Philos Stud (2021) 178:3443–3463
https://doi.org/10.1007/s11098-021-01608-1

Two notions of fusion and the landscape of extensionality

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Accepted: 19 January 2021 / Published online: 8 February 2021

Abstract There are two main ways in which the notion of mereological fusion is usually defined in the current literature in mereology which have been labelled 'Leśniewski fusion' and 'Goodman fusion'. It is well-known that, with Minimal Mereology as the background theory, every Leśniewski fusion also qualifies as a Goodman fusion. However, the converse does not hold unless stronger mereological principles are assumed. In this paper I will discuss how the gap between the two notions can be filled, focussing in particular on two specific sets of principles that appear to be of particular philosophical interest. The first way to make the two notions equivalent can be used to shed some interesting light on the kind of intuition both notions seem to articulate. The second shows the importance of a little-known mereological principle which I will call 'Mild Supplementation'. As I will show, the mereology obtained by adding Mild Supplementation to Minimal Mereology occupies an interesting position in the landscape of theories that are stronger than Minimal Mereology but weaker than what Achille Varzi and Roberto Casati have labelled 'Extensional Mereology'.

Keywords Mereology · Mereological fusion · Leśniewski fusion · Goodman fusion · Extensionality

1 Introduction

The notion of *mereological fusion* is central to many debates in contemporary metaphysics. Two main definitions of this notion have been proposed in the literature which, following Cotnoir (2018), we can call 'Leśniewski' and

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'Goodman' fusion, respectively.'[1,2] Let '$<$' stand for the relation of *proper parthood* (which I will be assuming to be a primitive notion), and let us stipulate that x is *part of* y just in case x is either a proper part of y or identical to y, and that *x and y overlap* just in case they have a common part:

$$\text{(Parthood)} \quad x \leq y =_{df} x < y \vee x = y$$

$$\text{(Overlap)} \quad Oxy =_{df} \exists z(z \leq x \wedge z \leq y)$$

By adopting the following abbreviations for ease of exposition (where 'yy' is a plural variable and '$\prec$' is the one-many 'one of' relation)

$$xOyy =_{df} \exists z(z \prec yy \wedge Oxz)$$

$$xx \leq y =_{df} \forall z(z \prec xx \rightarrow z \leq y)$$

('$xOyy$' and '$xx \leq y$' can be read as 'x overlaps the yy', and 'all the xx are part of y', respectively) the notions of Leśniewski fusion and Goodman fusion can be defined as follows:

$$\text{(L-def)} \quad xF^L yy =_{df} yy \leq x \wedge \forall z(z \leq x \rightarrow zOyy)$$

$$\text{(G-def)} \quad xF^G yy =_{df} \forall z(Ozx \leftrightarrow zOyy)$$

The mereological theory known in the literature as 'Minimal Mereology' (henceforth '**MM**') can be axiomatized by means of the following two principles—transitivity of proper parthood and Weak Supplementation:

$$(<\text{-Transitivity}) \quad \forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$$

$$\text{(WSP)} \quad \forall x \forall y(x < y \rightarrow \exists z(z \leq y \wedge \sim Ozx))$$

(notice that **MM** entails that proper parthood is a strict partial order, and so it is not only transitive but also irreflexive and asymmetric). As is well-known (see, for instance, Pietruszczak 2005: 216), in the presence of ($<$-Transitivity) every Leśniewski-fusion (or 'L-fusion' for short) qualifies as a Goodman-fusion (or 'G-fusion' for short):[3]

[1] The notion of Leśniewski fusion corresponds to the definition given by Leśniewski (1916). Notice that although for the ease of exposition I choose here to follow Cotnoir (2018) and use the label 'Goodman fusion', the second notion of fusion was also originally formulated by Leśniewski (1931) (in terms of the relations 'exterior to' and 'discrete from', respectively; many thanks to an anonymous referee of this Journal). The notion of Goodman fusion is also used by Leonard and Goodman (1940) and Goodman (1951). Hovda (2009) calls Goodman fusions 'type 1 fusions', and Leśniewski fusions 'type 2 fusions'. Varzi (2019) calls Goodman and Leśniewski fusions 'General Sums 2', and 'General Sums 3', respectively.

[2] A third definition of fusion that can be found in the literature is the notion of 'Algebraic' fusion which consists in taking the fusion of a plurality of entities to be their least upper bound with respect to parthood (see Hovda 2009: 61 and below: Sect. 2).

[3] *Proof.* Suppose that x L-fuses the yy. If some entity z overlaps x, then z and x have at least a part w in common. By (L-def), w overlaps the yy and so has a part j in common with some of the yy. By ($<$-Transitivity) it follows that j is also part of z, so that also z overlaps the yy. By generalization, it follows that if something overlaps x then it overlaps the yy. Suppose, instead, that some entity z overlaps the yy. Then, z has at least a part w in common with some of the yy. By (L-def) the yy are all parts of x. Since w is

$$\text{(L-to-G)} \quad \forall x \forall yy \big(xF^L yy \rightarrow xF^G yy \big)$$

However, even under the assumption of **MM**, we don't have that every G-fusion qualifies as an L-fusion, as one can easily appreciate from the model depicted in Fig. 1. In fact, as noted by Pietruszczak (2018: 147), Hovda (2009: 64–65) and Varzi (2009: 602), in Fig. 1 many pluralities of entities have a G-fusion without having an L-fusion. For instance, the plurality $[a, b, c, d]$ (that is, the plurality of entities that are identical to either *a*, *b*, *c*, or *d*)[4] is G-fused by both *a* and *b*, and yet there is no entity that is their L-fusion, given that no entity has all of them as parts (as required by the first conjunct of (L-def)). It follows, therefore, that the following principle—stating that something is an L-fusion if and only if it is a G-fusion—is *not* a theorem of **MM**:

$$\text{(EqLG)} \quad \forall x \forall yy \big(xF^L yy \leftrightarrow xF^G yy \big)$$

This naturally raises the question as to which pluralities of pairwise logically independent principles jointly entail (EqLG)—with **MM** in the background—and are entailed by it. In other words: let say that a plurality $S_1, \ldots, S_n$ of sentences (taken together) *are a way of expressing* (or, more simply, *express*) *the difference between L-fusions and G-fusions* just in case the following is true:

(Diff) (i) for every m and o $(1 \leq m/o \leq n)$, if, $m \neq o$ then $\mathbf{MM}, S_m \nvdash S_o$
(ii) $\mathbf{MM}, S_1, \ldots, S_n \vdash \text{(EqLG)}$ and, for every m $(1 \leq m \leq n)$, $\mathbf{MM}, \text{(EqLG)} \vdash S_m$

The question raised by the considerations just made is, thus, the following: which pluralities of sentences comply with (Diff) and, thus, 'express the difference' between L-fusions and G-fusions?

