



Forbidding Rank-Preserving Copies of a Poset

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Abstract

The maximum size, $La(n, P)$, of a family of subsets of $[n] = \{1, 2, \dots, n\}$ without containing a copy of P as a subposet, has been extensively studied. Let P be a graded poset. We say that a family \mathcal{F} of subsets of $[n] = \{1, 2, \dots, n\}$ contains a *rank-preserving* copy of P if it contains a copy of P such that elements of P having the same rank are mapped to sets of same size in \mathcal{F} . The largest size of a family of subsets of $[n] = \{1, 2, \dots, n\}$ without containing a rank-preserving copy of P as a subposet is denoted by $La_{rp}(n, P)$. Clearly, $La(n, P) \leq La_{rp}(n, P)$ holds. In this paper we prove asymptotically optimal upper bounds on $La_{rp}(n, P)$ for tree posets of height 2 and monotone tree posets of height 3, strengthening a result of Bukh in these cases. We also obtain the exact value of $La_{rp}(n, \{Y_{h,s}, Y'_{h,s}\})$ and $La(n, \{Y_{h,s}, Y'_{h,s}\})$, where $Y_{h,s}$ denotes the poset on $h + s$ elements $x_1, \dots, x_h, y_1, \dots, y_s$ with $x_1 < \dots < x_h < y_1, \dots, y_s$ and $Y'_{h,s}$ denotes the dual poset of $Y_{h,s}$, thereby proving a conjecture of Martin et. al. [10].

Keywords Posets · Rank preserving copy · P-free · Extremal number

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1 Introduction

In extremal set theory, many of the problems considered can be phrased in the following way: what is the size of the largest family of sets that satisfy a certain property. The very first such result is due to Sperner [15] which states that if \mathcal{F} is a family of subsets of $[n] = \{1, 2, \dots, n\}$ (we write $\mathcal{F} \subseteq 2^{[n]}$ to denote this fact) such that no pair $F, F' \in \mathcal{F}$ of sets are in inclusion $F \subsetneq F'$, then \mathcal{F} can contain at most $\binom{n}{\lfloor n/2 \rfloor}$ sets. This is sharp as shown by $\binom{[n]}{\lfloor n/2 \rfloor}$ (the family of all k -element subsets of a set X is denoted by $\binom{X}{k}$ and is called the k^{th} layer of X). If P is a poset, we denote by \leq_P the partial order acting on the elements of P . Generalizing Sperner’s result, Katona and Tarján [8] introduced the problem of determining the size of the largest family $\mathcal{F} \subseteq 2^{[n]}$ that does not contain sets satisfying some inclusion patterns. Formally, if P is a finite poset, then a subfamily $\mathcal{G} \subseteq \mathcal{F}$ is

- a (weak) copy of P if there exists a bijection $\phi : P \rightarrow \mathcal{G}$ such that we have $\phi(x) \subseteq \phi(y)$ whenever $x \leq_P y$ holds,
- a strong or induced copy of P if there exists a bijection $\phi : P \rightarrow \mathcal{G}$ such that we have $\phi(x) \subseteq \phi(y)$ if and only if $x \leq_P y$ holds.

A family is said to be P -free if it does not contain any (weak) copy of P and induced P -free if it does not contain any induced copy of P . Katona and Tarján started the investigation of determining

$$La(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]}, \mathcal{F} \text{ is } P\text{-free}\}$$

and

$$La^*(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]}, \mathcal{F} \text{ is induced } P\text{-free}\}.$$

The above quantities have been determined precisely or asymptotically for many classes of posets (see [6] for a nice survey), but the question has not been settled in general. Recently, Methuku and Pálvölgyi [11] showed that for any poset P , there exists a constant C_P such that $La(n, P) \leq La^*(n, P) \leq C_P \binom{n}{\lfloor n/2 \rfloor}$ holds (whereas the inequality $La(n, P) \leq |P| \binom{n}{\lfloor n/2 \rfloor}$ follows trivially from a result of Erdős [4]). However, it is still unknown whether the limits $\pi(P) = \lim_{n \rightarrow \infty} \frac{La(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}$ and $\pi^*(P) = \lim_{n \rightarrow \infty} \frac{La^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}$ exist. In all known cases, the asymptotics of $La(n, P)$ and $La^*(n, P)$ were given by “taking as many middle layers as possible without creating an (induced) copy of P ”. Therefore researchers of the area believe the following conjecture that appeared first in print in [7].

