Positivity (2010) 14:831–839
DOI 10.1007/s11117-010-0068-6

Positivity

Linear operators on the space of bounded continuous functions with strict topologies

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Received: 15 October 2009 / Accepted: 22 May 2010 / Published online: 23 June 2010

Abstract Let X be a completely regular Hausdorff space, and let $C_b(X)$ denote the Banach space of all real-valued bounded continuous functions on X. We study linear operators from $C_b(X)$ provided with the strict topology β_σ to a Banach space $(E, \|\cdot\|_E)$. In particular, we derive a Yosida–Hewitt type decomposition for weakly compact operators from $C_b(X)$ to E.

Keywords Space of bounded continuous functions · Strict topologies · Baire measures · σ-Dini topologies · Weakly compact operators · Yosida–Hewitt decomposition · Generalized DF-space

Mathematics Subject Classification (2000) 46A70 · 46E10 · 46E27 · 47B38

1 Introduction and terminology

For terminology concerning vector lattices we refer to [1]. We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L, K\rangle$. By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of natural and real numbers. From now on we assume that $(E, \|\cdot\|)$ is a real Banach space, and let E' and E'' stand for the Banach dual and the Banach bidual of E, respectively.

We assume that X is a completely regular Hausdorff space. Let $C_b(X)$ be the Banach space of all real-valued bounded continuous functions on X endowed with the supremum norm $\|\cdot\|_\infty$. Then the Banach dual $C_b(X)'$ of $C_b(X)$ with the natural

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order ($\Phi_1 \leq \Phi_2$ if $\Phi_1(u) \leq \Phi_2(u)$ for each $0 \leq u \in C_b(X)$) is a Dedekind complete Banach lattice. By $C_b(X)''$ we will denote the Banach bidual of $C_b(X)$.

Let $\mathcal{B}$ (respectively, $\mathcal{B}a$) be the algebra (respectively, σ-algebra) of Baire sets in X, which is the algebra (respectively, σ-algebra) generated by the class of all zero-sets of functions of $C_b(X)$. Let $M(X)$ stand for the space of all Baire measures on $\mathcal{B}$. Then $M(X)$ with the norm $\|\mu\| = |\mu|(X)$ (= the total variation of μ) and the natural order ($\mu_1 \leq \mu_2$ if $\mu_1(A) \leq \mu_2(A)$ for all $A \in \mathcal{B}$) is a Dedekind complete Banach lattice. Due to the Alexandroff representation theorem (see [20], [21, Theorem 5.1]) $C_b(X)'$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \mu \mapsto \Phi_\mu \in C_b(X)'$, where $\Phi_\mu(u) = \int_X u(x)d\mu$ for all $u \in C_b(X)$, and $\|\Phi_\mu\| = \|\mu\|$.

A functional $\Phi \in C_b(X)'$ is said to be σ*-additive* if $\Phi(u_n) \to 0$ for each sequence (u_n) in $C_b(X)$ such that $u_n(x) \downarrow 0$ for all $x \in X$. We will denote by $L_\sigma(C_b(X))$ the set of all σ-additive functionals on $C_b(X)$.

In the topological measure theory the so-called strict topologies on $C_b(X)$ are of importance (see [17,21] for more details). In this paper we will consider the strict topology β_σ on $C_b(X)$. Note that Sentilles [17] used for β_σ the name superstrict topology and denoted it by β_1.

Now we recall the concept of the strict topology β_σ on $C_b(X)$. Let βX stand for the Stone-Čech compactification of X. For $v \in C_b(X)$, $\overline{v}$ denotes its unique continuous extension to βX. For a compact subset Q of $\beta X \setminus X$ let $C_Q(X) = \{v \in C_b(X) : \overline{v}|_Q \equiv 0\}$. Let β_Q be the locally convex topology on $C_b(X)$ defined by the family of seminorms $\{p_v : v \in C_Q(X)\}$, where $p_v(u) = \sup_{x\in X} |u(x)v(x)|$ for $u \in C_b(X)$. We define the strict topology β_σ on $C_b(X)$ to be the inductive limit topology $\mathrm{Lin}\beta_Z$ of the topologies β_Z taken over the family $\mathcal{Z}$ of all zero sets such that $Z \subset \beta X \setminus X$ (see [17, pp. 314–315]).