The aim of this paper is not to list all the possible ways to express the difference between L-fusions and G-fusions. Rather, I will only discuss two ways to do so that appear to be of particular interest. In the literature, it is a well-known fact that, assuming ($<$-Transitivity), (EqLG) is equivalent to Strong Supplementation (and, thus, that (SSP) complies with (Diff)):

$$\text{(SSP)} \quad \forall x \forall y \Big(x \nleq y \rightarrow \exists z (z \leq x \wedge \sim Ozy) \Big)$$

(Pietruszczak 2005: 218, 2018: 142; Gruszczyński 2013: 140; Varzi 2016: section 4.3). As I will argue in Sect. 2, however, there is a way to reach this result which shows how (L-def) and (G-def) can be seen as articulating the same, highly plausible intuition concerning the notion of mereological fusion. In Sect. 3, I will argue that there is a second way of articulating the difference between L-fusions and G-fusions on the background of **MM** which reveals the importance of a little-known mereological principle that I will label 'Mild Supplementation'. As I will show,

Footnote 3 continued
part of one of the *yy*, we have, by ($<$-Transitivity), that *w* is part of *x*, so that *z* has a part in common with *x*. By generalization it follows that if something overlaps the *yy*, then it overlaps *x*. Therefore, something overlaps *x* if and only if it overlaps the *yy*, which means that *x* G-fuses the *yy*. ■

[4] Letting '$\imath xx.\phi xx$' stand for the plural definite description 'the *xx* that ϕ', '$[x_1, \ldots, x_n]$' is short for '$\imath xx.\forall z(z \prec xx \leftrightarrow (z = x_1 \vee \ldots \vee z = x_n))$'.

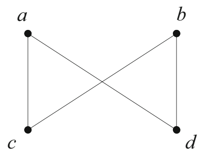

Fig. 1 Not every G-fusion is an L-fusion

Mild Supplementation will allow us to unearth a theory which appears to occupy an interesting position in the logical space lying between **MM** and the theory that Casati and Varzi (1999) call 'Extensional Mereology'.

2 L-fusions, G-fusions and strong supplementation

According to the definition of L-fusion, an entity x is an L-fusion of a plurality yy of entities just in case all the yy are part of x and every part of x overlaps at least one of the yy. Instead, according to the definition of G-fusion, an entity x is a G-fusion of a plurality yy of entities just in case something overlaps x if and only if it overlaps one of the yy:

$$\text{(L-def)} \quad xF^L yy =_{df} yy \leq x \wedge \forall z(z \leq x \rightarrow zOyy)$$

$$\text{(G-def)} \quad xF^G yy =_{df} \forall z(Ozx \leftrightarrow zOyy)$$

Consider, then, the second conjunct of (L-def).

(1) $\forall z(z \leq x \rightarrow zOyy)$

Notice that, under the assumption of **MM**, (1) is equivalent to[5]:

(2) $\forall z(Ozx \rightarrow zOyy)$

But (2) is just the left-to-right direction of (G-def). Consider, then, the right-to-left direction of (G-def):

(3) $\forall z(zOyy \rightarrow Ozx)$

(3) can be unpacked as:

(4) $\forall z(\exists w(w \prec yy \wedge Owz) \rightarrow Ozx)$

In turn, (4) is logically equivalent to:

[5] *Proof. (1) entails (2).* Suppose z overlaps x. Then, some entity w is part of both z and x. It follows, thus by (1) that w overlaps at least some of the yy. By the transitivity of parthood, every part of w is also a part of z, so that also z overlaps the yy. *(2) entails (1).* Suppose z is part of x. Then, z overlaps x (since by the reflexivity of parthood they have at least a part in common, that is, z itself). By (2), x also overlaps the yy. ■

(5) $\forall z(z \prec yy \rightarrow \forall w(Owz \rightarrow Owx))$

Let, then, an entity x be *covered by* an entity y ('$x \sqsubseteq y$') if and only if everything that overlaps x also overlaps y:

(6) $x \sqsubseteq y =_{df} \forall z(Ozx \rightarrow Ozy)$

The right-to-left direction of (G-def) can, then, be rewritten as follows:

(7) $\forall z(z \prec yy \rightarrow z \sqsubseteq x)$

Putting everything together, we have that, under the assumption of **MM**, (L-def) and (G-def) are equivalent to the following two definitions:

(L-def+) $xF^{L}yy =_{df} \forall z(z \prec yy \rightarrow z \leq x) \wedge \forall z(z \leq x \rightarrow zOyy)$

(G-def+) $xF^{G}yy =_{df} \forall z(z \prec yy \rightarrow z \sqsubseteq x) \wedge \forall z(z \leq x \rightarrow zOyy)$

Consider the following principle employing pre-theoretic notions of 'containment' and 'covering' (and in which 'fusion' can be taken to be a dummy term for an object that is made from a plurality of entities):

(Fusion) A an entity x is a fusion of a plurality yy of entities just in case (i) x 'contains' all the yy, and (ii) the yy 'cover' all of x.

(Fusion) strikes me as a highly intuitive principle. If x didn't 'contain' z and z was one of the yy, how could one say that x a fusion of *them*? At the same time, if some part of x wasn't completely 'covered' by the yy—if some part of x was *free of* the yy, so to speak—how could a fusion of the yy be identical to x? (L-def+) and (G-def+) allow one to appreciate at a glance how close the two definitions are. Both (L-def+) and (G-def+) consist, in fact, in a conjunction which can be seen as formulating the two conditions expressed by (Fusion). The 'covering condition' is expressed in the second conjunct of (L-def+) and (G-def+) in the same way: x is covered by the yy in the sense that every part of x overlaps at least one of the yy, so that no part of x is disjoint from—and, in this sense, 'free of'—all of them. We can call this sense of covering *plural covering* (or *many-one covering*). The difference between (L-def+) and (G-def+) concerns the way in which the 'containment condition' is expressed in their first conjunct. According to (L-def+), to contain an entity is to have it as a part, so that x can contain all the yy only if it has all of them as parts. Instead, according to (G-def+), to contain an entity is to cover it, in the sense of overlapping everything it overlaps (we can call this sense of covering *singular covering,* or *one–one covering*, for disambiguation). Therefore, (L-def+) and (G-def+) can be seen as disagreeing on how the pre-theoretical notion of (mereological) 'containment' should be properly expressed.[6]

[6] As Cotnoir (2017) notices, 'being in' seems to be said in many ways. In particular, Cotnoir distinguishes between *predicational* 'being in', according to which the relata of 'being in' are objects on the one hand and entities of a higher ontological category on the other, and *containment* 'being in'. The second divides in (i) set-theoretical (set-membership), (i) plural-logical (being one of), and (ii) mereological. The focus here is on the notion of mereological containment.

