Order
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Antichains in the Bruhat Order for the Classes $\mathcal{A}(n, k)$

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Abstract
Let $\boldsymbol{\mathcal{A}(n,k)}$ represent the collection of all $\boldsymbol{n \times n}$ zero-and-one matrices, with the sum of all rows and columns equalling $\boldsymbol{k}$. This set can be ordered by an extension of the classical Bruhat order for permutations, seen as permutation matrices. The Bruhat order on $\boldsymbol{\mathcal{A}(n,k)}$ differs from the Bruhat order on permutations matrices not being, in general, graded, which results in some intriguing issues. In this paper, we focus on the maximum length of antichains in $\boldsymbol{\mathcal{A}(n,k)}$ with the Bruhat order. The crucial fact that allows us to obtain our main results is that two distinct matrices in $\boldsymbol{\mathcal{A}(n,k)}$ with an identical number of inversions cannot be compared using the Bruhat order. We construct sets of matrices in $\boldsymbol{\mathcal{A}(n,k)}$ so that each set consists of matrices with the same number of inversions. These sets are hence antichains in $\boldsymbol{\mathcal{A}(n,k)}$. We use these sets to deduce lower bounds for the maximum length of antichains in these partially ordered sets.

Keywords (0,1)-matrices · Bruhat order · Inversions · Antichains

1 Introduction and Main Results

The (0, 1)-matrices, or matrices with only zeros and ones, occur naturally in a wide range of contexts, including mathematics, educational tests, ecological studies, and social networks. A special class amongst these are the (0, 1)-matrices with a prescribed non-increasing row sum vector R and a prescribed non-increasing column sum vector S. These classes are denoted by $\mathcal{A}(R, S)$, and they have been the subject of intensive study by H.J. Ryser, D.R. Fulkerson, R.M. Haber, and D. Gale, among others, since the 1950s (see [1–3, 12, 16], and the references therein). When R and S are constant vectors with n components equal to k, $k \leq n$, we write $\mathcal{A}(n, k)$ for $\mathcal{A}(R, S)$.

One of the fundamental results involving classes $\mathcal{A}(R, S)$ is the characterization, in terms of majorization, of the non-emptiness of these classes. That characterization was obtained independently by D. Gale [13], using the theory of network flows, and by H.J. Ryser [19], using induction and a direct combinatorial argument. In order to present this characterization, we have to introduce some new concepts. Let $R = (r_1, \ldots, r_m)$ be a partition, that is, a non-

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increasing vector of positive integers. We can identify R with its *Ferrers diagram*, which we can obtain by placing r_i left justified boxes in the $i\,th$ row, for $1 \leq i \leq m$. The conjugate partition of R, denoted by R^*, is the partition corresponding to the transposition of the Ferrers diagram of R.

The majorization order on partitions with the same sum states that $R = (r_1 \ldots, r_m)$ precedes $S = (s_1, \ldots, s_n)$, and we write $R \preccurlyeq S$ if

$$r_1 + \ldots + r_i \leq s_1 + \ldots + s_i,$$

for all $i = 1, \ldots, \min\{m, n\}$.

Theorem 1 (Gale-Ryser Theorem) *Let R and S be partitions with the same sum. Then, $\mathcal{A}(R, S) \neq \emptyset$ if and only if $S \preccurlyeq R^*$.*

This result guarantees the non-emptiness of any class $\mathcal{A}(n, k)$.

Let $R = (r_1, \ldots, r_n)$ be a partition and let $g_1, \ldots, g_t$ integers such that $g_1 > \ldots > g_t$ and $\{g_1, \ldots, g_t\} = \{r_1, \ldots, r_n\}$. From now on, we write $R = (g_1^{i_1}, \ldots, g_t^{i_t})$, where $i_1, \ldots, i_t$ are the multiplicities of $g_1, \ldots, g_t$.

