Order (2023) 40:455–468
https://doi.org/10.1007/s11083-022-09606-6

Order-Preserving Self-Maps of Complete Lattices

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Received: 20 October 2021 / Accepted: 24 May 2022 / Published online: 13 February 2023

Abstract
We study isotone self-maps of complete lattices and their fixed point sets, which are complete lattices contained as suborders, but not necessarily as subsemilattices. We develop a representation of such maps by means of relations and show how to navigate their fixed point lattices using a modification of the standard NEXT CLOSURE algorithm. Our approach is inspired by early work of Shmuely [8] and Crapo [1]. We improve and substantially extend our earlier publication [4].

Keywords Complete lattice · Order-preserving · Isotone · Fixed point

1 Introduction

The interest in *concept lattices* [6] has stimulated the creation of algorithms for generating lattices, and the availability of fast algorithms may conversely have contributed to the popularity of concept lattices. Moreover, concept lattices have easy representations either by a binary relation or by a set of implications, both of which can conveniently be used as input for the algorithms.

Although all complete lattices are isomorphic to concept lattices, they sometimes come in a form for which the above-mentioned algorithms are not easy to apply. There are, for example, many families of sets which form complete lattices when ordered by the subset relation $\subseteq$, but are neither closure nor kernel systems. We provide a relational representation for such lattices and adapt one of the standard algorithms accordingly.

Throughout the paper, $(L,\leq)$ will be some abstract complete lattice. The supremum and infimum of a subset $S \subseteq L$ will be denoted by $\bigvee S$ and $\bigwedge S$. The reader may assume, without much loss of generality, that $(L,\leq)$ is a powerset lattice $(\mathfrak{P}(M), \subseteq)$. We use the abstract setting because we find it more transparent.

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2 Representing relations

A mapping $\varphi : P \to Q$ between two ordered sets $(P,\leq)$ and $(Q,\leq)$ is called **order-preserving**[1] if $x \leq y$ always implies $\varphi(x) \leq \varphi(y)$.

Let $(P,\leq)$ be an ordered set and let $(L,\leq)$ be a complete lattice. An easy way to give examples of order-preserving maps $\varphi : P \longrightarrow L$ is to choose an arbitrary relation $\rhd \subseteq P \times L$ between P and L and to define for all $v \in P$

$$\varphi_{\rhd}(v) := \bigvee\{y \mid (x,y) \in \rhd, x \leq v\}.$$

$\varphi_{\rhd}$ obviously is order-preserving. It is also evident that conversely every order-preserving map $\varphi : (P,\leq) \longrightarrow (L,\leq)$ can be obtained via such a **representing relation**, since

$$\rhd_{\varphi} := \{(p,\varphi(p)) \mid p \in P\}$$

is a trivial solution.

For a more compact representation define $p \in P$ to be a **$\bigvee$-proper preimage** for φ if

$$\varphi(p) \neq \bigvee\{\varphi(v) \mid v < p\}$$

and denote the set of all such $\bigvee$-proper preimages by P_φ. We say that φ is **represented on its $\bigvee$-proper preimages** if

$$\varphi(v) = \bigvee\{\varphi(p) \mid p \in P_{\varphi}, p \leq v\} \tag{1}$$

holds for all $v \in P$. This is not always the case, a generalized finiteness condition is needed. The **descending chain condition (dcc)** requires that every non-empty subset has a minimal element.

Proposition 1 *Let $(P,\leq)$ be an ordered set and let $(L,\leq)$ be a non-trivial complete lattice. The following conditions are equivalent:*

1. *$(P,\leq)$ satisfies the dcc.*
2. *Every order-preserving mapping $\varphi : (P,\leq) \to (L,\leq)$ is represented on its $\bigvee$-proper preimages.*

Proof If there was an element violating equation (1) in an ordered set $(P,\leq)$ satisfying the dcc, then there also is a minimal such violating element, say, v. Every $w < v$ fulfills the equation

$$\varphi(w) = \bigvee\{\varphi(p) \mid p \in P_{\varphi}, p \leq w\}.$$

A violating element cannot be a $\bigvee$-proper preimage, and thus

$$\begin{aligned}\varphi(v) &= \bigvee\{\varphi(w) \mid w < v\}\\ &= \bigvee\{\bigvee\{\varphi(p) \mid p \in P_{\varphi}, p \leq w\} \mid w < v\}\\ &= \bigvee\{\varphi(p) \mid p \in P_{\varphi}, p \leq v\},\end{aligned}$$

showing that v does not violate the equation, a contradiction.

[1] Synonyms are **isotone** and **monotone**

For the converse assume that $(P,\leq)$ contains a non-empty subset without a minimal element. The order filter F generated by this subset has no minimal elements either. Mapping F to the largest and $P \setminus F$ to the smallest element of $(L,\leq)$ yields a mapping that is order-preserving, but has no $\bigvee$-proper preimages at all, since each $\bigvee$-proper preimage would be minimal in F. □

An interesting question is how to find the $\bigvee$-proper preimages for a given order-preserving mapping φ. This is easy when a representing relation $\triangleright$ of reasonable size is given, as the following proposition shows.