It is known that β_σ is a locally convex-solid topology and is a σ-Dini topology, that is, $u_n \to 0$ for β_σ whenever $u_n(x) \downarrow 0$ for all $x \in X$ (see [17, Theorem 6.2], [21, Theorem 11.16]). Then

$$(C_b(X), \beta_\sigma)' = \{\Phi_\mu : \mu \in M_\sigma(X)\} = L_\sigma(C_b(X)), \tag{1.1}$$

where $M_\sigma(X)$ stands for the space of σ-additive Baire measures (see [21, §4]). Moreover, $(C_b(X), \beta_\sigma)$ is a strong Mackey space, i.e., each countably $\sigma(M_\sigma(X), C_b(X))$-compact subset of $M_\sigma(X)$ is β_σ-equicontinuous (see [21, Theorem 11.5]). It follows that $\beta_\sigma = \tau(C_b(X), M_\sigma(X))$. It is known that β_σ is the finest locally convex topology on $C_b(X)$ which agrees with itself on $\|\cdot\|_\infty$-bounded (equivalently, β_σ-bounded) sets (see [17, Theorem 4.1], [21, Theorem 11.2]). This means that $(C_b(X), \beta_\sigma)$ is a generalized DF-space (see [15, §1]).

In this paper we study linear operators from $C_b(X)$ provided with β_σ to E. In particular, we derive a Yosida–Hewitt type decomposition for weakly compact operators from $C_b(X)$ to E.

2 Linear operators on $C_b(X)$ with the strict topology β_σ

It is section we study linear operators from $C_b(X)$ (provided with the strict topology β_σ) to a Banach space E. For a bounded linear operator $T : C_b(X) \longrightarrow E$

let $T' : E' \longrightarrow C_b(X)'$ denote its conjugate, i.e., $\langle u, T'(e') \rangle = \langle T(u), e' \rangle$ for $u \in C_b(X)$ and $e' \in E'$.

Definition 2.1 A bounded linear operator $T : C_b(X) \to E$ is said to be σ*-additive* if $\|T(u_n)\|_E \to 0$ for each sequence (u_n) in $C_b(X)$ such that $u_n(x) \downarrow 0$ for all $x \in X$.

Now we prove a characterization of σ-additive operators $T : C_b(X) \longrightarrow E$.

Proposition 2.1 *For a bounded linear operator $T : C_b(X) \longrightarrow E$ the following statements are equivalent:*

(i) $T'(E') \subset L_\sigma(C_b(X))$ *i.e.,* $e' \circ T \in L_\sigma(C_b(X))$ *for each* $e' \in E'$.
(ii) T *is* $(\sigma(C_b(X), M_\sigma(X)), \sigma(E, E'))$*-continuous.*
(iii) T *is* $(\beta_\sigma, \|\cdot\|_E)$*-continuous.*
(iv) T *is sequentially* $(\beta_\sigma, \|\cdot\|_E)$*-continuous.*
(v) T *is* σ*-additive.*

Proof (i)$\Longleftrightarrow$(ii) See [1, Theorem 9.26].

(ii)$\Longleftrightarrow$(iii) It is known that T is $(\sigma(C_b(X), M_\sigma(X)), \sigma(E, E'))$-continuous if and only if T is $(\tau(C_b(X), M_\sigma(X)), \|\cdot\|_E)$-continuous (see [1, Ex. 11, p. 149]). Since $\beta_\sigma = \tau(C_b(X), M_\sigma(X))$, the proof is complete.

(iii)$\Longrightarrow$(iv) It is obvious.

(iv)$\Longrightarrow$(v) Assume that T is sequentially $(\beta_\sigma, \|\cdot\|_E)$-continuous, and let (u_n) be a sequence in $C_b(X)$ such that $u_n(x) \downarrow 0$ for all $x \in X$. Since β_σ is a σ-Dini topology, we get $u_n \to 0$ for β_σ. Hence $\|T(u_n)\|_E \longrightarrow 0$.

(v)$\Longrightarrow$(i) It is obvious.

Let ξ be a linear topology on $C_b(X)$. Recall that a linear operator $T : C_b(X) \longrightarrow E$ is said to be $(\xi, \|\cdot\|_E)$*-weakly compact* if there exists a neighbourhood V of zero for ξ such that $T(V)$ is a relatively $\sigma(E, E')$-compact subset of E. From now on we will briefly say that T is weakly compact whenever T is $(\|\cdot\|_\infty, \|\cdot\|_E)$-weakly compact.

