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Stationary optical solitons with complex Ginzburg–Landau equation having nonlinear chromatic dispersion

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Abstract
The current work is on the retrieval of stationary soliton solutions to the complex Ginzburg–Landau equation that is studied with nonlinear chromatic dispersion having a plethora of nonlinear refractive index structures. The Jacobi's elliptic function approach is employed to recover doubly periodic waves which leads to soliton solutions when the limiting value of the modulus of ellipticity is reached.

Keywords Stationary solitons · Nonlinear chromatic dispersion · Complex Ginzburg–Landau equation

Mathematics Subject Classification 060.2310 · 060.4510 · 060.5530 · 190.3270 · 190.4370

1 Introduction

The analytics and the rapid advancement of the technology of optical solitons have portrayed a lasting impression in the telecommunications industry (Biswas 2009; Triki et al. 2012; Mirzazadeh et al. 2016; Arnous et al. 2017; Biswas and Alqahtani 2017; Biswas et al. 2018a, b; Biswas 2018; Arshed et al. 2019; Das et al. 2019; Yıldırım et al. 2020; Zayed et al. 2020, 2021; Yıldırım et al. 2020; Yan et al. 2020; Biswas et al. 2021; Biswas et al. 2021, 2018, 2012; Mirzazadeh et al. 2014; Liu et al. 2018; Biswas et al. 2016; Biswas and Arshed 2018; Biswas 2009; Liu et al. 2019; Bakodah et al. 2017; Zhou et al. 2014; Adem et al. 2020, 2020, 2021; Atai and Malomed 2001; Biswas and Konar 2006; Biswas and Khalique 2011, 2013; Biswas et al. 2018; Ekici et al. 2018, 2021; Geng and Li 2008; Guo and Zhou 2010; Kara 2021; Kudryashov 2019, 2020a, b, c, d, e, f, 2021a, b, c; Sonmezoglu et al. 2021; Sucu et al. 2021; Susanto and Malomed 2021; Yan 2006a, b; Zhang et al. 2010; Zhou et al. 2016; Zayed 2009; Malik et al. 2012). This gave way to a plethora of results and uncountable avenues for performance enhancement in this field. There are

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several models that govern the dynamical flow and of solitons across inter–continental distances. While the most visible model is the nonlinear Schrödinger's equation, it is often necessary to veer off to other models depending on the circumstantial situation. For example, dispersive solitons are governed by Schrödinger–Hirota equation or Fokas–Lenells equation and others.

Today's paper will address the complex Ginzburg–Landau equation (CGLE) (Biswas 2009; Triki et al. 2012; Mirzazadeh et al. 2016; Arnous et al. 2017; Biswas and Alqahtani 2017; Biswas et al. 2018a, b; Biswas 2018; Arshed et al. 2019; Das et al. 2019; Yıldırım et al. 2020; Zayed et al. 2020; Yıldırım et al. 2020; Yan et al. 2020; Biswas et al. 2021; Zayed et al. 2021; Biswas et al. 2021) that is also an alternative model that governs the soliton propelling dynamics for long distances. This model is studied with nonlinear chromatic dispersion (CD). Several forms of self-phase modulation (SPM) structures (Biswas and Konar 2006) are studied in the paper. Rough handling of fibers and other issues, such as environmental causes, may lead to CD being rendered nonlinear. In such a situation the solitons would become stationary and thus the information transfer for trans-continental and trans-oceanic distances would completely stall. This would lead to a catastrophic effect especially during COVID-19 times when the world is totally dependent on Internet activities. The analytical derivation of these stationary solitons for an abundant variety of SPM are displayed in the rest of the work. The stationary solitons are derived through an intermediary Jacobi's elliptic functions that approach soliton solutions when the modulus of ellipticity approaches its appropriate limit. The details are exhibited in the rest of the paper after a succinct intro to the model.

1.1 Governing model

The dimensionless form of CGLE with nonlinear CD reads as (Biswas 2009; Triki et al. 2012; Mirzazadeh et al. 2016; Arnous et al. 2017; Biswas and Alqahtani 2017; Biswas et al. 2018a, 2018b; Biswas 2018; Arshed et al. 2019; Das et al. 2019; Yıldırım et al. 2020; Zayed et al. 2020; Yıldırım et al. 2020; Yan et al. 2020; Biswas et al. 2021; Zayed et al. 2021; Biswas et al. 2021)

$$iq_t + a(|q|^n q)_{xx} + bF\left(|q|^2\right)q = \frac{1}{|q|^2 q^*}\left[\alpha |q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \qquad (1)$$

where a, b, α, β and γ are constants and F stands for the nonlinear function. The first term stands for the linear evolution term, while the coefficient of a is the nonlinear CD and the third term accounts for the generalized nonlinear term. Next, the terms with α, β and γ arise from the perturbation effects; in particular γ comes from the detuning effect. Also, in the model (1), the independent variables are x and t which are spatial and temporal coordinates. The dependent variable $q(x, t)$ is a complex-valued function which stands for the wave profile, $q^*(x, t)$ denotes the conjugate of $q(x, t)$ and finally $i = \sqrt{-1}$.