Proposition 2 *If $\triangleright$ is a representing relation for $\varphi : (P,\leq) \to (L,\leq)$, then the $\bigvee$-proper preimages of φ are precisely the elements $p \in P$ for which there esists a pair $(p, w) \in \triangleright$ such that*

$$w \nleq \bigvee\{u \mid (q,u) \in \triangleright, q < p\}.$$

Proof According to the definition, p is a $\bigvee$-proper preimage for φ iff

$$\varphi(p) \neq \bigvee\{\varphi(v) \mid v < p\}.$$

Since $\triangleright$ is a representing relation for φ, we can rewrite this condition to

$$\varphi(p) \neq \bigvee\{\bigvee\{u \mid (q,u) \in \triangleright, q \leq v\} \mid v < p\} = \bigvee\{u \mid (q,u) \in \triangleright, q < p\}.$$

Comparing this to

$$\varphi(p) = \bigvee\{u \mid (q,u) \in \triangleright, q \leq p\}$$

yields the claim of the proposition. □

Corollary 1 *If φ is represented on its $\bigvee$-proper preimages, then that representation is of minimum size.*

Proposition 2 also allows to check if two relations represent the same order-preserving mapping (assuming dcc): they must have the same $\bigvee$-proper preimages and produce the same images of these.

Example 1 Let $(P,\leq)$ and $(L,\leq)$ both be equal to the powerset lattice of the three-element set $\{a,b,c\}$. We give three examples E_1, E_2, and E_3 in terms of representing relations $\triangleright$, which we write in infix notation.

- Example E_1 is represented by the relation $\triangleright$ given as

$$\{a\} \triangleright \{a,b,c\}, \ \{b\} \triangleright \{b,c\}, \ \{c\} \triangleright \{b,c\}.$$

The resulting order-preserving map $\varphi_\triangleright$ is as follows:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\varphi_\triangleright(x)$	$\emptyset$	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$

$\{a\}$, $\{b\}$, and $\{c\}$ are indeed the $\bigvee$-proper preimages.

- Example E_2 is represented by the relation $\triangleright$ given as

$$\{a\} \triangleright \{a,b\},\ \{a,b,c\} \triangleright \{a,b,c\}.$$

$\varphi_\triangleright$ has the following values:

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\varphi_\triangleright(x)$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\{a,b\}$	$\emptyset$	$\{a,b,c\}$

$\{a\}$ and $\{a,b,c\}$ are the $\bigvee$-proper preimages.

- For example E_3 we use the powerset lattice of $\{a,b\}$ with

$$\emptyset \triangleright \{a\},\ \{a\} \triangleright \{a,b\},\ \{b\} \triangleright \{b\}$$

and obtain

x	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\varphi_\triangleright(x)$	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

The $\bigvee$-proper preimages are $\emptyset$, $\{a\}$, and $\{b\}$.

E_1, E_2, and E_3 will be used as separating examples in Theorem 1 below.

Proposition 3 *All proper preimages of a $\bigvee$-preserving map $\varphi : L_1 \to L_2$ between complete lattices $(L_1,\leq)$ and $(L_2,\leq)$ are $\bigvee$-irreducible.*

Proof Let $p \in V$ be a proper preimage for φ. Then $\varphi(p) \neq \bigvee\{\varphi(q) \mid q < p\}$ and thus, since φ is $\bigvee$-preserving, $\varphi(p) \neq \varphi(\bigvee\{q \mid q < p\})$. As a consequence we get that $p \neq \bigvee\{q \mid q < p\}$, which shows that p must be $\bigvee$-irreducible. □

3 Order-preserving self-maps

Definition 1 Let $(L,\leq)$ be a complete lattice. An order-preserving mapping $\varphi : L \to L$ is called

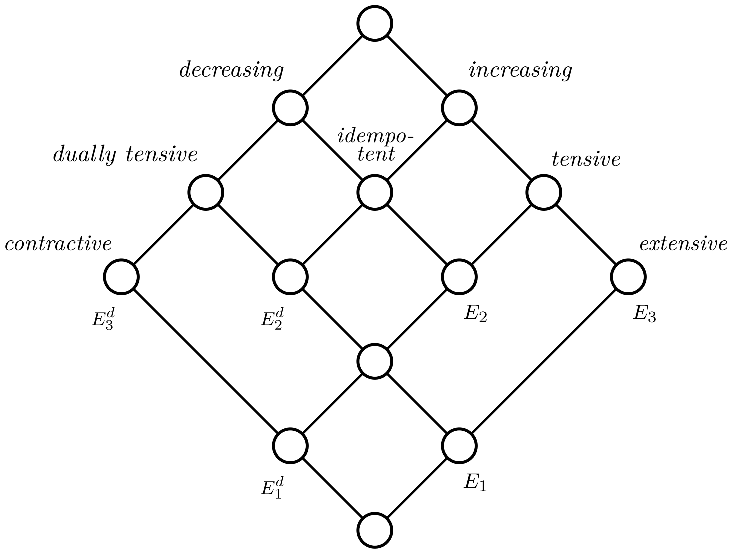

Fig. 1 The result of an attribute exploration [5] for order-preserving self-maps, see Theorem 1

