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On a Problem of Conrad on Riesz Space Structures

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Abstract
This paper is concerned with Riesz space structures on a lattice ordered abelian group, continuing a line of research conducted by the author and the collaborators Antonio Di Nola and Gaetano Vitale. First we prove a statement in a paper of Paul Conrad (given without proof) that every non-archimedean totally ordered abelian group has at least two Riesz space structures, if any. Then, as a main result, we prove that there is a non-archimedean lattice ordered abelian group with strong unit having only one Riesz space structure. This gives a solution to a problem posed in a paper of Conrad dating back to 1975. Then we combine these results and the categorial equivalence between lattice ordered abelian groups with strong unit and MV-algebras (due to Daniele Mundici) and the one between Riesz spaces with strong unit and Riesz MV-algebras (due to Di Nola and Ioana Leustean). By combining these tools, we prove that every non-semisimple totally ordered MV-algebra has at least two Riesz MV-algebra structures, if any, and that there is a non-semisimple MV-algebra with only one Riesz MV-algebra structure.

Keywords Lattice ordered abelian groups · MV-algebras · Riesz spaces · Riesz MV-algebras

1 Introduction

This paper belongs to a short list of papers on the relations between Riesz spaces and abelian lattice ordered groups (abelian ℓ-groups) on the one hand, and Riesz MV-algebras and MV-algebras on the other hand; the predecessors of this paper are [14] and [8].

For the structures subject of this paper see [15] and [20] for Riesz spaces, [2] and [3] for abelian ℓ-groups, and [5] and [17] for MV-algebras. There is no real survey on Riesz MV-algebras, but see [9] for some information.

Riesz spaces are expansions of lattice ordered abelian groups, and Riesz MV-algebras are expansions of MV-algebras. Riesz spaces are sometimes called with other names, for instance vector lattices in the paper [6] which is fundamental for our purposes, or K-lineal or semi-ordered linear spaces as reported in [15] (and as noticed by a referee).

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Riesz spaces are real vector spaces equipped with a lattice order. They have applications, for instance, in functional analysis [15, 20], dynamical systems [11], and economics [1]. It seems that they have been invented by Riesz in [18]. Important early contributions are those of Freudenthal [10] and Kantorovich [12]. Key examples are both the spaces of real valued functions on a topological space, and the L^p spaces on measure spaces.

Like real vector spaces are a particular case of abelian group (they have an abelian group reduct), Riesz spaces are a particular case of abelian ℓ-group (they have an abelian ℓ-group reduct).

In this paper we are interesting in developing the ideas of [6] on Riesz spaces. More precisely, we are interested in an unproved assertion and an open problem of [6] (which are actually strictly related to each other, since the former is a particular case of the latter).

The unproved assertion (page 336 of [6], last two lines) is that every totally ordered, nonarchimedean abelian group admits at least two Riesz space structures, if any. In this paper we prove this assertion in detail (Theorem 5).

The open problem we are interested in is the fourth and last problem in the final section of [6]. The problem is whether all nonarchimedean lattice ordered abelian groups have at least two Riesz structures (if any). For totally ordered abelian groups, this coincides with the assertion above. In this paper we solve the fourth problem in the negative with a counterexample.

As a related work, the first problem of [6] is whether all Riesz structures of a lattice ordered abelian group (if any) are isomorphic. This problem has been solved in the negative by [14], by showing a counterexample. In [8] the solution is refined by giving a totally ordered example. Note that already [19], example 11.54 gives an abelian ℓ-group with at least two Riesz structures, correcting an inaccuracy of [4], p. 349 and [9], Corollary 2.

We are not aware of papers about the second and third problems of [6]. However they look more technical than the other two and will be omitted in this paper.

I conclude the paper with some applications to Riesz MV-algebras. Recall that MV-algebras are generalized Boolean algebras which give the algebraic semantics of fuzzy Łukasiewicz logic, so they are applied to reasoning with uncertain information. Riesz MV-algebras are obtained by adding a multiplication by real values in the interval $[0, 1]$, intuitively a truncated vector space structure.

Mundici in [16] discovered a categorial equivalence between abelian ℓ-groups with strong unit and MV-algebras. The equivalence arises from the study of AF C^*-algebras in quantum mechanics and their K^0 group, which is a partially ordered abelian group, sometimes a lattice ordered one; when this is the case, the K^0-group is associated with an MV-algebra via the functor invented by Mundici. All countable MV-algebras arise in this way.