The class $\mathcal{A}(n, 1)$ consists of all permutation matrices of order n and thus corresponds with the symmetric group. This correspondence inspired Brualdi and Hwang to extend the classical Bruhat order on the symmetric group to any nonempty class $\mathcal{A}(R, S)$, (see [5]). For any $m \times n$ $(0, 1)$-matrix A, they define another $m \times n$ matrix, denoted by Σ_A, whose $(k, \ell)-$ entry is

$$\sigma_{k,\ell}(A) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} a_{i,j},$$

for all $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$.

If $A, C \in \mathcal{A}(R, S)$, then A precedes C in de Bruhat order, written $A \preccurlyeq_B C$ if $\sigma_{i,j}(A) \geq \sigma_{i,j}(C)$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Numerous studies have been conducted in recent years on a range of subjects connected to these new partially ordered sets. These studies include the description of minimal elements and the study of Bruhat order restrictions on subclasses of $\mathcal{A}(R, S)$, such as, for instance, the subclass of symmetric matrices (see [4, 6, 9]).

Given a finite poset $\mathcal{P}$, a chain is a completely ordered subset of $\mathcal{P}$, and an antichain is a subset of $\mathcal{P}$ whose elements are mutually unrelated. The cardinality of the largest antichain of $\mathcal{P}$ is called the *width* of $\mathcal{P}$, and the cardinality of the largest chain of $\mathcal{P}$ is called the *height*. Two classical results, the Dilworth theorem [10] and the Mirsky theorem [17], state a relationship between the width and the height of partially ordered sets. Dilworth's theorem states that the maximum number of elements in any antichain in a partially ordered set equals the minimum number of chains into which the set can be partitioned. Mirsky's theorem is a dual of this theorem and states that the maximum number of elements in any chain in a partially ordered set equals the minimum number of antichains into which the set can be partitioned.

The primary goal of this paper is to present new lower bounds for the maximal length of antichains of the matrix classes $\mathcal{A}(n, k)$ in the Bruhat order. This is an intriguing problem because these posets are not often graded. In fact, the authors of [4] provided an example showing that $\mathcal{A}(4, 2)$ is not graded. We denote by $w(n, k)$ the width of $\mathcal{A}(n, k)$ in the Bruhat order, and by $h(n, k)$ we denote the height. The theorems of Dilworth and Mirsky imply

$$h(n, k)w(n, k) \geq |\mathcal{A}(n, k)|,$$

where $|\mathcal{A}(n,k)|$ denotes the cardinality of $\mathcal{A}(n,k)$.

There have been some advances in recent years regarding the width and the height in the Bruhat order for the matrix classes $\mathcal{A}(2k,k)$ and $\mathcal{A}(n,2)$ (see [7, 8, 14, 15]). This paper focuses on the width of classes $\mathcal{A}(n,k)$ with the Bruhat order. For classes $\mathcal{A}(n,2)$ Ghebleh proved that:

Theorem 2 [15] *Let n be a positive integer, $n \geq 2$. Then*

$$w(n,2) \geq \begin{cases} \frac{n!}{2^{\frac{n}{2}}}, & \text{if } n \text{ is even} \\ \frac{(n-1)!}{2^{\frac{n-3}{2}}}, & \text{if } n \text{ is odd.} \end{cases}$$

Conflitti et al. [8] proved that k^8 is a lower bound for $w(2k,k)$. Ghebleh improved this result, showing that:

Theorem 3 [15] *Let k be a positive integer, and let $w(2k,k)$ be the width of the Bruhat order of $\mathcal{A}(2k,k)$. Then*

$$w(2k,k) \geq \begin{cases} \frac{(k!)^4}{4^k}, & \text{if } k \text{ is even} \\ \frac{((k-1)!)^4}{4^{k-3}}, & \text{if } k \text{ is odd.} \end{cases}$$

The following results are the paper's main results. We present lower bounds for the width of $\mathcal{A}(n,k)$ in the Bruhat order that, for the particular case of $\mathcal{A}(2k,k)$, are better than those stated in Theorem 3.

