Order
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Higman's Lemma is Stronger for Better Quasi Orders

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Abstract
We prove that Higman's lemma is strictly stronger for better quasi orders than for well quasi orders, within the framework of reverse mathematics. In fact, we show a stronger result: the infinite Ramsey theorem (for tuples of all lengths) is equivalent to the statement that any array $[\mathbb{N}]^{n+1} \to \mathbb{N}^n \times X$ for a well order X and $n \in \mathbb{N}$ is good, over the base theory RCA_0.

Keywords Higman's lemma · Better quasi order · Reverse mathematics · Nash-Williams' theorem · Ramsey's theorem

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1 Introduction

Let $\mathsf{Seq}(Q)$ denote the collection of finite sequences in Q. To refer to the entries and lengths of sequences, we stipulate that $\sigma \in \mathsf{Seq}(Q)$ is equal to $\langle \sigma_0, \ldots, \sigma_{l(\sigma)-1} \rangle$. Where the context suggests it, we identify $n \in \mathbb{N}$ with $\{0, \ldots, n-1\}$. When Q is a quasi order, we define $Q^{<\omega}$ as the quasi order with underlying set $\mathsf{Seq}(Q)$ and

$$\sigma \leq \tau \quad \Leftrightarrow \quad \begin{cases} \text{there is a strictly increasing } f : l(\sigma) \to l(\tau) \\ \text{with } \sigma_i \leq_Q \tau_{f(i)} \text{ for all } i < l(\sigma). \end{cases}$$

By Higman's lemma [1] we mean the statement that $Q^{<\omega}$ is a well quasi order whenever the same holds for Q.

As finite sequences in Q correspond to functions $n \to Q$, a natural generalization leads to transfinite sequences with ordinal numbers as lengths. The collection of transfinite sequences in Q need not be a well quasi order when Q is one (see the counterexample due to R. Rado [2]). To secure closure properties under infinitary constructions, C. Nash-Williams has introduced the more restrictive notion of better quasi order [3]. We refer to [4] for an introduction that uses the same notation as the present paper. Parts of the definition will also be recalled in the next section. By Nash-Williams' theorem we mean the statement that the collection of transfinite sequences in Q is a better quasi order whenever the same holds for Q (which is

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proved in [3]). The statement that $Q^{<\omega}$ is a better quasi order whenever the same holds for Q will be referred to as the generalized Higman lemma.

Reverse mathematics is a research program in logic, which aims to determine the minimal axioms that are needed to prove given theorems from various areas of mathematics (see the paper by H. Friedman [5] and the textbook by S. Simpson [6]). A classical result states that Higman's lemma is equivalent to an abstract set existence principle known as arithmetical comprehension, over the weak base theory RCA_0 (see [6, Theorem X.3.22]). Question 24 from a well-known list of A. Montalbán [7] asks about the precise strength of Nash-Williams' theorem. The latter is known to imply the principle of arithmetical transfinite recursion (which is considerably stronger than arithmetical comprehension), by a result of R. Shore [8] (see also [9]).

Conversely, A. Marcone [9] has shown that arithmetical transfinite recursion suffices to reduce Nash-Williams' theorem to the generalized Higman lemma. It remains open whether the latter can be proved by arithmetical transfinite recursion. A proof in ACA_0 (the extension of RCA_0 by arithmetical comprehension) had been suggested by P. Clote [10], but according to Marcone it could not be substantiated (see the paragraph after Conjecture 5.6 in [9]). In the present paper, we show that no such proof can exist: Arithmetical comprehension is known to be strictly weaker than the infinite Ramsey theorem for tuples of all lengths. We will prove that the latter is equivalent, over RCA_0, to the statement that all arrays $[\mathbb{N}]^{n+1} \to \mathbb{N}^n \times X$ for any well order X and all $n \in \mathbb{N}$ are good (see Theorem 7 below). The generalized Higman lemma is at least as strong as this statement (cf. Corollary 9). In Theorem 7, the direction from (ii) to (i) has been pointed out by Giovanni Soldà, for which the author is very grateful.

