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Dynamics of generalized time-fractional viscous-capillarity compressible fluid model

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Abstract
This analysis examines the time-fractional mixed hyperbolic-elliptic *p*-system of conservation laws by applying the new extended direct algebraic method. The *p*-system with generalized cubic van der Waals flux, and potential applications in the field of compressible isothermal viscosity-capillarity fluids, is investigated. In particular, this issue describes the longitudinal isothermal motion in elastic bars or fluids. A diverse periodic, kink, and singular soliton structures are extracted. The 3D dynamical behaviors and corresponding contour profiles of some obtained solitons are displayed. The fractional effects in the sense of Beta, M-truncated, and modified Riemann–Liouville, are discussed and illustrated. The method shows the straightforward, reliability, and efficiency for solving complex physical phenomena that is modeled by nonlinear partial differential equations.

Keywords Gas fluid · Capillarity · Traveling wave solutions · New extended direct algebraic method · Mixed hyperbolic-elliptic *p*-system · Fractional derivative

The research reported in this article will appear in a heavily modified version as a M.Sc. dissertation for the second author Qais M. M. Alomari.

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1 Introduction

Nonlinear partial differential equations (NPDEs) arise in numerous evolved scientific disciplines such as mathematical physics, fluid mechanics, nonlinear optics, ocean waves, engineering, atmospheric science, biology, chemistry, economics, etc (Farlow 2012; Ablowitz and Segur 1981; Hull 2018). In the past few years, numerous scientists and engineers have been interested in the search for novel solutions to differential equations (DEs), notably nonlinear evolved forms (NLEEs). Such equations represent the occurring of most nonlinear real-life phenomena. So, it is substantial to comprehend the structure of these equations to find new varieties of wave solutions and dynamical configurations. To derive these solutions, a number of effective, and powerful approaches have been developed to tackle the NLEEs. A few examples of most recent methods are the new auxiliary equation method (Rahman et al. 2023, 2023), unified auxiliary equation method (Tarla and Yilmazer 2022; Zayed et al. 2021), modified Sardar sub-equation method (Younas et al. 2022; Tao et al. 2022), $\boldsymbol{\varphi}^6$-model expansion method (Zhang et al. 2022; Safi Ullah et al. 2023), $\exp_a$-function method (Raheel et al. 2023, 2023), generalized Ricatti equation mapping method (GREMM) (Altawallbeh et al. 25022; Az-Zo'bi et al. 2022; Khan et al. 2022), Jacobi elliptic function expansion method (Khan et al. 2022; Az-Zo'bi et al. 2021), modified double Laplace transform decomposition method (Saifullah et al. 2021), adaptive moving mesh method (Almatrafi et al. 2021), improved tanh method Yokuş et al. (2022), rational (G'/G)-expansion scheme (Tarikul Islam et al. 2022), extended FAN sub-equation method (Badshah et al. 2023), modified $(1/G')$-expansion method (Ali Akbar et al. 2023), and the new extended direct algebraic method (NEDAM) (Tasnim et al. 2023; Rehman et al. 2023).

In physics and engineering, fluid dynamics depicts the flow of gases and liquids. The mixed hyperbolic–elliptic systems of conservation laws model dynamical phase transitions in solid dynamics as in the case of propagating phase boundaries. Also, they describe the stationary, transonic, and van der Waals flow in fluid dynamics. The one-dimensional regular viscos-capillarity compressible fluid of van der Waals system, also known by the p-system, that demonstrates the simple liquid–gas phase transition has the form (Slemrod 1983, 1984; Zahran et al. 2023),

$$\begin{aligned} &q_t - v_x = 0, \\ &v_t + (p(q))_x = \varsigma v_{xx} - \sigma v_{xxx}. \end{aligned} \tag{1}$$

In Eq. 1, the gas velocity is denoted by v, q is the specific volume, while p represents the van der Waals pressure. x and t are the Lagrangian space and time variables. The positive parameters ς and σ represent the strength viscous and capillarity coefficients respectively. Notify that the square of ς is proportional to σ.

For some material models, the system is of mixed type since the constitutive pressure may be non-monotone. That is, the eigenvalues $\lambda = \mp\sqrt{p'(q)}$, which are also known by wave speeds, differ according to the flow motion.The main crucial feature of nonlinear mixed-type systems is that shocks may occur when transitioning between different regions; hyperbolic with positive eigenvalues, boundary which is assumed to be smooth, and the elliptic region when $p'(q) > 0$. Such discontinues are routinely observed in transonic flow as mentioned above.

The existence and asymptotic stability of monotonic increasing traveling wave solutions of Eq. 1 was studied by Zhang et al. (2016). In Bedjaoui et al. (2005), authors investigated the existence and properties of non-monotonic traveling waves. With van der Waals pressure $p(q) = q - q^3$, the system describes the longitudinal isothermal motion. Such case has been discussed analytically by applying the Kudryashov simple equation method (Az-Zo'bi 2019), $\frac{G'}{G^2}$- expansion, and $e^{-\varphi(\varsigma)}$-expansion function methods (Bilal et al. 2021), Painleve analysis, and auxiliary equation mapping approaches (Akbar et al. 2021). Recently, the extended simple equation, Paul-Painleve approach, and He's variational iteration methods have been employed in Zahran et al. (2023). With $p(q) = q - (q-2)^3$, Eq. 1 is considered in Affouf and Caflisch (1991). The present study addresses Eq. 1 with general cubic van der Waals pressure

$$p(q) = \delta_0 + \delta_1 q + \delta_2 q^2 + \delta_3 q^3, \delta_j \in \mathbb{R}. \tag{2}$$

Upon conducting review through the published literature, the system has not been explored before. The NEDAM will be employed to tackle the aforementioned model.

The outline of our work is arranged as follows: in the coming section, Sect. 2, the use of extended direct algebraic scheme for processing (1+1)-NLEEs is discussed. Section 3 includes the application of the NEDAM to our generalized model. The existence sets of free parameters and constrains are listed. According to the analysis in Sect. 3, the wave soliton solutions for one set of constraints are listed in Sect. 4. The fractional issue of the considered model, subject to some recent developed fractional derivatives, is processed in Sect. 5. Dynamical behaviors of some obtained solitons are depicted, and the summarizing of entire article is discussed in Sect. 6.

2 The NEDAM

The NEDAM is a recent development in the field of NPDEs. This method covers many other existing schemes. As will be shown, the method depends on traveling wave to convert a given NPDE with polynomial nonlinearity into a nonlinear ordinary differential equations (NODE) that can be processed analytically. This approach provides an effective way to obtain exact solutions to a wide range of NPDEs with integer and fractional derivatives. In the recent last, this technique is considered to construct more general exact traveling wave solutions with different shapes. Here, we mention,the Tzitzeica, Dodd-Bullough-Mikhailor, and the Liouville equations (Mirhosseini-Alizamini et al. 2020), the Kerr-resonant nonlinear Schrödinger (NLS) equation (Tasnim et al. 2023; Javad Vahidi et al. 2021), the generalized non-linear Schrödinger model in metamaterials (Salathiel et al. 2019), the potential Kadomtsev-Petviashvili equation in shallow waters waves (Kurt et al. 2020), the 2D Kundu-Mukherjee-Naskar equation (Günerhan et al. 2020), cubic focusing and paraxial (NLS) equations in Kerr media (Hussain et al. 2022), the Biswas-Arshed equation (Munawar et al. 2021), the Zakhrov model in ionized plasma (Rehman et al. 2023), and the biofilm model (Iqbal et al. 2023).