Before moving further, notice that a third definition of the notion of fusion can be found in the literature, which following Cotnoir (2018) we can call 'Algebraic' fusion:

$$\text{(A-def)} \quad xF^{A}yy =_{df} \forall z(z \prec yy \rightarrow z \leq x) \wedge \forall z(\forall w(w \prec yy \rightarrow w \leq z) \rightarrow x \leq z)$$

(A-def) gets the 'containment part' of (Fusion) right (or, at least, in a way that is at least *prima facie* legitimate). The problem with (A-def) is the way it gets the 'covering part' of (Fusion). Consider, for instance, the model depicted in Fig. 2. In Fig. 2 *a* counts as an A-fusion of $[b, d]$. Yet there is clearly *more* to *a* than just *b* and *d* taken together, namely, *c*. *b* and *d* taken together clearly fail to cover all of *a*. Therefore, as long as we follow (Fusion) as our guiding intuition, the notion of A-fusion appears to fall short of being a live option to define the notion of mereological fusion.[7]

Also the notion of G-fusion may be subjected to some criticism. Consider, in fact, the model depicted in Fig. 1. *a* is a fusion of $[a, b, c, d]$ and yet it doesn't have *b* as a part. However, as Varzi (2009) claims, it may seem that

> […] no matter how exactly one defines the word, one should always expect a fusion to include, among its parts, *all* the things it fuses. (Varzi 2009: 602)

Even worse, *a* is a fusion of [*b*]—the 'improper' plurality of the things that are identical to *b*—without having it a part. Yet, it may seem plausible to object that

> surely a fusion is supposed to include, among its parts, at least *some* of the things it fuses. (Varzi 2009: 603)

Notice, first, that G-mereologists endorsing a notion of 'containment-as-covering' agree that a fusion must *contain* all the entities it fuses. What they deny is that, in order to be contained by their fusion, the entities fused must be *part* of the fusion. Therefore, simply replying that a fusion must contain all the entities it fuses *as parts* seems to be dialectically ineffective in this case. Second, G-mereologists appear to have an independent way to argue that in Fig. 1 *a* and *b* do contain each other, namely, by appealing to the idea that a fusion is 'nothing over and above' the plurality of its proper parts. This argument can be presented as follows:

(A1) For every *x* and *y* and *zz*, if *x* contains the *zz* and *y* is nothing over and above the *zz*, then *x* also contains *y*

(A2) In Fig. 1, *a* contains [*c,d*]

(A3) *b* is nothing over and above [*c,d*]

[7] Cotnoir and Bacon (2012: 196) argue that, if parthood is not assumed to be antisymmetric, then the notion of A-fusion may be adequate if paired with the following Complementation axiom which (in the presence of **MM**) is strictly stronger than both (WSP) and (SSP)):

$$\text{(Complementation)} \quad \forall x\forall y(x \nleq y \rightarrow \exists z\forall w(w \leq z \leftrightarrow (w \leq x \wedge \sim Owy))$$

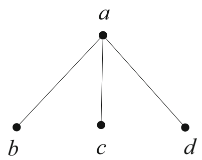

Fig. 2 *a* is an A-fusion of [*b*,*d*]

(A4) *Therefore*, *a* contains *b*

(A2) is something that also containment-as-parthood theorists like Varzi (2009) accept. Therefore, whether or not this argument is successful depends on whether G-theorists can appeal to a notion of nothing-over-and-aboveness that is both compatible with anti-extensionalism and such as to make (A1) and (A3) true.

Many philosophers share the intuition that a whole is nothing over and above the parts it fuses[8]: if I buy separately the four parts composing a certain lot of land, I don't need to *also* buy the lot they compose. Similarly, if I buy six cans of beer, I don't need to spend extra money to also buy the six-pack (Baxter 1988). Several ways to cash out the slippery notion of nothing-over-and-aboveness have been proposed in the literature.[9] However, many of them appear to be incompatible with anti-extensionalist models like the one depicted in Fig. 1. For instance, G-theorists cannot take a whole to be nothing over and above its parts in the sense that it is *literally identical* to its parts taken together (a thesis commonly known as 'Strong Composition as Identity').[10] In fact, if both *a* and *b* in Fig. 1 were identical to the plurality of their proper parts then (by symmetry and transitivity of identity) they themselves would be identical. Similarly, G-mereologists cannot take a composite entity to be nothing over and above its parts in the sense that it is *the only object* that is composed of those parts (Smid 2017: 2; 12, fn 2), as that would also be straightforwardly inconsistent with Fig. 1. More generally, the idea that anti-extensionalists cannot claim that entities sharing the same proper parts are nothing over and above them may indeed seem to have the ring of intuitiveness to it:

> if some *X*s compose two things, then wholes could not be "nothing over and above their parts." How could *distinct* things each be nothing over and above the *same* parts? (Sider 2007: 70; my italics)

Gilmore (2010: 181) formulates the principle that appears to be behind Sider's (2007) intuition as follows:

> 'P1 For any *X*s, any *y*, and any *z*, if *y* is nothing over and above the *X*s, and *z* is nothing over and above the *X*s, then $y = z$' (Gilmore 2010: 181)

[8] See, among many others, Lewis (1991), Varzi (2014), and for additional references Smid (2017).

[9] Smid (2017), for instance, recently distinguishes between five readings of '*x* is nothing over and above the *yy*': (i) *x* is not an additional commitment with respect to the *yy*, (ii) the existence of the *yy* is sufficient for the existence of x, (iii) *x* is the only object that is composed of the *yy*, (iv) *x* has no properties that are not reducible to the *yy*, (v) *x* is identical to the *yy*.

[10] See Cotnoir (2014) for an introduction to Composition as Identity.

As Gilmore argues, as intuitive as P1 may initially sound, there seem to be good reasons to doubt it. Consider, in fact, the following principle P1+ which extends P1 to pluralities of entities:

> 'P1+ For any *X*s, any *Y*s, and any *Z*s, if [the *Y*s are nothing over and above the *X*s, the *Z*s are nothing over and above the *X*s, and there are exactly as many of the *Y*s as there are of the *Z*s], then the *Y*s = the *Z*s.' (Gilmore 2010: 181)

If being nothing over and above is 'identity-like' as P1 suggests, it seems that also P1+ should be accepted as true. However, P1+ is intuitively *false*: both the rows and the columns in my chessboard (which are equal in number) are nothing over and above the squares of my chessboard, and yet they are not identical (the rows run horizontally, the columns vertically). It seems, therefore, that if we are 'free to endorse the nonidentity of pluralities […] each of which [are] nothing over the same things' we should also be 'free to endorse the nonidentity of single individuals [that are] nothing over and above the same things' (Gilmore 2010: 182).

Furthermore, there seems to be at least one possible way to account for the notion of 'being nothing over and above' that is both consistent with anti-extensionalist models like the one of Fig. 1 and with premises (A1) and (A3). The general idea behind this account can be briefly sketched as follows. Intuitively, certain entities seem to depend for their existence on other entities. In particular, composite entities are often claimed to depend for their existence on their proper parts. This notion of existential dependence can be understood by means of the notion of *metaphysical grounding*, in at least two ways.[11] The first one, proposed by Correia (2005) and Schnieder (2006), is that of taking an entity *x* to depend existentially on a plurality *yy* of entities if and only if the fact that *x* exists is grounded in some facts concerning the *yy*. Alternatively, one could follow Schaffer (2009) and *identify* this notion of dependence with a primitive, cross-categorical relation of grounding.[12, 13] In both cases, there seems to be at least some intuitive appeal to the idea that, if *x*'s existence is grounded in the *yy*, then *x owes* its existence to the *yy*, so that *x* and the *yy* are really the same 'portion of reality', or the same 'amount of being', so to speak.[14] This suggests the possibility of the following kind of grounding-based account of nothing-over-and-aboveness:

> (NOA) The *xx* are nothing over and above the *yy* if and only if either (i) the *xx* depend for their existence on the *yy*, (ii) the *yy* depend for their existence on

[11] For a general introduction on the notion of grounding see Correia and Schnieder (2012). On the idea that a whole is grounded in its parts see, among others, Cameron (2014) and Loss (2016).