The idea of our proof is to iterate an argument due to Marcone, which shows that arithmetical comprehension follows when the better quasi orders are closed under binary products (see Theorem 5.10 and Lemma 5.17 of [4]). In subsequent work, the author has pushed this idea even further than in the present paper, to obtain a result that entails a stronger bound on Higman's lemma: Let $\mathbf{n}$ be the antichain with n elements. As shown in [11], the statement that $\mathbf{3}$ is a better quasi order entails the main axiom of a system denoted by ACA_0^+, which is strictly stronger than the infinite Ramsey theorem. Now RCA_0 proves that $\mathbf{2}$ is a better quasi order (as shown by Marcone [4]) and that $\mathbf{3}$ embeds into $\mathbf{2}^{<\omega}$. Hence the generalized Higman lemma will also entail the main axiom of ACA_0^+. The present paper retains its interest, first, because Theorem 7 does only involve barriers of the form $[\mathbb{N}]^n$, while more complicated barriers are required for the argument in [11]. This means that the present paper yields additional information on the fine structure of better quasi orders (in the spirit of [12]). Secondly, Theorem 7 is sharp in the sense that it includes a reversal, in contrast to the results in [11]. Finally, we hope that the reader will find the present paper helpful as preparation for [11], just as it was an important stepping stone for the author.[1]

2 Well Foundedness Proofs Via Better Quasi Orders

To connect the generalized Higman lemma and the infinite Ramsey theorem, we will use the following transformations of linear orders. The definition employs notation for sequences that is explained at the beginning of the previous section.

[1] Note added in proof: After completion of this paper, the author has become aware that Harvey Friedman has proved closely related results in the context of his adjacent Ramsey theory. These are published in a paper with Florian Pelupessy [Independence of Ramsey theorem variants using ε_0, Proceedings of the American Mathematical Society 144:2 (2016) 853–860], which does not, however, make the connection with better quasi orders.

Definition 1 For a linear order X and finite sequences $\alpha, \beta \in \mathsf{Seq}(X)$, we put

$$j(\alpha,\beta) := \min\left(\left\{ j < \min\left(l(\alpha), l(\beta)\right) \mid \alpha_j \neq \beta_j \right\} \cup \left\{\min\left(l(\alpha), l(\beta)\right)\right\}\right).$$

On $\mathsf{Seq}(X)$ we consider the lexicographic comparisons given by

$$\alpha \prec \beta \quad \Leftrightarrow \quad \begin{cases} \text{either } j := j(\alpha,\beta) < \min\left(l(\alpha), l(\beta)\right) \text{ and } \alpha_j <_X \beta_j, \\ \text{or } j(\alpha,\beta) = l(\alpha) < l(\beta). \end{cases}$$

Let $\omega(X)$ be the linear order with lexicographic comparisons and underlying set

$$\omega(X) := \{\alpha \in \mathsf{Seq}(X) \mid \alpha_{l(\alpha)-1} \leq_X \ldots \leq_X \alpha_0\}.$$

Finally, we define iterations by stipulating $\omega_0^X := X$ and $\omega_{n+1}^X := \omega\left(\omega_n^X\right)$.

It may help to think of $\alpha \in \omega(X)$ as the Cantor normal form $\omega^{\alpha_0} + \ldots + \omega^{\alpha_{l(\alpha)-1}}$. Within RCA_0, the official definition of ω_n^X does not proceed by recursion on $n \in \mathbb{N}$. Instead, one first observes that iterated applications of Seq yield trees or terms with leaf labels or constant symbols from X. The relation $\prec$ between trees of height n and the set of trees in ω_n^X can then be determined by primitive recursion over the number of vertices, for all $n \in \mathbb{N}$ simultaneously. The following are equivalent over the base theory RCA_0, by results of A. Marcone and A. Montalbán [13] as well as C. Jockusch [14] and K. McAloon [15]:

(i) if X is a well order, then so is ω_n^X for every $n \in \mathbb{N}$,
(ii) for all $n \in \mathbb{N}$, the n-th Turing jump of any set exists,
(iii) the infinite Ramsey theorem holds for tuples of any length.

It is straightforward to conclude that each of these statements is strictly stronger than arithmetical comprehension (see, e. g., [16, Section 4]). In the following we shall relate (i) to (a weaker statement than) the generalized Higman lemma.

Given a linear order Y, we write $\omega + Y$ for the linear order with underlying set

$$\omega + Y := \{(0,n) \mid n \in \mathbb{N}\} \cup \{(1,y) \mid y \in Y\}$$

and $(0,n) \leq (0,n') < (1,y) \leq (1,y')$ for $n \leq n'$ in $\mathbb{N}$ and $y \leq y'$ in Y. When X is (isomorphic to) an order of the form $\omega + Y$, we have an element $0 := (0,0) \in X$ and a strictly increasing map $X \ni x \mapsto 1 + x \in X$ that is given by $1 + (0,n) := (0, 1+n)$ and $1 + (1,y) := (1,y)$, which yields $0 < 1 + x$ for any $x \in X$. The next definition and lemma provide a convenient characterization of the order from above.