Proposition 2.2 *For a linear operator $T : C_b(X) \longrightarrow E$ the following statements are equivalent:*

(i) T *is weakly compact and* $(\beta_\sigma, \|\cdot\|_E)$*-continuous.*
(ii) T *is* $(\beta_\sigma, \|\cdot\|_E)$*-weakly compact.*

Proof (i)$\Longrightarrow$(ii) Assume that T is weakly compact and $(\beta_\sigma, \|\cdot\|_E)$-continuous. Since β_σ-bounded subsets of $C_b(X)$ are $\|\cdot\|_\infty$-bounded, T transforms β_σ-bounded sets into relatively $\sigma(E, E')$-compact sets in E. But $(C_b(X), \beta_\sigma)$ is a generalized DF-space, so in view of [15, Theorem 3.1] T is $(\beta_\sigma, \|\cdot\|_E)$-weakly compact, as desired.

(ii)$\Longrightarrow$(i) Assume that T is $(\beta_\sigma, \|\cdot\|_E)$-weakly compact, i.e., there exists a neighbourhood V of zero for β_σ such that $T(V)$ is a relatively $\sigma(E, E')$-compact set in E. Hence $T(V) \subset B_E(r)$ $(= \{e \in E : \|e\|_E \le r\})$ for some $r > 0$. Then for $\varepsilon > 0$ we get $T(\frac{\varepsilon}{r} V) \subset B_E(\varepsilon)$, and it follows that T is $(\beta_\sigma, \|\cdot\|_E)$-continuous. Clearly T is weakly compact because β_σ is weaker than the $\|\cdot\|_\infty$-topology. □

Corollary 2.3 *Assume that a linear operator $T : C_b(X) \longrightarrow E$ is $(\beta_\sigma, \|\cdot\|_E)$-continuous. Then the following statements are equivalent:*

(i) *T is weakly compact.*
(ii) *T is $(\beta_\sigma, \|\cdot\|_E)$-weakly compact.*

Remark It is well known (see [17]) that the strict topologies β_t and β_τ on $C_b(X)$ both coincide with the Buck's original topology β (see [9]) in the locally compact case. Sentilles [16] showed that if T is a $(\beta, \|\cdot\|_E)$-continuous linear operator from $C_b(X)$ (X is locally compact) to E, then T is weakly compact if and only if T is $(\beta, \|\cdot\|_E)$-weakly compact.

Let $B(\mathcal{B})$ (resp. $B(\mathcal{B}a)$) denote the Banach space of all totally $\mathcal{B}$-measurable (respectively, totally $\mathcal{B}a$-measurable) functions $u : X \to \mathbb{R}$, provided with the uniform norm $\|\cdot\|_\infty$ (see [11, § 6]). $(B(\mathcal{B}a), \|\cdot\|_\infty)$ is a σ-Dedekind complete AM-space (see [3, Theorem 13.2]). Then $C_b(X) \subset B(\mathcal{B})$ (see [4, Lemma 1.2]) and one can inject isometrically $B(\mathcal{B})$ into $C_b(X)''$ by the mapping $\pi : B(\mathcal{B}) \longrightarrow C_b(X)''$, where for each $u \in B(\mathcal{B})$

$$\pi(u)(\Phi_\mu) = \int_X u(x)\mathrm{d}\mu \quad \text{for all } \mu \in M(X).$$

Let $i : E \to E''$ stand for the canonical isometry, i.e., $i(e)(e') = e'(e)$ for all $e \in E$ and $e' \in E'$. Moreover, let $j : i(E) \longrightarrow E$ denote the left inverse of i, i.e., $j \circ i = id_E$.

Now assume that $T : C_b(X) \to E$ is a weakly compact operator. Let $T'' : C_b(X)'' \longrightarrow E''$ denote the biconjugate of T. Then $T''(C_b(X)'') \subset i(E)$ (see [1, Theorem 17.2]) and we can define a representing measure $m : \mathcal{B} \to E$ by

$$m(A) := j\left(T''(\pi(\mathbb{1}_A))\right) \quad \text{for all } A \in \mathcal{B}$$

(here $\mathbb{1}_A$ stands for the characteristic function of a set A). Using the Alexandroff representation theorem (see [21, Theorem 5.1]) we obtain that for each $e' \in E'$, $e' \circ m \in M(X)$ and

$$e'(T(u)) = \int_X u(x)\, d(e' \circ m) \quad \text{for all } u \in C_b(X). \tag{2.1}$$