Theorem 4 *Let n and k be two positive integers with $k < n$ and n an even integer. Then,*

$$w(n,k) \geq \begin{cases} \left|\mathcal{A}\left(\frac{n}{2},\frac{k}{2}\right)\right|^2, & \text{if } k \text{ is even} \\ \left|\mathcal{A}\left(\frac{n}{2},\frac{k-1}{2}\right)\right| \cdot \left|\mathcal{A}\left(\frac{n}{2},\frac{k+1}{2}\right)\right|, & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 5 *Let n and k be two positive integers with $k < n$ and n an odd integer. Then,*

$$w(n,k) \geq \begin{cases} \left|\mathcal{A}\left(\frac{n-1}{2},\frac{k}{2}\right)\right| \cdot |\mathcal{A}(R,R)|, & \text{if } k \text{ is even} \\ \left|\mathcal{A}\left(\frac{n-1}{2},\frac{k-1}{2}\right)\right| \cdot \left|\mathcal{A}(R',R')\right|, & \text{if } k \text{ is odd,} \end{cases}$$

where $R = \left((\frac{k}{2})^{\frac{n-k-1}{2}}, (\frac{k-2}{2})^{\frac{k}{2}}\right)$ *and* $R' = \left((\frac{k+1}{2})^{\frac{n-k}{2}}, (\frac{k-1}{2})^{\frac{k-1}{2}}\right)$.

This paper is organised as follows: In the next section, we will present some auxiliary results that will be utilised in the third section to prove the main results. In the last section we compare these novel lower bounds with bounds established by Ghebleh for classes $\mathcal{A}(2k,k)$. We also discuss the methods used to derive the bounds and explain why these new bounds are superior to those of Ghebleh.

2 Auxiliary Results

As previously stated, the symmetric group is naturally identified with the class $\mathcal{A}(n, 1)$ of permutation matrices of order n. A permutation inversion corresponds in this sense to a pair of ones in the corresponding permutation matrix P, one to the top-right of the other. As a result, an inversion in P consists of two ones in the entries (i, j) and (k, l), such that $(i-k)(j-l) < 0$. Ghebleh adopted the same definition of inversion for any (0, 1)-matrix.

Definition 1 [14] Let $A = [a_{i,j}]$ be a (0, 1)-matrix. In A, an inversion is a pair of entries $a_{i,j}$ and $a_{k,l}$, both of which equal to 1, satisfying $(i-k)(j-l) < 0$. We denote the total number of inversions in A by $\nu(A)$.

Example 1 Let

$$A = [a_{i,j}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix has four inversions: $(a_{1,3}, a_{2,1})$, $(a_{1,3}, a_{2,2})$, $(a_{1,3}, a_{3,2})$, and $(a_{3,4}, a_{4,3})$. Hence $\nu(A) = 4$.

The next result states the monotonicity of the number of inversions with respect to the Bruhat order in $\mathcal{A}(R, S)$:

Theorem 6 [11] *Let $A, C \in \mathcal{A}(R, S)$ such that $A \preccurlyeq_B C$, $A \neq C$. Then $\nu(A) < \nu(C)$.*

Remark 7 From this Theorem, we can conclude that if $\nu(A) = \nu(C)$ for two distinct matrices A and C in $\mathcal{A}(n, k)$, then A and C are incomparable in the Bruhat order. Therefore, the set $\nu^{-1}(t)$ of all matrices $A \in \mathcal{A}(n, k)$ with $\nu(A) = t$ is an antichain in the Bruhat order of $\mathcal{A}(n, k)$.

Given a matrix A, the matrix obtained by flipping the columns of A in the right/left direction is referred to as the conjugate of A, and it is denoted by $\overline{A}$. Namely, if $A = [a_{i,j}]$ is an $m \times n$ matrix, then $\overline{A}$ is the matrix $B = [b_{i,j}]$, where $b_{i,j} = a_{i,n-j+1}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. A matrix A is said to be *self-conjugate* if $A = \overline{A}$.

The following Lemma is an elementary application of the inclusion-exclusion principle.

