

Gaussian-type upper bound for the evolution kernels on nilpotent meta-abelian groups

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Abstract We give a Gaussian-type upper bound for the transition kernels of the time-inhomogeneous diffusion processes on a nilpotent meta-abelian Lie group N generated by the family of time dependent second order left-invariant differential operators. These evolution kernels are related to the heat kernel for the left-invariant second order differential operators on higher rank NA groups.

Keywords Left invariant differential operators · Time-dependent parabolic operators · Brownian motion · Evolution kernel · Diffusion process · Meta-abelian nilpotent Lie groups · Borell–TIS inequality

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1 Introduction

Time-dependent parabolic equations and, in particular, the problem of finding the upper and lower bounds for their fundamental solutions has attracted considerable attention in recent years (see e.g. [5, 12–15, 31] and the monographs by Stroock and Varadhan [27], and van Casteren [4]). The aim of this paper is to get a Gaussian-

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type upper bound for the transition kernel of a particular kind of diffusion process (evolution) on a nilpotent meta-abelian group N . The type of the evolution equation considered here comes from the study of the heat equation on a class of solvable Lie groups, the so called higher rank NA groups which are, by definition, the semi-direct products of a nilpotent and abelian (with dimension greater than 1) groups (more on that in Sect. 1.4).

1.1 Our setting

In what follows we assume that the group N is meta-abelian

$$N = M \rtimes V,$$

where M and V are abelian Lie groups with the corresponding Lie algebras \mathfrak{m} and \mathfrak{v} . We consider a family of automorphisms $\{\Phi(a)\}_{a \in \mathbb{R}^k}$ of \mathfrak{n} , that leaves \mathfrak{m} and \mathfrak{v} invariant, where $a \mapsto \Phi(a)$ is a homomorphism of \mathbb{R}^k into $\text{Aut}(\mathfrak{n})$. Let \mathfrak{m} and \mathfrak{v} be spanned, respectively, by $\{Y_1, \dots, Y_{d_1}\}$ and $\{X_1, \dots, X_{d_2}\}$. We use these bases to identify \mathfrak{m} and \mathfrak{v} with \mathbb{R}^{d_1} and \mathbb{R}^{d_2} respectively. We also use the exponential mapping to identify M and V with \mathfrak{m} and \mathfrak{v} and thus with \mathbb{R}^{d_1} and \mathbb{R}^{d_2} respectively. For $x \in N$ we write $x = m(x)v(x) = mv = (m, v)$ where $m(x) = m \in M$ and $v(x) = v \in V$ denote the components of x in $M \rtimes V$.

Now we consider the action of an Lie abelian group $A = \mathbb{R}^k$ on N . We have a semi-direct product $S = N \rtimes A = N \rtimes \mathbb{R}^k$ with the multiplication in S given by

$$(x, a)(y, b) = (xy^a, a + b),$$

where, for $x = \exp X$, $X \in \mathfrak{n}$, the action of $a \in A = \exp A = \mathbb{R}^k$ on N is defined as

$$x^a = \exp(\Phi(a)X).$$

The group S is a solvable Lie group. The *rank* of S is, by definition, equal to $\dim A$. Similarly, for $g \in S$ we write $g = x(g)a(g) = xa = (x, a)$, where $x(g) = x \in N$ and $a(g) = a \in A$ denote the components of g in $N \rtimes A$. In what follows we identify the group A , its Lie algebra \mathfrak{a} , and \mathfrak{a}^* , the space of linear forms on \mathfrak{a} , with the Euclidean space \mathbb{R}^k endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|a\| = \langle a, a \rangle^{1/2}$. By $\|\cdot\|_\infty$ we denote the maximum norm $\|a\|_\infty = \max_{1 \leq j \leq k} |a_j|$.

Let σ be a continuous function from $[0, +\infty)$ to $A = \mathbb{R}^k$, and denote

$$\Phi^\sigma(t) = \Phi(\sigma(t)).$$

We assume also that

- (A1) in the $\{Y_i\}_{1 \leq i \leq d_1}$ basis on \mathfrak{m} , ad_X is lower triangular for all $X \in \mathfrak{v}$ and
- (A2) the restriction S^σ of Φ^σ to M considered as a linear operator on \mathfrak{m} is given in the $\{Y_i\}_{1 \leq i \leq d_1}$ basis by a $d_1 \times d_1$ lower triangular matrix:

$$S^\sigma(t) = \Phi^\sigma(t)|_M = [s_{ij}^\sigma]_{1 \leq i, j \leq d_1}.$$

Specifically, for $i \geq j$,

$$s_{ij}^\sigma(u) = h_{ij}^M(\sigma(u))e^{\xi_j(\sigma(u))},$$

where $h_{ij}^M \in \mathbb{R}[a_1, \dots, a_k]$ are polynomials in $a \in A = \mathbb{R}^k$ with $h_{jj}^M = 1$, for $1 \leq j \leq d_1$, and $\xi_1, \dots, \xi_{d_1} \in A^* = (\mathbb{R}^k)^*$.

(A3) The matrix

$$T^\sigma(t) = \Phi^\sigma(t)|_V = [t_{ij}^\sigma]_{1 \leq i, j \leq d_2}$$

is a $d_2 \times d_2$ lower triangular and, for $i \geq j$,

$$t_{ij}^\sigma(u) = h_{ij}^V(\sigma(u))e^{\vartheta_j(\sigma(u))},$$

where $h_{ij}^V \in \mathbb{R}[a_1, \dots, a_k]$ are polynomials in $a \in A = \mathbb{R}^k$ with $h_{jj}^V = 1$, for $1 \leq j \leq d_2$, and $\vartheta_1, \dots, \vartheta_{d_2} \in A^* = (\mathbb{R}^k)^*$.

1.2 Evolution kernel

Let, for $Z \in \mathfrak{n}$,

$$Z(t) = \Phi^\sigma(t)Z.$$

Let,

$$\mathcal{L}_N^\sigma(t) = \sum_{i=1}^{d_2} X_i(t)^2 + \sum_{j=1}^{d_1} Y_j(t)^2.$$

Now we consider the *evolution process* generated by $\mathcal{L}_N^\sigma(t)$. By $C(N)$ we denote the set of continuous functions on N . Let

$$C_\infty(N) = \left\{ f \in C(N) : \lim_{x \rightarrow \infty} f(x) \text{ exists} \right\}.$$

Let $d = \dim \mathfrak{n}$. For $X \in \mathfrak{n}$, we let \tilde{X} denote the corresponding right-invariant vector field. For a multi-index $I = (i_1, \dots, i_d)$, $i_j \in \mathbb{Z}^+$ and a basis X_1, \dots, X_d of the Lie algebra \mathfrak{n} we write $X^I = X_1^{i_1}, \dots, X_m^{i_m}$. For $\kappa, \ell = 0, 1, 2, \dots, \infty$ we define

$$C^{(\kappa, \ell)}(N) = \{f : \tilde{X}^I X^J f \in C_\infty(N) \text{ for every } |I| < \kappa + 1 \text{ and } |J| < \ell + 1\}$$

and

$$\|f\|_{(\kappa, \ell)}^0 = \sup_{|I|=\kappa, |J|=\ell} \|\tilde{X}^I X^J f\|_\infty,$$

$$\|f\|_{(\kappa, \ell)} = \sup_{|I| \leq \kappa, |J| \leq \ell} \|\tilde{X}^I X^J f\|_\infty.$$

In particular $C^{(0,2)}(N)$ with the norm $\|f\|_{(0,2)}$ is a Banach space. It is known (see [4, 19, 28]) that there exists the (unique) family of bounded operators $U_{s,t}^\sigma$ on C_∞ which satisfies

- (i) $U_{s,s}^\sigma = \text{Id}$, for all $s \geq 0$,
- (ii) $\lim_{h \rightarrow 0} U_{s,s+h}^\sigma f = f$ in $C_\infty(N)$,
- (iii) $U_{s,r}^\sigma U_{r,t}^\sigma = U_{s,t}^\sigma$, $0 \leq s \leq r \leq t$,
- (iv) $\partial_s U_{s,t}^\sigma f = -\mathcal{L}_N^\sigma(s) U_{s,t}^\sigma f$ for every $f \in C^{(0,2)}(N)$,
- (v) $\partial_t U_{s,t}^\sigma f = U_{s,t}^\sigma \mathcal{L}_N^\sigma(t) f$ for every $f \in C^{(0,2)}(N)$,
- (vi) $U_{s,t}^\sigma : C^{(0,2)}(N) \rightarrow C^{(0,2)}(N)$ for all $s \leq t$.