idempotent	if	$\varphi(x) = \varphi(\varphi(x))$	for all $x \in L$,
extensive	if	$x \leq \varphi(x)$	for all $x \in L$,
contractive[2]	if	$x \geq \varphi(x)$	for all $x \in L$,
tensive	if	$\varphi(x) = \varphi(x \wedge \varphi(x))$	for all $x \in L$,
increasing[3]	if	$\varphi(x) \leq \varphi(\varphi(x))$	for all $x \in L$, and
decreasing	if	$\varphi(x) \geq \varphi(\varphi(x))$	for all $x \in L$.

Theorem 1 *Figure 1 shows the logical hierarchy of the properties given in Definition 1. In particular, if $\varphi : L \to L$ is order-preserving, then the following statements hold (as well as their duals):*

1. *If φ is extensive, then φ is tensive.*
2. *If φ is tensive, then φ is increasing.*
3. *φ is idempotent iff φ is both increasing and decreasing.*
4. *If φ is idempotent and extensive, then φ is dually tensive.*

Moreover, there are examples of order-preserving mappings falsifying other implications, as indicated in the diagram.[4]

[2] A synonym is **intensive**

[3] Following Shmuely [8]. Tarski [9] uses “increasing” in the sense of “order-preserving”

[4] Assertions 1., 3. and 4. hold even without assuming that φ is order preserving, as one of the reviewers has pointed out.

Proof 1) If $x \leq \varphi(x)$, then $x \wedge \varphi(x) = x$ and thus $\varphi(x \wedge \varphi(x)) = \varphi(x)$. 2) From $x \wedge \varphi(x) \leq \varphi(x)$ we infer $\varphi(x) = \varphi(x \wedge \varphi(x)) \leq \varphi(\varphi(x))$. 3) is obvious. 4) If $x \leq \varphi(x)$ then $\varphi(x \vee \varphi(x)) = \varphi(\varphi(x)) = \varphi(x)$.

The separating examples E_1, E_2, E_3 are defined in Example 1. E_1^d, E_2^d, E_3^d are dual to E_1, E_2, E_3. □

In Example 1 above we have used infix notation for the relation $\rhd$, writing $u \rhd v$ instead of $(u, v) \in \rhd$. Note that $u \rhd v$ always implies $v \leq \varphi_{\rhd}(u)$, but that the converse does not hold in general. It will be convenient to have a short notation for this case as well. We therefore define

$$u \blacktriangleright v :\Longleftrightarrow v \leq \varphi_{\rhd}(u)$$

and keep in mind that $u \rhd v$ implies $u \blacktriangleright v$.

We say that $\rhd$ **entails** a relation $R \subseteq L \times L$, when $\rhd$ and $\rhd \cup R$ represent the same mapping.

Proposition 4 *$\rhd$ entails R if and only if $\blacktriangleright$ holds for every (r,s) $\in$ R.*

Proof For arbitrary $x \in L$ we have

$$\varphi_{\rhd \cup R}(x) = \varphi_{\rhd}(x) \vee \bigvee \{s \mid (r, s) \in R, r \leq x\}.$$

The two images are the same if

$$\bigvee \{s \mid (r, s) \in R, r \leq x\} \leq \varphi_{\rhd}(x),$$

and this holds if for all $(r,s) \in R$

$$r \leq x \implies s \leq \varphi_{\rhd}(x).$$

But $\blacktriangleright$ and $r \leq x$ imply $s \leq \varphi_{\rhd}(r) \leq \varphi_{\rhd}(x)$, showing that the condition is sufficient. It is also necessary, because if $s \nleq \varphi_{\rhd}(r)$ for some $(r,s) \in R$, then $\varphi_{\rhd}$ and $\varphi_{\rhd \cup R}$ are different since $s \leq \varphi_{\rhd \cup R}(r)$. □

Proposition 5 *The order-preserving mapping $\varphi_{\rhd}$ represented by $\rhd \subseteq L \times L$ is*

1. *extensive iff $x \blacktriangleright x$ holds for all $x \in L$,*
2. *increasing iff $\varphi_{\rhd}(x) \blacktriangleright \varphi_{\rhd}(x)$ holds for all $x \in L$, and*
3. *decreasing iff $u \blacktriangleright v$ and $v \rhd w$ together always imply $u \blacktriangleright w$.*