Lemma 1 [15] *Let $A \in \mathcal{A}(R, S)$, with $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$.Then*

$$\nu(A) + \nu(\overline{A}) = \binom{r_1 + r_2 + \ldots + r_m}{2} - \sum_{i=1}^{m} \binom{r_i}{2} - \sum_{j=1}^{n} \binom{s_j}{2}.$$

If $A \in \mathcal{A}(n, 2)$ is self-conjugate, then we get $\nu(A) = n^2 - \frac{3n}{2}$.

From this equality, we conclude that if n is odd, then there are no self-conjugate matrix in the class $\mathcal{A}(n, 2)$, but for n even, self-conjugate matrices were used in [15] to construct antichains.

In the next section we are going to build antichains in $\mathcal{A}(n, k)$ using the same ideas. To begin, if $A \in \mathcal{A}(n, k)$, then, by Lemma 1,

$$\nu(A) + \nu(\overline{A}) = \binom{kn}{2} - 2n\binom{k}{2},$$

and hence, if A is self-conjugated, then

$$\nu(A) = \frac{(kn)^2}{4} - \frac{2nk^2 - nk}{4}. \tag{1}$$

As a result, if n and k are odd, then $\mathcal{A}(n, k)$ contains no self-conjugate matrices. We also conclude that all self-conjugate matrices have the same number of inversions and thus, they form an antichain in the Bruhat order of $\mathcal{A}(n, k)$.

Another difficult issue with classes $\mathcal{A}(R, S)$ is determining the precise number of its elements. The exact number of elements in $\mathcal{A}(R, S)$ is represented by $|\mathcal{A}(R, S)|$. As Ryser predicted, "the exact number of them is undoubtedly an extremely intricate function of the row and column sums". Formulas that provide the exact number of (0, 1)-matrices in $\mathcal{A}(R, S)$ can be found in [20] or in [18]. A lower bound for the cardinality of the class $\mathcal{A}(n, k)$ was presented in [21]:

Proposition 8 [21] *Let n and k be two positive integers such that $k \leq n$. Then*

$$|\mathcal{A}(n, k)| \geq \frac{(n!)^k}{(k!)^n}.$$

This section concludes with the following result:

Proposition 9 *Let t and l be two positive integers such that $l \geq t$. Let $R = (t^{l-t}, (t-1)^t)$ and let $R' = ((t+1)^{l-t}, t^t)$. Then $\mathcal{A}(R, R) \neq \emptyset$, and if $l > t$, then $\mathcal{A}(R', R') \neq \emptyset$.*

Proof If $R = (t^{l-t}, (t-1)^t)$, then $R^* = (l^{t-1}, l-t)$. Since $l \geq t$, we conclude that $R \preccurlyeq R^*$, and according to Theorem 1, $\mathcal{A}(R, R) \neq \emptyset$. The proof that $\mathcal{A}(R', R') \neq \emptyset$ is similar. □

3 Proof of Theorems 4 and 5

This section is primarily concerned with the proofs of Theorems 4 and 5. As previously stated, a set of matrices in $\mathcal{A}(n, k)$ with an equal number of inversions forms an antichain in $\mathcal{A}(n, k)$ with the Bruhat order. We present antichain constructions in the Bruhat order of $\mathcal{A}(n, k)$, which are made up of matrices with a fixed number of inversions.

Proof of Theorem 4: We divide the proof into the two cases of the statement:

Case 1: *k is even.*

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} S & \overline{S} \\ T & \overline{T} \end{bmatrix} : S, T \in \mathcal{A}\left(\frac{n}{2}, \frac{k}{2}\right) \right\}.$$

Then, a matrix

$$A = \begin{bmatrix} S & \overline{S} \\ T & \overline{T} \end{bmatrix} \in \mathcal{B},$$

is a self-conjugated matrix of $\mathcal{A}(n, k)$. Therefore, by Eq. 1,

$$\nu(A) = \frac{(kn)^2}{4} - \frac{2nk^2 - nk}{4}.$$

We showed how to generate self-conjugated matrices of $\mathcal{A}(n, k)$ from any pair of matrices $S, T \in \mathcal{A}\left(\frac{n}{2}, \frac{k}{2}\right)$. A set of self-conjugate matrices, as previously demonstrated, is an antichain of $\mathcal{A}(n, k)$. Therefore,

$$w(\mathcal{A}(n, k)) \geq \left|\mathcal{A}\left(\frac{n}{2}, \frac{k}{2}\right)\right|^2.$$