The family $U_{s,t}^\sigma$ is called the *evolution* generated by $\mathcal{L}_N^\sigma(t)$. By $P_{t,s}^\sigma$ we denote the corresponding kernel

$$U_{s,t}^\sigma f(x) = \int_N P_{t,s}^\sigma(x; y) f(y) dy.$$

Since $\mathcal{L}_N^\sigma(t)$ commutes with left translation, the same is true for $U_{s,t}^\sigma$. Hence,

$$P_{t,s}^\sigma(x; y) = P_{t,s}^\sigma(e; x^{-1}y).$$

With a small abuse of notation we write

$$P_{t,s}^\sigma(x) = P_{t,s}^\sigma(e; x).$$

Hence, the operator $U_{s,t}^\sigma$ is a convolution operator with a probability measure (with a smooth density) $P_{t,s}^\sigma$,

$$U_{s,t}^\sigma f = f * P_{t,s}^\sigma.$$

We call $P_{t,s}^\sigma(x)$ or $P_{t,s}^\sigma(x; y)$ the *evolution kernel*. Sometimes $P_{t,s}^\sigma(x; y)$ is called the *transition kernel* since in probabilistic terms $P_{t,s}^\sigma(x; y)$ is the transition kernel for the time-dependent Markov process (or evolution), $\omega(t)$, on N defined by the operator $\mathcal{L}_N^\sigma(t)$. Probability that starting from x at time s the process $\omega(t)$ is in a given set $B \subset N$ is

$$\mathbf{P}_{s,x}(\omega(t) \in B) = \int_B P_{t,s}^\sigma(x; y) dy.$$

By (iii), for $s \leq r \leq t$,

$$P_{t,r}^\sigma * P_{r,s}^\sigma = P_{t,s}^\sigma.$$

1.3 Main result

Our aim is to estimate the evolution kernel $P_{t,s}^\sigma$. In order to do this, first we disintegrate the process $\omega(t)$ into the corresponding processes on M and V respectively. Specifically, let

$$\mathcal{L}_M^\sigma(t) = \sum_{j=1}^{d_1} Y_j(t)^2 \quad \text{and} \quad \mathcal{L}_V^\sigma(t) = \sum_{j=1}^{d_2} X_j(t)^2 \tag{1.1}$$

thought of as operators on M and V respectively.

For $v \in V$, let

$$\mathcal{L}_M^\sigma(t)^v = \sum_{j=1}^{d_1} (\text{Ad}(v)Y_j(t))^2. \tag{1.2}$$

Then the operator $\mathcal{L}_N^\sigma(t)$ is the *skew-product* of the above defined operators, i.e.,

$$\mathcal{L}_N^{\sigma(t)} f(m, v) = \mathcal{L}_V^\sigma(t) f(m, \cdot)|_v + \mathcal{L}_M^\sigma(t)^v f(\cdot, v)|_m, \quad t \in \mathbb{R}^+.$$

The time-dependent family of operators $\mathcal{L}_V^\sigma(t)$ gives rise to an evolution on $V = \mathbb{R}^{d_2}$ that is described by a kernel $P_{t,s}^{V,\sigma}$ which may be explicitly computed, since V is abelian. For $\eta \in C^\infty([0, +\infty), V)$ let

$$\mathcal{L}_M^\sigma(t)^\eta = \sum_{j=1}^{d_1} (\text{Ad}(\eta(t))Y_j(t))^2.$$

This family of operators gives rise to an evolution on $M = \mathbb{R}^{d_1}$ that is described by a kernel $P_{t,s}^{M,\sigma,\eta}$ which may also be explicitly computed (see Sect. 4).

One of our main tools is the following *skew-product formula* for $P_{t,s}^\sigma$ (which can be proved along the lines of [23, Theorem 1.2], where diagonal action of A on N was considered).

Theorem 1.1 For $m \in M$ and $v \in V$,

$$\begin{aligned} & \int_N P_{t,s}^\sigma(m, v; m', v') f(m', v') dm' dv' \\ &= \int \int_M P_{t,s}^{M,\sigma,\eta}(m; m') f(m', \eta(t)) dm' d\mathbf{W}_{s,v}^{V,\sigma}(\eta) \end{aligned}$$

where $\mathbf{W}_{s,v}^{V,\sigma}$ is the probability measure on the space $C([s, +\infty), V)$ generated by the diffusion process $\eta(t)$ starting from $v \in V$ at time s , with the generator $\mathcal{L}_V^\sigma(t)$.

A difficulty in applying the above formula is that the process $\eta(t)$ does not have independent coordinates. This difficulty is overcome with the help of Proposition 3.1

which gives the estimate for the joint probability of $\sup_{u \in [s, t]} \|\eta(u)\|_\infty$ and the position of the process η at time t , i.e., $\eta(t)$. This makes all the computation quite involved.

In order to state our main theorem we need to introduce some notation. Let, for $1 \leq j \leq d_2$,

$$V_j(\tau, t) = \max_{s \in [\tau, t]} (A_V^\sigma(\tau, s) - A_V^\sigma(\tau, s)A_V^\sigma(\tau, t)^{-1}A_V^\sigma(\tau, s))_{jj}, \tag{1.3}$$

where

$$A_V^\sigma(\tau, s) = 2 \int_\tau^s T^\sigma(u)T^\sigma(u)^* du. \tag{1.4}$$

Set

$$\begin{aligned} \mathcal{S}(\tau, t) &= \sum_{i \geq j} \int_\tau^t |s_{ij}^\sigma(u)|^2 du, & \mathcal{S}_\Pi(\tau, t) &= \prod_{j=1}^{d_1} \int_\tau^t e^{2\xi_j(\sigma(u))} du, \\ \mathcal{T}(\tau, t) &= \sum_{i \geq j} \int_\tau^t |t_{ij}^\sigma(u)|^2 du, & \mathcal{T}_\Pi(\tau, t) &= \prod_{j=1}^{d_2} \int_\tau^t e^{2\vartheta_j(\sigma(u))} du, \\ \mathcal{V}(\tau, t) &= \sum_{j=1}^{d_2} V_j(\tau, t). \end{aligned} \tag{1.5}$$

The main result is the following estimate.

Theorem 1.2 *For every $T > 0$ there are positive constants c_1, c_2, c_3 and a natural number k_o such that for all $T \geq t \geq \tau \geq 0$ and all $(m, v) \in N$,*

$$\begin{aligned} &P_{t, \tau}^\sigma(m, v) \\ &\leq c_1 \frac{\tilde{\Theta}(\tau, t, v) - \|v\|_\infty + 2}{\mathcal{S}_\Pi(\tau, t)^{1/2} \mathcal{T}_\Pi(\tau, t)^{1/2}} \exp\left(-\frac{c_2 \|v\|^2}{\mathcal{T}(\tau, t)} - \frac{c_3 \|m\|^2}{(\tilde{\Theta}(\tau, t, v) + 1)^{2k_o} \mathcal{S}(\tau, t)}\right) \\ &+ c_1 \frac{\|m\|^{\frac{1}{2k_o}}}{\mathcal{S}_\Pi(\tau, t)^{1/2} \mathcal{T}_\Pi(\tau, t)^{1/2}} \exp\left(-\frac{c_2 \|v\|^2}{\mathcal{T}(\tau, t)} - \frac{c_3 \|m\|^2}{(\tilde{\Theta}(\tau, t, v) + 1 + \|m\|^{\frac{1}{2k_o}})^{2k_o} \mathcal{S}(\tau, t)}\right) \\ &+ c_1 \mathcal{S}_\Pi(\tau, t)^{-1/2} \mathcal{T}_\Pi(\tau, t)^{-1/2} \mathcal{V}(\tau, t)^{1/2} \exp\left(-\frac{c_2 \|v\|^2}{\mathcal{T}(\tau, t)} - \frac{\|m\|^{1/k_o}}{2\mathcal{V}(\tau, t)}\right), \end{aligned} \tag{1.6}$$

where

$$\Theta(\tau, t, v) = \max_{s \in [\tau, t]} \|A_V^\sigma(\tau, s)A_V^\sigma(\tau, t)^{-1}v\|_\infty, \tag{1.7}$$

and

$$\tilde{\Theta}(\tau, t, v) = \Theta(\tau, t, v) + C \sum_{j=1}^n V_j(\tau, t)^{1/2}. \tag{1.8}$$

Remark In Sect. 7 we give explicit estimates for the quantities $\tilde{\Theta}(\tau, t, v)$ and $\mathcal{V}(\tau, t)$.

