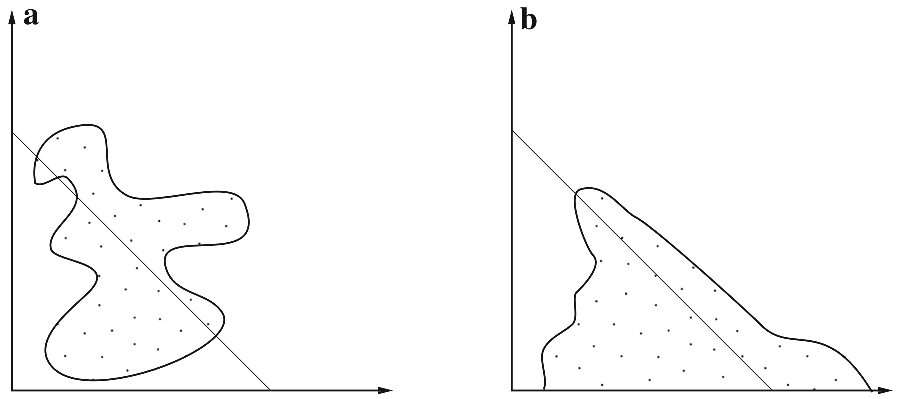

Fig. 1 **a** Set A before the symmetrisation. **b** Set $C = C_A$

Lemma 5 *For any Borel set* $A \subset \mathbb{R}^2_+$ *we introduce*

$$C = C_A := \bigcup_{t>0} \left\{ (x, y) \in S_t : y \leqslant \frac{f_A(t)}{\sqrt{2}} \right\}.$$

Then $\mu(C) = \mu(A)$ *and* $\mu(C + hT) \leqslant \mu(A + hT)$ *for every* $h > 0$.

Proof By the definition of C we get $f_A = f_C$, so $\mu(A) = \mu(C)$.

We will prove that for all $s, t, h > 0$ we have

$$\mathcal{H}_1\big(S_s \cap (A \cap S_t + hT)\big) \geqslant \mathcal{H}_1\big(S_s \cap (C \cap S_t + hT)\big). \tag{11}$$

This implies (since all the sets $S_s \cap (C \cap S_t + hT)$ are intervals with endpoints at $(s, 0)$) that $f_{A+hT} \geq f_{C+hT}$ and thus

$$\mu(A + hT) = \int_0^\infty \frac{1}{\sqrt{2}} f_{A+hT}(t) e^{-t} dt \geq \int_0^\infty \frac{1}{\sqrt{2}} f_{C+hT}(t) e^{-t} dt = \mu(C + hT).$$

Let us first consider the case $s \geq t$. It suffices to consider $h = s - t$, since for $h > s - t$ both sides of (11) do not change, while for $h < s - t$ both sides of (11) vanish. Let x be the point $(t - u, u)$, where u is the smallest possible non-negative number such that the point $(t - u, u)$ belongs to A (see Fig. 2a). Since $h = s - t$, we have

$$(A \cap S_t + he_2) \cup (x + S_h) \subset S_s \cap (A \cap S_t)^h,$$

where $e_2 = (0, 1)$.

Moreover, this inclusion becomes an equality if we replace A by C. Therefore

$$\begin{aligned} \mathcal{H}_1\big(S_s \cap (A \cap S_t + hT)\big) &\geqslant \mathcal{H}_1(A \cap S_t) + \sqrt{2}h = \mathcal{H}_1(C \cap S_t) + \sqrt{2}h \\ &= \mathcal{H}_1\big(S_s \cap (C \cap S_t)^h\big), \end{aligned}$$