Let

$$\widetilde{T} := j \circ T'' \circ \pi : B(\mathcal{B}) \longrightarrow E.$$

Then $\widetilde{T}$ is a bounded linear operator and

$$\widetilde{T}(u) = \int_X u(x)\mathrm{d}m \quad \text{for all } u \in B(\mathcal{B}),$$

and $\|\widetilde{T}\| = \|m\|(X)$ (= the semivariation of m) (see [10, Theorem 1.1.13]). Since $T'' : C_b(X)'' \longrightarrow E''$ is a weakly compact operator (see [1, Theorem 17.2]) and the mapping $j : i(E) \to E$ is $(\sigma(i(E), E'), \sigma(E, E'))$-continuous, we see that $\widetilde{T}$ is also weakly compact. It follows that m is strongly bounded (see [10, Theorem 6.1.1]). One can easily verify that

$$T(u) = \widetilde{T}(u) = \int_X u(x)\mathrm{d}m \quad \text{for all } u \in C_b(X).$$

It is well known that the σ-order continuous dual $B(\mathcal{B}a)^\sim_c$ of $B(\mathcal{B}a)$ can be identified with the Banach lattice $ca(\mathcal{B}a)$ of countably additive signed measures on $\mathcal{B}a$ throughout the lattice isomorphism $ca(\mathcal{B}a) \ni \nu \mapsto \Phi_\nu \in B(\mathcal{B}a)^\sim_c$, where $\Phi_\nu(u) = \int_X u(x)d\nu$ for all $u \in B(\mathcal{B}a)$ (see [3, Theorem 13.5]). Then the Mackey topology $\tau(B(\mathcal{B}a), ca(\mathcal{B}a))$ is a locally solid σ-Lebesgue topology on $B(\mathcal{B}a)$ (see [**?** , Ex. 18, p. 178]AB2) and $(B(\mathcal{B}a), \tau(B(\mathcal{B}a), ca(\mathcal{B}a)))$ is a generalized DF-space (see [13, § 4]).

It follows that

$$\tau(B(\mathcal{B}a), ca(\mathcal{B}a))|_{C_b(X)} \subset \beta_\sigma = \tau(C_b(X), M_\sigma(X)). \tag{2.2}$$

Indeed, let (u_n) be a sequence in $C_b(X)$ and $u_n(x) \downarrow 0$ for all $x \in X$. Then $u_n \downarrow 0$ in $B(\mathcal{B}a)$ (in the vector lattice sense); hence $u_n \to 0$ for $\tau(B(\mathcal{B}a), ca(\mathcal{B}a))$. This means that $\tau(B(\mathcal{B}a), ca(\mathcal{B}a))|_{C_b(X)}$ is a σ-Dini topology on $C_b(X)$. Since β_σ is the finest σ-Dini topology on $C_b(X)$ (see [[21, Theorem 11.16]), we conclude that the inclusion (2.2) holds.

Now we are ready to state the following result.

Proposition 2.4 *Let $T : C_b(X) \to E$ be a $(\beta_\sigma, \|\cdot\|_E)$-weakly compact linear operator, and let $m : \mathcal{B} \to E$ be its representing measure. Then m has a unique countably additive extension $\overline{m} : \mathcal{B}a \to E$ and the corresponding integration operator $T_{\overline{m}} : B(\mathcal{B}a) \to E$ is $(\tau(B(\mathcal{B}a), ca(\mathcal{B}a)), \|\cdot\|_E)$-weakly compact and $T_{\overline{m}}(u) = \widetilde{T}(u) = T(u)$ for $u \in C_b(X)$.*

Proof In view of Proposition 2.1 $e' \circ T \in L_\sigma(C_b(X))$ for each $e' \in E'$. Hence by (2.1) and (1.1) we obtain that $e' \circ m \in M_\sigma(X)$ for each $e' \in E'$. This means that $e' \circ m : \mathcal{B} \to \mathbb{R}$ is countably additive for each $e' \in E'$. Since $m : \mathcal{B} \to E$ is strongly bounded, by the Carathéodory–Hahn–Kluvanek extension theorem (see [10, Theorem 1.5.2]) m has a unique countably additive extension $\overline{m} : \mathcal{B}a \to E$. Next, by [13, Corollary 12] the corresponding integration operator $T_{\overline{m}} : B(\mathcal{B}a) \to E$ is $(\tau(B(\mathcal{B}a), ca(\mathcal{B}a)), \|\cdot\|_E)$-weakly compact.