Remark Gaussian estimates in \mathbb{R}^n for the fundamental solution of the time-dependent parabolic equations are usually obtained under the assumption that the operator is (uniformly) elliptic (see e.g. classical papers by Aronson [2] and Fabes and Stroock [11]). We do not require this condition and our estimate explicitly depends on the coefficients of the operator.

Remark If the action of A on N is diagonal, i.e., the polynomials in entries of matrices $S^\sigma(t)$ and $T^\sigma(t)$ [see the assumptions (A2) and (A3)] satisfy $h_{ij}^M = h_{ij}^V = 0$ for $i \neq j$ then all the quantities appearing in Theorem 1.2 can be easily computed. We get

$$V_j(\tau, t) = 2 \int_\tau^t e^{2\vartheta_j(\sigma(u))} du, \quad \mathcal{S}(\tau, t) = \sum_{j=1}^{d_1} \int_\tau^t e^{2\xi_j(\sigma(u))} du$$

and

$$\mathcal{V}(\tau, t) = 2 \sum_{j=1}^{d_2} \int_\tau^t e^{2\vartheta_j(\sigma(u))} du \quad \mathcal{T}(\tau, t) = \mathcal{V}(\tau, t)/2.$$

Finally,

$$\Theta(\tau, t, v) = \max_{s \in [\tau, t]} \int_\tau^s e^{2\vartheta_j(\sigma(u))} du \left(\int_\tau^t e^{2\vartheta_j(\sigma(u))} du \right)^{-1} \|v\|_\infty = \|v\|_\infty.$$

In this setting Theorem 1.2 simplifies and we obtain [23, Theorem 4.1].

1.4 Applications

Since the estimate given by Theorem 1.2, at first glance, seems to be quite technical and complicated it is worth to explain why this formula is important and where it can be used. First of all the estimate for $P_{t,s}^\sigma$, given by Theorem 1.2, can be applied in the analysis of left-invariant, second-order differential operators on the higher rank NA groups, i.e., the semi-direct product $N \rtimes \mathbb{R}^k$ as described above (at this moment we do not assume that $N = M \rtimes V$). Consider, for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$, the left-invariant differential operator of the form

$$\mathcal{L}_\alpha = \sum_{j=1}^{d_2} X_j(a)^2 + \sum_{j=1}^{d_1} Y_j(a)^2 + \Delta_\alpha, \tag{1.9}$$

where

$$\Delta_\alpha = \sum_{j=1}^k (\partial_{a_j}^2 - 2\alpha_j \partial_{a_j}).$$

In this setting properties of bounded harmonic functions on S is certainly of interest. Under some assumption on the drift vector α there exists a *Poisson kernel* ν for \mathcal{L}_α [6, 7]. That is, there is a C^∞ function ν on N such that every bounded \mathcal{L}_α -harmonic function F on S may be written as a *Poisson integral* against a bounded function f on $S/A = N$,

$$F(g) = \int_{S/A} f(gx)\nu(x)dx = \int_N f(x)\check{\nu}^a(x^{-1}x_o)dx, \quad g = (x_o, a),$$

where

$$\check{\nu}^a(x) = \nu(a^{-1}x^{-1}a)\chi(a)^{-1},$$

where χ is the modular function for left invariant Haar measure on S , i.e.,

$$\chi(g) = \det(\text{Ad}(g)).$$

Conversely the Poisson integral of any $f \in L^\infty(N)$ is a bounded \mathcal{L}_α -harmonic function.

It is known that the Poisson kernel ν is equal to $\lim_{t \rightarrow \infty} \pi_N(\mu_t)$, where $\pi_N(g) = x(g)$ is a projection from S onto N . To get some information on μ_t we use a well known formula which express T_t as a skew-product of the diffusion on N and A . For $f \in C_c(N \times \mathbb{R}^k)$ and $t \geq 0$,

$$T_t f(x, a) = \mathbf{E}_a U_{0,t}^\sigma f(x, \sigma_t) = \mathbf{E}_a (f *_N P_{t,0}^\sigma)(x, \sigma_t), \tag{1.10}$$

where the expectation \mathbf{E} is taken with respect to the distribution of the process σ_t (Brownian motion with drift) in \mathbb{R}^k generated by Δ_α . The operator $U_{0,t}^\sigma$ acts on the first variable of the function f (as a convolution operator). The idea of such a decomposition goes back to [16, 17, 29]. In the context of NA groups with $\dim A = 1$ this decomposition was used in [7–10], and later was generalized by the authors and applied for $\dim A > 1$, see e.g. [20, 22]. Note that Theorem 1.1 is a generalization of (1.10) to evolution operators.

Estimates for the Poisson kernel for the operator (1.9) were obtained by the authors in a series of papers [20–24]. However, in all these papers the action of A on N is diagonal. Thus Theorem 1.2 opens the door to consider non-diagonal actions. This is going to be the subject of our future research.

1.5 Structure of the paper

The outline of the rest of the paper is as follows. In Sect. 2 we state the formula for the evolution kernel in \mathbb{R}^n and recall the Borell–TIS inequality which is in Sect. 3 used in the proof of an appropriate estimate for $\mathbf{P}(\sup_{s \in [\tau, t]} \|\eta(s)\|_\infty \geq u \text{ and } \eta(t) \in B)$ for $u \in \mathbb{R}$ and $B \subset \mathbb{R}^n$. In Sects. 4 and 5 we study evolutions on M and V , respectively. Finally in Sect. 6 we give the proof of Theorem 1.2 and in Sect. 7 we give some estimates for quantities given in (1.7) and (1.8).

2 Preliminaries

2.1 Gaussian variables and fields

We follow the presentation in [1]. For \mathbb{R}^n -valued random variables X and Y their covariance matrix is defined as $\text{Cov}(X, Y) = \mathbf{E}(X - \mathbf{E}X)(Y - \mathbf{E}Y)^t$. An \mathbb{R}^n -valued random variable X is said to be *multivariate Gaussian* if for every non-zero $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, the real valued random variable $\langle \alpha, X \rangle = \sum_{i=1}^n \alpha_i X_i$ is Gaussian. In this case the density of X is given by the *multivariate normal density*

$$(2\pi)^{-n/2}(\det C)^{-1/2}e^{-\frac{1}{2}C^{-1}(x-m)\cdot(x-m)},$$

where $m = \mathbf{E}X$ and $C = \text{Cov}(X, X)$ is a positive semi-definite $n \times n$ covariance matrix. In this case we write $X \sim \mathcal{N}_n(m, C)$ or simply $X \sim \mathcal{N}(m, C)$.

Lemma 2.1 *Let $X \sim \mathcal{N}_n(m, C)$. Assume that $d < n$ and make the partition*

$$\begin{aligned} X &= (X^1, X^2) = ((X_1, \dots, X_d), (X_{d+1}, \dots, X_n)), \\ m &= (m_1, m_2) = ((m_1, \dots, m_d), (m_{d+1}, \dots, m_n)) \end{aligned}$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where C_{11} is a $d \times d$ -matrix. Then each $X^i \sim \mathcal{N}(m^i, C_{ii})$ and the conditional distribution of X^i given X^j is also Gaussian, with mean vector

$$m_{i|j} = m^i + C_{ij}C_{jj}^{-1}(X^j - m^j)$$

and covariance matrix

$$C_{i|j} = C_{ii} - C_{ij}C_{jj}^{-1}C_{ji}.$$

Proof See e.g. [1, p. 8]. □

A *random field* is a stochastic process, taking values in some space, usually in a Euclidean space, and defined over a parametric space T . A real valued *Gaussian process* is a random field f on a parameter set T for which the (finite dimensional) distributions of $(f_{t_1}, \dots, f_{t_n})$ are multivariate Gaussian for each $1 \leq n < +\infty$ and each $(t_1, \dots, t_n) \in T^n$.