- Conjecture 1.1** (i) For any poset P let $e(P)$ denote the largest integer k such that for any j and n the family $\cup_{i=1}^k \binom{[n]}{j+i}$ is P -free. Then $\pi(P)$ exists and is equal to $e(P)$.
- (ii) For any poset P let $e^*(P)$ denote the largest integer k such that for any j and n the family $\cup_{i=1}^k \binom{[n]}{j+i}$ is induced P -free. Then $\pi^*(P)$ exists and is equal to $e^*(P)$.

Let P be a graded poset with rank function ρ . Given a family \mathcal{F} , a subfamily $\mathcal{G} \subseteq \mathcal{F}$ is a rank-preserving copy of P if \mathcal{G} is a (weak) copy of P such that elements having the same rank in P are mapped to sets of same size in \mathcal{G} . More formally, $\mathcal{G} \subseteq \mathcal{F}$ is a rank-preserving copy of P if there is a bijection $\phi : P \rightarrow \mathcal{G}$ such that $|\phi(x)| = |\phi(y)|$ whenever $\rho(x) = \rho(y)$ and we have $\phi(x) \subseteq \phi(y)$ whenever $x \leq_P y$ holds. A family \mathcal{F} is rank-preserving P -free if it does not contain a rank-preserving copy of P . In this paper, we study the function

$$La_{rp}(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]}, \mathcal{F} \text{ is rank-preserving } P\text{-free}\}.$$

In fact, our problem is a natural special case of the following general problem introduced by Nagy [13]. Let $c : P \rightarrow [k]$ be a coloring of the poset P such that for any $x \in [k]$ the pre-image $c^{-1}(x)$ is an antichain. A subfamily $\mathcal{G} \subseteq \mathcal{F}$ is called a c -colored copy of P in \mathcal{F} if \mathcal{G} is a (weak) copy of P and sets corresponding to elements of P of the same color have the same size. Nagy investigated the size of the largest family $\mathcal{F} \subseteq 2^{[n]}$ which does not contain a c -colored copy of P , for several posets P and colorings c . Note that when c is the rank function of P , then this is equal to $La_{rp}(n, P)$. Nagy also showed that there is a constant C_P such that $La_{rp}(n, P) \leq C_P \binom{n}{\lfloor n/2 \rfloor}$.

A complete multi-level poset is a poset in which every element of a level is related to every element of another level. Note that any rank-preserving copy of a complete multi-level poset P is also an induced copy of P . In fact, in [14], Patkós determined the asymptotics of $La^*(n, P)$, for some complete multi-level posets P by finding a rank preserving copy of P .

By definition, for every graded poset P we have $La(n, P) \leq La_{rp}(n, P)$. Boehnlein and Jiang [1] gave a family of posets P showing that the difference between $La^*(n, P)$ and $La(n, P)$ can be arbitrarily large. Since their posets embed into a complete multi-level poset of height 3 in a rank-preserving manner, the above mentioned result of Patkós implies that for the same family of posets, $La_{rp}(n, P)$ can be arbitrarily smaller than $La^*(n, P)$. However, it would be interesting to determine if the opposite phenomenon can occur.

1.1 Our Results

1.1.1 Asymptotic Results

For a poset P its Hasse diagram, denoted by $H(P)$, is a graph whose vertices are elements of P , and xy is an edge if $x < y$ and there is no other element z of P with $x < z < y$. We call a poset, tree poset if $H(P)$ is a tree. A tree poset is called monotone increasing if it has a unique minimal element and it is called monotone decreasing if it has a unique maximal element. A tree poset is monotone if it is either monotone increasing or decreasing.

A remarkable result concerning Conjecture 1.1 is that of Bukh [2], who verified Conjecture 1.1 (i) for tree posets T by showing that $La(n, T) = (h(T) - 1 + o(1)) \binom{n}{\lfloor \frac{n}{2} \rfloor}$ where $h(T)$ denotes the height of T . In the following two results we strengthen his result in two cases, by showing that the answer is asymptotically the same, even if one only forbids rank-preserving copies of T (instead of forbidding all copies of T).