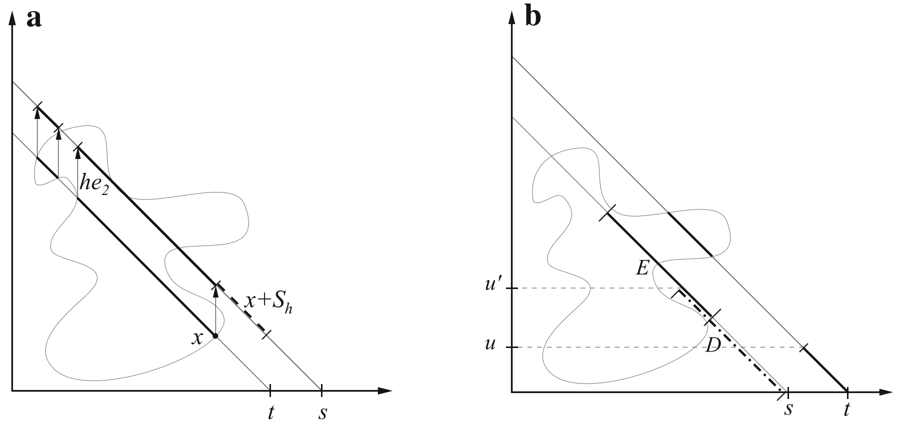

Fig. 2 **a** Case $s \geq t$. **b** Case $s < t$

which shows that (11) is satisfied in the case $t \leq s$.

Let us assume now that $t > s$. Again, it suffices to consider $h = t - s$. Suppose (11) does not hold. Let $u' \geq 0$ be such that $E := S_s \cap (A \cap S_t)^h$ has the same Hausdorff measure as $D := S_s \cap \{(y_1, y_2) : y_2 \leqslant u'\}$ (see Fig. 2b). Let u be given by $C \cap S_t = S_t \cap \{(y_1, y_2) : y_2 \leqslant u\}$. Since (11) does not hold, $u' < u \wedge s$ (note that in Fig. 2b we have $u' \geq u$, since this figure reflects the true situation, whereas we are arguing by contradiction). By the conclusion of the first case (in which we had $t \leqslant s$), we have

$$\begin{aligned}\mathcal{H}_1\big((S_s \setminus E + hT) \cap S_t\big) &\geqslant \mathcal{H}_1\big((S_s \setminus D + hT) \cap S_t\big)\\ &= \mathcal{H}_1\big(\{(y_1, y_2) :\ y_2 \geqslant u'\} \cap S_t\big),\end{aligned}$$

since the sets $S_s \setminus E$ and $S_s \setminus D$ are of the same Hausdorff measure.

Moreover, by the definition of the set E we get $(S_s \setminus E + hT) \cap S_t \subset S_t \setminus A$. Therefore

$$\begin{aligned}\mathcal{H}_1(A \cap S_t) &\leqslant \mathcal{H}_1\big(S_t \setminus \{(y_1, y_2) :\ y_2 \geqslant u'\}\big) = \mathcal{H}_1\big(S_t \cap \{(y_1, y_2) :\ y_2 \leqslant u'\}\big)\\ &< \mathcal{H}_1\big(S_t \cap \{(y_1, y_2) :\ y_2 \leqslant u\}\big) = \mathcal{H}_1(S_t \cap C),\end{aligned}$$

which contradicts the property $\mathcal{H}_1(A \cap S_t) = \mathcal{H}_1(S_t \cap C)$. Hence (11) is satisfied also in the case $s < t$. $\square$

Note that in higher dimensions the above proof works in the case $s \geq t$ (we only have to additionally use the Brunn–Minkowski inequality for an arbitrary set and a simplex). However, the same reasoning as above shows, that the analogue of (11) for $s < t$ holds if we consider $\mathbb{R}^n_+ \setminus D$ (where $\mathbb{R}^n_+ \setminus D$ has the same measure as C, and D is such that $C_D = D$) instead of C. Therefore (11) fails in general if $s < t$ and $n > 2$. The reason why (11) works for $n = 2$ is that S_t is an interval and therefore the sections of C and D (at the level t) are both intervals starting from an end point of S_t.