Definition 2 Assume X has the form $\omega + Y$. For $\alpha \in \omega(X)$ and $j \in \mathbb{N}$, we put

$$\overline{\alpha}_j := \begin{cases} 1 + \alpha_j & \text{if } j < l(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Given $\alpha, \beta \in \omega(X)$, we then set $c(\alpha,\beta) := \overline{\alpha}_j \in X$ with $j := j(\alpha,\beta)$.

Let us record the following basic facts.

Lemma 3 *Consider $\alpha, \beta, \gamma \in \omega(X)$ with X of the form $\omega + Y$. We have*

$$\alpha \prec \beta \quad \Leftrightarrow \quad \overline{\alpha}_j <_X \overline{\beta}_j \text{ with } j := j(\alpha,\beta).$$

When we have $\alpha \succ \beta$ and $j(\alpha,\beta) \leq j(\beta,\gamma)$, we get $c(\alpha,\beta) > c(\beta,\gamma)$ in X.

Proof The equivalence is checked by a case distinction between strict inequalities and equalities in $j(\alpha, \beta) \leq l(\alpha)$ and $j(\alpha, \beta) \leq l(\beta)$. To verify the remaining claim, put $i := j(\alpha, \beta) = j(\beta, \alpha)$ and $j := j(\beta, \gamma)$. Given $\alpha \succ \beta$, we get $c(\alpha, \beta) = \overline{\alpha}_i > \overline{\beta}_i$ by the equivalence. Due to $\beta \in \omega(X)$ and $i \leq j$, we also have $\overline{\beta}_i \geq \overline{\beta}_j = c(\beta, \gamma)$. □

When X has the form $\omega+Y$, so has ω_n^X for all numbers $n \in \mathbb{N}$. To confirm this for $n = m+1$, we note that ω_m^X contains a minimal element, which we denote by 0. If we have $m = 0$ and thus $\omega_m^X = X$, this element is given as above, while $m = k+1$ leads to $0 = \langle\rangle \in \omega(\omega_k^X) = \omega_m^X$. Now the elements $\langle 0, \ldots, 0\rangle \in \omega(\omega_m^X) = \omega_n^X$ form an initial segment isomorphic to $\mathbb{N}$. We can conclude that the previous considerations apply with ω_n^X at the place of X. In particular, we obtain elements $j(\alpha, \beta) \in \mathbb{N}$ and $c(\alpha, \beta) \in \omega_n^X$ for any $\alpha, \beta \in \omega_{n+1}^X = \omega(\omega_n^X)$.

Let $[\mathbb{N}]^n$ be the set of strictly increasing sequences $s \in \mathsf{Seq}(\mathbb{N})$ of length $l(s) = n$. Whenever we use this notation, we assume $n > 0$. For $s, t \in [\mathbb{N}]^n$ we declare

$$s \lhd t \quad :\Leftrightarrow \quad s_0 < t_0 \text{ and } s_{i+1} = t_i \text{ for all } i < n-1.$$

If we have $n > 1$, the second conjunct on the right does already entail $s_0 < s_1 = t_0$. For $n = 1$, the condition $s_0 < t_0$ allows us to identify $([\mathbb{N}]^1, \lhd)$ with the isomorphic structure $(\mathbb{N}, <)$. Given a quasi order Q, a map $f : [\mathbb{N}]^n \to Q$ is called good if there are $s \lhd t$ with $f(s) \leq_Q f(t)$. Otherwise it is called bad. The structures $([\mathbb{N}]^n, \lhd)$ are examples for the notion of barrier that appears in the definition of better quasi orders (see, e. g., [4]). To follow the present paper, it suffices to know that if Q is a better quasi order, then any $f : [\mathbb{N}]^n \to Q$ is good. We note that Q is a well quasi order precisely when this holds for $n = 1$. Over RCA_0, any map $f : [\mathbb{N}]^n \to Q$ into a well order Q is good (consider $s^k \lhd s^{k+1}$ with $s_i^k := k + i$ and exploit linearity).