Theorem 1.2 *Let T be any tree poset of height 2. Then we have*

$$La_{rp}(n, T) = \left(1 + O_T \left(\sqrt{\frac{\log n}{n}} \right) \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Theorem 1.3 *Let T be any monotone tree poset of height 3. Then we have*

$$La_{rp}(n, T) = \left(2 + O_T \left(\sqrt{\frac{\log n}{n}} \right) \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The lower bounds in Theorem 1.2 and Theorem 1.3 follow simply by taking one and two middle layers of the Boolean lattice of order n , respectively. It is a natural question to ask whether Bukh’s theorem can be strengthened similarly for other tree posets.

1.1.2 An Exact Result

The *dual* of a poset P is the poset P' on the same set with the partial order relation of P replaced by its inverse, i.e., $x \leq y$ holds in P if and only if $y \leq x$ holds in P' . Let $Y_{h,s}$ denote the poset on $h + s$ elements $x_1, \dots, x_h, y_1, \dots, y_s$ with $x_1 < \dots < x_h < y_1, \dots, y_s$ and let $Y'_{h,s}$ denote the dual of $Y_{h,s}$. Let $\Sigma(n, h)$ denote the number of elements on the h middle layers of the Boolean lattice of order n , so $\Sigma(n, h) = \sum_{i=1}^h \binom{n}{\lfloor \frac{n-h}{2} \rfloor + i}$.

Investigation on $La(n, Y_{h,s})$ was started by Thanh in [16], where asymptotic results were obtained. Thanh also gave a construction showing that $La(n, Y_{h,s}) > \Sigma(n, h)$, from which it easily follows that $La(n, Y'_{h,s}) > \Sigma(n, h)$ as well. Interestingly, De Bonis, Katona and Swanepoel [3] showed that if both $Y_{2,2}$ and $Y'_{2,2}$ are forbidden, then an exact result can be obtained: $La(n, \{Y_{2,2}, Y'_{2,2}\}) = \Sigma(n, 2)$. Later this was extended by Methuku and Tompkins [12], who proved $La(n, \{Y_{h,2}, Y'_{h,2}\}) = \Sigma(n, h)$, and $La^*(n, \{Y_{2,2}, Y'_{2,2}\}) = \Sigma(n, 2)$. Very recently, Martin, Methuku, Uzzell and Walker [10] and independently, Tompkins and Wang [17] showed that $La^*(n, \{Y_{h,2}, Y'_{h,2}\}) = \Sigma(n, h)$. We prove the following theorem which extends all of these previous results and proves a conjecture of [10].

Theorem 1.4 *For any pair $s, h \geq 2$ of positive integers, there exists $n_0 = n_0(h, s)$ such that for any $n \geq n_0$ we have*

$$La_{rp}(n, \{Y_{h,s}, Y'_{h,s}\}) = \Sigma(n, h).$$

The lower bound trivially follows by taking h middle layers of the Boolean lattice of order n . (Note that adding any extra set creates a rank-preserving copy of either $Y_{h,s}$ or $Y'_{h,s}$.) Moreover, any rank-preserving copy of $Y_{h,s}$ (respectively $Y'_{h,s}$) is also an induced copy of $Y_{h,s}$ (respectively $Y'_{h,s}$). Therefore, Theorem 1.4 implies that $La^*(n, \{Y_{h,s}, Y'_{h,s}\}) = La(n, \{Y_{h,s}, Y'_{h,s}\}) = \Sigma(n, h)$.

Remark One wonders if the condition $h \geq 2$ is necessary in Theorem 1.4. Katona and Tarján [8] proved that $La(n, \{Y_{1,2}, Y'_{1,2}\}) = \binom{n}{n/2}$ if n is even and $La(n, \{Y_{1,2}, Y'_{1,2}\}) = 2\binom{n-1}{(n-1)/2} > \binom{n}{n/2}$ if n is odd. The following construction shows that no matter how little we weaken the condition of being $\{Y_{1,2}, Y'_{1,2}\}$ -free, there are families strictly larger than $\binom{n}{n/2}$ even in the case n is even. Let us define

$$\mathcal{F}_{2,3} = \left\{ F \in \binom{[n]}{n/2 + 1} : n - 1, n \in F \right\} \cup \left\{ F \in \binom{[n]}{n/2} : |F \cap \{n - 1, n\}| \leq 1 \right\}.$$

Observe that $\mathcal{F}_{2,3}$ is $\{Y_{1,2}, Y'_{1,3}\}$ -free and its size is $\binom{n-2}{n/2-1} + \left(\binom{n}{n/2} - \binom{n-2}{n/2-2} \right) > \binom{n}{n/2}$.