Proof The first two claims are immediate from the definitions of being extensive ($x \leq \varphi_{\rhd}(x)$) or increasing ($\varphi_{\rhd}(x) \leq \varphi_{\rhd}(\varphi_{\rhd}(x))$). The third requires a few more words: The condition of being decreasing is $\varphi_{\rhd}(\varphi_{\rhd}(x)) \leq \varphi_{\rhd}(x)$, which obviously is equivalent to

$$v \leq \varphi_{\rhd}(u), v \rhd w \implies w \leq \varphi_{\rhd}(u).$$

This is, up to notation, exactly the condition given in the proposition. □

We say that $\rhd \subseteq L \times L$ is an **upward** relation iff

$$x \rhd y \implies x \leq y.$$

Proposition 6 *Let (L,≤) be a complete lattice and $\varphi : L \to L$ be order-preserving. φ has an upward representing relation if and only if φ is tensive.*

Proof If v,x,y are elements of L with $x \leq v$, $x \leq v$, $x \rhd y$,, and $x \leq y$, then $x \leq y \leq \varphi(v)$ and thus $x \leq v \wedge \varphi(v)$. If $\rhd$ is upward, then $x \rhd y$ always implies $x \leq y$, and thus

$$\{x \mid x \leq v, \exists_y\, x \rhd y\} = \{x \mid x \leq v \wedge \varphi(v), \exists_y\, x \rhd y\}.$$

A consequence is that $\varphi(v) = \varphi(v \wedge \varphi(v))$ holds for all $v \in L$, i.e., φ is tensive. For the converse we prove that if φ is tensive, then the relation

$$\rhd := \{(x \wedge \varphi(x), \varphi(x)) \mid x \in L\},$$

which obviously is upward, represents φ. Recall that for arbitrary $y \in L$

$$\varphi_{\rhd}(y) = \bigvee\{\varphi(x) \mid x \wedge \varphi(x) \leq y\}.$$

From $y \wedge \varphi(y) \leq y$ and $y \wedge \varphi(y) \rhd \varphi(y)$ we conclude that $\varphi_{\rhd}(y) \geq \varphi(y)$. If $x \wedge \varphi(x) \leq y$ holds for some x, then $\varphi(x) = \varphi(x \wedge \varphi(x)) \leq \varphi(y)$ (since φ is order-preserving and tensive), and thus $\varphi_{\rhd}(y) \leq \varphi(y)$. This proves $\varphi_{\rhd} = \varphi$. □

4 Closed and fixed points

Definition 2 If $\varphi : L \to L$ is a mapping and $x \in L$, then we say that x is a **fixed** point of φ iff $\varphi(x) = x$, and that x is a **closed** point of φ iff $\varphi(x) \leq x$.

Proposition 7 *Every fixed point is closed. If φ is order-preserving and increasing, and x is closed, then $\varphi(x)$ is fixed.*

Proof The first statement is obvious. Suppose that x is closed, i.e., that $x \geq \varphi(x)$. Then $\varphi(x) \geq \varphi(\varphi(x)) \geq \varphi(x)$, when φ is order-preserving and increasing. We conclude that $\varphi(x) = \varphi(\varphi(x))$ and thus $\varphi(x)$ is fixed. □

The proposition may suggest a pairing between fixed and closed elements. But note for example that when φ is the function which maps everything to the least element of (L,≤), then *every* element of (L,≤) is closed, but *only the least* element is fixed.

A function that is both idempotent and order-preserving is called a **closure operator** on (L,≤) if it is extensive, and is a **kernel operator** if contractive. The set of fixed points of a closure operator is called a **closure system**. It is well known that the closure systems are precisely the $\bigwedge$-subsemilattices. Each complete meet-subsemilattice of a complete lattice is itself a complete lattice, because the join operation can be expressed in terms of the meet operation: the join of a subset S equals the meet of all upper bounds of S. However, this join operation usually is not identical with the join in the original complete lattice. The meet-subsemilattice therefore is a complete lattice, but *not* a complete sublattice in general. In a closure system of sets, for example, the join of two elements is usually not given by their set union, but by the closure of this union. Thus a closure system, ordered by set inclusion, is a complete lattice, but not necessarily a sublattice.

The fixed points of a kernel operator are closed under arbitrary joins and thus form a $\bigvee$-subsemilattice, also called a **kernel system**. Again we get the second operation from the first, so that each kernel system also is a complete lattice.