Given $s \lhd t$ in $[\mathbb{N}]^n$, it is standard to define $s \cup t \in [\mathbb{N}]^{n+1}$ as the sequence with

$$(s \cup t)_i := \begin{cases} s_0 & \text{when } i = 0, \\ s_i = t_{i-1} & \text{when } 0 < i < n, \\ t_{n-1} & \text{when } i = n. \end{cases}$$

Note that $s \cup t$ is strictly increasing, which relies on $s_0 < t_0$ when we have $n = 1$. Any element of $[\mathbb{N}]^{n+1}$ can be uniquely written as $s \cup t$ with $s \lhd t$ in $[\mathbb{N}]^n$. When we use the notation $s \cup t$, we always assume $s \lhd t$. Let us observe that $r \cup s \lhd s' \cup t$ in $[\mathbb{N}]^{n+1}$ entails that we have $s = s'$.

To show that there can be no strictly decreasing sequence $f : \mathbb{N} \to \omega_n^X$, we now construct maps f_k with increasingly complex domain but ever simpler codomain. For $\sigma = \langle \sigma_0, \ldots, \sigma_{k-1}\rangle \in \mathbb{N}^k$ and $j \in \mathbb{N}$ we write $\sigma^\frown j := \langle \sigma_0, \ldots, \sigma_{k-1}, j\rangle \in \mathbb{N}^{k+1}$.

Definition 4 Consider $f : \mathbb{N} \to \omega_n^X$ for X of the form $\omega + Y$. To define

$$f_k : [\mathbb{N}]^{k+1} \to \mathbb{N}^k \times \omega_{n-k}^X$$

by recursion on $k \leq n$, we stipulate $f_k(r) := \langle f_k^0(r), f_k^1(r)\rangle$ with

$$f_0^0(\langle i\rangle) := \langle\rangle \in \mathbb{N}^0, \qquad f_{k+1}^0(s \cup t) := f_k^0(s)^\frown j\big(f_k^1(s), f_k^1(t)\big),$$
$$f_0^1(\langle i\rangle) := f(i), \qquad f_{k+1}^1(s \cup t) := c\big(f_k^1(s), f_k^1(t)\big).$$

The family of functions f_k can be seen as a single function on sequences, as k is determined by the length of the argument. This single function can be constructed by a recursion over subsequences, which is available in the base theory RCA_0.

Let us declare that $(\sigma_0, \ldots, \sigma_{k-1}, \alpha) \leq (\tau_0, \ldots, \tau_{k-1}, \beta)$ holds in $\mathbb{N}^k \times \omega^X_{n-k}$ if we have $\alpha \preceq \beta$ in ω^X_{n-k} as well as $\sigma_i \leq \tau_i$ in $\mathbb{N}$ for all $i < k$. The following observation is the crucial step in our argument.

Lemma 5 *If f_k is bad, then so is f_{k+1} (in the situation of Definition 4).*

Proof Aiming at a contradiction, we assume

$$f_{k+1}(r \cup s) \leq f_{k+1}(s \cup t) \tag{$\star$}$$

with $r \cup s \lhd s \cup t$ in $[\mathbb{N}]^{k+2}$. From ($\star$) we can, first, conclude that $f^0_k(r) \leq f^0_k(s)$ holds in $\mathbb{N}^k$ (i.e., componentwise). Given that f_k is bad and that we have $r \lhd s$, we must thus have $f^1_k(r) \succ f^1_k(s)$ in $\omega^X_{n-k} = \omega(\omega^X_{n-(k+1)})$. Now ($\star$) does, secondly, entail $j(f^1_k(r), f^1_k(s)) \leq j(f^1_k(s), f^1_k(t))$. We obtain $c(f^1_k(r), f^1_k(s)) \succ c(f^1_k(s), f^1_k(t))$ due to Lemma 3. But by ($\star$) we do, finally, get the converse inequality as well. □

In the following result, one may take $X = Z$ when Z itself has an initial segment that is isomorphic to $\mathbb{N}$. Otherwise, we can always put $X := \omega + Z$, which is a well order whenever the same holds for Z, provably in RCA_0.