[12] See Schnieder (2020) for some discussion of these two grounding-based approaches to ontological dependence.

[13] Schaffer (2010) himself defends the idea that the cosmos is prior to its parts. However, this choice is clearly independent from the choice of identifying existential dependence with a primitive grounding relation.

[14] On the idea that, in general, the grounded is the same portion of reality of its grounds see Schaffer (2016). On the general idea that grounding entails nothing-over-and-aboveness see Fine (2001: 15–16; 2012: 39) and Schaffer (2009: 353).

the *xx*, or (iii) for some *zz*, both the *xx* and the *yy* depend for their existence on the *zz*.

According to (NOA), in order for the *xx* to really ‘add something’ to the *yy* in the required sense, the portion of reality in which the *xx* consist must not be completely contained in the portion of reality in which the *yy* consist. Therefore, in order to be something over and above the *yy* the *xx* cannot stand in a grounding relation to the *yy* or share a common ground with them. Given (NOA), (A3) follows directly from the assumption that *b* is grounded in [*c,d*], while (A1) appears to have at least the ring of plausibility to it. Suppose, in fact, that an entity *x* contains the *zz* and that the existence of a certain other entity *y* is grounded in the *zz*. In this case, it seems plausible to say that—in virtue of containing the ‘ontological root’ of *y* (as we may call it)—*x* also contains *y*.[15]

Both this kind of account of the notion of being nothing over and above in terms of grounding and the general idea that anti-extensionalism is compatible with nothing-over-and-aboveness may, of course, be challenged. Be that as it may, however, the foregoing considerations seem to show at least that the case against the notion of G-fusion is not as tight as it may appear at first sight and certainly not as straightforward as the case against the notion of A-fusion. Therefore, we seem to have at least some *prima facie* reason to conclude that, *pace* Sider (2007) and Varzi (2009), both the notion of L-fusion and the notion of G-fusion can be taken to be admissible notions of fusion.

Let us now return to the issue concerning the difference between L-fusions and G-fusions. It is a theorem of **MM** that parthood entails covering[16]:

(8) $\forall x \forall y (x \leq y \rightarrow x \sqsubseteq y)$

It is, thus, straightforward to observe from the first conjuncts of (L-def+) and (G-def+) that (assuming **MM**) if something is an L-fusion (of a certain plurality of entities) it is also a G-fusion (of those entities):

(9) $\forall x \forall yy \left(xF^{L}yy \rightarrow xF^{G}yy \right)$

[15] Skiles (2015: 739–741) has argued that the idea that grounding entails nothing-over-and-aboveness is compatible with ‘grounding contingentism’ (namely, the idea that grounds do not necessitate what they ground). Notice, however, that G-mereologists don’t appear to be forced to endorse grounding contingentism. In fact, at least under the assumption of Correia’s and Schnieder’s account of ontological dependence, entities that share the same proper parts can existentially depend on them by being grounded in *different facts* about them. For instance, the fact that *a* exists may be grounded simply in the fact that *c* exists and the fact that *d* exist, taken together, while the fact that *b* exists may be grounded in the fact that *c* and *d* are arranged in a certain way *R*. Therefore, when *c* and *d* cease to be *R*-arranged the fact that only *b* ceases to exist (while *a* keeps existing) doesn’t appear to be problematic for ‘grounding necessitarians’.

[16] *Proof*: Suppose that *x* is a part of *y* and assume that *z* overlaps *x*. There is, therefore, some entity *w* that is part of both *z* and *x*. So *w* is a part of *x* and *x* is a part of *y*. By the transitivity of parthood, *w* is a part of *y*. Therefore, *z* also overlaps *y*. By generalization, everything that overlaps *x* also overlaps *y* or, in other words, *x* is covered by *y*. ■

In the same way, it is also straightforward to see what principle can fill *by itself* the gap between the two notions. In fact, if an entity x is a G-fusion of some entities yy, then although the yy cover all of x (as required by the second conjunct of (L-def+)) we only have that if something is one of the yy, it is only guaranteed to be *covered* by x and not to be *part* of it. It is, thus, sufficient to add to the mix the principle according to which if x is covered by y, then x is part of y—

(10) $\forall x \forall y (x \sqsubseteq y \rightarrow x \leq y)$

—in order to guarantee that every G-fusion is also an L-fusion:

(11) $\forall x \forall yy \left(xF^G yy \rightarrow xF^L yy \right)$

Notice that if we unpack (10) we get.

(12) $\forall x \forall y (\forall z (Ozx \rightarrow Ozy) \rightarrow x \leq y)$

which, given **MM**, is equivalent to the Strong Supplementation principle[17]

$$\text{(SSP)} \quad \forall x \forall y \left(x \nleq y \rightarrow \exists z (z \leq x \wedge \sim Ozy) \right)$$

Therefore, **MM** and (SSP) jointly entail (EqLG):

(13) **MM**, (SSP) $\vdash$ (EqLG)

Similarly, it can be proved that (EqLG) and **MM** jointly entail (SSP).

(14) **MM**, (EqLG) $\vdash$ (SSP)

Proof Suppose that x is covered by y. Clearly, y covers itself. Therefore, each of the $[x, y]$ (that is, the plurality of x and y taken together) is covered by y. On the other hand, every part of y overlaps at least some of the $[x, y]$, since y clearly overlaps itself. Therefore y is a G-fusion of $[x, y]$. From (EqLG) it follows that y is also an L-fusion of the $[x, y]$, so that x is a part of y. We have, thus, proved (10), and namely that, for every x and every y, if x is covered by y, then x is a part of y. As we have just seen above, given **MM** (10) is equivalent to (SSP). ■

We can, thus, conclude that (SSP) is a way to express the difference between L-fusions and G-fusions on the background of **MM**:

(Difference 1) **MM**, (SSP) $\vdash$ (EqLG) and **MM**, (EqLG) $\vdash$ (SSP)

This result shouldn't come as a surprise: the fact that, in the presence of (SSP), (L-def) and (G-def) are equivalent is well-known in the literature.[18] It is, however, the *way* in which we reached this result that is particularly interesting. In fact, what

[17] See Pietruszczak (2018: 90; 2020: 34–35).

[18] See, for instance, Pietruszczak (2005: 216–8), Cotnoir and Bacon (2012: 195–6), and Varzi (2019: 4.3).

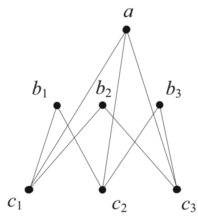

Fig. 3 A model of (EPP) that is not a model of (EqLG)

we have done in this section is to unpack both definitions of mereological fusion so as to make a 'containment' part (requiring the fusion to 'contain' in some sense all the things it fuses) and a 'covering' part (requiring the things fused to completely 'cover' the fusion) explicit. This allowed us to interpret the difference between the two notions of fusion as a different way of articulating the notion of containment at play in the definition of fusion: *containment-as-parthood* and *containment-as-covering*. At that point, the 'gap' between the two notions emerged very naturally as the requirement that *covering entail parthood*, which is just a different way to express (SSP). Therefore, in this section we haven't just proved that (SSP) is a way to express the difference between L-fusions and G-fusions. We have also provided a seemingly intuitive explanation as to *why* that is the case, namely because (SSP) functions as a bridge principle between the two different notions of containment in play.