To execute the NEDAM, consider the generic one-dimensional NLPDE:

$$\phi\left(v, v_t, v_x, v_{tt}, v_{tx}, v_{xx}, \dots\right) = 0, \tag{3}$$

where ϕ is a polynomial of the empirical function v and its total partial derivatives. The concept of new extended direct algebraic algorithm follows the listed steps.

Step 1. Assume that

$$v(x,t) = v(\zeta), \zeta = x + \nu t, \tag{4}$$

with $\nu \neq 0$ is the wave's speed, to convert Eq 2 into the NODE

$$\varphi\left(v, v', v'', \ldots\right) = 0, \tag{5}$$

where $v' = \frac{dv}{d\zeta}, \ldots$.

Step 2. The NEDAM Assumes the solution of Eq. 5 by the formal polynomial in $\vartheta(\zeta)$ given by

$$v(\zeta) = \sum_{j=0}^{m} c_j \vartheta^j(\zeta), c_m \neq 0. \tag{6}$$

In Eq. 6: c_j's are real constants to be identified, and the function $\vartheta(\zeta)$ holds the generalized auxiliary NODE

$$\vartheta'(\zeta) = ln(b)\left(\alpha + \beta\,\vartheta(\zeta) + \gamma\,\vartheta^2(\zeta)\right), b \in \mathbb{R}^+ - \{1\}, \alpha, \beta, \gamma \in \mathbb{R}. \tag{7}$$

Many solutions sets of Eq. 7 were derived. Through the generalized trigonometric and hyperbolic functions

$$\begin{aligned} &\sin_b(\zeta) = \frac{rb^{i\zeta} - sb^{-i\zeta}}{2i}, \cos_b(\zeta) = \frac{rb^{i\zeta} + sb^{-i\zeta}}{2}, \tan_b(\zeta) = \frac{\sin_b(\zeta)}{\cos_b(\zeta)}, \ldots \\ &\sinh_b(\zeta) = \frac{rb^{\zeta} - sb^{-\zeta}}{2}, \cosh_b(\zeta) = \frac{rb^{\zeta} + sb^{-\zeta}}{2}, \tanh_b(\zeta) = \frac{\sinh_b(\zeta)}{\cosh_b(\zeta)}, \ldots \end{aligned} \tag{8}$$

and with arbitrary constants $r, s > 0$, the various families of solutions are summarized as follows:

Family 1. For $\beta^2 - 4\alpha\gamma < 0$ and $\gamma \neq 0$, a class of periodic, singular, and combined periodic-singular solitons is obtained as follows:

$$\vartheta_1(\zeta) = -\frac{\beta}{2\gamma} + \frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\tan_b\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2}\zeta\right), \tag{9}$$

$$\vartheta_2(\zeta) = -\frac{\beta}{2\gamma} + \frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\cot_b\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2}\zeta\right), \tag{10}$$

$$\begin{aligned} \vartheta_3(\zeta) = &-\frac{\beta}{2\gamma} \\ &+ \frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\left(\tan_b\left(\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right) \pm \sqrt{rs}\,\sec_b\left(\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right)\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \vartheta_4(\zeta) = &-\frac{\beta}{2\gamma} \\ &+ \frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\left(-\cot_b\left(\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right) \pm \sqrt{rs}\,\csc_b\left(\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right)\right), \end{aligned} \tag{12}$$

$$\vartheta_5(\zeta) = -\frac{\beta}{2\gamma} + \frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{4\gamma}\left(\tan_b\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{4}\zeta\right) - \cot_b\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{4}\zeta\right)\right). \tag{13}$$

Family 2. For $\beta^2 - 4\alpha\gamma > 0$ and $\gamma \neq 0$, a class of dark, combined bright-dark, and dark-singular solitons is obtained as follows:

$$\vartheta_6(\zeta) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\tanh_b\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2}\zeta\right), \tag{14}$$

$$\vartheta_7(\zeta) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\coth_b\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\gamma\right)}}{2}\zeta\right), \tag{15}$$

$$\vartheta_8(\zeta) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\left(-\tanh_b\left(\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right) \pm \mathrm{i}\sqrt{rs}\ sech_b\left(\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right)\right), \tag{16}$$

$$\vartheta_9(\zeta) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\left(-\coth_b\left(\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right) \pm \sqrt{rs}\ csch_b\left(\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}\zeta\right)\right), \tag{17}$$

$$\vartheta_{10}(\zeta) = -\frac{\beta}{2\gamma} - \frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{2\gamma}\left(\tanh_b\left(\frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{4}\zeta\right) \pm \coth_b\left(\frac{\sqrt{\left(\beta^2 - 4\alpha\gamma\right)}}{4}\zeta\right)\right). \tag{18}$$

Family 3. For $\alpha\gamma > 0$ and $\beta = 0$, a sub-case of **Family 1** is obtained as follows:

$$\vartheta_{11}(\zeta) = \sqrt{\frac{\alpha}{\gamma}}\tan_b\left(\sqrt{\alpha\gamma}\zeta\right), \tag{19}$$

$$\vartheta_{12}(\zeta) = -\sqrt{\frac{\alpha}{\gamma}}\cot_b\left(\sqrt{\alpha\gamma}\zeta\right), \tag{20}$$

$$\vartheta_{13}(\zeta) = \sqrt{\frac{\alpha}{\gamma}}\Big(\tan_b\big(2\sqrt{\alpha\gamma}\zeta\big) \pm \sqrt{rs}\ \sec_b\big(2\sqrt{\alpha\gamma}\zeta\big)\Big), \tag{21}$$

$$\vartheta_{14}(\zeta) = \sqrt{\frac{\alpha}{\gamma}}\Big(-\cot_b\big(2\sqrt{\alpha\gamma}\zeta\big) \pm \sqrt{rs}\ \csc_b\big(2\sqrt{\alpha\gamma}\zeta\big)\Big), \tag{22}$$

$$\vartheta_{15}(\zeta) = \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left(\tan_b\left(\frac{\sqrt{\alpha\gamma}}{2}\zeta\right) - \cot_b\left(\frac{\sqrt{\alpha\gamma}}{2}\zeta\right)\right). \tag{23}$$

Family 4. For $\alpha\gamma < 0$ and $\beta = 0$, a sub-case of **Family 2** is obtained as follows:

$$\vartheta_{16}(\zeta) = -\sqrt{-\frac{\alpha}{\gamma}}\tanh_b\big(\sqrt{-\alpha\gamma}\zeta\big), \tag{24}$$

$$\vartheta_{17}(\zeta) = -\sqrt{-\frac{\alpha}{\gamma}}\coth_b\big(\sqrt{-\alpha\gamma}\zeta\big), \tag{25}$$

$$\vartheta_{18}(\zeta) = \sqrt{-\frac{\alpha}{\gamma}}\Big(-\tan_b\big(2\sqrt{-\alpha\gamma}\zeta\big) \pm \mathrm{i}\sqrt{rs}\ sech_b\big(2\sqrt{-\alpha\gamma}\zeta\big)\Big), \tag{26}$$

$$\vartheta_{19}(\zeta) = \sqrt{-\frac{\alpha}{\gamma}}\Big(-\coth_b\big(2\sqrt{-\alpha\gamma}\zeta\big) \pm \sqrt{rs}\ csch_b\big(2\sqrt{-\alpha\gamma}\zeta\big)\Big), \tag{27}$$

$$\vartheta_{20}(\zeta) = -\frac{1}{2}\sqrt{-\frac{\alpha}{\gamma}}\left(\tanh_b\left(\frac{\sqrt{-\alpha\gamma}}{2}\zeta\right) + \coth_b\left(\frac{\sqrt{-\alpha\gamma}}{2}\zeta\right)\right). \tag{28}$$