2 Mathematical analysis

To extract stationary solutions to (1), initial assumption (Adem et al. 2020, 2020, 2021; Biswas and Khalique 2011, 2013; Biswas et al. 2018, Ekici et al. 2018, 2021; Sonmezoglu et al. 2021; Sucu et al. 2021)

$$q(x,t)=\phi(x)e^{i\lambda t} \tag{2}$$

is considered. Here the constant λ is the wave number. Substituting (2) into (1), it is reached that

$$\begin{aligned} &-(\gamma+\lambda)\phi^2+b\phi^2G(\phi^2)-2(\alpha-2\beta)(\phi')^2+an(n+1)\phi^n(\phi')^2\\ &-2\alpha\phi\phi''+a(n+1)\phi^{n+1}\phi''=0. \end{aligned} \tag{3}$$

Equation (3) will now be analyzed according to the type of nonlinear media in next subsections.

2.1 Kerr law

For this nonlinearity,

$$F(s)=s. \tag{4}$$

Thus, (1) becomes

$$iq_t+a(|q|^nq)_{xx}+b|q|^2q=\frac{1}{|q|^2q^*}\left[\alpha|q|^2(|q|^2)_{xx}-\beta\{(|q|^2)_x\}^2\right]+\gamma q. \tag{5}$$

For $n=1$, Eq. (5) can be integrated. Thus, Eq. (5) simplifies to

$$iq_t+a(|q|q)_{xx}+b|q|^2q=\frac{1}{|q|^2q^*}\left[\alpha|q|^2(|q|^2)_{xx}-\beta\{(|q|^2)_x\}^2\right]+\gamma q \tag{6}$$

and Eq. (3) changes to

$$-(\gamma+\lambda)\phi^2+b\phi^4-2(\alpha-2\beta)(\phi')^2+2a\phi(\phi')^2-2\alpha\phi\phi''+2a\phi^2\phi''=0. \tag{7}$$

Generalized (G'/G)-expansion approach will now be applied to deal with (7). To kick off, suppose Eq. (7) possess the solution as (Zayed 2009; Malik et al. 2012)

$$\phi(x)=\sum_{i=0}^{N}\alpha_i\left(\frac{G'}{G}\right)^i \tag{8}$$

where $G=G(x)$ holds

$$[G'(x)]^2=e_2G^4(x)+e_1G^2(x)+e_0 \tag{9}$$

that is called Jacobi elliptic (JE) equation. Here α_i, e_0, e_1 and e_2 are the arbitrary constants that need to be fixed such that $\alpha_n\neq 0$. The solutions of Eq. (9) are presented as follows (Zayed 2009; Malik et al. 2012):