2.2 Gaussian inequalities

The following powerful inequality was discovered independently, and was proved in very different ways, by Borell [3] and Tsirelson et al. [30]. Following [1] we call the following inequality Borell–TIS inequality.

Theorem 2.2 (Borell–TIS inequality) *Let f_t be a centered Gaussian process, almost surely bounded on T . Write $|f|_T = \sup_{t \in T} f_t$. Then $\mathbf{E}|f|_T < +\infty$ and, for all $u > 0$,*

$$\mathbf{P}(|f|_T - \mathbf{E}|f|_T > u) \leq e^{-u^2/2\sigma_T^2},$$

where

$$\sigma_T^2 = \sup_{t \in T} \mathbf{E}f_t^2.$$

Proof For the proof see the original papers [3,30] or [1]. □

Immediately, we get the following

Corollary 2.3 *Let f_t be a centered Gaussian process, almost surely bounded on T . Then for all $u > \mathbf{E}|f|_T$,*

$$\mathbf{P}(|f|_T > u) \leq e^{-(u-\mathbf{E}|f|_T)^2/2\sigma_T^2}.$$

2.3 Evolution equation in \mathbb{R}^n

Let

$$L(t) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(t) \partial_i \partial_j + \sum_{j=1}^n \delta_j(t) \partial_j, \tag{2.1}$$

where $\partial_i = \partial_{x_i}$ and $a(t) = [a_{ij}(t)]$ is a symmetric, positive definite matrix and the a_{ij} and δ_j belong to $C([0, \infty), \mathbb{R})$. For $s > t$, let $P_{t,s}$ be the evolution kernel generated by $L(t)$. Let, for $1 \leq i, j \leq n$,

$$\begin{aligned} A_{s,t} &= [A_{ij}(s, t)] = \left[\int_s^t a_{ij}(u) du \right], \\ D_{s,t} &= [D_j(s, t)] = \left[\int_s^t \delta_j(u) du \right]. \end{aligned} \tag{2.2}$$

Proposition 2.4 *The evolution kernel $P_{t,s}$ corresponding to the operator $L(t)$ defined in (2.1) is given by*

$$P_{t,s}(x) = (2\pi)^{-\frac{n}{2}} (\det A_{s,t})^{-\frac{1}{2}} e^{-\frac{1}{2}(A_{s,t}^{-1}(x-D_{s,t})) \cdot (x-D_{s,t})}.$$

Proof See e.g. [23, Proposition 2.9] □

3 Main probabilistic estimate

Consider the operator $L(t)$, defined in (2.1), without the drift vector $\delta(t) = (\delta_1(t), \dots, \delta_n(t))$, i.e,

$$L(t) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(t) \partial_i \partial_j. \tag{3.1}$$

Let $b_s = (b_s^1, \dots, b_s^n)$ be the stochastic process generated by the operator $L(t)$. Define, for $v \in \mathbb{R}^n$,

$$B_\varepsilon(v) = \prod_{j=1}^n B_\varepsilon^1(v_j) \quad \text{and} \quad B_\varepsilon^1(v_j) = [v_j - \varepsilon/2, v_j + \varepsilon/2],$$

and let $\|\cdot\|_\infty$ denote the ℓ^∞ -norm on \mathbb{R}^n , i.e., for a vector $y \in \mathbb{R}^n$, $\|y\|_\infty = \max_{1 \leq j \leq n} |y_j|$.

The distribution of the process b_t starting at time τ from v , i.e., $b_\tau = v$, is denoted by $\mathbf{P}_{\tau,v}(\cdot)$. This is a probability measure on the space of trajectories $b_t \in C([0, \infty), \mathbb{R}^n)$.

Proposition 3.1 *Let b_t be the process generated by $L(t)$ defined in (3.1). For every $T > 0$ there exists a constant $C > 0$ such that, for every $\varepsilon > 0$, $u \geq 0$, $v \in \mathbb{R}^n$, and all $T \geq t \geq \tau \geq 0$, the following estimate holds,*

$$\begin{aligned} & (\det A_{\tau,t})^{\frac{1}{2}} \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \|b_s\|_\infty > u \text{ and } b_t \in B_\varepsilon(v) \right) \\ & \leq C \int_{B_\varepsilon(v)} e^{-(u - \Theta(\tau,t,a) - C \sum_{j=1}^n V_j(\tau,t)^{1/2})^2 / 2\mathcal{V}(\tau,t)} e^{-\frac{1}{2} A_{\tau,t}^{-1} a \cdot a} da \end{aligned}$$

for all $u > \sup_{a \in B_\varepsilon(v)} \Theta(\tau, t, a) + C \sum_{j=1}^n V_j(\tau, t)^{1/2}$, where

$$\Theta(\tau, t, a) = \sup_{s \in [\tau,t]} \|A_{\tau,s} A_{\tau,t}^{-1} a\|_\infty$$

and

$$V_j(\tau, t) = \max_{s \in [\tau,t]} (A_{\tau,s} - A_{\tau,s} A_{\tau,t}^{-1} A_{\tau,s})_{jj}, \quad \mathcal{V}(\tau, t) = \sum_{j=1}^n V_j(\tau, t).$$

Proof Let τ be fixed. To simplify notation we write, for $s \geq \tau$,

$$A_s = [A_{ij}(\tau, s)],$$

where $[A_{ij}]$ is the matrix defined in (2.2). We write

$$\begin{aligned} & \mathbf{P}_{\tau,0} \left(\sup_{s \in [0,t]} \|b_s\|_\infty > u \text{ and } b_t \in B_\varepsilon(v) \right) \\ & = \int_{B_\varepsilon(v)} \mathbf{P}_{\tau,0} \left(\sup_{s \in [0,t]} \|b_s\|_\infty > u \mid b_t = a \right) P_{t,\tau}(a) da. \end{aligned} \tag{3.2}$$

Now we estimate the conditional probability under the integral sign. For $s \leq t$ and fixed, consider $2n$ -dimensional random vector

$$(b_s, b_t) = (b_s^1, \dots, b_s^n, b_t^1, \dots, b_t^n) \sim \mathcal{N}_{2n}(0, C).$$

By Proposition 2.4, $\text{Cov}(b_s, b_s) = [A_{ij}(0, s)] = A_s$. Since the process b_s has independent increments we get

$$\text{Cov}(b_s, b_t) = \text{Cov}(b_s, b_s + b_t - b_s) = \text{Cov}(b_s, b_s) = A_s.$$

By $\text{Cov}(b_t, b_s) = \text{Cov}(b_s, b_t)^t$ we get,

$$C = \begin{bmatrix} A_s & A_s \\ A_s & A_t \end{bmatrix}.$$

By Lemma 2.1 the conditional distribution of b_s given b_t is Gaussian with mean vector

$$m_{s|t} = A_s A_t^{-1} b_t$$

and covariance matrix

$$C_{s|t} = A_s - A_s A_t^{-1} A_s.$$

Let $b_s(a) = (b_s^1(a), \dots, b_s^n(a))$ denote the process whose distribution is the conditional distribution of b_s given $b_t = a$. In this notation

$$\mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \|b_s\|_\infty > u \mid b_t = a \right) = \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} |b_s^j(a)| > u \right). \tag{3.3}$$

Let

$$\tilde{b}_s(a) := b_s(a) - \mathbf{E}_0 b_s(a) = b_s(a) - A_s A_t^{-1} a.$$

Clearly,

$$\tilde{b}_s(a) \sim \mathcal{N}_n(0, C_{s|t}). \tag{3.4}$$

We continue (3.3) as follows.

$$\begin{aligned} & \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} |b_s^j(a)| > u \right) \\ &= \mathbf{P}_0 \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} |\tilde{b}_s^j(a) + (A_s A_t^{-1} a)_j| > u \right) \end{aligned}$$

$$\begin{aligned} &\leq \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} |\tilde{b}_s^j(a)| + \sup_{s \in [\tau,t]} \|A_s A_t^{-1} a\|_\infty > u \right) \\ &= \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} |\tilde{b}_s^j(a)| > u - \Theta(\tau, t, a) \right). \end{aligned}$$

Thus, by symmetry of $\tilde{b}_s(a)$ (see (3.4)),

$$\mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} |b_s^j(a)| > u \right) \leq 2\mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \tilde{b}_s^j(a) > u - \Theta(\tau, t, a) \right). \tag{3.5}$$