Family 5. For $\beta = 0$ and $\gamma = \alpha$, a sub-case of **Family 1**, as well as **Family 3**, is obtained as follows:

$$\vartheta_{21}(\zeta) = \tan_b(\alpha\zeta), \tag{29}$$

$$\vartheta_{22}(\zeta) = -\cot_b(\alpha\zeta), \tag{30}$$

$$\vartheta_{23}(\zeta) = \tan_b(2\alpha\zeta) \pm \sqrt{rs}\ \sec_b(2\alpha\zeta), \tag{31}$$

$$\vartheta_{24}(\zeta) = -\cot_b(2\alpha\zeta) \pm \sqrt{rs}\ \csc_b(2\alpha\zeta), \tag{32}$$

$$\vartheta_{25}(\zeta) = \frac{1}{2}\left(\tan_b\left(\frac{\alpha}{2}\zeta\right) - \cot_b\left(\frac{\alpha}{2}\zeta\right)\right). \tag{33}$$

Family 6. For $\beta = 0$ and $\gamma = -\alpha$, a sub-case of **Family 2**, as well as **Family 4**, is obtained as follows

$$\vartheta_{26}(\zeta) = -\tanh_b(\alpha\zeta), \tag{34}$$

$$\vartheta_{27}(\zeta) = -\coth_b(\alpha\zeta), \tag{35}$$

$$\vartheta_{28}(\zeta) = -\tanh_b(\alpha\zeta) \pm \mathrm{i}\sqrt{rs}\ sech_b(2\alpha\zeta), \tag{36}$$

$$\vartheta_{29}(\zeta) = -\coth_b(2\alpha\zeta) \pm \sqrt{rs}\ csch_b(2\alpha\zeta), \tag{37}$$

$$\vartheta_{30}(\zeta) = -\frac{1}{2}\left(\tanh_b\left(\frac{\alpha}{2}\zeta\right) + \coth_b\left(\frac{\alpha}{2}\zeta\right)\right). \tag{38}$$

Family 7. For $\beta^2 = 4\alpha\gamma$, a singular soliton is obtained:

$$\vartheta_{31}(\zeta) = -2\alpha\frac{((\ln(b))\ \beta\zeta + 2)}{(\ln(b))\ \beta^2\ \zeta}. \tag{39}$$

Family 8. For $\alpha = k\beta(k \neq 0)$ and $\gamma = 0$, a bright-like soliton is obtained:

$$\vartheta_{32}(\zeta) = b^{\beta\zeta} - k. \tag{40}$$

Family 9. For $\beta = \gamma = 0$, a linear bright-like soliton is obtained::

$$\vartheta_{33}(\zeta) = (\ln(b))\ \alpha\,\zeta. \tag{41}$$

Family 10. For $\beta = \alpha = 0$, a singular soliton is obtained:

$$\vartheta_{34}(\zeta) = \frac{-1}{(\ln(b))\ \gamma\,\zeta}. \tag{42}$$

Family 11. For $\alpha = 0$ and $\beta \neq 0$, a class of dark (kink and anti-kink) solitons is obtained as follows:

$$\vartheta_{35}(\zeta) = -\frac{p\beta}{\gamma\left(\cosh_b(\beta\zeta) - \sinh_b(\beta\zeta) + r\right)}. \tag{43}$$

$$\vartheta_{36}(\zeta) = -\frac{\beta\left(\sinh_b(\beta\zeta) + \cosh_b(\beta\zeta)\right)}{\gamma\left(\sinh_b(\beta\zeta) + \cosh_b(\beta\zeta) + s\right)}. \tag{44}$$

Family 12. For $\gamma = k\beta(k \neq 0)$ and $\alpha = 0$, a combined bright-singular soliton is obtained:

$$\vartheta_{37}(\zeta) = \frac{r\ b^{k\zeta}}{1 - k\ r\ b^{k\zeta}}. \tag{45}$$

Step 3. By Inserting $\vartheta(\zeta)$ (Eq. 6), along with Eq. 7, into Eq. 5, the positive integer m can be determined by employing the balance principle process between the orders of highest nonlinear and higher derivative terms in the resulting equation.

Step 4. Substituting the value of m into the obtained equation in the previous step, we get a polynomial of $\vartheta(\zeta)$. Collecting the coefficients of $\vartheta(\zeta)^j, j = 0, 1, 2, \ldots$, gives an algebraic system of unknown coefficients c_j, ν, α, β, γ, and $\delta_l, l = 0, \ldots, 3$.

Step 5. Finally, by solving the system with predetermined coefficients, and making backward substitutions completes the determinatio n of exact closed-form solutions for the considered problem in Eq. 3.

3 Mathematical analysis

In order to handle our problem in Eq. 1 along with the generalized pressure considered in Eq. 2, the new extended direct algebraic approach, discussed in the previous part, will be employed. By considering the wave's transform in Eq. 4, the following system of NODEs is executed

$$\begin{aligned} &vq'(\zeta) - v'(\zeta) = 0,\\ &vv'(\zeta) + p(q(\zeta))' = \varsigma v''(\zeta) + \sigma v^{(3)}(\zeta). \end{aligned} \tag{46}$$

Integrating once gives

$$\begin{aligned} &vq(\zeta) - v(\zeta) = 0,\\ &\delta_3 q(\zeta)^3 + \delta_2 q(\zeta)^2 + \delta_1 q(\zeta) - \sigma v''(\zeta) - \varsigma v'(\zeta) + vv(\zeta)' = 0. \end{aligned} \tag{47}$$

Now, by inserting the first part into the other, we get

$$-v\sigma q''(\zeta) - v\varsigma q'(\zeta) + \delta_0 + \left(\delta_1 + v^2\right)q(\zeta) + \delta_2 q(\zeta)^2 + \delta_3 q(\zeta)^3 = 0. \tag{48}$$

Achieving a balance between the highest-order derivative and the nonlinear term that emerge in vv'' and v^2v', implies

$$2m + 2 = 3m + 1 \Rightarrow m = 1.$$

Therefore, the formal solution of Eq. 48 is given by

$$q(\zeta) = c_1\vartheta(\zeta) + c_0, c_1 \neq 0. \tag{49}$$

Substituting into Eq. 48 through Eq. 7, gathering the coefficients with the same power of ϑ, and making them vanishes result the following set of simultaneous algebraic equations:

$$\begin{aligned} &\vartheta^0 : \alpha c_1 v \log(b)(\beta\sigma\log(b) - \varsigma) + c_0\left(\delta_1 + v^2\right) + c_0^3\delta_3 + c_0^2\delta_2 + \delta_0 = 0.\\ &\vartheta^1 : v\left(\sigma\log^2(b)\left(2\alpha\gamma + \beta^2\right) - \beta\varsigma\log(b) + v\right) + 3c_0^2\delta_3 + 2c_0\delta_2 + \delta_1 = 0.\\ &\vartheta^2 : \gamma v\log(b)(3\beta\sigma\log(b) - \varsigma) + c_1\left(3c_0\delta_3 + \delta_2\right) = 0.\\ &\vartheta^3 : 2\gamma^2 v\sigma\log^2(b) + c_1^2\delta_3 = 0. \end{aligned} \tag{50}$$

To extract the soliton solutions for the system under consideration, the Mathematica package is used to solve the obtained system. That is, to reach the values of free coefficients and regarding constraints. the following nontrivial and non-duplicate sets of parameters are obtained.