Case 2: *k is odd.*
Let

$$\mathcal{B} = \left\{ \begin{bmatrix} C & \overline{S} \\ S & \overline{C} \end{bmatrix} : C \in \mathcal{A}\left(\frac{n}{2}, \frac{k-1}{2}\right) \text{ and } S \in \mathcal{A}\left(\frac{n}{2}, \frac{k+1}{2}\right) \right\}.$$

We will show that all matrices in $\mathcal{B}$ have the same number of inversions. Let $A \in \mathcal{B}$. Then, there are matrices $C \in \mathcal{A}\left(\frac{n}{2}, \frac{k-1}{2}\right)$ and $S \in \mathcal{A}\left(\frac{n}{2}, \frac{k+1}{2}\right)$ such that

$$A = \begin{bmatrix} C & \overline{S} \\ S & \overline{C} \end{bmatrix}.$$

To count the number of inversions of A first notice that $\nu(C) + \nu(\overline{C})$ is constant for any C as it happens with $\nu(S) + \nu(\overline{S})$.
Any nonzero entry of C does not result in an inversion when combined with any nonzero entry of $\overline{C}$. Any nonzero entry of S produces inversion with all nonzero entries of $\overline{S}$, resulting in more $(\frac{n(k+1)}{4})^2$ inversions.
The nonzero entries in the ith row of $\overline{S}$ produce inversions when combined with all the nonzero entries of C that lie in the rows $i+1, \ldots, \frac{n}{2}$. Since each row of $\overline{S}$ has $\frac{k+1}{2}$ nonzero entries and each row of C has $\frac{k-1}{2}$ nonzero entries, we have more

$$\sum_{i=1}^{\frac{n}{2}} \frac{(k-1)(k+1)}{4}\left(\frac{n}{2} - i\right)$$

inversions. Observe that this number is independent of the choice of S and C.
The nonzero entries in the jth column of $\overline{S}$, produce inversions when combined with any nonzero entries of $\overline{C}$ that lie in the columns $1, \ldots, j-1$. Since each column of S has $\frac{k+1}{2}$ nonzero entries and each column of C has $\frac{k-1}{2}$ nonzero entries, these entries produce

$$\sum_{i=1}^{\frac{n}{2}} \frac{(k-1)(k+1)}{4}(i-1).$$

inversions.
The nonzero entries in the ith row of S produce inversions with the nonzero entries of $\overline{C}$ that lie in rows $1, \ldots, i-1$, and the nonzero entries in the jth column of S produce inversions with the nonzero entries of C that lie in columns $j+1, \ldots, \frac{n}{2}$. This results in

$$\left(\sum_{i=1}^{\frac{n}{2}} \frac{(k-1)(k+1)}{4}\left(\frac{n}{2} - i\right)\right) + \left(\sum_{i=1}^{\frac{n}{2}} \frac{(k-1)(k+1)}{4}(i-1)\right)$$

additional inversions. Therefore the total number of inversions of A is

$$\begin{aligned}\nu(A) = \nu(C) + \nu(\overline{C}) + \nu(S) + \nu(\overline{S}) + (\tfrac{n(k+1)}{4})^2 + \\ 2\left(\sum_{i=1}^{\frac{n}{2}} \tfrac{(k-1)(k+1)}{4}(\tfrac{n}{2} - i) + \sum_{i=1}^{\frac{n}{2}} \tfrac{(k-1)(k+1)}{4}(i-1)\right).\end{aligned}$$

We demonstrated how to construct a family of matrices on $\mathcal{A}(n,k)$ with the same number of inversions, using any $S \in \mathcal{A}\left(\frac{n}{2}, \frac{k+1}{2}\right)$ and $C \in \mathcal{A}\left(\frac{n}{2}, \frac{k-1}{2}\right)$. Consequently, this set is an antichain of $\mathcal{A}(n,k)$, and hence,

$$w(n,k) \geq \left|\mathcal{A}\left(\frac{n}{2}, \frac{k-1}{2}\right)\right| \cdot \left|\mathcal{A}\left(\frac{n}{2}, \frac{k+1}{2}\right)\right|.$$