This shows that closure systems are not the only subsets yielding order-embedded complete lattices. In fact, the following is well known[5]:

Lemma 1 *A subset of a complete lattice $(L,\leq)$, with the induced order, is a complete lattice if and only if it is the image of an idempotent order-preserving function $\varphi : L \to L$.*

Proof Suppose that $\mathcal{F} = \{\varphi(x) \mid x \in L\}$ for some idempotent order-preserving function $\varphi : L \to L$. We claim that for any subfamily $\mathcal{S} \subseteq \mathcal{F}$ the element $\varphi(\bigwedge \mathcal{S})$ is the infimum of $\mathcal{S}$ in $\mathcal{F}$. Clearly $\bigwedge \mathcal{S} \leq s$ holds for every $s \in \mathcal{S}$. Since φ is order-preserving, we get that $\varphi(\bigwedge \mathcal{S}) \leq \varphi(s) = s$ for all $s \in \mathcal{S}$, which shows that $\varphi(\bigwedge \mathcal{S})$ is a lower bound of $\mathcal{S}$. But any lower bound b of $\mathcal{S}$ must satisfy $b \leq s$ for all $s \in \mathcal{S}$ and therefore $b \leq \bigwedge \mathcal{S}$. If $b \in \mathcal{F}$, then $b = \varphi(b) \leq \varphi(\bigwedge \mathcal{S})$, as desired.

For the converse suppose that $\mathcal{F} \subseteq L$ is a complete lattice and define a function $\varphi : L \to L$ by $\varphi(x) := \sup_{\mathcal{F}}\{f \in \mathcal{F} \mid f \leq x\}$ (where $\sup_{\mathcal{F}}$ denotes the supremum in $\mathcal{F}$). This function is clearly idempotent and order-preserving, and its image is $\mathcal{F}$. □

Lemma 1 adds a kind of converse to the celebrated Knaster-Tarski theorem [7, 9], which states that the set of fixed points of any order-preserving function on a complete lattice is itself a complete lattice:

Corollary 2 *A subset $\mathcal{F} \subseteq L$ of a complete lattice $(L,\leq)$, with the induced order, is a complete lattice if and only if $\mathcal{F}$ is the set of fixed points of some order-preserving function.*

The second part of the proof of Lemma 1 is more informative than required, since the function which was used not only is order-preserving and idempotent, but has an additional property:

Proposition 8 *The function which was used in the proof of Lemma 1,*

$$x \mapsto \varphi(x) := \sup_{\mathcal{F}}\{f \in \mathcal{F} \mid f \leq x\},$$

is tensive.

Proof If $f \leq x$ and $f \in \mathcal{F}$, then $f \leq \varphi(x)$ and so $f \leq x \wedge \varphi(x)$. Thus

$$\{f \in \mathcal{F} \mid f \leq x\} \subseteq \{f \in \mathcal{F} \mid f \leq x \wedge \varphi(x)\},$$

which implies that $\varphi(x) \leq \varphi(x \wedge \varphi(x))$. Since φ is order-preserving, we conclude equality. □

[5] Crapo [1] cites Duffus and Rival [2], while Shmuely [8] cites older notes by Crapo.

A simple consequence of the Knaster-Tarski result which we shall use is

Proposition 9 *If $(L,\leq)$ is a complete lattice, $\varphi : L \to L$ is order-preserving, and $x \in L$ is an element for which $x \leq \varphi(x)$, then there is a least fixed point of φ that is greater or equal to x.*

Proof Note that since φ is order-preserving, the set $\uparrow x := \{y \in L \mid x \leq y\}$ is mapped into itself by φ: when $y \geq x$, then $\varphi(y) \geq \varphi(x) \geq x$. But $\uparrow x$ is a complete lattice as well, to which the Knaster-Tarski result can be applied. So there is a least fixed point of φ in $\uparrow x$. □

Lemma 2 *If $\varphi : L \to L$ is order-preserving and increasing, then for each $x \in L$ there is a least closed element $\overline{\varphi}(x) \geq x$, and there is a least fixed element $\widehat{\varphi}(x) \geq \varphi(x)$. Both $\overline{\varphi}$ and $\widehat{\varphi}$ are order preserving, in fact, $\overline{\varphi}$ is a closure operator. $\widehat{\varphi}$ has the same fixed points as φ. If φ is tensive, then so is $\widehat{\varphi}$.*

Proof For the first claim define a function $\rho(x) := x \vee \varphi(x)$. Clearly ρ is order-preserving and extensive, so by Proposition 9 there is a least fixed point y of ρ which is greater or equal to x. But the fixed points of ρ are precisely the closed points of φ, and thus y also is the least closed point of φ which is greater or equal to x. The second claim follows again from Proposition 9, assuming that the function φ is increasing. Proving that $\overline{\varphi}$ is a closure operator and that $\widehat{\varphi}$ is order-preserving is straightforward.

If x is a fixed point of φ, then x also is the smallest fixed point greater or equal to $\varphi(x) = x$, i.e., $x = \widehat{\varphi}(x)$. Conversely if $x = \widehat{\varphi}(x)$, then x is a fixed point of φ by definition.