Denote

$$\begin{aligned} \Xi(\tau, t, a) &= \mathbf{E}_{\tau,0} \sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \tilde{b}_s^j(a), \\ \sigma_t^2 &= \sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \mathbf{E}_{\tau,0}(\tilde{b}_s^j(a))^2. \end{aligned} \tag{3.6}$$

By Corollary 2.3, with $T = \{1, \dots, n\} \times [\tau, t]$,

$$\mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \tilde{b}_s^j(a) > u - \Theta(\tau, t, a) \right) \leq e^{-(u - \Theta(\tau, t, a) - \Xi(\tau, t, a))^2 / 2\sigma_t^2},$$

for all $u > \Theta(\tau, t, a) + \Xi(\tau, t, a)$. Taking together the above estimate, (3.3), (3.5), and putting the resulting upper bound into (3.2) we get

$$\begin{aligned} &\mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \|b_s\|_\infty > u \text{ and } b_t \in B_\varepsilon(v) \right) \\ &\leq 2 \int_{B_\varepsilon(v)} e^{-(u - \Theta(\tau, t, a) - \Xi(\tau, t, a))^2 / 2\sigma_t^2} P_{t,\tau}(a) da \\ &\leq C(\det A_t)^{-\frac{1}{2}} \int_{B_\varepsilon(v)} e^{-(u - \Theta(\tau, t, a) - \Xi(\tau, t, a))^2 / 2\sigma_t^2} e^{-\frac{1}{2} A_t^{-1} a \cdot a} da, \end{aligned} \tag{3.7}$$

for all $u > \sup_{a \in B_\varepsilon(v)} \Theta(\tau, t, a) + \Xi(\tau, t, a)$. We used Proposition 2.4 in the last inequality.

Let us estimate the quantities introduced in (3.6). The coordinate process $\tilde{b}_s^j(a)$ is a Gaussian process and, by (3.4),

$$\tilde{b}_s^j(a) \sim \mathcal{N}_1(0, v_j(s)), \quad \text{where } v_j(s) = v_j(\tau, s) = (A_s - A_s A_t^{-1} A_s)_{jj}. \tag{3.8}$$

In this notation

$$V_j(\tau, t) = \max_{s \in [\tau,t]} v_j(s).$$

It follows from [26, (1.1)] that for every $\eta > 0$ and $T > 0$ there is $c = c_{\eta,T}$ such that for every $u > 0$ and all $T \geq t \geq \tau \geq 0$,

$$\mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \tilde{b}_s^j(a) > u \right) \leq c e^{-(1-\eta)u^2/2\sigma_t^2}.$$

Hence, taking $\eta = 1/2$,

$$\begin{aligned} \mathbf{E}_{\tau,0} \sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \tilde{b}_s^j(a) &= \int_0^\infty \mathbf{P}_{\tau,0}(\sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} \tilde{b}_s^j(a) > u) du \\ &\leq c \int_0^\infty e^{-u^2/4\sigma_t^2} du \leq c \int_{\mathbb{R}} e^{-u^2/4\sigma_t^2} du \leq c\sigma_t. \end{aligned}$$

Hence,

$$\mathbf{E}(\tau, t, a) \leq c\sigma_t.$$

By (3.8),

$$\sigma_t^2 = \sup_{s \in [\tau,t]} \max_{1 \leq j \leq n} v_j(\tau, s) \leq \sum_{j=1}^n V_j(\tau, t) = \mathcal{V}(\tau, t).$$

Hence, for $u > \sup_{a \in B_\varepsilon(v)} \Theta(\tau, t, a) + C \sum_{j=1}^n V_j(\tau, t)^{1/2}$, we can rewrite (3.7) as follows,

$$\begin{aligned} &(\det A_t)^{\frac{1}{2}} \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \|b_s\|_\infty > u \text{ and } b_t \in B_\varepsilon(v) \right) \\ &\leq C \int_{B_\varepsilon(v)} e^{-(u-\Theta(\tau,t,a)-C \sum_{j=1}^n V_j(\tau,t)^{1/2})^2/2\mathcal{V}(\tau,t)} e^{-\frac{1}{2}A_t^{-1}a \cdot a} da. \end{aligned}$$

Hence, the result follows. □

With the notation as in Proposition 3.1 we have immediately the following

Corollary 3.2 *For every $T > 0$ there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and all $T \geq t \geq \tau \geq 0$ the following estimate holds,*

$$\begin{aligned} &\varepsilon^{-n} (\det A_{\tau,t})^{\frac{1}{2}} \mathbf{P}_{\tau,0} \left(\sup_{s \in [\tau,t]} \|b_s\|_\infty > u \text{ and } b_t \in B_\varepsilon(v) \right) \\ &\leq C \sup_{a \in B_\varepsilon(v)} e^{-(u-\Theta(\tau,t,a)-C \sum_{j=1}^n V_j(\tau,t)^{1/2})^2/2 \sum_{j=1}^n V_j(\tau,t)} e^{-\frac{1}{2}A_{\tau,t}^{-1}a \cdot a} \end{aligned}$$

for all $u > \sup_{a \in B_\varepsilon(v)} \Theta(\tau, t, a) + C \sum_{j=1}^n V_j(\tau, t)^{1/2}$.

4 Evolution on M

We choose coordinates y_i for M for which Y_i corresponds to $\partial_i = \partial_{y_i}$, $1 \leq i \leq d_1$. Let $\eta \in C([0, \infty), V)$ and consider the evolution on M generated by the operator

$$\mathcal{L}_M^\sigma(t)^\eta = \sum_{j=1}^{d_1} (\text{Ad}(\eta(t))Y_j(t))^2.$$

Then

$$\text{Ad}(\eta(t))Y_j(t) = \text{Ad}(\eta(t))\Phi^\sigma(t)Y_j = \sum_{k=1}^{d_1} \psi_{j,k}(t)Y_k,$$

and consequently,

$$\begin{aligned} \text{Ad}(\eta(t))Y_j(t) &= \sum_{j=1}^{d_1} (\text{Ad}(\eta(t))Y_j(t))^2 = \sum_{k,l=1}^{d_1} \sum_{j=1}^{d_1} \psi_{k,j}(t)\psi_{l,j}(t)Y_kY_l \\ &= \sum_{k,l=1}^{d_1} (\psi(t)\psi(t)^*)_{kl}Y_kY_l, \end{aligned}$$

where $\psi(t) = [\psi_{i,j}(t)]$ is the matrix of $\text{Ad}(\eta(t))\Phi^\sigma(t)|_M$. Thus the matrix $[a_{ij}]$ from (2.1) for the operator $\mathcal{L}_M^\sigma(t)^\eta$ is

$$a_M^{\sigma,\eta}(t) = 2[\text{Ad}(\eta(t))S^\sigma(t)][\text{Ad}(\eta(t))S^\sigma(t)]^*,$$

where the adjoint is in the y_j coordinates. Let

$$A_M^{\sigma,\eta}(s, t) = \int_s^t a_M^{\sigma,\eta}(u) du.$$

For a $d \times d$ invertible matrix A we set

$$B(A)(x) = \frac{1}{2}A^{-1}x \cdot x \quad \text{and} \quad \mathcal{D}(A) = (2\pi)^{-\frac{d}{2}}(\det A)^{-\frac{1}{2}}.$$

It follows from Proposition 2.4 that the evolution kernel $P_{t,s}^{M,\sigma,\eta}$ for the operator $\mathcal{L}_M^\sigma(t)^\eta$ is Gaussian, and in our notation, is given by

$$P_{t,s}^{M,\sigma,\eta}(m) = \mathcal{D}(A_M^{\sigma,\eta}(t, s))e^{-B(A_M^{\sigma,\eta}(t,s))(m)}, \quad m \in M = \mathbb{R}^{d_1}$$

and the corresponding transition kernel is,

$$P_{t,s}^{M,\sigma,\eta}(m; m') = P_{t,s}^{M,\sigma,\eta}(m - m'), \quad m^1, m^2 \in M = \mathbb{R}^{d_1}.$$

For a matrix A the operator norm, that is the norm A considered as a liner operator from $\ell^2(\mathbb{R}^n) \rightarrow \ell^2(\mathbb{R}^n)$ is denoted by $\|A\|$. We will need the following two simple lemmas.