Case	e_0	e_1	e_2	$G(x)$	$G'(x)$
1	1	$-(1+k^2)$	k^2	$\operatorname{sn} x$	$\operatorname{cn} x \operatorname{dn} x$
2	1	$-(1+k^2)$	k^2	$\operatorname{cd} x$	$-(1-k^2)\operatorname{sd} x \operatorname{nd} x$
3	$1-k^2$	$2k^2-1$	$-k^2$	$\operatorname{cn} x$	$-\operatorname{sn} x \operatorname{dn} x$
4	k^2-1	$2-k^2$	-1	$\operatorname{dn} x$	$-k^2 \operatorname{sn} x \operatorname{cn} x$
5	k^2	$-(k^2+1)$	1	$\operatorname{ns} x$	$-\operatorname{ds} x \operatorname{cs} x$
6	k^2	$-(k^2+1)$	1	$\operatorname{dc} x$	$(1-k^2)\operatorname{nc} x \operatorname{sc} x$
7	$-k^2$	$2k^2-1$	$1-k^2$	$\operatorname{nc} x$	$\operatorname{sc} x \operatorname{dc} x$
8	-1	$2-k^2$	k^2-1	$\operatorname{nd} x$	$k^2 \operatorname{sd} x \operatorname{cd} x$
9	$1-k^2$	$2-k^2$	1	$\operatorname{cs} x$	$-\operatorname{ns} x \operatorname{ds} x$
10	1	$2-k^2$	$1-k^2$	$\operatorname{sc} x$	$\operatorname{nc} x \operatorname{dc} x$
11	1	$2k^2-1$	$k^2(k^2-1)$	$\operatorname{sd} x$	$\operatorname{nd} x \operatorname{cd} x$
12	$k^2(k^2-1)$	$2k^2-1$	1	$\operatorname{ds} x$	$-\operatorname{cs} x \operatorname{ns} x$
13	$\frac{1}{4}$	$\frac{1}{2}(1-2k^2)$	$\frac{1}{4}$	$\operatorname{ns} x \pm \operatorname{cs} x$	$-\operatorname{ds} x \operatorname{cs} x \mp \operatorname{ns} x \operatorname{ds} x$
14	$\frac{1}{4}(1-k^2)$	$\frac{1}{2}(1+k^2)$	$\frac{1}{4}(1-k^2)$	$\operatorname{nc} x \pm \operatorname{sc} x$	$\operatorname{sc} x \operatorname{dc} x \pm \operatorname{nc} x \operatorname{dc} x$
15	$\frac{k^2}{4}$	$\frac{1}{2}(k^2-2)$	$\frac{1}{4}$	$\operatorname{ns} x \pm \operatorname{ds} x$	$-\operatorname{ds} x \operatorname{cs} x \mp \operatorname{cs} x \operatorname{ns} x$
16	$\frac{k^2}{4}$	$\frac{1}{2}(k^2-2)$	$\frac{k^2}{4}$	$\operatorname{sn} x \pm i \operatorname{cn} x$	$\operatorname{dn} x \operatorname{cn} x \mp i \operatorname{sn} x \operatorname{dn} x$
17	0	1	-1	$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
18	0	1	1	$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$
19	0	-1	1	$\sec x$	$\sec x \tan x$
20	0	0	1	$\frac{1}{\xi}$	$-\frac{1}{\xi^2}$
21	0	$-(1+k^2)$	k^2	$\operatorname{sn} x$	$\operatorname{cn} x \operatorname{dn} x$

Here, the modulus of JE functions is stood for by $k\,(0 < k < 1)$ and $i = \sqrt{-1}$.

Balancing ϕ^4 with $\phi\left(\phi'\right)^2$ or $\phi^2\phi''$ leads to $N = 2$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2. \tag{10}$$

Inserting (10) along with (9) into (7), one recovers a polynomial in G^j, $G'G^j$ $(j = \pm 1, \pm 2, \ldots)$. Equating each coefficient of the polynomial obtained to zero and then overcoming the resulting systems yields

$$b = \frac{80a^2e_1}{\alpha}, \quad \alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{\alpha}{4ae_1}, \quad \beta = \frac{3\alpha}{4}, \quad \lambda = -\gamma + 5\alpha e_1 + \frac{12\alpha e_0 e_2}{e_1}, \tag{11}$$

$$b = -\frac{20a^2e_1}{\alpha}, \quad e_0 = 0, \quad \alpha_0 = -\frac{\alpha}{a}, \quad \alpha_1 = 0, \quad \alpha_2 = \frac{\alpha}{ae_1}, \quad \beta = \frac{\alpha}{4}, \quad \lambda = -\gamma - 12\alpha e_1. \tag{12}$$

Substituting (11) into (10) and employing (2) gives

$$q(x,t) = -\frac{\alpha}{4ae_1}\left(\frac{G'}{G}\right)^2 \exp\left[i\left(-\gamma + 5\alpha e_1 + \frac{12\alpha e_0 e_2}{e_1}\right)t\right]. \tag{13}$$

Next, solutions for the model under consideration (6) are attained as below:

If $e_0 = 1$, $e_1 = -\left(k^2+1\right)$, $e_2 = k^2$,

$$q(x,t)=\frac{\alpha}{4a\left(k^2+1\right)}\operatorname{cs}^2x\operatorname{dn}^2x\exp\left[-i\left(\gamma+5\alpha\left(k^2+1\right)+\frac{12\alpha k^2}{k^2+1}\right)t\right] \tag{14}$$

or

$$q(x,t)=\frac{\alpha\left(1-k^2\right)^2}{4a\left(k^2+1\right)}\operatorname{sc}^2x\operatorname{nd}^2x\exp\left[-i\left(\gamma+5\alpha\left(k^2+1\right)+\frac{12\alpha k^2}{k^2+1}\right)t\right]. \tag{15}$$

For $e_0=1-k^2, e_1=2k^2-1, e_2=-k^2$,

$$q(x,t)=-\frac{\alpha}{4a\left(2k^2-1\right)}\operatorname{sc}^2x\operatorname{dn}^2x\exp\left[i\left(-\gamma+5\alpha\left(2k^2-1\right)+\frac{12\alpha k^2\left(k^2-1\right)}{2k^2-1}\right)t\right]. \tag{16}$$