Set 1. For arbitrary $c_1 \neq 0$, and $\gamma \neq 0$, we get

$$
\begin{aligned}
\nu &= -\frac{c_1^2\delta_3}{2\gamma^2\sigma\log^2(b)},\\
c_0 &= \frac{1}{6}\left(\frac{c_1(3\beta\sigma\log(b)-\varsigma)}{\gamma\sigma\log(b)} - \frac{2\delta_2}{\delta_3}\right),\\
\delta_0 &= \frac{\left(\gamma\delta_2\sigma\log(b)-c_1\delta_3\varsigma\right)\left(4\gamma^2\delta_2^2\sigma^2\log^2(b)+c_1^2\delta_3^2\left(\varsigma^2-9\sigma^2\log^2(b)\left(\beta^2-4\alpha\gamma\right)\right)+4\gamma c_1\delta_3\delta_2\sigma\varsigma\log(b)\right)}{108\gamma^3\delta_3^2\sigma^3\log^3(b)},\\
\delta_1 &= \frac{4\gamma^4\delta_2^2\sigma^2\log^4(b)-c_1^2\delta_3^2\left(\gamma^2\log^2(b)\left(3\sigma^2\log^2(b)\left(\beta^2-4\alpha\gamma\right)+\varsigma^2\right)+3c_1^2\delta_3\right)}{12\gamma^4\delta_3\sigma^2\log^4(b)}.
\end{aligned}
\tag{51}
$$

Set 2. For $\varrho = 3\sigma^2\log^2(b)\left(\beta^2-4\alpha\gamma\right)+\varsigma^2$, we get

$$
\begin{aligned}
\nu &= \frac{\delta_3\varrho-\sqrt{\delta_3\left(48\delta_2^2\sigma^2+\delta_3\left(\varrho^2-144\delta_1\sigma^2\right)\right)}}{12\delta_3\sigma},\\
c_1 &= \mp\frac{\sqrt{\gamma^2\log^2(b)\left(\sqrt{\delta_3\left(48\delta_2^2\sigma^2+\delta_3\left(\varrho^2-144\delta_1\sigma^2\right)\right)}-\delta_3\varrho\right)}}{\sqrt{6}\delta_3},\\
c_0 &= \frac{\gamma\nu\log(b)(\varsigma-3\beta\sigma\log(b))-c_1\delta_2}{3c_1\delta_3},\\
\delta_0 &= \alpha c_1\nu\log(b)(\varsigma-\beta\sigma\log(b)) - c_0\left(\delta_1+\nu^2\right)+c_0^3\left(-\delta_3\right)-c_0^2\delta_2.
\end{aligned}
\tag{52}
$$

Set 3. For $c_0 = 0$, and $\varsigma \neq 3\beta\sigma\log(b)$, we get

$$
\begin{aligned}
\nu &= -\frac{2\delta_2^2\sigma}{\delta_3(\varsigma-3\beta\sigma\log(b))^2},\\
c_1 &= -\frac{2\gamma\delta_2\sigma\log(b)}{\delta_3(\varsigma-3\beta\sigma\log(b))},\\
\delta_0 &= \frac{4\alpha\gamma\delta_2^3\sigma^2\log^2(b)(\varsigma-\beta\sigma\log(b))}{\delta_3^2(\varsigma-3\beta\sigma\log(b))^3},\\
\delta_1 &= \frac{2\delta_2^2\sigma\left(\delta_3\log(b)(\varsigma-3\beta\sigma\log(b))^2\left(\sigma\log(b)\left(2\alpha\gamma+\beta^2\right)-\beta\varsigma\right)-2\delta_2^2\sigma\right)}{\delta_3^2(\varsigma-3\beta\sigma\log(b))^4}.
\end{aligned}
\tag{53}
$$

In what follow, the corresponding solitons will be derived.

4 Soliton solutions

By backward substitution along with Eq. 49, wave's traveling Eq. 4, and the mentioned solutions of Eq. 7, we list the dynamical behaviors through formal soliton solutions of gas volume, $q(x,t)$, according to the sets of parameters in Eq. 51. The velocity of gas, $v(x,t)$, can be derived easily by putting $v = \nu q$.

For **Set 1** in Eq. 51, the following solutions are received:

1. When $\Delta = \beta^2 - 4\alpha\gamma < 0$:

$$q_{1,1}(x,t)=\frac{1}{6}\left(\frac{c_1\left(3\sqrt{-\Delta}\sigma\ln(b)\tan_b\left(\frac{1}{2}\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)-\varsigma\right)}{\gamma\sigma\ln(b)}-\frac{2\delta_2}{\delta_3}\right), \quad (54)$$

$$q_{1,2}(x,t)=\frac{1}{6}\left(-\frac{c_1\left(3\sqrt{-\Delta}\sigma\ln(b)\cot_b\left(\frac{1}{2}\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)+\varsigma\right)}{\gamma\sigma\ln(b)}-\frac{2\delta_2}{\delta_3}\right), \quad (55)$$

$$q_{1,3}(x,t)=\frac{1}{6}\left(\frac{c_1\left(3\sqrt{-\Delta}\sigma\ln(b)\left(\tan_b\left(\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)\pm\sqrt{rs}\sec_b\left(\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)\right)-\varsigma\right)}{\gamma\sigma\ln(b)}-\frac{2\delta_2}{\delta_3}\right), \quad (56)$$

$$q_{1,4}(x,t)=\frac{1}{6}\left(\frac{c_1\left(3\sqrt{-\Delta}\sigma\ln(b)\left(-\cot_b\left(\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)\pm\sqrt{rs}\csc_b\left(\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)\right)-\varsigma\right)}{\gamma\sigma\ln(b)}-\frac{2\delta_2}{\delta_3}\right), \quad (57)$$

$$q_{1,5}(x,t)=\frac{\sigma\ln(b)\left(3c_1\delta_3\sqrt{-\Delta}\left(\tan_b\left(\frac{1}{4}\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)-\cot_b\left(\frac{1}{4}\sqrt{-\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\log^2(b)}\right)\right)\right)-4\gamma\delta_2\right)-2c_1\delta_3\varsigma}{12\gamma\delta_3\sigma\ln(b)}. \quad (58)$$