Proof of Theorem 5: We divide the proof into the two cases of the statement:

Case 1: *k is even.*

Let l and t be two positive integers such that $n = 2l + 1$ and $k = 2t$. Since $n > k$ we have $l \geq t$. Let $R = (t^{l-t}, (t-1)^t)$. According to Proposition 9, $\mathcal{A}(R, R) \neq \emptyset$. Let $L = [0 \ldots 0\, 1 \ldots 1]$ be the row matrix with l entries, t of which are 1 and the rest are 0. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} C & L^T & T \\ \overline{L} & 0 & L \\ \overline{T} & L^T & \overline{C} \end{bmatrix} : C \in \mathcal{A}(l, t) \quad \text{and} \quad T \in \mathcal{A}(R, R) \right\}.$$

Then $\mathcal{B} \subseteq \mathcal{A}(n, k)$. We will prove that all matrices in $\mathcal{B}$ have the same number of inversions which implies that $\mathcal{B}$ is an antichain in $\mathcal{A}(n, k)$. Let $A \in \mathcal{B}$. Then there exits a matrix $C \in \mathcal{A}(l, t)$ and a matrix $T \in \mathcal{A}(R, R)$ such that

$$A = \begin{bmatrix} C & L_1 & T \\ L_4 & 0 & L_2 \\ \overline{T} & L_3 & \overline{C} \end{bmatrix},$$

where $L_1 = L_3 = L^T$, $L_2 = L$ and $L_4 = \overline{L}$.
First, A has $\nu(T) + \nu(\overline{T}) + \nu(C) + \nu(\overline{C})$ inversions in the submatrices $C, \overline{C}, T$ and $\overline{T}$. According to Lemma 1,

$$\nu(T) + \nu(\overline{T}) + \nu(C) + \nu(\overline{C}) = \binom{lt}{2} + \binom{lt-t}{2} - 4l\binom{t}{2} + 2t\left(\binom{t}{2} - \binom{t-1}{2}\right). \tag{2}$$

This summation has the same value for any $C \in \mathcal{A}(l, t)$ and any $T \in \mathcal{A}(R, R)$.
The nonzero entries of L_1 produce inversions when combined with the nonzero entries of L_4. The identical is true for the nonzero entries of L_2 and L_3. Therefore, these entries result in a total of

$$2t^2 \tag{3}$$

inversions.
Each nonzero entry of T, when combined with all nonzero entries of $\overline{T}$ and all nonzero entries of L_4 and L_3, results in an inversion. Because matrices T and $\overline{T}$ have $(l-t)t + (t-1)t = lt - t$ nonzero entries and matrices L_4 and L_3 have t nonzero entries, a total of

$$(lt-t)^2 + 2t(lt-t) = (lt-t)(lt-t+2t) = (lt-t)(lt+t) \tag{4}$$

inversions are produced by these entries.
The nonzero entries in the ith row of T produce inversions with the nonzero entries of C and with the nonzero entries of L_1 that lie in the rows $i+1$ to l. If $i \in \{1, \ldots, l-t\}$, then the ith row of T has t entries equal to 1, that produces a total of

$$\sum_{k=i+1}^{l-t} t^2 + \sum_{k=l-t+1}^{l} t(t+1)$$

inversions when combined with the nonzero entries of C and L_1. If $i \in \{l-t+1,\dots,l\}$, then the ith row of T has $t-1$ entries equal to 1 that produces a total of

$$\sum_{k=i+1}^{l}(t-1)(t+1)$$

inversions when combined with the nonzero entries of C and L_1. Therefore, the nonzero entries of T produce a total of

$$\sum_{i=1}^{l-t}\left(\sum_{k=i+1}^{l-t}t^2+\sum_{k=l-t+1}^{l}t(t+1)\right)+\sum_{i=l-t+1}^{l}\left(\sum_{k=i+1}^{l}(t-1)(t+1)\right) \quad (5)$$