Later Di Nola and Leustean in [9] lifted the Mundici's equivalence to Riesz spaces with strong unit and Riesz MV-algebras.

Using these categorial equivalences one can transfer our results on Riesz spaces and abelian ℓ-groups to Riesz MV-algebras and MV-algebras.

2 Preliminaries

Let us denote as usual the natural numbers by $\mathbb{N}$, the positive natural numbers by $\mathbb{N}^+$, the integers by $\mathbb{Z}$, the rationals by $\mathbb{Q}$, the reals by $\mathbb{R}$ and the positive reals by $\mathbb{R}^+$.

An abelian ℓ-group is a structure

$$(G, +, 0, -, \wedge, \vee)$$

where $(G, +, 0, -)$ is an abelian group, $(G, \wedge, \vee)$ is a lattice and the compatibility equations hold: $x + (y \wedge z) = (x + y) \wedge (x + z)$ and $x + (y \vee z) = (x + y) \vee (x + z)$.

These equations imply that if $x \leq y$ then $x + z \leq y + z$.

A strong unit of an ℓ-group G is an element u such that for every x there is $n \in \mathbb{N}$ such that $x \leq nu$.

Given $g \in G$ its absolute value is $|g| = g \vee -g$. Absolute values are always positive (or zero if $g = 0$).

The abelian ℓ-groups are an equational variety (in the language above) and, by [2], the variety of abelian ℓ-groups is generated by $\mathbb{Z}$. So the following properties of $\mathbb{Z}$ extend to every abelian ℓ-group:

Lemma 1

$$x = (x \vee 0) + (x \wedge 0) \tag{1}$$

$$|x| = (x \vee 0) - (x \wedge 0) \tag{2}$$

$$|x + y| \leq |x| + |y| \tag{3}$$

An element $x \geq 0$ of an abelian ℓ-group is said to be well above an element $y \geq 0$, written $x \gg y$, if $x \geq ny$ for every $n \in \mathbb{N}$.

An abelian ℓ-group is archimedean if there are no $x, y \geq 0$ with $x \gg y$ unless $y = 0$.

Two elements $x, y \geq 0$ are called archimedean equivalent if there is $n \in \mathbb{N}$ such that $nx \leq y$ and $ny \leq x$.

Now let us turn to Riesz spaces.

A Riesz space is a structure

$$(G, +, 0, -, \rho, \wedge, \vee)$$

such that $(G, +, 0, -, \rho)$ is a real vector space, with $\rho : \mathbb{R} \times G \to G$, $(G, +, 0, -, \wedge, \vee)$ is an abelian ℓ-group, and $\rho(r, x \vee y) = \rho(r, x) \vee \rho(r, y)$ for every $r \in \mathbb{R}^+$.

The map ρ is called a Riesz space structure on the abelian ℓ-group $(G, +, 0, -, \wedge, \vee)$.

The equations of Riesz spaces imply that, if $x \geq 0$ and $r \in \mathbb{R}^+$, we have $\rho(r, x) \geq 0$.

As explained in [13], Riesz spaces are a variety in the language $(+, 0, -, \rho_r, \wedge, \vee)$ where $\rho_r(x) = \rho(r, x)$ for every $r \in \mathbb{R}$, and the variety of Riesz spaces is generated by $\mathbb{R}$.

As groups, Riesz spaces have the following well known properties:

Lemma 2 *Every Riesz space G is a divisible and torsion free abelian group.*

By the previous lemma, if $q \in \mathbb{Q}$ and $q = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ we can write $qx = m((1/n)x)$ and every Riesz space has a vector space structure over $\mathbb{Q}$ restricting the Riesz space (and independent of the Riesz structure).

Moreover we give the following estimate:

Lemma 3 *Let G be a Riesz space. If $x \in G$ and ρ, ρ' are two Riesz structures on G, and $r \in \mathbb{R}$, then*

$$|\rho(r, x) - \rho'(r, x)| \ll |x|.$$

Proof Suppose $x \geq 0$ and $n \in \mathbb{N}$. Let $q \leq r \leq q'$ be two rationals with $q' - q \leq 1/n$. Then

$$\rho(r, x) - \rho'(r, x) \leq \rho(q', x) - \rho'(q, x) = q'x - qx = (q' - q)x \leq (1/n)x$$

likewise

$$\rho'(r, x) - \rho(r, x) \leq (1/n)x$$

and passing to the absolute value

$$|\rho'(r,x) - \rho(r,x)| \leq (1/n)x.$$

Since n is arbitrary we have the thesis for $x \geq 0$.