2 Proofs

Using Chernoff’s inequality, it is easy to show (see for example [7]) that the number of sets $F \subset [n]$ of size more than $n/2 + 2\sqrt{n \log n}$ or smaller than $n/2 - 2\sqrt{n \log n}$ is at most

$$O\left(\frac{1}{n^{3/2}} \binom{n}{n/2}\right). \tag{1}$$

Thus in order to prove 1.2 and 1.3, we can assume the family only contains sets of size more than $n/2 - 2\sqrt{n \log n}$ and smaller than $n/2 + 2\sqrt{n \log n}$.

2.1 Proof of 1.2: Trees of Height Two

The proof of 1.2 follows the lines of a reasoning of Bukh’s [2]. The new idea is that we count the number of related pairs between two fixed levels as detailed in the proof below.

Let \mathcal{F} be a T -free family of subsets of $[n]$ and let the number of elements in T be t . Using (1), we can assume \mathcal{F} only contains sets of sizes in the range $[n/2 - 2\sqrt{n \log n}, n/2 + 2\sqrt{n \log n}]$. A pair of sets $A, B \in \mathcal{F}$ with $A \subset B$ is called a 2-chain in \mathcal{F} . It is known by a result of Kleitman [9] that the number of 2-chains in \mathcal{F} is at least

$$\left(|\mathcal{F}| - \binom{n}{\lfloor \frac{n}{2} \rfloor} \right) \frac{n}{2}. \tag{2}$$

For any i such that $n/2 - 2\sqrt{n \log n} \leq i \leq n/2 + 2\sqrt{n \log n}$, let $\mathcal{F}_i := \mathcal{F} \cap \binom{[n]}{i}$.

Claim 2.1 For any $i < j$, the number of 2-chains $A \subset B$ with $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ is at most $(t - 2)(|\mathcal{F}_i| + |\mathcal{F}_j|)$.

Proof Suppose otherwise, and construct an auxiliary graph G whose vertices are elements of \mathcal{F}_i and \mathcal{F}_j , and two vertices form an edge of G if the corresponding elements form a 2-chain. This implies that G contains more than $(t - 2)(|\mathcal{F}_i| + |\mathcal{F}_j|)$ edges, so it has average degree more than $2(t - 2)$. One can easily find a subgraph G' of G with minimum degree at least $t - 1$, into which we can greedily embed any tree with t vertices. So in particular, we can find T in G' which corresponds to a rank-preserving copy of T into \mathcal{F} , a contradiction. \square

Claim 2.1 implies that the total number of 2-chains in \mathcal{F} is at most

$$\sum_{n/2 - 2\sqrt{n \log n} \leq i < j \leq n/2 + 2\sqrt{n \log n}} (t - 2)(|\mathcal{F}_i| + |\mathcal{F}_j|) = (t - 2)(4\sqrt{n \log n}) |\mathcal{F}|.$$

Combining this with (2), and simplifying we get

$$|\mathcal{F}| \left(1 - 8(t - 2)\sqrt{\frac{\log n}{n}} \right) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Rearranging, we get

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + O_T \left(\sqrt{\frac{\log n}{n}} \right) \right)$$

as desired.

2.2 Proof of 1.3: Monotone Trees of Height Three

First note that it is enough to prove the statement for $T = T_{r,3}$ the monotone increasing tree poset of height three where all elements, except its leaves (i.e., its elements on the top level) have degree r . Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of sets which does not contain any rank-preserving copies of $T_{r,3}$. Using (1) we can assume that for any set $F \in \mathcal{F}$ we have $|F - n/2| \leq 2\sqrt{n \log n}$.

We will prove that for such a family,

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)! \leq (2 + O_r(1/n))n! \tag{3}$$

holds. This is enough as dividing by $n!$ yields

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq (2 + O_r(1/n))$$

and hence the statement of the theorem will follow.