inversions when combined with the nonzero entries of C and L_1.
The nonzero entries in the jth column of T produce inversions with the nonzero entries of $\overline{C}$ and with the nonzero entries of L_2 that lie in the columns 1 to $j-1$. If $j \in \{1,\dots,l-t\}$, then the jth column of T has t entries equal to 1 that produces

$$\sum_{k=1}^{j-1}t^2$$

inversions when combined with the nonzero entries of $\overline{C}$. If $j \in \{l-t+1,\dots,l\}$, then the jth column of T has $t-1$ entries equal to 1 that produces

$$\sum_{k=1}^{l-t}(t-1)t+\sum_{k=l-t+1}^{j-1}(t-1)(t+1)$$

inversions when combined with the nonzero entries of $\overline{C}$ and L_2. Therefore, the nonzero entries of T produce a total of

$$\sum_{j=1}^{l-t}\left(\sum_{k=1}^{j-1}t^2\right)+\sum_{j=l-t+1}^{l}\left(\sum_{k=1}^{l-t}(t-1)t+\sum_{k=l-t+1}^{j-1}(t-1)(t+1)\right) \quad (6)$$

inversions when combined with the nonzero entries of $\overline{C}$ and L_2.
Each nonzero entry of $\overline{T}$ produce inversions when combined with the nonzero entries of L_1 and L_2 resulting in more

$$2(lt-t)t \quad (7)$$

inversions.
The nonzero entries in the ith row of $\overline{T}$ also produce inversions with the nonzero entries of $\overline{C}$ and L_3 that lie in the rows 1 to $i-1$. If $i \in \{1,\dots,l-t\}$, then the ith row of $\overline{T}$ has t entries equal to 1 that produces a total of

$$\sum_{k=1}^{i-1}t^2$$

inversions when combined with the nonzero entries $\overline{C}$. If $i \in \{l-t+1,\dots,l\}$, then the ith row of $\overline{T}$ has $t-1$ entries equal to 1 that produces a total of

$$\sum_{k=1}^{l-t}(t-1)t+\sum_{k=l-t+1}^{i-1}(t-1)(t+1)$$

inversions when combined with the nonzero entries of $\overline{C}$ and L_3. Therefore, the nonzero entries of $\overline{T}$ produce a total of

$$\sum_{i=1}^{l-t}\left(\sum_{k=1}^{i-1} t^2\right) + \sum_{i=l-t+1}^{l}\left(\sum_{k=1}^{l-t}(t-1)t + \sum_{k=l-t+1}^{i-1}(t-1)(t+1)\right) \tag{8}$$

inversions when combined with the nonzero entries of $\overline{C}$ and L_3.
The nonzero entries in the jth column of $\overline{T}$ also produce inversions when combined with the nonzero entries of C and with the nonzero entries of L_4 that lie in the columns $j+1$ to l. These entries give rise a total of

$$\sum_{j=1}^{l-t}\left(\sum_{k=j+1}^{l-t}(t-1)(t+1) + \sum_{k=l-t+1}^{l}(t-1)t\right) + \sum_{j=l-t+1}^{l}\left(\sum_{k=j+1}^{l} t^2\right) \tag{9}$$

inversions.
Hence, by adding Eqs. 2, 3, 4, 5, 6, 7, 8, and 9 we obtain the total number of inversions of A, which has the same value regardless the choice of $C \in \mathcal{A}(l,t)$ and $T \in \mathcal{A}(R,R)$ due to the fact that it only depends on l and t. Therefore,

$$w(n,k) \geq |\mathcal{B}| = |\mathcal{A}(l,t)| \,.\, |\mathcal{A}(R,R)| = \left|\mathcal{A}\left(\frac{n-1}{2}, \frac{k}{2}\right)\right| . \,|\mathcal{A}(R,R)| \,.$$