Likewise if $x \leq 0$ we obtain

$$|\rho(r,x) - \rho'(r,x)| = |\rho(r,-x) - \rho'(r,-x)| \leq (1/n)(-x).$$

If $x \in G$ is not positive or negative, then anyway one can write $x = (x \vee 0) + (x \wedge 0)$. So, using the subadditivity of absolute value one has

$$\begin{aligned} |\rho(r,x) - \rho'(r,x)| &= |\rho(r,x\vee 0) + \rho(r,x\wedge 0) - \rho'(r,x\vee 0) - \rho'(r,x\wedge 0)| \\ &\quad |\rho(r,x\vee 0) - \rho'(r,x\vee 0) + \rho(r,x\wedge 0) - \rho'(r,x\wedge 0)| \\ &\leq |\rho(r,x\vee 0) - \rho'(r,x\vee 0)| + |(\rho(r,x\wedge 0) - \rho'(r,x\wedge 0))| \\ &\leq (1/n)(x\vee 0) - (1/n)(x\wedge 0) = (1/n)|x|. \end{aligned}$$

The last equality follows from Lemma 1. Since n and x are arbitrary we conclude the proof. □

The following proposition is known:

Proposition 4 *Every archimedean Riesz space has only one Riesz space structure.*

Proof It follows from Lemma 3. □

Conrad in [6] asks about the converse of this proposition: if a Riesz space has only one Riesz structure, is it archimedean? In this paper we give a negative answer by exhibiting an example of non-archimedean Riesz space with only one Riesz space structure. The example is quite natural and is inspired by McNaughton functions, a fundamental kind of functions in MV-algebra theory, where they are used to build free MV-algebras.

A further section of the paper is devoted to MV-algebras and Riesz MV-algebras.

MV-algebras are a kind of generalization of Boolean algebras, where sum and product are not necessarily idempotent.

More precisely, an MV-algebra is a structure $(A, \oplus, 0, \neg)$ such that $(A, \oplus, 0)$ is a commutative monoid, $x \oplus \neg 0 = \neg 0$, $\neg\neg x = x$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. We can think of $\oplus$ as a truncated sum and of $\neg$ as a negation.

A useful symbol is $x \ominus y = \neg(\neg x \oplus y)$ (a kind of truncated difference).

The real interval [0, 1] is an MV-algebra with $x \oplus y = min(x + y, 1)$ and $\neg x = 1 - x$.

MV-algebras are partially ordered by saying that $x \leq y$ if there is z with $y = x \oplus z$.

An infinitesimal in an MV-algebra is an element $x \neq 0$ such that $nx \leq \neg x$ for every $n \in \mathbb{N}$. Intuitively this means $x \leq 1/n$ for every n, but $1/n$ need not exist in an MV-algebra.

An MV-algebra is called semisimple if it does not contain infinitesimals.

A Riesz MV-algebra is a structure $(A, \{\nabla_r\}_{r\in[0,1]})$ where A is an MV-algebra, moreover $\nabla_r(x \ominus y) = \nabla_r x \ominus \nabla_r y$, $\nabla_{r\ominus s} x = \nabla_r x \ominus \nabla_s x$, and $\nabla_r \nabla_s x = \nabla_{rs} x$. Finally, $\nabla_1 x = x$.

By [16] there is an equivalence Γ between the category of MV-algebras and the category of abelian ℓ-groups with strong unit. This equivalence has been extended to Riesz MV-algebras and Riesz spaces with strong unit in [9].

In particular, if (G, u) is a lattice ordered abelian group with strong unit, then $\Gamma(G, u)$ is an MV-algebra where the domain is the interval $[0, u]$ in G and the operations are $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$.

For instance, $[0, 1] = \Gamma(\mathbb{R}, 1)$ and $\{0, 1\} = \Gamma(\mathbb{Z}, 1)$.

3 The Totally Ordered Case

We begin by giving a complete proof of the result stated in [6], page 336, without proof:

Theorem 5 *(originally from [6]) Every non-archimedean totally ordered Riesz space G has at least two Riesz space structures.*

Proof Let G be a non Archimedean totally ordered real vector space. Let us call levels the Archimedean equivalence classes, which are ordered by Archimedean domination.