2. When $\Delta=\beta^2-4\alpha\gamma>0$:

$$q_{1,6}(x,t)=\frac{1}{6}\left(-\frac{c_1\left(3\sqrt{\Delta}\sigma\ln(b)\tanh_b\left(\frac{1}{2}\sqrt{\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+\varsigma\right)}{\gamma\sigma\ln(b)}-\frac{2\delta_2}{\delta_3}\right), \quad (59)$$

$$q_{1,7}(x,t)=\frac{1}{6}\left(-\frac{c_1\left(3\sqrt{\Delta}\sigma\ln(b)\coth_b\left(\frac{1}{2}\sqrt{\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+\varsigma\right)}{\gamma\sigma\ln(b)}-\frac{2\delta_2}{\delta_3}\right), \quad (60)$$

$$q_{1,8}(x,t)=\frac{1}{6}\left(-\frac{2\delta_2}{\delta_3}+\frac{c_1\left(-\varsigma+3\sqrt{\Delta}\sigma\ln(b)\left(-\tanh_b\left(\sqrt{\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\pm i\sqrt{rs}\operatorname{sech}_b\left(\sqrt{\Delta}\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right)\right)}{\gamma\sigma\ln(b)}\right), \quad (61)$$

$$q_{1,9}(x,t) = \frac{1}{6}\left(-\frac{2\delta_2}{\delta_3} + \frac{c_1\left(-\varsigma + 3\sqrt{\Delta}\sigma\ln(b)\left(-\coth_b\left(\sqrt{\Delta}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) \pm i\sqrt{rs}\operatorname{csch}_b\left(\sqrt{\Delta}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right)\right)}{\gamma\sigma\ln(b)}\right), \tag{62}$$

$$q_{1,10}(x,t) = -\frac{\sigma\ln(b)\left(3c_1\delta_3\sqrt{\Delta}\left(\tanh_b\left(\frac{1}{4}\sqrt{\Delta}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) + \coth_b\left(\frac{1}{4}\sqrt{\Delta}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right) + 4\gamma\delta_2\right) + 2c_1\delta_3\varsigma}{12\gamma\delta_3\sigma\ln(b)}. \tag{63}$$

3. When $\beta = 0$, and $\alpha\gamma > 0$:

$$q_{1,11}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{\alpha c_1\tan_b\left(\sqrt{\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)}{\sqrt{\alpha\gamma}} - \frac{\delta_2}{3\delta_3}, \tag{64}$$

$$q_{1,12}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} - \frac{\alpha c_1\cot_b\left(\sqrt{\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)}{\sqrt{\alpha\gamma}} - \frac{\delta_2}{3\delta_3}, \tag{65}$$

$$q_{1,13}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{\alpha c_1\left(\tan_b\left(\sqrt{\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right) \pm \sqrt{rs}\sec_b\left(\sqrt{\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right)\right)}{\sqrt{\alpha\gamma}} - \frac{\delta_2}{3\delta_3}, \tag{66}$$

$$q_{1,14}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{\alpha c_1\left(-\cot_b\left(\sqrt{\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right) \pm \sqrt{rs}\csc_b\left(\sqrt{\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right)\right)}{\sqrt{\alpha\gamma}} - \frac{\delta_2}{3\delta_3}, \tag{67}$$

$$q_{1,15}(x,t) = \frac{1}{6}\left(-\frac{c_1\left(3\sigma\sqrt{\alpha\gamma}\ln(b)\left(\cot_b\left(\frac{1}{2}\sqrt{\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) - \tan_b\left(\frac{1}{2}\sqrt{\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right) + \varsigma\right)}{\gamma\sigma\ln(b)} - \frac{2\delta_2}{\delta_3}\right). \tag{68}$$

4. When $\beta = 0$, and $\alpha\gamma < 0$:

$$q_{1,16}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{\alpha c_1\tanh_b\left(\sqrt{-\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)}{\sqrt{-\alpha\gamma}} - \frac{\delta_2}{3\delta_3}, \tag{69}$$

$$q_{1,17}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{\alpha c_1\coth_b\left(\sqrt{-\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)}{\sqrt{-\alpha\gamma}} - \frac{\delta_2}{3\delta_3}, \tag{70}$$

$$q_{1,18}(x,t) = -\frac{c_1\left(\varsigma - 6\sigma\sqrt{-\alpha\gamma}\ln(b)\left(-\tanh_b\left(\sqrt{-\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right) \pm i\sqrt{rs}\,\mathrm{sech}_b\left(\sqrt{-\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right)\right)\right)}{6\gamma\sigma\ln(b)} - \frac{\delta_2}{3\delta_3}, \tag{71}$$

$$q_{1,19}(x,t) = -\frac{c_1\left(\varsigma - 6\sigma\sqrt{-\alpha\gamma}\ln(b)\left(-\coth_b\left(\sqrt{-\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right) \pm \sqrt{rs}\,\mathrm{csch}_b\left(\sqrt{-\alpha\gamma}\left(2x - \frac{c_1^2\delta_3 t}{\gamma^2\sigma\ln(b)^2}\right)\right)\right)\right)}{6\gamma\sigma\ln(b)} - \frac{\delta_2}{3\delta_3}, \tag{72}$$

$$q_{1,20}(x,t) = \frac{1}{6}\left(-\frac{c_1\left(3\sigma\sqrt{-\alpha\gamma}\ln(b)\left(\tanh_b\left(\frac{1}{2}\sqrt{-\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) + \coth_b\left(\frac{1}{2}\sqrt{-\alpha\gamma}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right) + \varsigma\right)}{\gamma\sigma\ln(b)} - \frac{2\delta_2}{\delta_3}\right). \tag{73}$$

5. When $\beta = 0$, and $\alpha = \gamma$:

$$q_{1,21}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + c_1\tan_b\left(\gamma\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) - \frac{\delta_2}{3\delta_3}, \tag{74}$$

$$q_{1,22}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} - c_1\cot_b\left(\gamma\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) - \frac{\delta_2}{3\delta_3}, \tag{75}$$

$$q_{1,23}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + c_1\left(\tan_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right) \pm \sqrt{rs}\sec_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right)\right) - \frac{\delta_2}{3\delta_3}, \tag{76}$$

$$q_{1,24}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + c_1\left(-\cot_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right) \pm \sqrt{rs}\csc_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right)\right) - \frac{\delta_2}{3\delta_3}, \tag{77}$$

$$q_{1,25}(x,t) = \frac{1}{6}\left(-\frac{c_1\left(3\sqrt{\gamma^2}\sigma\ln(b)\left(\cot_b\left(\frac{1}{2}\sqrt{\gamma^2}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) - \tan_b\left(\frac{1}{2}\sqrt{\gamma^2}\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right) + \varsigma\right)}{\gamma\sigma\ln(b)} - \frac{2\delta_2}{\delta_3}\right). \tag{78}$$

6. When $\beta = 0$, and $\alpha = -\gamma$:

$$q_{1,26}(x,t) = -\frac{c_1\left(6\gamma\sigma\ln(b)\tanh_b\left(\gamma\left(x - \frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right) + \varsigma\right)}{6\gamma\sigma\ln(b)} - \frac{\delta_2}{3\delta_3}, \tag{79}$$

$$q_{1,27}(x,t) = -\frac{c_1\left(6\gamma\sigma\ln(b)\coth_b\left(\gamma\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+\varsigma\right)}{6\gamma\sigma\ln(b)} - \frac{\delta_2}{3\delta_3}, \tag{80}$$

$$q_{1,28}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + c_1\left(-\tanh_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right) \pm i\sqrt{rs}\,\mathrm{sech}_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right)\right) - \frac{\delta_2}{3\delta_3}, \tag{81}$$

$$q_{1,29}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + c_1\left(-\coth_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right) \pm \sqrt{rs}\,\mathrm{csch}_b\left(\frac{2\gamma^2 x - \frac{c_1^2\delta_3 t}{\sigma\ln(b)^2}}{\gamma}\right)\right) - \frac{\delta_2}{3\delta_3}, \tag{82}$$

$$q_{1,30}(x,t) = \frac{1}{6}\left(-\frac{c_1\left(3\gamma\sigma\ln(b)\left(\tanh_b\left(\frac{1}{2}\gamma\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+\coth_b\left(\frac{1}{2}\gamma\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)\right)+\varsigma\right)}{\gamma\sigma\ln(b)} - \frac{2\delta_2}{\delta_3}\right). \tag{83}$$