3 Mild Supplementation and Extensionality

The model of Fig. 1 was used to show that, if **MM** is the only mereological assumption in the background, then not every G-fusion is an L-fusion. Figure 1 is a counterexample to Extensionality of Proper Parthood:

$$(\text{EPP})\forall x \forall y(\exists z(z < x) \rightarrow (\forall z(z < x \leftrightarrow z < y) \rightarrow x = y))$$

This may lead one to suppose that (EPP) is another way to express the difference between the two notion of fusions. However, the model depicted in Fig. 3 clearly shows that this is not the case. Consider, in fact, the plurality $[b_1,b_2,b_3]$ (the '*b*s'). *a* is clearly a G-fusion of the *b*s, as it covers all of them and it is covered by them (taken together). However, no entity in Fig. 3 is an L-fusion of the *b*s, as no entity in Fig. 3—not even *a*—has all the *b*s as parts. Figure 3 is a model of (EPP). Therefore, (EPP) is not enough to fill the gap between L-fusions and G-fusions.

In Fig. 3 *a* would qualify as a fusion of the *b*s if it had all of the *b*s as parts. In that case each of the *b*s would be a *proper* part of *a*. This is, thus, what may appear to be odd about Fig. 3[19]: *a* covers each of the *b*s, none of the *b*s covers *a*, and yet

[19] Pietruszczak (2018: 145) uses this model to show that **MM**, what we may call 'G-Universalism' (that is, the principle according to which every non-empty set of entities has a G-fusion—where in this case the notion of G-fusion is defined in terms of sets, instead of pluralities) and the principle Extensionality of Overlap (see Sect. 4 below) do not entail either (SSP) or 'L-Universalism'. Hovda (2009: 71) uses this

none of the bs is a proper part of a. Therefore, it is sufficient to add the following principle to **MM** (which I will label 'Mild Supplementation') to ensure that in Fig. 3 a is also an L-fusion of the bs[20]:

$$\text{(MSP)} \quad \forall x \forall y((x \sqsubseteq y \wedge y \not\sqsubseteq x) \rightarrow x < y)$$

or, in its contrapositive form

$$\text{(MSPc)} \quad \forall x \forall y(x \not< y \rightarrow (x \not\sqsubseteq y \vee y \sqsubseteq x))$$

which is equivalent to the following formulation:

$$\text{(MSPc2)} \quad \forall x \forall y(x \not< y \rightarrow (x \sqsubseteq y \rightarrow y \sqsubseteq x))$$

Notice that, given **MM**, (SSP) is equivalent to the following principle:

$$\text{(SSP2)} \quad \forall x \forall y(x \not< y \rightarrow (x \sqsubseteq y \rightarrow x = y))$$

Therefore, for every x and y such that x is not a proper part of y and yet it is covered by y we have that, while (SSP2) demands that x be *identical* to y (thus excluding both models like the one depicted in Figs. 1 and in 3), (MSPc2) requires only that x *cover* y (thus excluding only models like the one in Fig. 3 and leaving the door open to failure of extensionality).

(MSP) and (EPP) are independent principles. Figure 1 is a model of (MSP) but not of (EPP), while Fig. 3 is a model of (EPP) but not of (MSP). Furthermore, as Figs. 1 and 3 witness, both (MSP) and (EPP) are weaker than (SSP). However, (under the assumption of **MM**) (SSP) entails both (MSP) and (EPP), while (MSP) and (EPP) jointly entail (SSP):

(15) a. **MM**, (SSP) $\vdash$ (MSP)
b. **MM**, (SSP) $\vdash$ (EPP)
c. **MM**, (MSP), (EPP) $\vdash$ (SSP)

The proofs of both (15a) and (15b) are straightforward.[21] The proof of (15c) can be presented as follows. First, we prove that **MM** and (MSP) jointly entail the principle '(OPP)' according to which entities overlapping the same entities have the same proper parts

Footnote 19 continued
model to argue against the notion of G-fusion. His argument is the following. Classical Mereology can be axiomatized by just two axioms: (i) the transitivity of parthood and (ii) the existence of a *unique* L-fusion of any plurality of entities. Instead, it is not sufficient to assume that parthood is transitive and that every plurality of entities has a unique G-fusion to get Classical Mereology, as Fig. 3 shows.

[20] Notice that (MSP) is the right-to-left reading of Goodman's (1951: 49) definition of proper part when expressed solely in terms of overlap.

[21] *Proof of* (15a): Suppose x is covered by y. By (SSP), x is part of y. Suppose, furthermore, that y is not covered by x. It follows from Leibniz's Law that x is different from y. Therefore, x is a proper part of y. We can, thus, conclude that if x is covered by y and y isn't covered by x, then x is a proper part of y. ■

Proof of (15b): Suppose x and y are two composite entities with the same proper parts. Therefore, every part of x overlaps y, and every of y overlaps x. It follows from (SSP) that x is part of y and y is part of x. By the anti-symmetry of parthood it follows that x is identical to y. ■

$$\text{(OPP)} \quad \forall z(Ozx \leftrightarrow Ozy) \rightarrow \forall z(z < y \leftrightarrow z < x)$$

(16) $\mathbf{MM}, \text{(MSP)} \vdash \text{(OPP)}$

Proof Suppose that x and y overlap the same entities. Suppose, for *reductio*, that some entity w is a proper part of only one of them (say, of x). w is not a proper part of y. It follows from (MSP) that either w is not covered by y or y is covered by w. Since w is a proper part of x, everything that overlaps w also overlaps x (by the transitivity of parthood). But we are assuming that everything that overlaps x also overlaps y. Therefore, everything that overlaps w overlaps y so that w *is* covered by y. Therefore, y is covered by w, so that everything that overlaps y also overlaps w. w is a proper part of x. It follows by (WSP) that some entity k is part of x and doesn't overlap w. Therefore, k doesn't overlap y. But we are assuming that x and y overlap the same entities, so that k doesn't overlap x either. Yet, k is a proper part of x and so it overlaps x. *Contradiction*! Therefore, for every x and y, if x and y overlap the same entities they have the same proper parts. ■

Second, we show that **MM**, (MSP), and (EPP) jointly entail (SSP):

(17) $\mathbf{MM}, \text{(MSP)}, \text{(EPP)} \vdash \text{(SSP)}$

Proof Suppose that x is not part of y and that yet every part of x overlaps y. It follows by the transitivity of parthood that everything that overlaps x also overlaps y or, in other words, that x is covered by y. Since x is not a part of y, x is not a proper part of y. By (MSP) we have that either x is not covered by y or y is covered by x. y is, thus, covered by x. x and y cover each other. By (OPP) they have the same proper parts. Suppose that x is an atom (the reasoning being similar in the case of y). x is, thus, a part of y that is different from y. Therefore, x is a proper part of y. But x and y have the same proper parts, so that x is a proper part of itself, *contra* the irreflexivity of proper parthood. Therefore, x and y are two different composite entities with the same proper parts. By (EPP), they are identical. *Contradiction*! Therefore, some part of x doesn't overlap y. By generalization it follows that, for every x and y, if x is not a part of y, then some part of x doesn't overlap y. ■

From (15) and (Difference-1) it follows, thus, that (MSP) and (EPP) taken together are a further way to express the difference between L-fusions and G-fusions:

(Difference 2)

$\mathbf{MM}, \text{(MSP)} \nvdash \text{(EPP)}$ and $\mathbf{MM}, \text{(EPP)} \nvdash \text{(MSP)}$
$\mathbf{MM}, \text{(MSP)}, \text{(EPP)} \vdash \text{(EqLG)}$
$\mathbf{MM}, \text{(EqLG)} \vdash \text{(MSP)}$ and $\mathbf{MM}, \text{(EqLG)} \vdash \text{(EPP)}$

4 Between minimal mereology and extensional mereology

Casati and Varzi (1999) call 'Extensional Mereology' ('**EM**') a theory that is equivalent to **MM**+(SSP) (and, thus, to **MM**+(EPP)+(MSP)). They justify the label 'extensional' with the fact that **EM** rules out countermodels to (EPP) (Casati and Varzi 1999: 40). Other extensionality principles that can be found in the literature are the principles 'Extensionality of Overlap', 'Uniqueness of G-Fusion', and 'Uniqueness of L-Fusion':

$$(\text{EO}) \quad \forall x \forall y(\forall z(Ozx \leftrightarrow Ozy) \rightarrow x = y)$$

$$(\text{UGF}) \quad \forall x \forall y\left(\exists zz\left(xF^{G}zz \wedge yF^{G}zz\right) \rightarrow x = y\right)$$

$$(\text{ULF}) \quad \forall x \forall y\left(\exists zz\left(xF^{L}zz \wedge yF^{L}zz\right) \rightarrow x = y\right)$$

In addition we also have the principle that Varzi (2008) calls 'Extensionality of Composition':

(EC) 'If x and y are composed of the same things, then $x = y$' (Varzi 2008: 109).

where

(C) 'x is composed of the zs $=_{\text{df}}$ x is a [fusion] of the zs and the zs are pairwise disjoint (i.e., no two of them have any parts in common)' (Varzi 2008: 109; see also van Inwagen 1990: 29)

Letting '$\mathbb{D}xx$' stand for 'the xx are pairwise disjoint'

$$(\mathbb{D}\text{-def}) \quad \mathbb{D}xx =_{df} \forall x \forall y((x \prec xx \wedge y \prec xx \wedge x \neq y) \rightarrow \sim Oxy)$$

we can define two notions of composition in this sense, namely, Goodman-composition (or G-composition) and Leśniewski-composition (or L-composition):

$$(\text{GC}) \quad xxC^{G}y =_{df} yF^{G}xx \wedge \mathbb{D}xx$$

$$(\text{LC}) \quad xxC^{L}y =_{df} yF^{L}xx \wedge \mathbb{D}xx$$

In turn, (GC) and (LC) allow us to state two different versions of Varzi's principle (EC):

$$(\text{EGC}) \quad \forall x \forall y\left(\exists zz\left(zzC^{G}x \wedge zzC^{G}y\right) \rightarrow x = y\right)$$

$$(\text{ELC}) \quad \forall x \forall y\left(\exists zz\left(zzC^{L}x \wedge zzC^{L}y\right) \rightarrow x = y\right)$$

Most of the ways in which (with **MM** in the background) (SSP), (EPP), (EO), (UGF), (ULF), (EGC), and (ELC) relate to each other are well-known in the literature. It may be useful to briefly review them in turn[22]:

[22] See, among others, Varzi (2008) and Pietruszczak (2018: 174; 2020: 37–42).

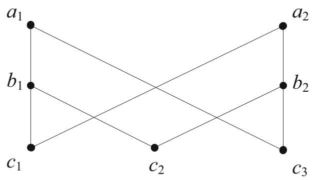

Fig. 4 A model of (EPP) that is not a model of either (EO), (UGF), (ULF), (EGC), or (ELC)

(a) (EPP) doesn't entail any of the other principles. For instance, the model depicted in Fig. 4 is a model of (EPP) but not a model of either (EO), (UGF), (ULF), (EGC), or (ELC). In fact, although a_1 and a_2 are different, they (i) overlap the same entities (*contra* (EO) and (ii) are both G-fusions and L-fusions of the *c*s (*contra* (UGF), (ULF), (EGC), and (ELC)).

(b) **MM** and (EO) jointly entail (EPP)[23]:

(18) $\mathbf{MM}, (\mathrm{EO}) \vdash (\mathrm{EPP})$

(c) **MM**+(EO), **MM**+(UGF), and **MM**+(ULF) are logically equivalent. One elegant way to prove this is due to Pietruszczak (2018: 85–86, 144) and can be briefly reformulated within this framework as follows.[24] First, it is proved from ($<$-Transitivity) that if every part of x overlaps y, then x is an L-fusion of the plurality of entities that are parts of both x and y:[25 26]

(19) $\forall z(z \leq x \rightarrow Ozy) \rightarrow xF^L(\imath xx.\forall z(z \prec xx \leftrightarrow (z \leq x \wedge z \leq y))$

[23] *Proof.* Suppose that x and y are composite entities having the same proper parts. Suppose, furthermore, that some z overlaps, say, x but not y (the reasoning being identical if we assume that something overlaps y but not x). Then, there is some w that is a common part of z and x but is disjoint from y. x has all of its proper parts in common with y, and so it clearly overlaps y. Therefore, z must be a proper part of x. But we are supposing that every proper part of x is also a proper part of y. Therefore, z is also a proper part of y and, thus, overlaps y. *Contradiction*! Therefore, everything that overlaps x also overlaps y. By (EO) it follows that x is identical to y. Therefore, if x and y are composite entities having the same proper parts, they are identical. ■ See also Pietruszczak (2018: 174; 2020: 42).

[24] As Pietruszczak (2018: 85–86, 144) shows, it is sufficient to assume that parthood is transitive in order to prove that (EO), (UGF) and (ULF) are equivalent. Notice, furthermore, that **MM**+(EO), **MM**+(UGF), **MM**+(ULF) are all equivalent to **POS**+(EO), **POS**+(UGF), **POS**+(ULF), where in this context **POS** can be taken to be the mereology axiomatised by ($<$-Transitivity) and ($<$-Irreflexivity) and entailing thus that proper parthood is a strict partial order (see on this Pietruszczak 2018: 86, 144; 2020: 46, 58).

[25] Recall that '$\imath xx.\phi xx$' stands for the plural definite description 'the xx that ϕ' (see footnote 4).

[26] *Proof* (see Pietruszczak 2018: 85). Suppose that every part of x overlaps y and let the zz be the entities that are parts of both x and y. The zz are all part of x, so that they comply with the first conjunct of (L-def). Consider an arbitrary part k of x. k overlaps y and so there is some j such that j is part of both k and y. By ($<$-transitivity) j is part of both x and y and so it is one of the zz. Therefore, k overlaps one of the zz. By generalization, every part of x overlaps one of the zz, so that they comply also with the second conjunct of (L-def). It follows that x is an L-fusion of the entities that are parts of both x and y. ■

Given (19), (EO) can be shown to be equivalent to (ULF), under the assumption of **MM**[27]:

(20) a. **MM**, (EO) $\vdash$ (ULF)
b. **MM**, (ULF) $\vdash$ (EO)

Finally, (20) can be used to prove that, given **MM**, (UGF) and (ULF) are equivalent[28]:

(21) **MM**,(UGF) $\vdash$ (ULF) and **MM**, (ULF) $\vdash$ (UGF)

(d) **EM** isn't entailed by either **MM**+(EO) or **MM**+(ELC):

(22) **MM**, (EO) $\nvdash$ **EM** and **MM**, (ELC) $\nvdash$ **EM**

For instance, the model depicted in Fig. 3—which is a counter-model to **EM** —is a model of both **MM**+(EO) and **MM**+(ELC).