Let L be a nonzero level in G. Let us define

$$L^+ = \{x \in G | level(x) \geq L\}$$
$$L^- = \{x \in G | level(x) < L\}$$

Let $m \in G \cap L^-$, $m \neq 0$ and $M \in G \cap L^+$. Note that m and M are linearly independent over $\mathbb{R}$, so there is a basis B of G as a real vector space containing m and M.

Let B_0 be a basis of L^- containing m, let C be a complement space of L^- in G containing M, B_1 a basis of C containing M and $B = B_0 \cup B_1$.

We need the following lemma:

Lemma 6 *There is a nonlinear group isomorphism $f : \mathbb{R}M \to \mathbb{R}m$.*

Proof Let $b \in \mathbb{R}$ such that M and bM are linearly independent elements of $\mathbb{R}M$ over $\mathbb{Q}$. Let x be linearly independent from $\{m, bm\}$ in $\mathbb{R}m$. Let B_1 be a base of $\mathbb{R}M$ extending $\{M, bM\}$ and B_2 a base of $\mathbb{R}m$ extending $\{m, x\}$. There is a group isomorphism $f : \mathbb{R}M \to \mathbb{R}m$ sending the first base B_1 to the second base B_2. In this isomorphism, $f(bM) = x \neq bm = bf(M)$. So, f is not linear. □

By the previous lemma we can fix a nonlinear group isomorphism $f : \mathbb{R}M \to \mathbb{R}m$. We define a function $\tau : G \to G$ as follows.

Fix $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ $(i = 0, \ldots, n)$ and $b_i \in B \setminus \{M\}$ $(i = 1, \ldots, n)$. Define

$$\tau(a_0M + a_1b_1 + \ldots + a_nb_n) = a_0M + f(a_0M) + a_1b_1 + \ldots + a_nb_n.$$

τ is well defined on G since B is a basis of G.

Let us list some properties of τ.

Lemma 7 *τ is a group endomorphism of G.*

Proof This follows because f is a group homomorphism. □

Lemma 8 *τ is increasing.*

Proof Since τ is a group endomorphism, it is enough to check that τ preserves positivity.

Suppose $g > 0$. Then

$$g = a_0M + a_1b_1 + \ldots + a_nb_n$$

for some $b_1, \ldots, b_n \in B \setminus \{M\}$ and for some $a_0, \ldots, a_n \in \mathbb{R}$.

If $a_0 = 0$, then $\tau(g) = g > 0$.

Suppose $a_0 \neq 0$. Then g is a sum of a nonzero combination C of elements of $L^+ \cap B$ and finitely many elements of L^-. By construction of B, one has $C \in L^+$, so C is positive, and $\tau(g) > 0$ since it is obtained from a positive element of L^+ by adding finitely many elements in L^-. □

Lemma 9 *τ is injective.*

Proof Since τ is a group endomorphism, it is enough to check that its kernel is zero.

Suppose $\tau(g) = 0$. Then

$$g = a_0 M + a_1 b_1 + \ldots + a_n b_n$$

for some $b_1, \ldots, b_n \in B \setminus \{M\}$ and for some $a_0, \ldots, a_n \in \mathbb{R}$.

If $a_0 = 0$, then $\tau(g) = g = 0$.

On the other hand, $a_0 \neq 0$ is impossible. In fact, if $a_0 \neq 0$, then $\tau(g) = a + b$, where $a \in L^+$ and $b \in L^-$. So $a = -b$, but two elements of different archimedean levels cannot be equal. □

Lemma 10 *τ is surjective.*

Proof One has $G = V + W$, where V is generated by $\{m, M\}$ and W is generated by $B \setminus \{m, M\}$.

τ sends V into V and W identically into W. Let $g \in G$. Then $g = aM + bm + w$, where $a, b \in \mathbb{R}$ and $w \in W$.