Observe that $\sum_{F \in \mathcal{F}} |F|!(n - |F|)!$ is the number of pairs (F, \mathcal{C}) where $F \in \mathcal{F} \cap \mathcal{C}$ and \mathcal{C} is a maximal chain in $[n]$. We will use the chain partitioning method introduced in [5]. For any $G \in \mathcal{F}$ we define \mathbf{C}_G to be the set of maximal chains \mathcal{C} in $[n]$ such that the smallest set of $\mathcal{C} \cap \mathcal{F}$ is G .

To prove (3) it is enough to show that for any fixed $G \in \mathcal{F}$ the number of pairs (F, \mathcal{C}) with $F \in \mathcal{F} \cap \mathcal{C}$, $\mathcal{C} \in \mathbf{C}_G$ is at most $(2 + O_r(1/n))|\mathbf{C}_G|$. We count the number of these pairs (F, \mathcal{C}) in three parts.

Firstly, the number of pairs where either $F = G$ or F is the second smallest element of $\mathcal{F} \cap \mathcal{C}$ is at most $2|\mathbf{C}_G|$ (there might be chains in \mathbf{C}_G with $\mathcal{C} \cap \mathcal{F} = \{G\}$).

Let us consider the following sub-partition of \mathbf{C}_G . For any $G \subsetneq G' \in \mathcal{F}$ let $\mathbf{C}_{G,G'}$ denote the set of maximal chains \mathcal{C} such that G and G' are the smallest and second smallest sets in $\mathcal{F} \cap \mathcal{C}$, respectively. Observe that $|\mathbf{C}_{G,G'}| = m_G \cdot m_{G,G'} \cdot (n - |G'|)!$, where m_G is the number of chains from \emptyset to G that do not contain any other sets from \mathcal{F} and $m_{G,G'}$ is the number of chains from G to G' that do not contain any other sets from \mathcal{F} .

Secondly, let us now count the pairs (F, \mathcal{C}) such that $F \in \mathcal{F} \cap \mathcal{C}$, $\mathcal{C} \in \mathbf{C}_{G,G'}$ and there are less than r^2 sets $F' \in \mathcal{F}$ with $|F'| = |F|$, $G' \subsetneq F'$. To this end, let us fix G' and count such pairs (F, \mathcal{C}) . All sets in \mathcal{F} have size at most $n/2 + 2\sqrt{n \log n}$ and at least $n/2 - 2\sqrt{n \log n}$, so $|G'| \geq n/2 - 2\sqrt{n \log n}$. For a set $F \supseteq G'$ the number of chains in $\mathbf{C}_{G,G'}$ that contain F is $m_G m_{G,G'} \cdot (|F| - |G'|)!(n - |F|)!$, thus we obtain that the number of such pairs is at most

$$\begin{aligned} \sum_{i=1}^{4\sqrt{n \log n}} r^2 m_G m_{G,G'} \cdot i!(n - |G'| - i)! &\leq 2r^2 m_G m_{G,G'} (n - |G'| - 1)! \\ &= \frac{2r^2}{n - |G'|} |\mathbf{C}_{G,G'}| \leq \frac{5r^2}{n} |\mathbf{C}_{G,G'}|. \end{aligned}$$

Summing this for all G' we obtain that the total number of such pairs (F, \mathcal{C}) of this second type is at most $\frac{5r^2}{n} |\mathbf{C}_G|$.

Finally, let us count the pairs (F, \mathcal{C}) with $F \in \mathcal{C} \cap \mathcal{F}$, $\mathcal{C} \in \mathbf{C}_{G,G'}$ and there are at least r^2 many sets $F' \in \mathcal{F}$ with $G' \subsetneq F'$, $|F'| = |F|$. To this end we group some of the $\mathbf{C}_{G,G'}$'s together. Let

$$\mathbf{C}_{G,k} := \cup_{G':|G'|=k} \mathbf{C}_{G,G'}, \quad \mathcal{F}_{G,k} := \{G' \in \mathcal{F} : G \subseteq G', |G'| = k\}$$

and let us introduce the function $f_{G,k} : \mathcal{F}_{G,k} \rightarrow [n]$ by