(e) Finally, **EM** entails (EO)[29]:

(23) **EM** $\vdash$ (EO)

In addition to (a)-(e) we clearly have that **MM**+(ULF) entails both (ELC) and (EGC):

[27] *Proof.* (see Pietruszczak 2018: 86). *(20a)*. Assume (EO) and suppose that both *x* and *y* L-fuse the *zz* and that *w* overlaps *x*. Therefore, *w* has a part *k* in common with *x*. By (L-def), *k* overlaps the *zz*. *k* has thus a part *j* in common with some of the *zz*. Each of the *zz* is part of *y*. By ($<$-transitivity), *j* is part of both *w* and *y*. Therefore, *w* overlaps *y*. It follows that everything that overlaps *x* also overlaps *y*. By symmetry of reasoning we also have that everything that overlaps *y* overlaps *x*. It follows from (EO) that *x* is identical to *y*. *(20b)*. Assume (ULG) and suppose that *x* and *y* overlap the same entities. It follows that every part of *x* overlaps *y* and every part of *y* overlaps *x*. By (19) we have that both *x* and *y* are an L-fusion of the same plurality of entities, namely, the entities that are part of both *x* and *y*. By (ULG) it follows that *x* and *y* are identical. ∎

[28] *Proof* (see Pietruszczak 2018: 144). *Left-to-right*. Assume (UGF) and suppose that both *x* and *y* are an L-fusion of the *zz*. By (L-to-G) (Sect. 1) it follows that they are a G-fusion of the *zz* so that, by (UGF), they are identical. *Right-to-left*. Assume (ULF) and suppose that both *x* and *y* are a G-fusion of the *zz*. By (G-def) it follows that if something overlaps either *x* or *y*, it overlaps the *zz*, and that if something overlaps the *zz*, then it overlaps both *x* and *y*. Therefore, *x* and *y* overlap the same entities. It follows, thus, from (EO)—which, by (20), is equivalent to (ULF)—that *x* and *y* are identical. ∎

[29] *Proof.* Suppose that *x* and *y* overlap the same entities and yet they are different. By the anti-symmetry of parthood, either *x* is not a part of *y* or *y* is not a part of *x*. Suppose that *x* is not a part of *y*. By (SSP) there is a part of *x* that doesn't overlap *y*. But every part of *x* clearly overlaps *x*, so that this means that something overlaps *x* without overlapping *y*. *Contradiction*! The same kind of reasoning applies if we suppose that *y* is not a part of *x*. Therefore, *x* and *y* are identical. ∎ See, also, Pietruszczak (2018: 92–93) for a proof that **EM** entails (ULG).

(24) **MM**, (ULF) $\vdash$ (ELC) and **MM**, (UGF) $\vdash$ (EGC)

Also, it follows directly from the fact that **MM** entails (L-to-G) (see Sect. 1) that (EGC) and **MM** jointly entail (ELC):

(25) **MM**, (EGC) $\vdash$ (ELC)

Interestingly, given the axiom of choice it can also be proved that (ELC) and **MM** jointly entail (UGF):

(26) **MM**, (ELC) $\vdash$ (UGF)

Proof *Part I*. Suppose *a* and *b* are both G-fusions of the *zz* (which may or may not be pairwise disjoint). Let *W* be the non-empty set of parts that *a* and *b* have in common,[30] *R* be a well-order on *W*,[31] e_1 be the least element of *W* under *R*, and $\mathbb{S}$ be a subset of *W* that is defined as follows:

(i) $e_1 \in \mathbb{S}$;
(ii) $\forall x((x \in W \wedge (\forall y(yRx \wedge y \in \mathbb{S}) \rightarrow Dxy)) \rightarrow x \in \mathbb{S})$(an item in *W* is in $\mathbb{S}$ whenever every *R*-smaller item in $\mathbb{S}$ is disjoint from it);
(iii) nothing else is a member of $\mathbb{S}$.[32]

30 We can prove that *W* is non-empty as follows. It follows from the definition of G-fusion that *a* and *b* overlap the same entities. Since overlap is reflexive, *a* and *b* overlap and have, thus, some part in common. ■

31 The existence of *R* is guaranteed by the Well-Ordering Theorem, which is equivalent to the axiom of choice (see e.g. Moschovakis 2006: 112).

32 The fact that $\mathbb{S}$ is well-defined can be shown as follows. Let *f* be a function such that, for every part *x* of *b*, $f(x) = 1$ if it can be decided, given the definition of $\mathbb{S}$, whether *x* is a member of $\mathbb{S}$ ($f(x) = 0$ otherwise). We have that:

(a) $f(e_1) = 1$
(b) $\forall x((\forall y(yRx \rightarrow f(y) = (1)) \rightarrow f(x) = 1)$

Proof. (*a*) From (i) we have that $e_1 \in \mathbb{S}$. Therefore, $f(e_1) = 1$. (*b*) Assume that for every *y*, such that *yRx*, $f(y) = 1$. It follows that for every *y*, such that *yRx*, it can be decided, given the definition of $\mathbb{S}$, whether *x* is a member of $\mathbb{S}$ or not. From (ii) and (iii) in the definition of $\mathbb{S}$ we have, thus, that, if for every *y*, such that *yRx* and $y \in \mathbb{S}$, *x* is disjoint from *y*, then $x \in \mathbb{S}$, otherwise $x \notin \mathbb{S}$. Therefore, $f(x) = 1$]. By the Transfinite Induction Theorem (see e.g. Moschovakis 2006: 94) it follows from (*a*) and (*b*) that, for every part *x* of *b*, $f(x) = 1$. Therefore, for every part *x* of *b*, the definition of $\mathbb{S}$ allows us to decide whether *x* is a member of $\mathbb{S}$ or not, so that $\mathbb{S}$ is well-defined. ■

We have, thus, the following: (*a*) e_1 is disjoint from any other member of $\mathbb{S}$[33]; (b) any two other members *c* and *d* of $\mathbb{S}$ are disjoint.[34] Therefore, the members of $\mathbb{S}$ are pairwise disjoint.[35]

Part II. Suppose that *z* is a part of *a*. If *z* is a member of $\mathbb{S}$, then it clearly overlaps some member of $\mathbb{S}$ (by $\leq$-reflexivity). Suppose that *z* is not a member of $\mathbb{S}$. Since *a* and *b* are G-fusions of the same plurality of entities it follows from (G-def) that they overlap the same entities. By the reflexivity of parthood, *z* overlaps *a*. Therefore, *z* also overlaps *b* and has, thus, a part *v* in common with *b*. By the transitivity of parthood *v* is a member of *W* (namely, the set of entities that are parts of both *a* and *b*; see *Part I*). If *v* is also a member of $\mathbb{S}$, then *z* clearly overlaps some member of $\mathbb{S}$ (by $\leq$-reflexivity). If *v* is not a member of $\mathbb{S}$, it follows from the definition of $\mathbb{S}$ that some member *w* of $\mathbb{S}$ is such that *wRv* and *v* overlaps *w*. Since *v* is part of *z*, the part that *v* has in common with *w* is also part of *z* (by $\leq$-transitivity) so that *z* overlaps *w*. Therefore, in each of these cases *z* overlaps some member of $\mathbb{S}$. By generalization, we have that every part of *a* overlaps some member of $\mathbb{S}$. By symmetry of reasoning, we can also conclude that every part of *b* overlaps some member of $\mathbb{S}$.