Now we are ready to prove the main theorem. Its proof clarifies, how to replace the set C_A by a *trapezoid*. This reasoning fails in higher dimensions too. Also the induction over n does not work, since a section parallel to the hyperplane $\mathrm{lin}(e_1, \ldots, e_{n-1})$ of a connected set does not have to be connected.

Proof of of Theorem 1 Due to Lemma 3 and Remark 3 it suffices to prove $\nu(A + hT) \geq \nu(B_A + hT) - Lh^2$ for connected bounded compact sets A and for sufficiently (depending on A) small $h > 0$. By Lemma 5 it suffices to prove that for sufficiently small h the inequality $\nu(C + hT) \geq \nu(B_C + hT) - Lh^2$ holds for $C = C_A$. Let $f := f_A = f_C$.

Note that for every Borel set A and $h > 0$ we have $\nu(A + he_1) = \nu(A + he_2) = e^{-h}\nu(A)$. Moreover, if $A - he_1 \subset \mathbb{R}^2_+$ (or $A - he_2 \subset \mathbb{R}^2_+$), then $\nu(A - he_1) = e^h\nu(A)$ (or $\nu(A - he_2) = e^h\nu(A)$, respectively). We will use this observation throughout the proof.

Recall that C is compact and connected. Therefore, if for every $u > 0$ we have $f(u) < \sqrt{2}u$ and $\mathrm{supp}\, f \subset (0, \infty)$, then there exists $\varepsilon > 0$ such that $f(u) < \sqrt{2}(u - \varepsilon)$ for every $u > \varepsilon$ and $f(u) = 0$ for every $u \leq \varepsilon$ (this means that C does not intersect the strip $[0, \varepsilon) \times [0, \infty)$). Hence for every $h \in (0, \varepsilon)$ we have $(C - he_1) \subset C^h \cap \mathbb{R}^2_+$, where $e_1 = (1, 0)$, so

$$\nu(C^h) \geq \nu(C - he_1) = \nu(C)e^h = \nu(D)e^h \geq \nu(D + hT),$$

where $\nu(D) = \nu(C)$ and D is the complement of rB^n_1, and the last inequality follows since $D + hT \subset D - he_1$. Therefore we can restrict our attention to the case in which there exists $u \geq 0$ such that $f(u) = \sqrt{2}u$. Let u be the smallest value for which $\sqrt{2}u = f(u)$ (the minimal u exists since C is compact).

Let $a \leq u \leq b$ be such that $\nu(uB^2_1 \setminus aB^2_1) = \nu(C \cap uB^2_1)$ and $\nu(bB^2_1 \setminus uB^2_1) = \nu(C \setminus uB^2_1)$. In other words we pick such a and b, that the *trapezoid* between a and u has the same measure as C below u (and similarly the *trapezoid* between u and b has the same measure as C above u). We will show that

$$\nu\big(C \cup (\mathbb{R}^2_+ \setminus uB^2_1) + hT\big) \geq \nu\big(\mathbb{R}^2_+ \setminus (a - h)B^n_1\big) \quad \text{for } 0 < h < \min\{h_0, a\} \tag{12}$$

and

$$\nu\left(C \cup uB^2_1 + hT\right) \geq \nu\big((b + h)B^2_1\big) - Lh^2 \quad \text{for } h > 0, \tag{13}$$

where $h_0 := \min\{\max\{\lambda : \nu_1(R_\lambda) \leq \nu_1([a, u])\}, h_1\}$ and $R_\lambda := \{t \in (\lambda, u) : t - \frac{f(t)}{\sqrt{2}} < \lambda\}$, and ν_1 is the marginal distribution of ν, i.e. the exponential measure on the half-line, and h_1 is such that $\mathrm{supp}\, f \subset (h_1, \infty)$. By Lemma 2, inequalities (12) and (13) will finish the proof of the theorem (we will see below that $h_0 > 0$ if $a \neq 0$), since (12) says that the hT neighbourhood of C below u is not less than the hT neighbourhood of the *trapezoid* between a and u (and similarly (13) gives us an analogous estimate above u, up to a term Lh^2).