Lemma 4.1 *Let A be a positive semi-definite matrix. Then*

$$B(A)(x) \geq \|x\|^2 / (2\|A\|).$$

Proof See e.g. [22, Lemma 4.1] □

Lemma 4.2 *Let K and D be square matrices and let*

$$A = \begin{bmatrix} K & B \\ C & D \end{bmatrix}.$$

If $\det K \neq 0$ then $\det A = \det K \det(D - CK^{-1}B)$.

Proof See e.g. [32]. □

Now we prove an upper bound on $\mathcal{D}(A^{\sigma,\eta}(s, t))$ that is independent of η generalizing [22, Lemma 4.2].

Lemma 4.3 *There is a constant $C > 0$ such that*

$$\mathcal{D}(A_M^{\sigma,\eta}(s, t)) \leq C \left(\prod_{i=1}^{d_1} \int_s^t s_{ii}^\sigma(u)^2 du \right)^{-1/2},$$

where $s_{ij}^\sigma(t)$ are the entries of the matrix $S^\sigma(t)$ defined in (1.1).

Proof By the assumptions (A1) and (A2) on p. 2 the operator $\text{ad}_X: \mathfrak{m} \rightarrow \mathfrak{m}$ is lower triangular for all $X \in \mathfrak{v}$ and $\Phi^\sigma(t)|_M = S^\sigma(t)$, where $S^\sigma(t)$ is linear operator on \mathfrak{m} that is lower triangular in the Y_i basis. We omit the t and σ dependence for the sake of simplicity. In the coordinates defined by the Y_i basis,

$$\text{ad}_X = \begin{bmatrix} X_o & 0 \\ v^t & 0 \end{bmatrix}, \quad \text{Ad}_x = e^{\text{ad}_X} = \begin{bmatrix} e^{X_o} & 0 \\ v(X)^t & 1 \end{bmatrix}, \quad \text{where } x = \exp X,$$

where the X_o is $(d_1 - 1) \times (d_1 - 1)$ -matrix and v is a $(d_1 - 1) \times 1$ -column vector.

Then

$$\text{Ad}_x S = e^{\text{ad}_X} \begin{bmatrix} S_0 & 0 \\ S_1 & s_{d_1 d_1} \end{bmatrix} = \begin{bmatrix} e^{X_o} S_0 & 0 \\ v(X)^t S_0 + S_1 & s_{d_1 d_1} \end{bmatrix} =: \begin{bmatrix} e^{X_o} S_0 & 0 \\ F^t & s_{d_1 d_1} \end{bmatrix}. \quad (4.1)$$

Then

$$\text{Ad}_x S (\text{Ad}_x S)^t = \begin{bmatrix} e^{X_o} S_0 S_0^t e^{X_o^t} & G \\ G^t & s_{d_1 d_1}^2 + |F|^2 \end{bmatrix},$$

where

$$G = e^{X_o} S_o F = e^{X_o} S_o (S_o^t v(X) + S_1^t).$$

Hence,

$$A^{\sigma,\eta}(s, t) = \begin{bmatrix} A_o & B \\ B^t & A + E \end{bmatrix},$$

where

$$\begin{aligned} A_o &= \int_s^t e^{X_o(u)} S_o(u) S_o(u)^t e^{X_o(u)^t} du, & B &= \int_s^t G(u) du, \\ A &= \int_s^t s_{d_1 d_1}^2(u) du, & E &= \int_s^t |F(u)|^2 du. \end{aligned}$$

From Lemma 4.2,

$$\begin{aligned} \det A^{\sigma,\eta}(s, t) &= (\det A_o)(A + E - B^t A_o^{-1} B) \\ &= (\det A_o)A + (\det A_o)(E - B^t A_o^{-1} B) \\ &= (\det A_o)A + \det \begin{bmatrix} A_o & B \\ B^t & E \end{bmatrix}. \end{aligned}$$

The determinant on the right is non-negative since it is the $s_{d_1 d_1} = 0$ case of formula (4.1). Hence,

$$\det A_M^{\sigma,\eta}(s, t) \geq A(\det A_o).$$

Our result follows by induction. □

Now we estimate the operator norm of the matrix

$$A_M^{\sigma,\eta}(s, t) = \int_s^t \text{Ad}(\eta(u)) S^\sigma (\text{Ad}(\eta(u)) S^\sigma(u))^t du. \tag{4.2}$$

Recall that we assume that S^σ is lower triangular [assumption (A2)]. Specifically, for $i \geq j$,

$$s_{ij}^\sigma(u) = h_{ij}^M(\sigma(u)) e^{\xi_j(\sigma(u))},$$

where h_{ij}^M are polynomials in $a \in A = \mathbb{R}^k$, and $h_{jj}^M = 1$.

Lemma 4.4 *Let $\eta = \eta(u) = (\eta_1(u), \dots, \eta_{d_2}(u)) \in \mathbb{R}^{d_2} = V$ be a continuous function. There exist constants $C > 0$ and $k_o \in \mathbb{N}$, such that*

$$\|A_M^{\sigma,\eta}(s, t)\| \leq C(1 + \Lambda^\eta(s, t))^{2k_o} \sum_{i \geq j} \int_0^t |s_{ij}^\sigma(u)|^2 du,$$

where

$$\Lambda^\eta(s, t) = \sup_{s \leq u \leq t} \|\eta(u)\|_\infty.$$

Proof We note first that for $X \in \mathfrak{n}$,

$$\text{Ad}_X \big|_{\mathfrak{m}} = \sum_{\ell=0}^{k_o} (\text{ad}_X \big|_{\mathfrak{m}})^\ell / \ell!$$

Hence,

$$\|\text{Ad}_X \big|_{\mathfrak{m}}\| \leq C(1 + \|\text{ad}_X\|)^{k_o} \leq C'(1 + \|X\|)^{k_o}.$$

Since all norms on finite dimensional vector space are equivalent we get

$$\|S^\sigma(u)\|^2 \leq C \sum_{i \geq j} |s_{ij}^\sigma(u)|^2. \tag{4.3}$$

Our result follows by bringing the norm inside the integral in (4.2). □

5 Evolution on V

Now we consider the evolution process $\eta(t)$ on V generated by

$$\mathcal{L}_V^\sigma(t) = \sum_{j=1}^{d_2} X_j(t)^2 = \sum_{j=1}^{d_2} (T^\sigma(t)X_j)^2$$

(see (1.1) on p. 4). The matrix $a_t = [a_{ij}(t)]$ defined in (2.1) is equal to

$$a_V^\sigma(t) = 2T^\sigma(t)T^\sigma(t)^*.$$

Let

$$A_V^\sigma(s, t) = \int_s^t a_V^\sigma(u) du.$$

One of the differences between this setting and the case of meta-abelian groups considered in [23] is that the coordinates $\eta_j(t)$, $j = 1, \dots, d_2$, of the process $\eta(t)$ are no longer independent.

Notice that exactly in the same way as in the proof of Lemma 4.3 we can show that

$$(\det A_V^\sigma(s, t))^{-1/2} \leq C \left(\prod_{j=1}^{d_2} \int_s^t e^{2\vartheta_j(\sigma(u))} du \right)^{-1/2}. \tag{5.1}$$

We have the following, analogous to (4.3), inequality

$$\|T^\sigma(u)\|^2 \leq C \sum_{i \geq j} |t_{ij}^\sigma(u)|^2$$

which implies (as in Lemma 4.4)

$$\|A_V^\sigma(s, t)\| \leq C \sum_{i \geq j} \int_s^t |t_{ij}^\sigma(u)|^2 du. \tag{5.2}$$

Hence in the notation introduced in (1.3) and (1.5), using Lemma 4.1, Corollary 3.2 reads,