We prove that there are $x, y \in \mathbb{R}$ such that $\tau(xM + ym + w) = g$. In fact, it is enough to find a pair of reals (x, y) such that $\tau(xM + ym) = aM + bm$. Now $\tau(xM + ym) = xM + f(xM) + ym$. Suppose $f(xM) = z_x m$. Then one has $xM + z_x m + ym = aM + bm$, so $x = a$ and $y = b - z_a$ is a solution. □

Lemma 11 *The inverse of τ is increasing.*

Proof This is to say that $\tau(g) \geq 0$ implies $g \geq 0$. But this holds because G is totally ordered and τ is increasing. □

Lemma 12 *τ is nonlinear.*

Proof By definition of τ we have

$$\tau(rM) = rM + f(rM).$$

It is enough to prove that $\tau(rM) - rM$ is nonlinear. But $\tau(rM) - rM = f(rM)$ and f is a nonlinear group isomorphism. □

Summing up the previous lemmas, τ is a nonlinear automorphism of the ordered real vector space G. So, as suggested in [6], we have in G another Riesz space structure given by the multiplication

$$r \star g = \tau^{-1} r(\tau g).$$

□

4 The Main Result

The following result solves the fourth problem of [6] and is the main result of the paper.

Theorem 13 *There is a non-archimedean Riesz space G (with strong unit) having only one Riesz space structure.*

Proof Let $\mathbb{R}^*$ be a nonprincipal ultrapower of $\mathbb{R}$. Let B be the set of all functions from $[0,1]^{\mathbb{N}}$ to $\mathbb{R}^*$.

Let F be the set of all bounded functions from $[0,1]^{\mathbb{N}}$ to $\mathbb{R}$ which depend only on finitely many variables.

Let ϵ be a positive infinitesimal in $\mathbb{R}^*$. Let G be the Riesz space generated by F and the constant ϵ in B. Now B, F, G are Riesz spaces and G is the Riesz space we are looking for.

I summarize some commutation (or anticommutation) properties in Riesz spaces, where I abbreviate $\rho(r,x)$ by rx:

Lemma 14 *For every x, y, z in a Riesz space and for every $r, r' \in \mathbb{R}$ one has*

$$\begin{aligned} x + (y \wedge z) &= (x+y) \wedge (x+z) \\ x + (y \vee z) &= (x+y) \vee (x+z) \\ 0x &= 0 \\ r(x \wedge y) &= rx \wedge ry, r > 0 \\ r(x \vee y) &= rx \vee ry, r > 0 \\ r(x \wedge y) &= rx \vee ry, r < 0 \\ r(x \vee y) &= rx \wedge ry, r < 0 \\ r(x+y) &= rx + ry \\ r(r'x) &= (rr')x \end{aligned} \tag{4}$$

Proof These equations hold because the variety of Riesz spaces is generated by $\mathbb{R}$ and they are easy to check in $\mathbb{R}$. □

More explicitly G has the following structure:

Lemma 15 *The elements of G are the finite lattice combinations of sums $f + r\epsilon$, where $f \in F$ and $r \in \mathbb{R}$.*

Proof Let T be the set of the lattice combinations of sums $f + r\epsilon$. Clearly every Riesz subspace of B containing F and ϵ contains T. So it is enough to show that T is a Riesz space.

First, $0 \in T$ trivially, and by definition, T is closed under the lattice operations $\wedge$ and $\vee$.

To prove that T is closed under sum, I have to show that if $t, t' \in T$ then $t + t' \in T$. To this aim, if u is a Riesz space polynomial in F and ϵ, let us denote by $n(u)$ the number of lattice operations in u.

Closure under sum can be proved by induction on $n(t+t')$. In fact, if $n(t+t') = 0$ then clearly $t + t' \in T$. If $n(t+t') > 0$ then at least one of t, t' begins with $\circ$, where $\circ$ is one of $\wedge, \vee$. By symmetry we can suppose $t' = u \circ v$. Now $t + t' = t + (u \circ v) = (t+u) \circ (t+v)$ by Lemma 14. But $t+u$ and $t+v$ have less lattice operators than $t+t'$, so we can apply the inductive hypothesis and find $t+u \in T$ and $t+v \in T$, so $t+t' = (t+u) \circ (t+v) \in T$. This completes the inductive proof.