7. When $\beta^2 = 4\alpha\gamma$:

$$q_{1,31}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{2\gamma c_1\sigma\ln(b)}{c_1^2\delta_3 t - 2\gamma^2\sigma x\ln(b)^2} - \frac{\delta_2}{3\delta_3}. \tag{84}$$

8. When $\alpha = \beta = 0$:

$$q_{1,32}(x,t) = -\frac{c_1\varsigma}{6\gamma\sigma\ln(b)} + \frac{2\gamma c_1\sigma\ln(b)^2}{\log(b)\left(c_1^2\delta_3 t - 2\gamma^2\sigma x\ln(b)^2\right)} - \frac{\delta_2}{3\delta_3}. \tag{85}$$

9. When $\alpha = 0$, and $\beta \neq 0$:

$$q_{1,33}(x,t) = \frac{1}{6}\left(\frac{c_1\left(-\frac{6\beta r}{-\sinh_b\left(\beta\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+\cosh_b\left(\beta\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+r} - \frac{\varsigma}{\sigma\ln(b)} + 3\beta\right)}{\gamma} - \frac{2\delta_2}{\delta_3}\right), \tag{86}$$

$$q_{1,34}(x,t)=\frac{1}{6}\left(\frac{c_1\left(\frac{6\beta s}{\sinh_b\left(\beta\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+\cosh_b\left(\beta\left(x-\frac{c_1^2\delta_3 t}{2\gamma^2\sigma\ln(b)^2}\right)\right)+s}-\frac{\varsigma}{\sigma\ln(b)}-3\beta\right)}{\gamma}-\frac{2\delta_2}{\delta_3}\right). \tag{87}$$

10. When $\alpha=0$, and $\gamma=k\beta$:

$$q_{1,35}(x,t)=c_1\left(\frac{1}{\frac{b^{\frac{c_1^2\delta_3 t}{2\beta k^2\sigma\ln(b)^2}-\beta x}}{r}-k}-\frac{\varsigma}{6\beta k\sigma\ln(b)}+\frac{1}{2k}\right)-\frac{\delta_2}{3\delta_3}. \tag{88}$$

5 The fractional effects

The non integer derivative operators are undeveloped and straightforward. Nevertheless, fractional calculus is a key tool for modelling numerous phenomena raised across various scientific fields (Podlubny 1999; Hilfer 2000; Kilbas 2010; Singh et al. 2022; Zhang and Shu 2021; Anastassiou 2022; Wang et al. 2022). In the recent past, several kinds of fractional order derivatives appeared. The unconventional fractional operators, as in the case of Jumarie's modified Riemann–Liouville derivative (MRLFD) (Jumarie 2006), local fractional transform (LFD) (Yang et al. 2015), Beta (BFD) (Rahman et al. 2023, 2023; Atangana et al. 2016) and M-truncated (MTFD)derivatives (Rahman et al. 2023, 2023; Sousa et al. 2017), show the applicability in the traveling-wave theory of NLEEs. Such derivatives agree the majority of conventional Newtonian definitions and properties of derivative, namely, linearity, product, quotient, and chain rules.

Here, we limit the fractional effects on wave propagation of our model to those obtained by Beta, M-truncated, and modified Riemann–Liouville fractional derivatives. The basics used definitions and properties of these operators are as following:

Definition 1 (BFD) The ρ-order Beta fractional derivative of a function $q:[\tau,\infty)\to\mathbb{R}$, $\tau>0$, is defined as (Atangana et al. 2016)

$${}_0^A D_t^{\rho}(q(t))=\lim_{h\to 0}\frac{q\left(t+h\left(t+\frac{1}{\Gamma(\rho)}\right)^{1-\rho}\right)-q(t)}{h}. \tag{89}$$

Theorem 1 *Assume that q and p are β-differentiable with* $\rho\in(0,1]$, *the following properties are satisfied*:

1. *The BDF is linear.*

2. ${}_0^A D_t^{\rho}(C)=0, C\in\mathbb{R}$

3. ${}_0^A D_t^{\rho}(q(t).p(t)) = p(t){}_0^A D_t^{\rho}(q(t)) + q(t){}_0^A D_t^{\rho}(p(t))$.

4. ${}_0^A D_t^{\rho}(q(t).p(t)) = p(t){}_0^A D_t^{\rho}(q(t)) + q(t){}_0^A D_t^{\rho}(p(t))$, *provided* $p(t) \neq 0$.

5. ${}_0^A D_t^{\rho}(q(p(t))) = p'(t){}_0^A D_t^{\rho}(q(p(t)))$, *provided that* $p(t)$ *is differentiable, and* $q(t)$ *is* ρ*-differentiable on the range of* p.

6. ${}_0^A D_t^{\rho}(q(t)) = \left(t + \frac{1}{\Gamma(\rho)}\right)^{1-\rho} q'(t)$, *provided that* $q(t)$ *is differentiable*.

Definition 2 (MTFD) The ρ-order M-truncated fractional derivative of a function $q : [\tau, \infty) \to \mathbb{R}, \tau > 0$, is defined as (Sousa et al. 2017)

$$
{}_i D_M^{\rho,\eta}(q(t)) = \lim_{h\to 0} \frac{q\left({}_i E_\eta(h t^{-\rho})t\right) - q(t)}{h}, \tag{90}
$$

where, $\eta > 0$, and

$$
{}_i E_\eta(s) = \sum_{j=0}^{i} \frac{s^j}{\Gamma(j\eta + 1)}, \eta > 0, s \in \mathbb{C}, \tag{91}
$$

defines the one-parameter truncated Mittag-Leffler function. The MTFD satisfies the first five properties in Theorem 1. In addition, the following property:

$$
{}_i D_M^{\rho,\eta}(q(t)) = \frac{t^{1-\rho}}{\Gamma(\eta + 1)} q'(t), \tag{92}
$$

is verified. The M-truncated fractional operator is considered as a generalization of the conformable fractional derivative (Az-Zo'bi et al. 2021).

Definition 3 (MRLFD) Suppose $q(x,t), t \in \mathbb{R}$ is continuous. The ρ-order modified Riemann–Liouville derivative is given by

$$
D_t^{\rho}(q(t)) = \begin{cases} \frac{1}{\Gamma(-\rho)} \frac{d}{dt} \int\limits_0^t \frac{q(\tau)-q(0)}{(t-\tau)^{\rho+1}} d\tau, & \rho < 0, \\ \frac{1}{\Gamma(1-\rho)} \frac{d}{dt} \int\limits_0^t \frac{q(\tau)-q(0)}{(t-\tau)^{\rho}} d\tau, & 0 < \rho < 1, \\ \left(q^{(\rho-n)}(t)\right)^{(n)}, & 1 \le n \le \rho < n+1. \end{cases} \tag{93}
$$

The MRLFD satisfies classical linearity, product, and chain rules of integer-order derivative. Moreover, for $\varrho > 0$, we have

$$
D_t^{\rho}(t^{\varrho}) = \frac{\Gamma(1+\varrho)}{\Gamma(1+\xi-\rho)} t^{\varrho-\rho}. \tag{94}
$$