Now let $u \in C_b(X) \subset B(\mathcal{B}) \subset B(\mathcal{B}a)$. Then there exists a sequence (s_n) of $\mathcal{B}$-simple functions such that $\|s_n - u\|_\infty \to 0$. Hence we get $T_{\overline{m}}(u) = \lim_n T_{\overline{m}}(s_n) = \lim_n T_m(s_n) = \widetilde{T}(u) = T(u)$. Thus the proof is complete. □

3 Yosida–Hewitt type decomposition for weakly compact operators on $C_b(X)$

There are many versions and generalizations of the classical Yosida–Hewitt decomposition theorem [22] in diverse setting, e.g. for vector-valued and group valued measures (see [7,19,14,18,12]), operators between vector-lattices and order-weakly compact operators from vector lattices to Banach spaces (see [5,6]). Brooks and Wright [8] obtained a Yosida–Hewitt type decomposition for weakly compact operators on a von Neumann algebra $\mathcal{M}$. In this section we derive a Yosida–Hewitt type decomposition for weakly compact operators from $C_b(X)$ to a Banach space E.

Since $M(X)$ is a Dedekind complete vector lattice and $M_\sigma(X)$ is a band of $M(X)$ (see [21, Theorem 7.2]), we see that $M_\sigma(X)$ is a projective band of $M(X)$ (see [[1, Theorem 3.8]). Thus we obtain the following Yosida–Hewitt decomposition

$$M(X) = M_\sigma(X) \oplus M_{pfa}(X),$$

where $M_{pfa}(X)$ (=$M_\sigma(X)^d$ - the disjoint complement of $M_\sigma(X)$ in $M(X)$) stands for the space of purely finitely additive members of $M(X)$. Hence

$$C_b(X)' = L_\sigma(C_b(X)) \oplus L_{pfa}(C_b(X)),$$

where $L_{pfa}(C_b(X))(= L_\sigma(C_b(X))^d$—the disjoint complement of $L_\sigma(C_b(X))$ in $C_b(X)'$) stands for the space of *purely finitely additive* functionals in $C_b(X)'$. Since $C_b(X)$ is a AM-space, $M(X)$ is a AL-space. This means that $\|\mu\| = \|\mu_c\| + \|\mu_p\|$ and $\|\Phi_\mu\| = \|\Phi_{\mu_c}\| + \|\Phi_{\mu_p}\|$ when $\mu = \mu_c + \mu_p$ with $\mu_c \in M_\sigma(X)$ and $\mu_p \in M_{pfa}(X)$.

Remark Note that if X is pseudocompact, then β_σ coincides with the $\|\cdot\|_\infty$-topology and hence $M_\sigma(X) = M(X)$. Then $L_\sigma(C_b(X)) = C_b(X)'$ and $L_{pfa}(C_b(X)) = \{0\}$.

Definition 3.1 A bounded linear operator $T : C_b(X) \longrightarrow E$ is said to be *purely finitely additive* if $e' \circ T \in L_{pfa}(C_b(X))$ for each $e' \in E'$.

Now we are in position to prove a Yosida–Hewitt type decomposition for weakly compact operators from $C_b(X)$ to E.

Theorem 3.1 *Let $T : C_b(X) \longrightarrow E$ be a weakly compact operator and let $m : \mathcal{B} \longrightarrow E$ be its representing measure. Then*

(i) *m can be uniquely decomposed as $m = m_c + m_p$, where $m_c : \mathcal{B} \longrightarrow E$ and $m_p : \mathcal{B} \longrightarrow E$ are strongly bounded measures, and $e' \circ m_c \in M_\sigma(X)$ (hence m_c has a unique countably additive extension $\overline{m_c} : \mathcal{B}a \longrightarrow E$) and $e' \circ m_p \in M_{pfa}(X)$ for each $e' \in E'$.*

(ii) *T can be uniquely decomposed as $T = T_1 + T_2$, where T_1 and T_2 are weakly compact operators, T_1 is σ-additive (hence T_1 is $(\beta_\sigma, \|\cdot\|_E)$-weakly compact) and T_2 is purely finitely additive, and*

$$T_1(u) = \int_X u(x)\mathrm{d}m_c \text{ and } T_2(u) = \int_X u(x)\mathrm{d}m_p \text{ forall } u \in C_b(X).$$

In consequence,

$$T(u) = \int_X u(x)\mathrm{d}m = \int_X u(x)\mathrm{d}m_c + \int_X u(x)\mathrm{d}m_p \quad \text{for all } u \in C_b(X).$$