Proposition 6 *[RCA_0] Consider a linear order Z with an embedding into a linear order X that has the form $\omega + Y$. If all maps $[\mathbb{N}]^{n+1} \to \mathbb{N}^n \times X$ are good, then ω^Z_n is a well order. In particular, this follows when $\mathbb{N}^n \times X$ is a better quasi order.*

Proof One readily constructs an embedding of ω^Z_n into ω^X_n. So it suffices to show that the latter is a well order. Towards a contradiction, we assume that $f : \mathbb{N} \to \omega^X_n$ is strictly decreasing. Let f_k for $k \leq n$ be given as in Definition 4. The map f_0 is bad by our assumption on f. In view of the previous lemma, we can use induction to conclude that $f_n : [\mathbb{N}]^{n+1} \to \mathbb{N}^n \times X$ is bad, against the assumption of the proposition. Concerning formalization in RCA_0, we note that the induction statement is Π^0_1 (and an even simpler induction over subsequences would also be possible). □

We now derive the promised equivalence. As noted in the introduction, the direction from (ii) to (i) has been pointed out by Giovanni Soldà. Similar applications of Ramsey's theorem have been considered by Marcone [4].

Theorem 7 *The following are equivalent over RCA_0:*

(i) all arrays $[\mathbb{N}]^{n+1} \to \mathbb{N}^n \times X$ for any well order X and all $n \in \mathbb{N}$ are good,
(ii) the infinite Ramsey theorem holds for tuples of all lengths.

Proof By the previous proposition and the paragraph that precedes it, (i) entails that ω^Z_n is a well order for any well order Z and all $n \in \mathbb{N}$. Statement (ii) follows by the results that were cited in the paragraph after Definition 1. For the converse we assume (ii) and derive (i) by contradiction. Suppose that $f : [\mathbb{N}]^{n+1} \to \mathbb{N}^n \times X$ is a bad array for some well order X. Write $f(s) = (f_0(s), \ldots, f_n(s))$ with $f_i(s) \in \mathbb{N}$ for $i < n$ and $f_n(s) \in X$. Recall that elements of $[\mathbb{N}]^{n+2}$ can be uniquely written as $s \cup t$ for $s, t \in [\mathbb{N}]^{n+1}$ with $s \lhd t$. Given that f is bad, we may consider

$$g : [\mathbb{N}]^{n+2} \to \{0, \ldots, n\} \quad \text{with} \quad g(s \cup t) := \min\{i \leq n \mid f_i(s) \not\leq f_i(t)\}.$$

The infinite Ramsey theorem yields a single $i \leq n$ and an infinite set $Y \subseteq \mathbb{N}$ such that any $s, t \in [Y]^{n+1}$ with $s \lhd t$ validate $g(s \cup t) = i$ and hence $f_i(s) \not\leq f_i(t)$. So f_i restricts to a bad array $[Y]^{n+1} \to Z$ with $Z := \mathbb{N}$ if $i < n$ and $Z := X$ otherwise. This is impossible since the well order Z is a better quasi order, as observed before Definition 4 or also by Lemma 3.1 of [4]. □

Let us note that statement (i) allows for some variation:

Remark 8 The proof of the previous theorem shows that the equivalence remains valid if we strengthen (i) by admitting all arrays $[\mathbb{N}]^m \to Q_0 \times \ldots \times Q_n$ into quasi orders Q_i with the property that any array $[\mathbb{N}]^m \to Q_i$ is good. In other words, we may view (i) as a closure property under products of arbitrary finite length.

As mentioned in the introduction, Higman's lemma for well quasi orders is equivalent to arithmetical comprehension, the main axiom of the theory ACA_0. We can now conclude that the extension to better quasi orders is strictly stronger. Note that an improved bound on the generalized Higman's lemma can be obtained from the results of [11], as explained in the introduction.

Corollary 9 *Over* RCA_0*, the infinite Ramsey theorem (for tuples of all lengths) follows from the generalized Higman lemma. So the latter cannot be proved in* ACA_0*.*

Proof To justify the first sentence of the corollary, we show that the generalized Higman lemma entails statement (i) from the previous theorem. Given an arbitrary well order X, let Y be the order $\omega + X$ or some other well order into which $\mathbb{N}$ and X embed. As above, Y is a better quasi order, provably in RCA_0. By the generalized Higman lemma, it follows that $Y^{<\omega}$ is a better quasi order. But the latter embeds Y^{n+1} and hence $\mathbb{N}^n \times X$ for any $n \in \mathbb{N}$. The remaining claim follows since the infinite Ramsey theorem for tuples of all lengths is unprovable in ACA_0, as noted in the paragraph after Definition 1. □

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Declarations

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