Finally, assume that φ is tensive. By definition, $\widehat{\varphi}(x \wedge \varphi(x))$ is the least fixed point of φ greater or equal to $\varphi(x \wedge \varphi(x))$. But when φ is tensive, the latter equals $\varphi(x)$, and therefore $\widehat{\varphi}(x \wedge \varphi(x)) = \widehat{\varphi}(x)$. But since $\varphi(x) \leq \widehat{\varphi}(x)$, we get $\widehat{\varphi}(x) = \widehat{\varphi}(x \wedge \varphi(x)) \leq \widehat{\varphi}(x \wedge \widehat{\varphi}(x)) \leq \widehat{\varphi}(x)$, which concludes the proof. □

Lemma 3 *If φ is order-preserving and increasing, then for all $x \in L$*

$$\widehat{\varphi}(x) = \overline{\varphi}(\varphi(x)).$$

Proof $\widehat{\varphi}(x)$ is fixed and therefore closed, and contains $\varphi(x)$, thus $\widehat{\varphi}(x) \geq \overline{\varphi}(\varphi(x))$. It remains to show that $\widehat{\varphi}(x) \leq \overline{\varphi}(\varphi(x))$. Proposition 7 yields that $\varphi(\overline{\varphi}(\widehat{\varphi}(x)))$ is fixed and less or equal to $\overline{\varphi}(\varphi(x))$. The proof is complete if we show that this fixed element contains $\varphi(x)$, because that forces it to be equal to $\widehat{\varphi}(x)$ (which is the *least* such fixed point). But from $\varphi(x) \leq \overline{\varphi}(\varphi(x))$ and the fact that φ is increasing and order-preserving we conclude that $\varphi(x) \leq \varphi(\varphi(x)) \leq \varphi(\overline{\varphi}(\varphi(x)))$. □

Corollary 3 *Let $\rhd$ be an upward relation over $(L,\leq)$. Then the function $\widehat{\varphi}_{\rhd} : L \to L$ defined by*

$\widehat{\varphi}_{\rhd}(x)$ is the least fixed point of $\varphi_{\rhd}$ greater or equal to $\varphi_{\rhd}(x)$,

is idempotent, order-preserving, and tensive and has the same fixed points as $\varphi_{\rhd}$. Moreover,

$$\widehat{\varphi}_{\rhd}(x) = \overline{\varphi}_{\rhd}(\varphi_{\rhd}(x)) \textit{ for all } x \in L.$$

The corollary shows how an **order embedded complete lattice**, i.e., a subset which, with the induced order, also is a complete lattice, can be constructed from an upward relation. We summarize our findings in the following theorem.

```
for all g ∈ G in reverse order do
        if g ∈ A then A := A \ {g}
        else
            B := φ(A ∪ {g})
            if g is the smallest element of B \ A then return B
return ⊥.
```

Fig. 2 The NEXT CLOSURE algorithm, from [5]

Theorem 2 *Let $(L,\leq)$ be a complete lattice and let $\mathcal{F} \subseteq L$ be an order embedded complete lattice. Then there is an upward relation $\rhd$ over $(L,\leq)$ such that $\mathcal{F}$ is the set of fixed points of $\varphi_{\rhd}$. Conversely it holds for every upward relation $\rhd$ over $(L,\leq)$ that the set of fixed points of $\varphi_{\rhd}$ is an order embedded complete lattice. In both cases the fixed points of $\varphi_{\rhd}$ are exactly the images of $\widehat{\varphi}_{\rhd}$.*

5 The NEXT FIXED POINT algorithm

Many years ago the author suggested a simple algorithm [3] for finding all closed sets of a given closure operator ϕ on a (finite, linearly ordered) set G. One starts with the closure $A := \phi(\emptyset)$ of the empty set and then repeats the procedure shown in Fig. 2, using the output of each application as the input of the next one, until it returns $\perp$.

The algorithm is extremely useful for browsing and navigating in closure systems. And since it is so simple, many variations and generalizations have been invented, see [5].

It is easy to generalize the algorithm to closure operators on complete lattices, not only powerset lattices. It therefore seems natural to ask if a modification of NEXT CLOSURE can be used for generating all images of any given idempotent, order-preserving, and tensive function. If such a mapping is given as a "black box" only, then unfortunately, the answer is "no". Our pessimism is prompted by the following example:

Example 2 Let $A \subseteq L$ be an antichain in a complete lattice $(L,\leq)$, let 0_L be the least and and 1_L be the greatest element of $(L,\leq)$, and let f be an element of A. The function

$$\varphi(x) := \begin{cases} f & \text{if } x = f \\ 1_L & \text{if } a < x \text{ for some } a \in A \\ 0_L & \text{else} \end{cases}$$

is idempotent, order-preserving, and tensive.