Proof We have $C_b(X)' = L_\sigma(C_b(X)) \oplus L_{pfa}(C_b(X))$ and $\|\Phi\| = \|\Phi_1\| + \|\Phi_2\|$, where $\Phi_1 \in L_\sigma(C_b(X))$ and $\Phi_2 \in L_{pfa}(C_b(X))$. Thus we have natural projections

$$P_k : C_b(X)' \longrightarrow C_b(X)',$$

where $P_k(\Phi) = \Phi_k$ and $\|P_k\| \leq 1$ $(k = 1, 2)$. Now we can consider the conjugate operators

$$P_k' : C_b(X)'' \longrightarrow C_b(X)''$$

defined by $P_k'(V)(\Phi) = V(P_k(\Phi))$ for $V \in C_b(X)''$ and $\Phi \in C_b(X)'$. One can observe that $(P_k' \circ \pi)(u) = \pi(u) \circ P_k$ for all $u \in B(\mathcal{B})$. Define linear operators $(k = 1, 2)$

$$\widetilde{T}_k := j \circ T'' \circ P_k' \circ \pi : B(\mathcal{B}) \longrightarrow E.$$

Since $T'' : C_b(X)'' \longrightarrow E''$ is a weakly compact operator with $T''(C_b(X)'') \subset i(E)$, and the mapping j is $(\sigma(i(E), E')), \sigma(E', E))$-continuous, we obtain that $\widetilde{T}_k$ are weakly compact.

Let $m_c(A) := \widetilde{T}_1(\mathbb{1}_A)$ and $m_p(A) := \widetilde{T}_2(\mathbb{1}_A)$ for $A \in \mathcal{B}$. Hence $m_c : \mathcal{B} \longrightarrow E$ and $m_p : \mathcal{B} \longrightarrow E$ are strongly bounded measures and $\|m_c\|(X) = \|\widetilde{T}_1\|$ and $\|m_p\|(X) = \|\widetilde{T}_2\|$. Moreover, for all $u \in B(\mathcal{B})$ we have

$$\widetilde{T}_1(u) = \int_X u(x)\mathrm{d}m_c, \quad \widetilde{T}_2(u) = \int_X u(x)\mathrm{d}m_p. \tag{3.1}$$

Note that $\widetilde{T}(u) = \widetilde{T}_1(u) + \widetilde{T}_2(u)$ for all $u \in B(\mathcal{B})$. Hence

$$m(A) = m_c(A) + m_p(A) \quad \text{for all } A \in \mathcal{B}.$$

For $k = 1, 2$ define $T_k = \widetilde{T}_k|_{C_b(X)} : C_b(X) \longrightarrow E$. Then $T(u) = T_1(u) + T_2(u)$ for all $u \in C_b(X)$ and T_1 and T_2 are weakly compact.

For each $e' \in E'$ and all $u \in C_b(X)$ we have

$$\begin{aligned}(e' \circ T_k)(u) = e'(T_k(u)) &= \left(\left(T'' \circ P_k' \circ \pi\right)(u)\right)(e') \\ &= \left(T''(\pi(u) \circ P_k)\right)(e') \\ &= (\pi(u) \circ P_k)(T'(e')) \\ &= \pi(u)\left(P_k(e' \circ T)\right) \\ &= P_k(e' \circ T)(u).\end{aligned}$$

Since $e' \circ T \in C_b(X)' = L_\sigma(C_b(X)) \oplus L_{pfa}(C_b(X))$, we get $e' \circ T_1 \in L_\sigma(C_b(X))$ and $e' \circ T_2 \in L_{pfa}(C_b(X))$. In view of Proposition 2.1, T_1 is σ-additive and T_2 is purely finitely additive. Moreover, by Proposition 2.2, T_1 is $(\beta_\sigma, \|\cdot\|_E)$-weakly compact. The uniquenes of the decomposition $T = T_1 + T_2$ follows from the uniqueness of the decomposition $e' \circ T = e' \circ T_1 + e' \circ T_2$ for each $e' \in E'$. Moreover, in view of (3.1) for $u \in C_b(X)$ we have

$$\left(e' \circ T_1\right)(u) = \int_X u(x) \mathrm{d}\left(e' \circ m_c\right) \text{ and } \left(e' \circ T_2\right)(u) = \int_X u(x) \mathrm{d}\left(e' \circ m_p\right).$$

Hence $e' \circ m_c \in M_\sigma(X)$ and $e' \circ m_p \in M_{pfa}(X)$. By Proposition 2.4 m_c has a unique countably additive extension $\overline{m_c} : \mathcal{B}a \longrightarrow E$. □

Acknowledgments The author is grateful to the referee for the valuable remarks.

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