Finally, T is closed under multiplication by any $r \in \mathbb{R}$ since the latter commutes (or anti-commutes) with lattice operations and rational linear combinations, in the sense of Lemma 14. In particular taking $r = -1$, T is closed under additive inverse, so is a group. □

From the previous lemma we derive:

Lemma 16 *All functions in G are bounded by a real number and depend only on a finite number of components.*

Proof By Lemma 15, every element of G is a Riesz polynomial in F and ϵ, so we can proceed by induction on the length of this polynomial. □

Since every element of G is bounded by a real number, the constant function 1 is a strong unit of G. Moreover $0 < \epsilon \ll 1$, so G is not archimedean. Finally, by Lemma 15, all values of a function $g \in G$ have the form $g(x) = r + s\epsilon$, where $r, s \in \mathbb{R}$ and the possible pairs (r, s) are finitely many for each g.

Given $x \in \mathbb{R}^*$, let $st(x)$ be the standard part of x and $ns(x) = x - st(x)$, the nonstandard part of x, if it exists. For every value of a function in G, the nonstandard part exists and is a real multiple of ϵ. Moreover from Lemma 15 we derive a crucial result:

Lemma 17 *For every $g \in G$, the values of g have only finitely many nonstandard parts.*

Proof By Lemma 15 there is a finite set $C(g) \subseteq F \times \mathbb{R}$ such that g is a lattice combination of $f + s\epsilon$ with $(f, s) \in C(g)$. Let $g(x)$ be a value of g. Then $g(x) = f(x) + s\epsilon$ where $f(x), s \in \mathbb{R}$. So, for some f, we have $(f, s) \in C(g)$ and therefore s ranges over a finite set. □

The standard Riesz structure $(\rho_r)_{r\in\mathbb{R}}$ on G sends $f \in F$ to rf and ϵ to $r\epsilon$.

Suppose for a contradiction there is another Riesz structure $\rho' \neq \rho$ on G.

We begin with a technical lemma.

Lemma 18 *Let $f \in F$, $r \in \mathbb{R}$ and $x \in [0, 1]^{\mathbb{N}}$. If $f(x) = 0$ then $\rho'(r, f)(x) = 0$.*

Proof If $f \geq 0$ everywhere then for every $q \in \mathbb{Q}$, $\rho'(q, f)(x) = qf(x) = 0$. If $r \in R$ there are $q', q'' \in \mathbb{Q}$ with $q' < r < q''$, so $\rho'(q', f) \leq \rho'(r, f) \leq \rho'(q'', f)$, hence $\rho'(r, f)(x) = 0$.

Likewise if $f \leq 0$ then $\rho'(r, f)(x) = 0$.

If f is arbitrary then $f = (f \vee 0) + (f \wedge 0)$ and $(f \vee 0)(x) = (f \wedge 0)(x) = 0$, and we have $f \vee 0 \geq 0$ and $f \wedge 0 \leq 0$, so by the previous cases

$$\rho'(r, f)(x) = \rho'(r, f \vee 0)(x) + \rho'(r, f \wedge 0)(x) = 0 + 0 = 0.$$

□

Corollary 19 *Let $f, g \in F$, $r \in \mathbb{R}$ and $x \in [0, 1]^{\mathbb{N}}$. If $f(x) = g(x)$ then $\rho'(r, f)(x) = \rho'(r, g)(x)$.*

Proof It follows by applying the previous lemma to $f - g$. □

Suppose there is a function $f \in F$ such that $\rho'(r, f) \neq rf$. By Eq. 1 we can suppose $f \geq 0$. Moreover we can suppose $f \geq 1$, since either $\rho'(r, 1) \neq r$ or $\rho'(r, f+1) \neq r(f+1)$.

Then $\rho'(r, f) - rf$ is a nonzero infinitesimal by Lemma 3, so there is $a \in [0, 1]^{\mathbb{N}}$ such that $\rho'(r, f)(a) = rf(a) + s\epsilon$ where $s \in \mathbb{R}$, $s \neq 0$.

In particular,

$$ns(\rho'(r, f)(a)) = s\epsilon.$$

Let us enumerate the components of a : $a = (a_0, a_1, a_2, \ldots)$.

Since f and $\rho'(r, f)$ belong to F, there is $n \in \mathbb{N}$ such that $f(x)$ and $\rho'(r, f)(x)$ depend only on the first n coordinates of x.

For every $q \in [0, 1] \cap \mathbb{Q}$ let me write $x_q = (a_0, \ldots, a_{n-1}, q, a_{n+1}, \ldots)$. Then for every q, we have

$$ns(\rho'(r, f)(x_q)) = s\epsilon.$$

Let me define the function $h : [0, 1]^{\mathbb{N}} \to \mathbb{R}$ such that

$$h(x) = x_n f(a).$$

Note that $h \in F$.