Let us first show (12) (Fig. 3a is attached for the reader's convenience). If $a = 0$ or $a = u$, we have nothing to prove. Suppose therefore that $u > a > 0$ and $h < h_0$. Note that $h_0 > 0$, since $0 < a < u$ and for every $0 < t < u$ we have $f(t) < \sqrt{2}t$.

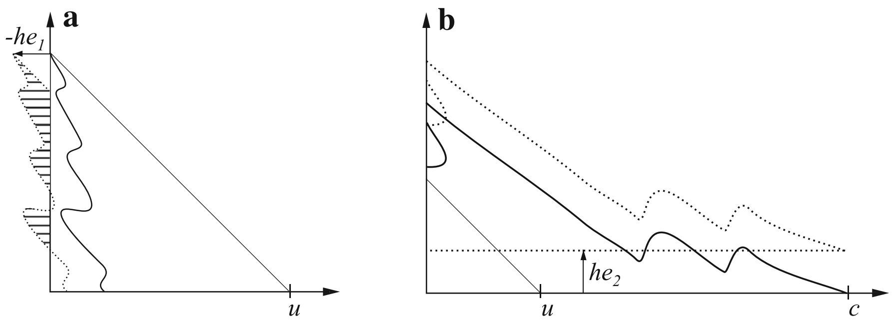

Fig. 3 **a** Proof of (12). **b** Proof of (13)

Moreover,

$$\big((C \cap uB_1^2 - he_1\big) \cap \mathbb{R}_+^2\big) \cup \big(\mathbb{R}_+^2 \backslash (u-h)B_1^n\big) \subset \big(C \cup \big(\mathbb{R}_+^2 \backslash uB_1^2\big)\big) + hT$$

and, since $h \le h_1$, the set $\mathbb{R}_+^2 \setminus (C \cap uB_1^2 - he_1)$ (see the hatched set in Fig. 3a) is contained in the set $\bigcup_{\delta=0}^{h} \big(\{0\} \times (R_h - h) + \big(-\delta, \delta\big)\big)$, which is the translation by the vector $-he_1$ of a set of measure $h\nu_1(R_h)$. By the definition of h_0 we know that for $h \le h_0$ we have $\nu_1(R_h) \le \nu_1([a, u])$. Therefore

$$\begin{aligned} \nu\big((C \cup (\mathbb{R}_+^2 \backslash uB_1^2)) + hT\big) & \\ &\ge \nu\big(\mathbb{R}_+^2 \backslash (u-h)B_1^2\big) + e^h \nu(C \cap uB_1^n) - e^h h \cdot \nu_1(R_h) \\ &\ge \nu\big(\mathbb{R}_+^2 \backslash (u-h)B_1^2\big) + e^h \nu(uB_1^n \backslash aB_1^n) - e^h h \cdot \nu_1([a, u]) \\ &= \nu\big(\mathbb{R}_+^2 \backslash (a-h)B_1^n\big), \end{aligned}$$

what yields inequality (12).

We will prove inequality (13) (Fig. 3b may be helpful to follow the estimates). Note that the fact that C is connected implies that supp f is connected, and let $c :=$ sup supp f. Obviously $c \ge b$ and supp $f \cup [0, u] = [0, c]$. Moreover we have $((C \cup uB_1^2) + he_2) \cup [0, c] \times [0, h] \subset (C \cup uB_1^2) + hT$, so

$$\begin{aligned} \nu\big((C \cup uB_1^2\big) + hT\big) &\ge \nu\big(\big(\big(C \cup uB_1^2\big) + he_2\big) \cup [0, c] \times [0, h]\big) \\ &\ge \nu\big(\big(C \cup uB_1^2\big) + he_2) + \nu([0, b] \times [0, h]\big) \\ &= \nu(bB_1^n + he_2) + \nu([0, b] \times [0, h]\big) \ge \nu\big((b+h)B_1^n\big) - Lh^2, \end{aligned}$$

where L is an absolute constant. The proof of the theorem is finished. □

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