$$\begin{aligned} f_{G,k}(G') := \left\{ j : \exists \text{ distinct } F_1, F_2, \dots, F_{r^2} \in \mathcal{F}, \text{ such that } G' \subseteq F_i, |F_i| = j \right. \\ \left. \text{for all } i = 1, 2, \dots, r^2 \right\}, \end{aligned}$$

that is the set of the levels that contains at least r^2 many elements from \mathcal{F} above G' . Observe that for any distinct $G'_1, G'_2, \dots, G'_r \in \mathcal{F}_{G,k}$ we have $\cap_{i=1}^r f_{G,k}(G'_i) = \emptyset$. Indeed, if $j \in \cap_{i=1}^r f_{G,k}(G'_i) \neq \emptyset$, then one could extend $G, G'_1, G'_2, \dots, G'_r$ to a rank-preserving copy of $T_{r,3}$ such that all sets corresponding to leaves of $T_{r,3}$ are of size j .

Note that by the assumption on the set sizes of \mathcal{F} , the function $f_{G,k}$ maps to $[n/2 - 2\sqrt{n \log n}, n/2 + 2\sqrt{n \log n}]$, so its range has size at most $4\sqrt{n \log n}$. As every maximal

chain contains exactly one set of size j (not necessarily contained in \mathcal{F}), we obtain that the number of pairs (F, \mathcal{C}) with $F \in \mathcal{F} \cap \mathcal{C}, \mathcal{C} \in \mathbf{C}_{G,k}$ is at most

$$m_G \cdot 4\sqrt{n \log n}(r - 1)(k - |G|)!(n - k)! \tag{4}$$

Indeed, if the size j of F is fixed, then j belongs to $f_{G,k}(G')$ for at most $r - 1$ sets $G' \in \mathcal{F}_{G,k}$, so for this particular j the number of pairs is at most $m_G \cdot (r - 1)(k - |G|)!(n - k)!$.

Summing up (4) for all $k > |G|$ we obtain that the number of pairs (F, \mathcal{C}) of this third type is at most

$$\begin{aligned} \sum_{k=|G|+1}^{n/2+2\sqrt{n \log n}} m_G \cdot 4\sqrt{n \log n}(r - 1)(k - |G|)!(n - k)! &\leq \frac{8(r - 1)\sqrt{n \log n}}{n - |G|} m_G(n - |G|)! \\ &\leq \frac{17(r - 1)\sqrt{n \log n}}{n} |\mathbf{C}_G|. \end{aligned}$$

Adding up the estimates on the number of pairs (F, \mathcal{C}) of these 3 types, completes the proof.

2.3 Proof of 1.4: $\{Y_{h,s}, Y'_{h,s}\}$ -free Families

Let $\mathcal{F} \subset 2^{[n]}$ be a family not containing a rank-preserving copy of $Y_{h,s}$ or $Y'_{h,s}$. First, we will introduce a weight function. For every $F \in \mathcal{F}$, let $w(F) = \binom{n}{|F|}$. For a maximal chain \mathcal{C} , let $w(\mathcal{C}) = \sum_{F \in \mathcal{C} \cap \mathcal{F}} w(F)$ denote the weight of \mathcal{C} . Let \mathbf{C}_n denote the set of maximal chains in $[n]$. Then

$$\frac{1}{n!} \sum_{\mathcal{C} \in \mathbf{C}_n} w(\mathcal{C}) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathbf{C}_n} \sum_{F \in \mathcal{C} \cap \mathcal{F}} w(F) = \frac{1}{n!} \sum_{F \in \mathcal{F}} |F|!(n - |F|)!w(F) = |\mathcal{F}|.$$

This means that the average of the weight of the full chains equals the size of \mathcal{F} . Therefore it is enough to find an upper bound on this average. We will partition \mathbf{C}_n into some parts and show that the average weight of the chains is at most $\Sigma(n, h)$ in each of the parts. Therefore this average is also at most $\Sigma(n, h)$, when calculated over all maximal chains, which gives us $|\mathcal{F}| \leq \Sigma(n, h)$.