For every (rational) $q \in [0, 1]$, we have $h(x_q) = qf(x_q)$. By Corollary 19, we have

$$ns(\rho'(r, h)(x_q)) = ns(\rho'(r, qf)(x_q)) = ns(q\rho'(r, f)(x_q)) = q \times ns(\rho'(r, f)(x_q)) = qs\epsilon.$$

So, the values of $\rho'(r, h)$ have infinitely many nonstandard parts (at least all $qs\epsilon$ for $q \in [0, 1] \cap \mathbb{Q}$), hence by Lemma 17, the function $\rho'(r, h)$ cannot belong to G. This is a contradiction.

This shows $\rho'(r, f) = rf$ for every $f \in F$. Since ρ' and ρ coincide on F and on the infinitesimals (by Lemma 3), they coincide everywhere in G by Lemma 15. □

5 Applications to MV-Algebras

The results of the previous sections on Riesz spaces and lattice ordered abelian groups can be transferred to MV-algebras thanks to the Mundici functor Γ and the Di Nola-Leustean functor, which we call also Γ.

First we need three well known lemmas:

Lemma 20 *Let G be an abelian ℓ-group, and let $g, h \in G$. Let $n \in \mathbb{N}$. Then $n(g \wedge h) = ng \wedge nh$.*

Lemma 21 *Let G be an abelian ℓ-group, and let $g, h \in G$. Let $m \in \mathbb{N}$. If $mg \leq mh$ then $g \leq h$.*

Lemma 22 *Suppose G is an abelian ℓ-group, u is a strong unit of G and $A = \Gamma(G, u)$. Then:*

1. *G is archimedean if and only if A is semisimple.*
2. *G is totally ordered if and only if A is totally ordered.*

Now the following lemma comes from [9], Proposition 3:

Lemma 23 *Let A be an MV-algebra and $A = \Gamma(G, u)$. Every Riesz MV-algebra structure on A can be extended to a Riesz space structure on G.*

Here is a further lemma.

Lemma 24 *If A is a semisimple MV-algebra, then A has at most one Riesz MV-algebra structure.*

Proof Suppose $A = \Gamma(G, u)$ is semisimple. Then G is semisimple. By Lemma 23, every Riesz MV-algebra structure on A extends to G. Now the thesis follows from Lemma 22. □

From the results of this and the previous sections we derive two corollaries on MV-algebras:

Corollary 25 *Every totally ordered, non-semisimple MV-algebra A has at least two Riesz MV-algebra structures.*

Proof Let $A = \Gamma(G, u)$. Then G is totally ordered and non archimedean. So G has at least two Riesz space structures ρ_1, ρ_2. Restricting the two structures to A, the two restricted structures cannot be equal, since by [9], proof of Proposition 1, every element of G is a finite sum of elements of A and their opposites. □

Corollary 26 *There is a non-semisimple MV-algebra A with only one Riesz MV-algebra structure.*

Proof Let $A = \Gamma(G, u)$ where G is the abelian ℓ-group constructed in Theorem 13 and u is the constant function with value 1. Note that u is a strong unit of G. By Lemma 23, every Riesz MV-algebra structure on A can be extended to a Riesz space structure on G. But G has only one Riesz space structure. So, the Riesz MV-algebra structure on A is also unique. □

6 Conclusion

We think that the relation between abelian ℓ-groups and their Riesz space structures can be expanded upon. The same can be said for MV-algebras and Riesz MV-algebras, thanks to the categorial equivalences mentioned earlier in this paper. There are many interesting classes of MV-algebras or abelian ℓ-groups. For these classes one can ask whether a Riesz structure can exist, and if this is the case, how many. For instance, one can consider Riesz spaces of the form $G = \Gamma(\mathbb{R}\ lex\ H, (1, 0))$, where H is a Riesz space and lex denotes lexicographic product of groups. These Riesz spaces have been studied in [7] and correspond to local Riesz MV-algebras, the Riesz MV-algebras with only one maximal ideal. They inherit from H the natural Riesz structure $\rho(r, (s, g)) = (rs, rg)$. One can ask whether other Riesz structures, not inherited by H, are present in G.

Finally, it would be interesting to complete the list of problems of [6] by solving the second and the third.

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