Proposition 5.1 *For every $T > 0$ there exist constants $C, c > 0$ such that, for all $T \geq t \geq \tau \geq 0$, all $u > 0$, and all $\varepsilon > 0$, the following estimate holds,*

$$\begin{aligned} &\varepsilon^{-n} \mathbf{W}_0^{V, \sigma} \left(\sup_{s \in [\tau, t]} \|\eta(s)\|_\infty \geq u \text{ and } \eta(t) \in B_\varepsilon(v) \right) \\ &\leq C \mathcal{T}_\Pi(\tau, t)^{-1/2} \sup_{a \in B_\varepsilon(v)} e^{-(u - \Theta(\tau, t, a) - C \sum_{j=1}^{d_2} V_j(\tau, t)^{1/2})^2 / 2\mathcal{V}(\tau, t)} \\ &\quad \times \exp\left(-\frac{c\|a\|^2}{T(\tau, t)}\right), \end{aligned}$$

for all $u > \sup_{a \in B_\varepsilon(v)} \Theta(\tau, t, a) + C \sum_{j=1}^{d_2} V_j(\tau, t)^{1/2}$, where

$$\Theta(\tau, t, v) = \max_{s \in [\tau, t]} \|A_V^\sigma(\tau, s)A_V^\sigma(\tau, t)^{-1}v\|_\infty.$$

6 Proof of Theorem 1.2

In this section we estimate the transition kernel for the evolution on $N = M \times V$,

$$P_{t, \tau}^\sigma(m, v) = P_{t, \tau}^\sigma(0, 0; m, v), \quad t \geq \tau.$$

Proof of Theorem 1.2 We allow the constants C and D to change from line to line. By Lemmas 4.1 and 4.3, for $t \geq \tau$ and $m, m' \in M$,

$$\begin{aligned}
 P_{t,\tau}^{M,\sigma,\eta}(m, m') &= \mathcal{D}(A_M^{\sigma,\eta}(\tau, t))e^{-B(A_M^{\sigma,\eta}(\tau, t))(m-m')} \\
 &\leq C\mathcal{S}_\Pi(\tau, t)^{-1/2}e^{-\frac{\|m-m'\|^2}{2\|A_M^{\sigma,\eta}(\tau, t)\|}}.
 \end{aligned}
 \tag{6.1}$$

By Theorem 1.1, for $m, m' \in M$ and $v, v' \in V$,

$$\begin{aligned}
 \int_V P_{t,\tau}^\sigma(m, v; m', v')\psi(v')dv' &= \int P_{t,\tau}^{M,\sigma,\eta}(m, m')\psi(\eta(t)) d\mathbf{W}_{\tau,v}^{V,\sigma}(\eta) \\
 &\leq C\mathcal{S}_\Pi(\tau, t)^{-1/2} \int \psi(\eta(t))e^{-\frac{\|m-m'\|^2}{2\|A_M^{\sigma,\eta}(\tau, t)\|}} d\mathbf{W}_{\tau,v}^{V,\sigma}(\eta).
 \end{aligned}$$

Let

$$F(m, \sigma, \eta) = \exp\left(-\frac{D\|m\|^2}{(1 + \Lambda^\eta(s, t))^{2k_\sigma}\mathcal{S}(\tau, t)}\right),$$

where

$$\Lambda^\eta(s, t) = \sup_{s \leq u \leq t} \|\eta(u)\|_\infty.$$

Then, by Lemma 4.4,

$$\mathcal{T}_\Pi(\tau, t)^{1/2} \int P_{t,\tau}^\sigma(m, v)\psi(v)dv \leq C \int F(m, \sigma, \eta)\psi(\eta(t)) d\mathbf{W}_{\tau,0}^{V,\sigma}(\eta). \tag{6.2}$$

For $v \in \mathbb{R}^{d_2}$ given and $\varepsilon > 0$, let

$$\psi_\varepsilon(\cdot) = \varepsilon^{-n}\mathbf{1}_{B_\varepsilon(v)}(\cdot),$$

where

$$B_\varepsilon(v) = \prod_{j=1}^{d_2} B_\varepsilon^1(v_j) \quad \text{and} \quad B_\varepsilon^1(v_j) = [v_j - \varepsilon/2, v_j + \varepsilon/2].$$

We will estimate (6.2) with ψ_ε in place of ψ as ε tend to zero.

Let $\mathbf{E}_{\tau,v}^\eta$ denote expectation with respect to the distribution $d\mathbf{W}_{\tau,v}^{V,\sigma}(\eta)$ of η in the space of trajectories ($\eta(\tau) = v \in V$). For $\ell = 1, 2, \dots$, define the sets of paths in V ,

$$\mathcal{A}_\ell = \left\{ \eta : \ell - 1 \leq \Lambda^\eta(\tau, t) = \sup_{u \in [\tau, t]} \|\eta(u)\|_\infty < \ell \right\}.$$

The integral on the right in (6.2) can be written as an infinite sum and estimated as follows

$$\begin{aligned}
& \sum_{\ell=1}^{\infty} \mathbf{E}_{\tau,0}^{\eta} F(m, \sigma, \eta) \psi_{\varepsilon}(\eta(t)) \mathbf{1}_{\mathcal{A}_{\ell}}(\eta) \\
& \leq \sum_{\ell=1}^{\infty} \exp\left(-\frac{D\|m\|^2}{(1+\ell)^{2k_o} \mathcal{S}(\tau, t)}\right) \mathbf{E}_{\tau,0}^{\eta} \psi_{\varepsilon}(\eta(t)) \mathbf{1}_{\mathcal{A}_{\ell}}(\eta) \\
& \leq \sum_{\ell=1}^{\infty} \exp\left(-\frac{D\|m\|^2}{\ell^{2k_o} \mathcal{S}(\tau, t)}\right) \mathbf{E}_{\tau,0}^{\eta} \psi_{\varepsilon}(\eta(t)) \mathbf{1}_{\mathcal{A}_{\ell}}(\eta), \tag{6.3}
\end{aligned}$$

for some $k_o \geq 1$.

To simplify notation we introduce

$$\begin{aligned}
c_{\ell} &= \exp\left(-\frac{D\|m\|^2}{\ell^{2k_o} \mathcal{S}(\tau, t)}\right), \\
\mathcal{E}_{\ell}(\varepsilon) &= \mathbf{E}_{\tau,0}^{\eta} \psi_{\varepsilon}(\eta(t)) \mathbf{1}_{\mathcal{A}_{\ell}}(\eta) = \varepsilon^{-n} \mathbf{W}_{\tau,0}^{V,\sigma}(\eta \in \mathcal{A}_{\ell} \text{ and } \eta(t) \in B_{\varepsilon}(v)).
\end{aligned}$$

Let $v \neq 0$ and choose $\varepsilon/2 < \|v\|_{\infty}$. If $\eta \in \mathcal{A}_{\ell}$ and $\eta(t) \in B_{\varepsilon}(v)$ then $\|\eta(t)\|_{\infty} \geq \|v\|_{\infty} - \varepsilon/2$. Hence,

$$\mathcal{E}_{\ell} = 0 \quad \text{for } \ell < \|v\|_{\infty} - \varepsilon/2. \tag{6.4}$$

Let

$$I = I(\varepsilon) = \sum_{\ell=1}^{\infty} c_{\ell} \mathcal{E}_{\ell}(\varepsilon),$$

and set

$$\ell_{\varepsilon}^* = \left\lceil \sup_{a \in B_{\varepsilon}(v)} \tilde{\Theta}(\tau, t, a) \right\rceil.$$

Obviously, $\lceil \tilde{\Theta}(\tau, t, v) \rceil > \|v\|_{\infty} - \varepsilon/2$ for sufficiently small ε . Since $\ell_{\varepsilon}^* \geq \lceil \tilde{\Theta}(\tau, t, v) \rceil$ we get that $\ell_{\varepsilon}^* > \|v\|_{\infty} - \varepsilon/2$ for sufficiently small ε . Hence,

$$I = I_1 + I_2 := \sum_{\|v\|_{\infty} \leq \ell \leq \ell_{\varepsilon}^*} c_{\ell} \mathcal{E}_{\ell}(\varepsilon) + \sum_{\ell = \ell_{\varepsilon}^*}^{\infty} c_{\ell} \mathcal{E}_{\ell}(\varepsilon).$$