In this example it is tedious to determine the fixed points by repeated invocation of φ. Since the number of antichains may be exponential[6] in the size of L, it seems difficult to find an algorithm which determines the fixed point f reasonably fast. Stronger assumptions are needed. What we shall assume is that an upward representation of reasonable size for φ is given.[7]

To a given upward relation $\rhd \subseteq L \times L$ we associate an extensive operator, which then will be iterated to compute a closure. This operator will be set valued. Its base set, named G in Fig. 2, will be $\rhd$ (considered as a set of pairs).

[6] For example, the Dedekind numbers in case that L is a powerset lattice.
[7] At this stage of the development, we are not yet studying the exact complexities.

Definition 3 Let $\rhd \subseteq L \times L$ be an upward relation over $(L,\leq)$. We define two mappings $S : \mathfrak{P}(\rhd) \to L$ and $T : L \to \mathfrak{P}(\rhd)$ as follows: for $A \subseteq \rhd$ and for $w \in L$ let

$$S(A) := \bigvee\{v \mid (u,v) \in A\}, \qquad T(w) := \{(u,v) \in \rhd \mid u \leq w\},$$

and finally their composition,

$$A^+ := T(S(A)).$$

Proposition 10

1. *$A \mapsto A^+$ is order-preserving and extensive (wrt. the $\subseteq$-order).*
2. *$S \circ T = \varphi_\rhd$.*
3. *$(T(w))^{\overbrace{+\cdots\cdots+}^{n\ times}} = T(\varphi_\rhd{}^n(w))$ holds for all $w \in L$ and all $n > 0$.*
4. *The fixed points[8] of the mapping $A \mapsto A^+$ form a closure system.*

Proof 1) If $A \subseteq B$, then obviously $S(A) \leq S(B)$ and $T(S(A)) \subseteq T(S(B))$, so that $T \circ S$ is $\subseteq$-preserving. If $(u,v) \in A$, then $u \leq v \leq S(A)$ and thus $(u,v) \in T(S(A))$, which equals A^+. 2) is immediate from the definition of $\varphi_\rhd$. 3) $(T(w))^+ = T(S(T(w))) = T(\varphi_\rhd(w))$ according to 2), and substituting $\varphi_\rhd(w)$ for w iterates. 4) This is generally true for any order-preserving extensive mapping $f : (L,\leq) \to (L,\leq)$: Let I be some index set, let $A_i, i \in I$, be fixed points, and let $B := \bigwedge_{i\in I} A_i$. From $B \leq A_i$ we infer $f(B) \leq f(A_i) = A_i$, and therefore $f(B) \leq \bigwedge_{i\in I} A_i$. But since f is extensive, we also have $f(B) \geq \bigwedge_{i\in I} A_i$, which concludes the proof, since it shows that $f(B) = B$. ☐

Theorem 3 *The mappings S and T, restricted to the fixed point sets of the operator $A \mapsto A^+$ and of $\varphi_\rhd$, induce mutually inverse bijections.*

Proof If $A = A^+$ then $S(A) = S(A^+) = S(T(S(A))) = \varphi_\rhd(S(A))$. Conversely, if $w = \varphi_\rhd(w)$ and $A = T(w)$, then $A = T(w) = T(\varphi_\rhd(w)) = (T(w))^+ = A^+$. Moreover, if $A = A^+$, then $T(S(A)) = A^+ = A$, and if $w = \varphi_\rhd(w)$, then $S(T(w)) = \varphi_\rhd(w) = w$, according to Proposition 10, (2). ☐

Now we have everything we need to compute the lattice of fixed points of an order-preserving mapping $\varphi_\rhd : (L,\leq) \to (L,\leq)$, provided it is given by an upward representig relation $\rhd$. We use the Next Closure algorithm for the closure system which is in bijective corespondence to (in fact, isomorphic to) the lattice of fixed points of $\varphi_\rhd$. Of course some finiteness condition is required, and we assume that $\rhd$ is finite. If so, then the closure operator ϕ, which is associated to this closure system, is easy to describe: it is computed by applying the set operator repeatedly until a fixed point is reached.

$$A \mapsto A^+ \mapsto A^{++} \mapsto \ldots \mapsto A^{\overbrace{+\cdots\cdots+}^{n\ times}} = A^{\overbrace{+\cdots\cdots+}^{n\ times}} := \phi(A).$$

The process terminates after finitely many steps because each application of $A \mapsto A^+$ that does not produce a fixed point $A = A^+$ requires a previously unused element $(u,v) \in \rhd$

[8] i.e., the sets $A \subseteq \rhd$ with $A = A^+$

with $u \leq S(A)$ and $v \nleq S(A)$. We expect that the idea behind the well-known LINCLOSURE algorithm can be carried over to this recursive construction.

We conclude the section with an example.