Case 2: *k is odd.*
Let l and t be two positive integers such that $n = 2l+1$ and $k = 2t+1$. Since $n > k$ we have $l > t$. Let $R' = ((t+1)^{l-t}, t^t)$. According to Proposition 9, $\mathcal{A}(R', R') \neq \emptyset$. Let $L = [0\ldots0\,1\ldots1]$ be the row matrix with l entries, t of which are 1 and the rest are 0. Let

$$\mathcal{B}' = \left\{ \begin{bmatrix} C & L^T & T \\ \overline{L} & 1 & L \\ \overline{T} & L^T & \overline{C} \end{bmatrix} : C \in \mathcal{A}(l,t) \quad \text{and} \quad T \in \mathcal{A}(R', R') \right\}.$$

Then $\mathcal{B} \subseteq \mathcal{A}(n,k)$. Proceeding as in the previous case we conclude that $\mathcal{B}'$ is an antichain of $\mathcal{A}(n,k)$. Therefore,

$$w(n,k) \geq \left|\mathcal{B}'\right| = |\mathcal{A}(l,t)| \,.\, \left|\mathcal{A}(R', R')\right| = \left|\mathcal{A}\left(\frac{n-1}{2}, \frac{k-1}{2}\right)\right| . \left|\mathcal{A}(R', R')\right| .$$

Using Theorem 4 and Proposition 8 we can state the following result:

Corollary 1 *Let n and k be two positive integers with $k < n$ and n an even integer. Then,*

$$w(n,k) \geq \begin{cases} \dfrac{(\frac{n}{2}!)^k}{(\frac{k}{2}!)^n}, & \text{if } k \text{ is even} \\[2ex] \dfrac{(\frac{n}{2}!)^k}{((\frac{k-1}{2}!)(\frac{k+1}{2}!))^{\frac{n}{2}}}, & \text{if } k \text{ is odd} \end{cases} .$$

We may use this corollary in the next section to compare these new bounds to the bounds obtained by Ghebleh in Theorem 3. Such a comparison is obviously only possible in the classes $\mathcal{A}(2k,k)$.

4 Concluding Remarks

We present antichain constructions in the Bruhat order of the classes $\mathcal{A}(n, k)$, where n and k are positive integers. The main tool in our constructions of antichains in the Bruhat order of $\mathcal{A}(n, k)$ is the fact that if two matrices in this class have the same number of inversions, then they are incomparable in the Bruhat order.

It is important to compare these new lower bounds for the width of $\mathcal{A}(2k, k)$ with those obtained by Ghebleh in Theorem 3. Examining the proofs of Ghebleh, we observe that the construction of an antichain in $\mathcal{A}(2k, k)$ just requires an antichain D in the Bruhat order of $\mathcal{A}(k, 2)$. Theorem 2 determines a lower bound for $|D|$, and Ghebleh proves that $A(2k, k)$ has an antichain of length $|D|^4$, which leads to Theorem 3. In our approach, we do not employ antichains of any class to find a lower bound for the width of $\mathcal{A}(2k, k)$. Instead, we use the entire class $\mathcal{A}\left(k, \frac{k}{2}\right)$ if k is even or the classes $\mathcal{A}\left(k, \frac{k-1}{2}\right)$ and $\mathcal{A}\left(k, \frac{k+1}{2}\right)$ if k is odd. This allows to improve the Ghebleh bounds in most situations as we can see in the next example:

Example 2 In this example, we present lower bounds for the width of $\mathcal{A}(16, 8)$ and $\mathcal{A}(14, 7)$. Using Theorem 4 we get

$$w(16, 8) \geq \frac{(8!)^8}{(4!)^{16}} = 576\,480\,100\,000\,000,$$

and

$$w(14, 7) \geq \frac{(7!)^7}{(3! \times 4!)^7} = 64\,339\,296\,875.$$

If we apply Theorem 3, then we obtain

$$w(16, 8) \geq \frac{(8!)^4}{(4)^8} = 40\,327\,580\,160\,000,$$

and

$$w(14, 7) \geq \frac{(6!)^4}{(4)^3} = 1\,049\,760\,000,$$

so in these cases our result outperforms Ghebleh's.

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Declarations

Conflict of Interests The author does not have any conflict of interest to declare.

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