We estimate I_1 . Since

$$\mathcal{E}_{\ell}(\varepsilon) \leq \mathcal{E}'(\varepsilon),$$

where

$$\mathcal{E}'(\varepsilon) = \varepsilon^{-n} \mathbf{W}_{\tau,0}^{V,\sigma}(\eta(t) \in B_{\varepsilon}(v)),$$

we get, by (5.1) and (5.2),

$$\begin{aligned}
 I_1 &\leq \mathcal{E}'(\varepsilon) \sum_{\|v\|_\infty \leq \ell \leq \ell_\varepsilon^*} \exp\left(-\frac{D\|m\|^2}{\ell^{2k_o}\mathcal{S}(\tau, t)}\right) \\
 &\leq c(\ell_\varepsilon^* - \|v\|_\infty + 1)\mathcal{T}_\Pi(\tau, t)^{-1/2} \exp\left(-\frac{c\|v\|^2}{\mathcal{T}(\tau, t)}\right) \exp\left(-\frac{D\|m\|^2}{(\ell_\varepsilon^*)^{2k_o}\mathcal{S}(\tau, t)}\right).
 \end{aligned}$$

Now we consider I_2 . In this case to estimate $\mathcal{E}_\ell(\varepsilon)$ we use Proposition 5.1,

$$\begin{aligned}
 I_2 &\leq C\mathcal{T}_\Pi(\tau, t)^{-1/2} \exp\left(-\frac{c\|v\|^2}{\mathcal{T}(\tau, t)}\right) \\
 &\quad \times \sum_{\ell=\ell_\varepsilon^*}^\infty \exp\left(-\frac{D\|m\|^2}{\ell^{2k_o}\mathcal{S}(\tau, t)}\right) \sup_{a \in B_\varepsilon(v)} \exp\left(-\frac{(\ell - \tilde{\Theta}(\tau, t, a))^2}{2\mathcal{V}(\tau, t)}\right).
 \end{aligned}$$

Now we estimate the series above. We split the sum above into two parts: $\ell_\varepsilon^* \leq \ell \leq \ell_\varepsilon^* + \|m\|^{1/2k_o}$ and $\ell > \ell_\varepsilon^* + \|m\|^{1/2k_o}$, and estimate the corresponding parts by the following two terms [we note that if $\varepsilon \rightarrow 0$ then $\ell_\varepsilon^* \rightarrow \tilde{\Theta}(\tau, t, v)$]:

$$\|m\|^{\frac{1}{2k_o}} \exp\left(-\frac{D\|m\|^2}{(\ell_\varepsilon^* + \|m\|^{\frac{1}{2k_o}})^{2k_o}\mathcal{S}(\tau, t)}\right)$$

and

$$\begin{aligned}
 &\sum_{\ell \geq \ell_\varepsilon^* + \|m\|^{1/2k_o}} \exp\left(-\frac{(\ell - \sup_{a \in B_\varepsilon(v)} \tilde{\Theta}(\tau, t, a))^2}{2\mathcal{V}(\tau, t)}\right) \\
 &\leq \sum_{\ell=1}^\infty \exp\left(-\frac{(\ell + 1 + \|m\|^{1/2k_o})^2}{2\mathcal{V}(\tau, t)}\right) \\
 &\leq \int_{\|m\|^{1/2k_o}}^\infty e^{-r^2/2\mathcal{V}(\tau, t)} dr \leq \sqrt{2}\sqrt{\mathcal{V}(\tau, t)} e^{-\frac{\|m\|^{1/k_o}}{2\mathcal{V}(\tau, t)}}.
 \end{aligned}$$

The theorem follows. □

7 Estimates for quantities (1.8) and (1.7)

In order to apply the estimate given in Theorem 1.2 into some particular problem one needs to control the quantities (1.8) and (1.7) appearing there. The aim of this section is to give some estimates for them which make the bound in Theorem 1.2 explicit.

We will need the following classical bounds for the norm of the inverse matrices which is due to Richter [25] (see also [18] for a different proof). Recall that for a matrix A , $\|A\|$ stands for the operator norm $\ell^2 \rightarrow \ell^2$.

Theorem 7.1 *Let A be a nonsingular $n \times n$ -matrix. Then*

$$(n^{(n-2)/2} |\det A|^{-1} \|A\|)^{1/(n-1)} \leq \|A^{-1}\| \leq n^{-(n-2)/2} |\det A|^{-1} \|A\|^{n-1}.$$

For the matrix $A_V^\sigma(\tau, t)$, defined in (1.4) on p. 5, we have by (5.2),

$$\|A_V^\sigma(\tau, t)\| \leq \mathcal{T}(\tau, t),$$

where $\mathcal{T}(\tau, t)$ is defined in (1.5). Also note that $\mathcal{T}(\tau, t)$ is an increasing function of t . Hence, the following is a corollary from Theorem 7.1. Notation below is as in (1.5).

Lemma 7.2 *There is a constant $c > 0$ such that for every $t \geq \tau \geq 0$ and for every $v \in \mathbb{R}^{d_2}$,*

$$\Theta(\tau, t, v) \leq cd_2^{-(d_2-2)/2} \mathcal{T}_\Pi(\tau, t)^{-1} \mathcal{T}(\tau, t)^{d_2} \|v\|.$$

Proof By Theorem 7.1,

$$\begin{aligned} \Theta(\tau, t, a) &= \sup_{s \in [\tau, t]} \|A_V^\sigma(\tau, s) A_V^\sigma(\tau, t)^{-1} v\|_\infty \\ &\leq cd_2^{-(d_2-2)/2} \sup_{s \in [\tau, t]} \|A_V^\sigma(\tau, s)\| |\det A_V^\sigma(\tau, t)|^{-1} \|A_V^\sigma(\tau, t)\|^{d_2-1} \|v\| \\ &\leq cd_2^{-(d_2-2)/2} \sup_{s \in [\tau, t]} \mathcal{T}(\tau, s) |\det A_V^\sigma(\tau, t)|^{-1} \mathcal{T}(\tau, t)^{d_2-1} \|v\| \\ &\leq cd_2^{-(d_2-2)/2} |\det A_V^\sigma(\tau, t)|^{-1} \mathcal{T}(\tau, t)^{d_2} \|v\|. \end{aligned}$$

Now (5.1) finishes the proof. □

Lemma 7.3 *There is a positive constant c such that for every $t \geq \tau \geq 0$ and for every $v \in \mathbb{R}^{d_2}$,*

$$\tilde{\Theta}(\tau, t, v) \leq c \mathcal{T}_\Pi(\tau, t)^{-1} \mathcal{T}(\tau, t)^{d_2} \|v\| + c(\mathcal{T}(\tau, t) + \mathcal{T}(\tau, t)^{d_2+1} \mathcal{T}_\Pi(\tau, t)^{-1})^{1/2}. \tag{7.1}$$

Proof Since all norms on the finite dimensional vector space are equivalent we get, by (1.3), for every j ,

$$\begin{aligned} 0 \leq V_j(\tau, t)^{1/2} &\leq c \left(\max_{s \in [\tau, t]} \|A_V^\sigma(\tau, s) - A_V^\sigma(\tau, s) A_V^\sigma(\tau, t)^{-1} A_V^\sigma(\tau, s)\| \right)^{1/2} \\ &\leq c \left(\max_{s \in [\tau, t]} \|A_V^\sigma(\tau, s)\| + \max_{s \in [\tau, t]} \|A_V^\sigma(\tau, s)\| \max_{s \in [\tau, t]} \|A_V^\sigma(\tau, t)^{-1} A_V^\sigma(\tau, s)\| \right)^{1/2}. \end{aligned}$$

From the proof of Lemma 7.2

$$\max_{s \in [\tau, t]} \|A_V^\sigma(\tau, s) A_V^\sigma(\tau, t)^{-1}\| \leq d_2^{-(d_2-2)/2} \mathcal{T}_\Pi(\tau, t)^{-1} \mathcal{T}(\tau, t)^{d_2}.$$

Thus

$$0 \leq V_j(\tau, t)^{1/2} \leq c(\mathcal{T}(\tau, t) + \mathcal{T}(\tau, t)^{d_2+1} \mathcal{T}_\Pi(\tau, t)^{-1})^{1/2}.$$

Hence the sum $\sum_j V_j(\tau, t)^{1/2}$ has the same (with a different constant) estimate. This together with the estimate obtained in Lemma 7.2 finish the proof. \square

From the proof of Lemma 7.3 we have the following corollary .

Lemma 7.4 *There is $c > 0$ such that for all $t \geq \tau > 0$,*

$$\mathcal{V}(\tau, t) \leq c(\mathcal{T}(\tau, t) + \mathcal{T}(\tau, t)^{d_2+1} \mathcal{T}_\Pi(\tau, t)^{-1}).$$

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