Example 3 Let $(L,\leq) = \{0 < 1 < 2 < 3\} \times \{0 < 1 < 2 < 3\}$ be the direct product of two 4-element chains, and let $\rhd$ be given as follows:

$(3,1) \rhd (3,2)$	$(3,0) \rhd (3,0)$	$(2,2) \rhd (3,2)$	$(2,1) \rhd (2,1)$
$(1,3) \rhd (2,3)$	$(1,2) \rhd (1,2)$	$(1,0) \rhd (1,0)$	$(0,3) \rhd (0,3)$

For the linear order of the base set $\rhd$, as required by the NEXT CLOSURE algorithm, we use the order in the table. For a shorter notation, we leave out parentheses. Writing *abcd* instead of $((a,b),(c,d))$, we obtain

$$\rhd = \{3132 < 3030 < 2232 < 2121 < 1323 < 1212 < 1010 < 0303\}.$$

This relation represents the following order-preserving map $\varphi_\rhd$:

x	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$
$\varphi_\rhd(x)$	$(\mathbf{0},\mathbf{0})$	$(0,0)$	$(0,0)$	$(\mathbf{0},\mathbf{3})$	$(\mathbf{1},\mathbf{0})$	$(1,0)$	$(\mathbf{1},\mathbf{2})$	$(2,3)$
x	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$	$(3,0)$	$(3,1)$	$(3,2)$	$(3,3)$
$\varphi_\rhd(x)$	$(1,0)$	$(\mathbf{2},\mathbf{1})$	$(3,2)$	$(3,3)$	$(\mathbf{3},\mathbf{0})$	$(3,2)$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{3})$

This table is so small that we can easily read off the final result: $\varphi_\rhd$ has eight fixed points. They are marked in boldface.

Next we show how the same result is obtained by the NEXT CLOSURE algorithm. We skip the first three steps where that algorithm computes the closed sets $\varnothing$, $\{0303\}$, and $\{1010\}$, which correspond to the fixed points (0,0), (0,3), and (1,0). We continue from there and show how the next two fixed points are computed:

$A = \{1010\}$, $g = 0303$: $S(\{1010,0303\}) = 10 \vee 03 = 13$. $A^+ = T(13) = \{1323,1212,1010,0303\}$. Since A^+ contains an element smaller than g, no value for B is returned and we continue with
$A = \{1010\}$, $g = 1010$: Here $g \in A$, so g is removed from A.
$A = \varnothing$, $g = 1212$: $S(\{1212\}) = 12$, $(A \cup \{g\})^+ = T(12) = \{1212,1010\}$. Since $S(\{1212,1010\}) = S(\{1212\})$, we have reached the closure $B := \{1212,1010\}$, and 1212 is indeed the smallest element of $B \setminus A$. B is returned as a new closed set. The corresponding fixed point of $\varphi_\rhd$ is (1,2). We continue with $A := B$.
$A = \{1212,1010\}$, $g = 0303$: $S(A \cup \{g\}) = 12 \vee 10 \vee 03 = 13$. $T(13)$ contains the element 1323, which is smaller than g, so nothing is returned. Instead, we try the next smaller g.
$A = \{1212,1010\}$, $g = 1010$: Here $g \in A$, so g is removed from A.
$A = \{1212\}$, $g = 1212$: Here $g \in A$, so g is removed from A.
$A = \varnothing$, $g = 1323$: $S(A \cup \{g\}) = 23$, $T(23)$ contains 2232, which is smaller than g. Nothing is returned.

$A = \emptyset$, $g = 2121$: $S(A \cup \{g\}) = 21$, $T(21) = \{2121, 1010\}$, $S(T(21)) = S(2121)$ is a closed set in which g is the smallest element. Therefore $B := \{2121, 1010\}$ is returned. The corresponding fixed point is (2,1).

Continuing in the same way yields the remaining three fixed points.

6 Discussion

Apart from closure and kernel systems, there are many "lattices of sets", i.e., families of sets which form complete lattices, when ordered by set inclusion. We have studied those and, more generally, subsets of arbitrary complete lattices which, endowed with the induced order, are complete lattices themselves. We have shown that each such complete lattice can be described by an "upward" relation, in a way which is very similar to the representation of closure systems by implications. The NEXT CLOSURE algorithm can be tweaked to work with this representation, so that we were able to give an algorithm for generating such lattices. We did not discuss complexity questions in detail, because we expect that there is room for improvements. Many mathematical questions remain open. For example, we did not investigate how to construct an upward representation from a given "non-upward" one.

An important question is if embedded complete lattices have a natural and useful interpretation. The work of Shmuely [8] gives interesting hints. Her *$u - v$-connections* generalize Galois connections and seem to be related to what we construct. One might hope that these can be derived from formal contexts with additional, meaningful structure. Our results may help to study more substantial examples.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability No empirical datasets were generated or analysed during the current study.

Declarations

Conflict of Interests The author has no relevant financial or non-financial interests to disclose.

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