Part III. Let the *cc* be the plurality of entities that are in $\mathbb{S}$. Each of the *cc* is part of both *a* and *b*. As we just proved in *Part II*, every part of either *a* or *b* overlaps the *cc*. Therefore, both *a* and *b* are an L-fusion of the *cc*. The *cc* are pairwise disjoint. It follows, thus, from (ELC) that *a* and *b* are identical.[36] ■

It follows from (21), (24), and (26) that (*pace* Varzi 2008: 110–111),[37] given **MM**, (EO), (ELC), (EGC), (UGF), and (ULF) are *all* equivalent:

(27) **MM**+(EO)=**MM**+(ELC)=**MM**+(EGC)=**MM**+(UGF)=**MM**+(ULF)

[33] *Proof*. Suppose *x* is a member of $\mathbb{S}$ that is different from e_1. By (iii), *x* must satisfy condition (ii). So every member of $\mathbb{S}$ that is *R*-smaller than *x* is disjoint from *x*. By (i) e_1 is in $\mathbb{S}$. Being the least element of $\mathbb{S}$ under *R*, e_1 is *R*-smaller than *x*, so that e_1 is disjoint from *x*.■

[34] *Proof*. By assumption, both *c* and *d* must satisfy (ii). Since *R* is well-order, we have that either *cRd* or *dRc*. Suppose *cRd* (the other case being similar). It follows from (ii) that *d* is disjoint from *c*. ■

[35] Many thanks to Scott Dixon and Stephan Krämer for feedback on this part of the proof.

[36] This proof was inspired by a somewhat similar proof (showing that, under the assumption of classical mereology, every G-fusion is a G-composition) discussed by Ballast (2020: §6.3).

[37] Varzi (2008: 110–111) claims that—even assuming **MM** and, thus, (WSP) (which he takes to be a principle that 'expresses a minimal requirement which any relation must satisfy (besides reflexivity, anti-symmetry and transitivity) if it is to qualify as parthood at all'; Varzi 2008: 110)—(ELC) (which he labels '(EC)') *doesn't* entail (ULF) (which he labels '(UC)'). In order to argue for this claim he presents an 'infinite atomless model' (depicted in the figure labelled 'Fig. 2' at p. 110) for which (ELC) is true but (ULF) is false. As Varzi himself notices, (ELC) is only *vacuously* true in the model, given that in the model 'everything overlaps everything' (p. 110). However, this means that in the model in question *no* proper part of *any* composite entity complies with (WSP), as every proper part of every composite entity *x* overlaps *all* of *x*'s parts. Therefore, it is *false* that 'both models [presented at p. 110, including the model depicted in Fig. 2] satisfy […] *weak supplementation*' (Varzi 2008: 110).

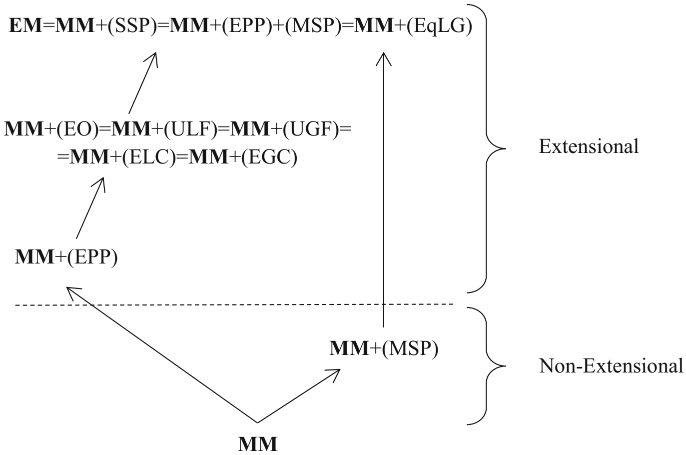

Fig. 5 Some of the theories between **MM** and **EM**

Notice, finally, that **MM**+(EO) doesn't entail **MM**+(MSP):

(28) **MM**, (EO) $\nvdash$ **MM**+(MSP)

For instance, the model depicted in Fig. 3 is a model of **MM**+(EO) but not of **MM**+(MSP).

Therefore, it can be concluded that that **MM**+(MSP) appears to occupy an interesting place among the theories under consideration (see Fig. 5).[38] In fact, while **MM**+(MSP) is a non-extensional mereology that is stronger than **MM**,[39] the only extensional mereology (among those under consideration) that is stronger than **MM**+(MSP) is **EM**.

5 Conclusion

[38] The diagram depicted in Fig. 5 is not meant to be exhaustive. Consider, for instance, that if we add Simons's (1987: 28) 'Proper Parts Principle' (PPP) to **MM**+(EO)

(PPP) $\forall x \forall y((\exists z(z<x) \land \forall z(z<x \rightarrow z<y) \rightarrow x \leq y)$

we obtain a system that is stronger than **MM**+(EO) but weaker than **EM** (see Pietruszczak 2020: 42) (many thanks to an anonymous referee of this Journal).

[39] **MM**+(MSP) is not the strongest mereology containing **MM** but not (EPP). For instance, **MM**+(MSP) could be extended by the addition of the following 'artificially weaker' version of (EPP):

(EPP3) $\forall x \forall y(\exists v \exists w \exists z(v<x \land w<x \land z<x \land v \neq w \land w \neq z \land v \neq z) \rightarrow (\forall z(z<x \leftrightarrow z<y) \rightarrow x=y))$

(many thanks to an anonymous referee for this Journal).

In this paper I have addressed the question concerning the 'difference'—with **MM** in the background—between the notion of Leśniewski fusion and the notion of Goodman fusion. Although it is well known that the Strong Supplementation principle is sufficient to fill this gap, I have argued that this fact can be proved in a way that sheds some interesting light on the relationship between the two notions of fusion. I have also shown how the difference between Leśniewski fusions and Goodman fusions can be broken down into two logically independent components, namely, the well-known principle Extensionality of Proper Parthood and the lesser-known Mild Supplementation. Finally, the theory combining Minimal Mereology and Mild Supplementation has also emerged as a non-extensional theory that occupies an interesting position in the logical space of theories that are stronger than Minimal Mereology but weaker than Casati and Varzi's (1999) Extensional Mereology.[40]

Funding For the work on this paper I received funding from the DFG project *The Structure of Fundamentality* (project number 448954791). Open Access funding enabled and organized by Projekt DEAL.

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[40] Many thanks to Stephan Krämer and to three anonymous referees for this Journal for very helpful feedback that helped improve the paper. Special thanks to Scott Dixon for discussion on these and related topics.

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