The time-fractional version of the p–system in Eq. 1 is considered. The NEDAM is employed with the following modifications of wave transformation Eq. 4:

$$
\zeta = x + \frac{\nu}{\rho}\left(t + \frac{1}{\Gamma(\rho)}\right)^{\rho}, \tag{95}
$$

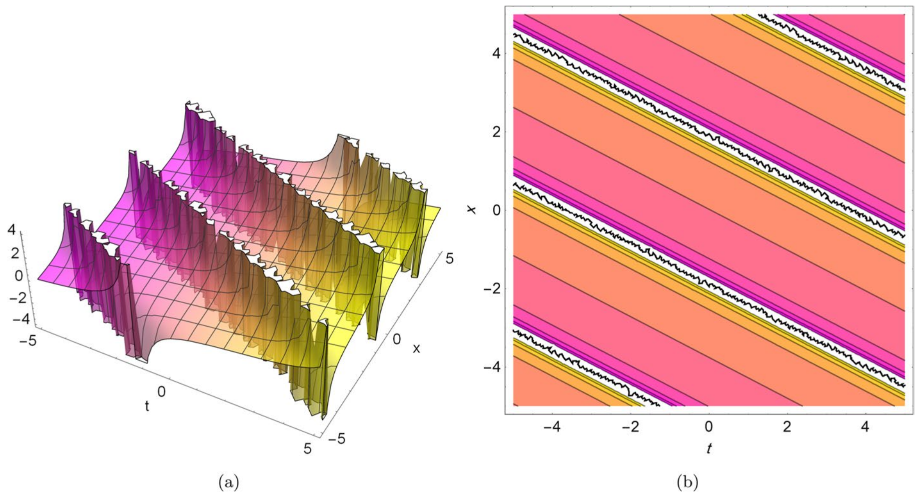


Fig. 1 The 3D **a** dynamical behavior and regarding, **b** contour plots of the gas volume in Eq. 54

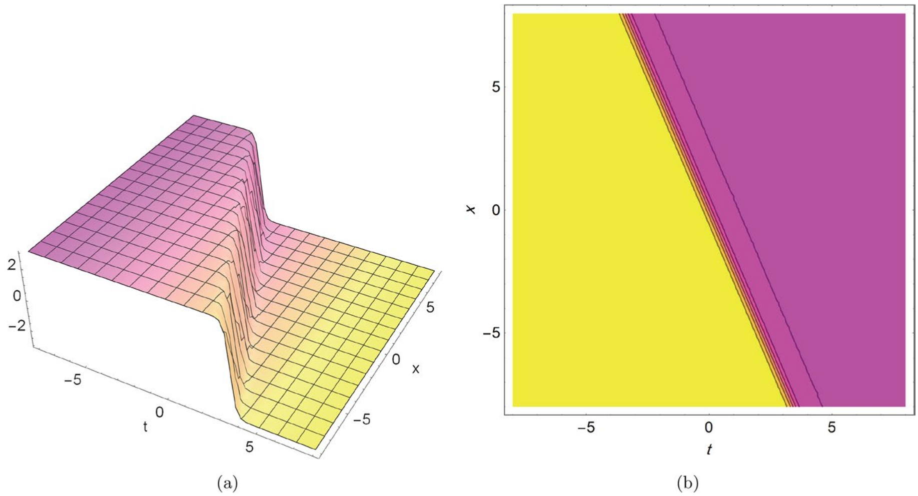


Fig. 2 The 3D **a** dynamical behavior and regarding, **b** contour plots of the gas volume in Eq. 86

$$\zeta = x + \nu \frac{\Gamma(\eta+1)}{\rho} t^{\rho}, \tag{96}$$

and,

$$\zeta = x + \nu \frac{t^{\rho}}{\Gamma(\rho+1)}, \tag{97}$$

for the BFD, MTFD, and MRLFD respectively. The rest of solving steps carry out identically as listed in Sect. 4. Also, the soliton solution will be as above with replacing the parameter ζ by its corresponding value in each fractional derivative case. The assumption

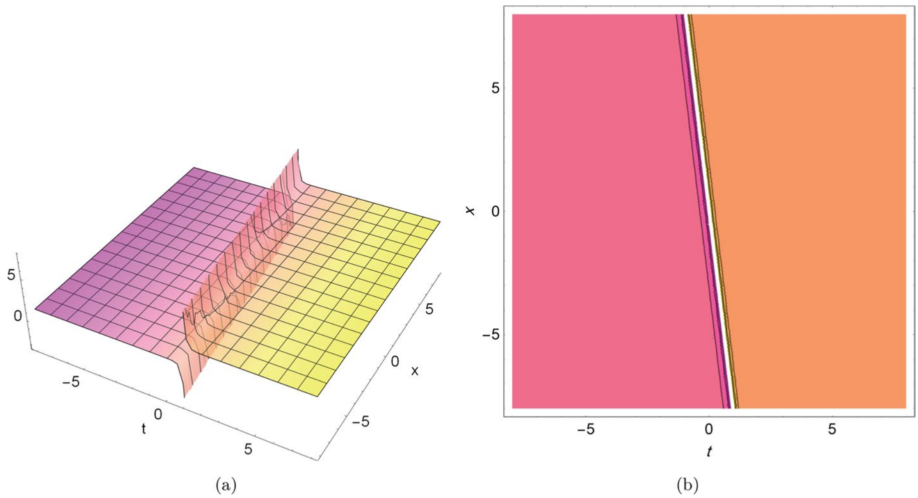


Fig. 3 The 3D **a** dynamical behavior and regarding, **b** contour plots of the gas volume in Eq. 88

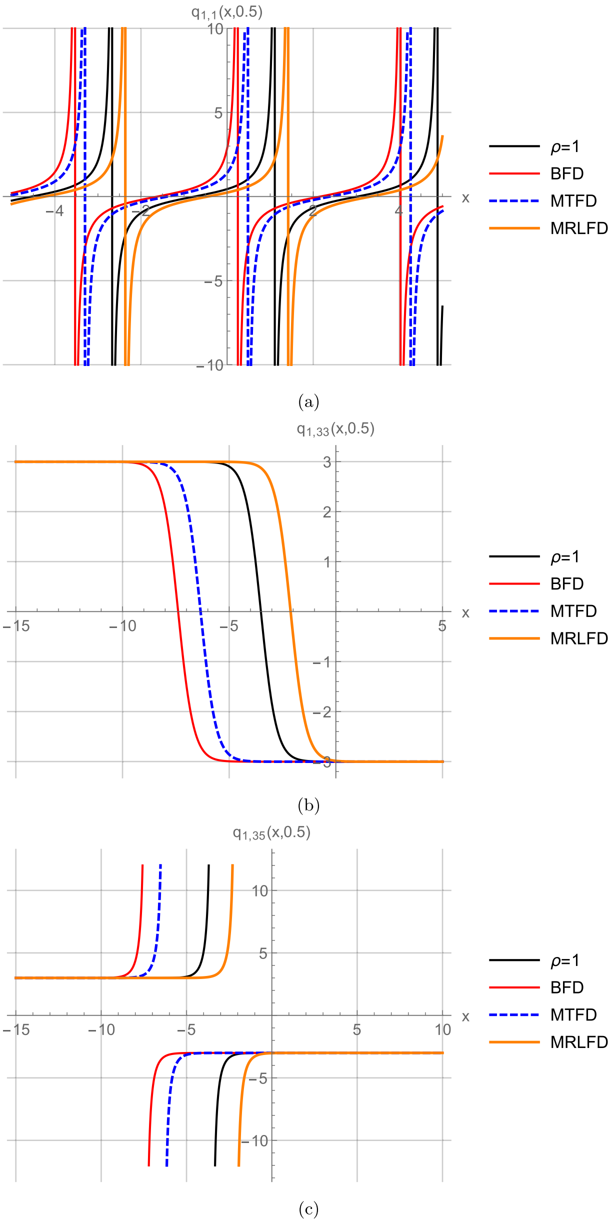
of wave transformation in the local fractional derivative (LFD) situation, the parameter ζ is postulated as in Eq. 96.