Let $\mathcal{G} = \{F \in \mathcal{F} \mid \exists P, Q \in \mathcal{F} \setminus \{F\}, P \subset F \subset Q\}$. Let $A_1 \subset A_2 \subset \dots \subset A_{h-1}$ be $h - 1$ different sets of \mathcal{G} . Then we define $\mathbf{C}(A_1, A_2, \dots, A_{h-1})$ as the set of those chains that contain all of A_1, A_2, \dots, A_{h-1} and these are the $h - 1$ smallest elements of \mathcal{G} in them. We also define \mathbf{C}_- as the set of those chains that contain at most $h - 2$ elements of \mathcal{G} . Then the sets of the form $\mathbf{C}(A_1, A_2, \dots, A_{h-1})$ together with \mathbf{C}_- are pairwise disjoint and their union is \mathbf{C}_n .

Now we will show the average weight within each of these sets of chains is at most $\Sigma(n, h)$. This is easy to see for \mathbf{C}_- . If $\mathcal{C} \in \mathbf{C}_-$, then $|\mathcal{C} \cap \mathcal{F}| \leq h$, since every element of $\mathcal{F} \cap \mathcal{C}$ except for the smallest and the greatest must be in \mathcal{G} . Therefore $w(\mathcal{C}) \leq \Sigma(n, h)$ for every $\mathcal{C} \in \mathbf{C}_-$, which trivially implies

$$\sum_{\mathcal{C} \in \mathbf{C}_-} w(\mathcal{C}) \leq |\mathbf{C}_-| \Sigma(n, h).$$

Now consider some sets $A_1 \subset A_2 \subset \dots \subset A_{h-1}$ in \mathcal{G} such that $\mathbf{C}(A_1, A_2, \dots, A_{h-1})$ is non-empty. We will use the notations $\mathbf{C}(A_1, A_2, \dots, A_{h-1}) = \mathcal{Q}$, $|A_1| = \ell_1$ and $n - |A_{h-1}| = \ell_2$ for simplicity. Note that the chains in \mathcal{Q} do not contain any member of \mathcal{F} of size between $|A_1|$ and $|A_{h-1}|$ other than the sets A_2, A_3, \dots, A_{h-2} . Such a set would be in \mathcal{G} (since it contains A_1 and is contained in A_{h-1}), therefore its existence would contradict the

minimality of $\{A_1, A_2, \dots, A_{h-1}\}$. The chains in \mathcal{Q} must also avoid all subsets of A_1 that are in \mathcal{G} for the same reason.

Let N_1 denote the number of chains between \emptyset and A_1 that avoid the elements of \mathcal{G} (except for A_1). Let N_2 denote the number of chains between A_1 and A_{h-1} that contain the sets A_2, A_3, \dots, A_{h-2} , but no other element of \mathcal{F} . Then $|\mathcal{Q}| = N_1 N_2 \ell_2!$.

Now we will investigate how much the sets of certain sizes can contribute to the sum

$$\sum_{\mathcal{C} \in \mathcal{Q}} w(\mathcal{C}). \tag{5}$$

The sets A_1, A_2, \dots, A_{h-1} appear in all chains of \mathcal{Q} , so their contribution to the sum is

$$|\mathcal{Q}| \sum_{i=1}^{h-1} w(A_i) = |\mathcal{Q}| \sum_{i=1}^{h-1} \binom{n}{|A_i|} \leq |\mathcal{Q}| \Sigma(n, h - 1).$$

We have already seen that there are no other sets of \mathcal{F} in these chains with a size between $|A_1|$ and $|A_{h-1}|$.

If $\ell_1 < \frac{n}{2} - 2\sqrt{n \log n}$, then (by (1)) the contribution coming from the subsets of A_1 is trivially at most

$$|\mathcal{Q}| \sum_{i=0}^{\ell_1-1} \binom{n}{i} = |\mathcal{Q}| O\left(\binom{n}{n/2} \frac{1}{n^{3/2}}\right).$$

The contribution coming from supersets of A_{h-1} is similarly small if $\ell_2 < \frac{n}{2} - 2\sqrt{n \log n}$. From now on we consider the cases when $\ell_1 \geq \frac{n}{2} - 2\sqrt{n \log n}$ and $\ell_2 \geq \frac{n}{2} - 2\sqrt{n \log n}$.

There are no s supersets of A_{h-1} of equal size in \mathcal{F} , since these would form a rank-preserving copy of $Y_{h,s}$ together with the sets A_1, A_2, \dots, A_{h-1} and some set $P \in \mathcal{F}, P \subset A_1$. (Such a set exists, since $A_1 \in \mathcal{G}$.)