6 Discussion and conclusion

In this article, the (1+1) dimensional non-monotone *p*-system with general cubic nonlinear viscosity-capillarity van der Waals pressure is presented, and studied analytically for the first time. The system arises in fluid dynamics, especially in the gas longitudinal isothermal transition. The recent developed traveling-wave technique, named the extended direct algebraic method (NEDAM)(Almatrafi et al. 2021; Tasnim et al. 2023; Mirhosseini-Alizamini et al. 2020; Javad Vahidi et al. 2021; Salathiel et al. 2019; Kurt et al. 2020; Günerhan et al. 2020; Hussain et al. 2022; Munawar et al. 2021; Iqbal et al. 2023), is applied to derive various types of soliton solutions for the considered model. This method is a generalization of the well-known generalized Riccati equation mapping method (Altawallbeh et al. 25022; Az-Zo'bi et al. 2022). Within two nonempty hyperbolic and elliptic regions, the gas transition ill-posedness causes an instability while pursuing numeric, or numeric-analytical solutions. To overcome such cases, the closed form analytical solutions gained the researchers' interests.

Through the listed solutions of the auxiliary equation Eq. 7, which can be derived by the ansatz method and normal techniques for special cases, the NEDAM guarantees conditional existence of periodic, singular, dark, mixed bright-dark, and combined soliton solutions of nonlinear evolution equations (NLEEs). In our case, three different sets of existence constrains have been derived by the aid of symbolic computation MATHEMATICA. Many other sets can be derived. For the set of parameters in Eq. 51, all possible solitons were typed in Sect. 4. The 3D propagation dynamical behavior of some derived solitons and the corresponding contours are figured as follows:

Fig. 4 The 2D time-fractional effect on **a** $q_{1,1}(x, 0.5)$, Eq. 54, **b** $q_{1,33}(x, 0.5)$, Eq. 86, and **c** $q_{1,35}(x, 0.5)$, Eq. 88, with $\rho = 0.3$, and $P = 0$



1. Figure 1 shows the dynamical behavior of gas volume $q_{1,1}(x, t)$, Eq. 54. The periodic wave propagation is obtained subject to $b = 2, \gamma = 2, \alpha = 0.75, \beta = 0.5, \sigma = 1, \varsigma = 0.1, c_1 = 1, \delta_2 = 0, \delta_3 = -2$, and $r = s = 1$.

2. The anti-kink dark soliton solution of gas volume $q_{1,31}(x, t)$, Eq. 86, is plotted in Fig. 2 for $b = 2, \beta = 4, \gamma = 2, \sigma = 1, \varsigma = 0.1, c_1 = 3, \delta_2 = 0.1, \delta_3 = -1$, and $r = s = 2$.

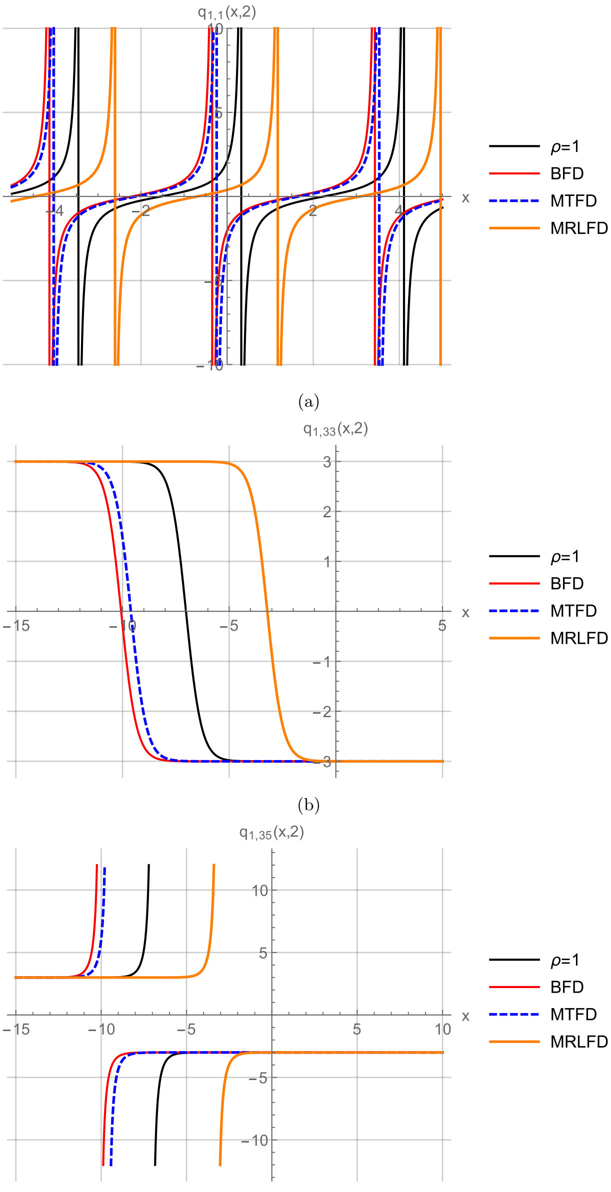


Fig. 5 The 2D time-fractional effect on **a** $q_{1,1}(x, 2)$, Eq. 54, **b** $q_{1,33}(x, 2)$, Eq. 86, and **c** $q_{1,35}(x, 2)$, Eq. 88, with $\rho = 0.3$, and $P = 0$

3. The singular-kink dark soliton solution of gas volume $q_{1,35}(x, t)$, Eq. 88, is depicted in Fig. 3 for $b = 2, \beta = 1, k = 0.5, \sigma = 0.5, \varsigma = 2, c_1 = -1, \delta_2 = 1, \delta_3 = -1$, and $r = s = 2$.

4. Figure 4 shows the 2D (0.3)-order time-fractional effects of BFD, MTFD, and MRLFD for the solitons in Figs. 1, 2, and 3 respectively for $t = 0.5$. For $t = 2$ and $\rho = 0.3$, the comparison of considered fractional derivatives with the first-order derivative of obtained solitons in Figs. 1, 2, and 3 is illustrated.

Since $\gamma \neq 0$, The mixed bright-dark soliton can't be obtained for this set of parameters. Many other periodic, singular, kink, and kink-like structures can be constructed with the free choices of unconditional parameters. These portraits clearly show the stability of solutions on the indicated domains. Also, they will be helpful to conclude the physical properties of the considered model that allow specialists to draw conclusions in an efficient way (Fig. 5).

To generalize our work, due to the high importance of fractional calculus, the time-fractional versions, in the Beta, M-truncated and modified Riemann–Liouville senses, are analyzed, compared, and illustrated.

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Declarations

Conflict of interest We declare that we have no competing interest in publishing this article.

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