A superset of A_{h-1} of size $n - i$ appears in $|\mathcal{Q}| \binom{\ell_2}{i}^{-1}$ chains of \mathcal{Q} . Therefore the total contribution to the sum (5) by supersets of A_{h-1} is at most

$$\begin{aligned} |\mathcal{Q}| w([n]) + \sum_{i=1}^{\ell_2-1} |\mathcal{Q}| \binom{\ell_2}{i}^{-1} (s-1) \binom{n}{n-i} &\leq |\mathcal{Q}| + |\mathcal{Q}| (s-1) \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{\ell_2-1} \binom{\ell_2}{i}^{-1} \\ &= |\mathcal{Q}| \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right). \end{aligned}$$

There are no s subsets of A_1 of equal size in \mathcal{F} , since these would form a rank-preserving copy of $Y'_{h,s}$ together with the sets A_1, A_2, \dots, A_{h-1} and some set $Q \in \mathcal{F}, A_{h-1} \subset Q$. (Such a set exists, since $A_{h-1} \in \mathcal{G}$.)

A subset of A_1 of size i appears in at most $\binom{\ell_1}{i}^{-1} \ell_1! N_2 \ell_2!$ chains of \mathcal{Q} . Therefore the total contribution to the sum (5) by subsets of A_1 is at most

$$\begin{aligned} \ell_1! N_2 \ell_2! w(\emptyset) + \sum_{i=1}^{\ell_1-1} \binom{\ell_1}{i}^{-1} \ell_1! N_2 \ell_2! (s-1) \binom{n}{i} \\ \leq \ell_1! N_2 \ell_2! + \ell_1! N_2 \ell_2! (s-1) \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{\ell_1-1} \binom{\ell_1}{i}^{-1} \end{aligned} \tag{6}$$

$$= \ell_1! N_2 \ell_2! \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right). \tag{7}$$

We will show that if n is large and $\ell_1 \geq \frac{n}{2} - 2\sqrt{n \log n}$ then most chains between \emptyset and A_1 avoid the elements of \mathcal{G} , therefore N_1 is close to $\ell_1!$. There are at most $s - 1$ sets of \mathcal{G} on any level (otherwise a rank-preserving copy of $Y'_{h,s}$ would be formed), and $\emptyset \notin \mathcal{G}$. There are $\ell_1! \binom{\ell_1}{i}^{-1}$ chains between \emptyset and A_1 containing a set of size i . Therefore

$$\ell_1! - N_1 \leq (s - 1) \sum_{i=1}^{\ell_1-1} \ell_1! \binom{\ell_1}{i}^{-1} = \ell_1! O\left(\frac{1}{n}\right).$$

This means that for large enough n , we have $\ell_1! \leq 2N_1$. Then (6) can be continued as

$$\ell_1! N_2 \ell_2! \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right) \leq 2N_1 N_2 \ell_2! \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right) = |\mathcal{Q}| \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right).$$

To summarize, we found that the contribution to the sum (5) from the subsets of A_1 and the supersets of A_{h-1} is at most

$$|\mathcal{Q}| \binom{n}{\lfloor \frac{n}{2} \rfloor} O_s\left(\frac{1}{n}\right).$$

For large enough n this is smaller than $|\mathcal{Q}| (\Sigma(n, h) - \Sigma(n, h - 1))$, which means that

$$\sum_{\mathcal{C} \in \mathcal{Q}} w(\mathcal{C}) \leq |\mathcal{Q}| \Sigma(n, h).$$

This completes the proof.

Remark We had to use a weighting technique in the above proof because the usual Lubell method (proving that $\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} \leq h$, and deducing $|\mathcal{F}| \leq \Sigma(n, h)$ from that) does not work for this problem. To see this, let $h \geq 3$, $n \geq 2h$ and consider the following set system:

$$\mathcal{F} = \{F \in [n] \mid |F| \leq h - 2 \text{ or } |F| \geq n - h + 2\}.$$

For $s \geq 2^{h-2}$ this set system is $Y_{h,s}$ -free and $Y'_{h,s}$ -free (even in the original sense, not necessarily in the rank-preserving sense). However, we have $\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} = 2(h-1) > h$.

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