When $e_0=k^2-1, e_1=2-k^2, e_2=-1$,

$$q(x,t)=-\frac{\alpha k^4}{4a\left(2-k^2\right)}\operatorname{sd}^2x\operatorname{cn}^2x\exp\left[i\left(-\gamma+5\alpha\left(2-k^2\right)+\frac{12\alpha\left(1-k^2\right)}{2-k^2}\right)t\right]. \tag{17}$$

Whenever $e_0=k^2, e_1=-\left(k^2+1\right), e_2=1$,

$$q(x,t)=\frac{\alpha}{4a\left(k^2+1\right)}\operatorname{ds}^2x\operatorname{cn}^2x\exp\left[-i\left(\gamma+5\alpha\left(k^2+1\right)+\frac{12\alpha k^2}{k^2+1}\right)t\right] \tag{18}$$

or

$$q(x,t)=\frac{\alpha\left(1-k^2\right)^2}{4a\left(k^2+1\right)}\operatorname{sc}^2x\operatorname{nd}^2x\exp\left[-i\left(\gamma+5\alpha\left(k^2+1\right)+\frac{12\alpha k^2}{k^2+1}\right)t\right]. \tag{19}$$

In the case of $e_0=1, e_1=2k^2-1, e_2=k^2\left(k^2-1\right)$,

$$q(x,t)=-\frac{\alpha}{4a\left(2k^2-1\right)}\operatorname{cd}^4x\operatorname{ns}^2x\exp\left[i\left(-\gamma+5\alpha\left(2k^2-1\right)+\frac{12\alpha k^2\left(k^2-1\right)}{2k^2-1}\right)t\right]. \tag{20}$$

For the case $e_0=\frac{k^2}{4}, e_1=\frac{1}{2}\left(k^2-2\right), e_2=\frac{1}{4}$,

$$q(x,t)=-\frac{\alpha}{2a\left(k^2-2\right)}\operatorname{cs}^2x\exp\left[i\left(-\gamma+\frac{5\alpha\left(k^2-2\right)}{2}+\frac{3\alpha k^2}{2\left(k^2-2\right)}\right)t\right]. \tag{21}$$

If $e_0=\frac{k^2}{4}, e_1=\frac{1}{2}\left(k^2-2\right), e_2=\frac{k^2}{4}$,

$$q(x,t)=\frac{\alpha}{2a\left(k^2-2\right)}\operatorname{dn}^2x\exp\left[i\left(-\gamma+\frac{5\alpha\left(k^2-2\right)}{2}+\frac{3\alpha k^4}{2\left(k^2-2\right)}\right)t\right]. \tag{22}$$

Here, the solutions from (14) to (22) represent JE function solutions to the model.

Next, for $e_0=0, e_1=1, e_2=-1$, dark soliton (Fig. 1) is

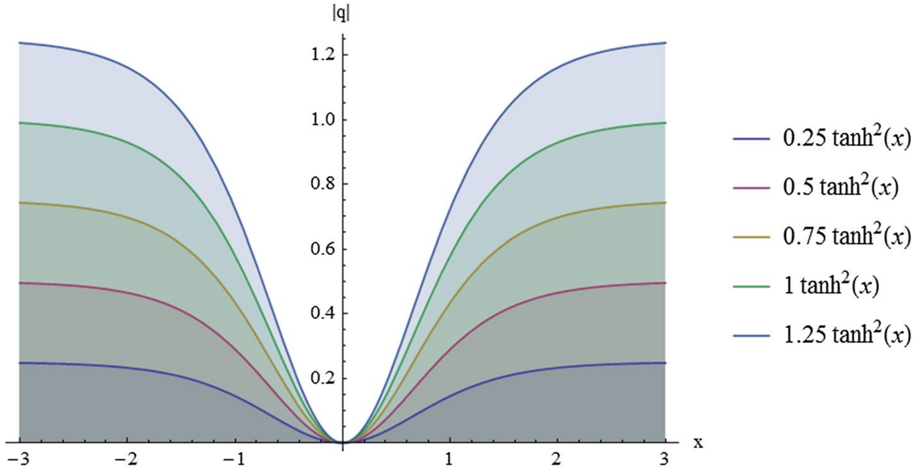

Fig. 1 Profile of dark soliton (23) for $a = -1$ and $\alpha = 1, 2, 3, 4, 5$, respectively

$$q(x,t) = -\frac{\alpha}{4a}\tanh^2 x \exp\left[i(-\gamma + 5\alpha)t\right]. \tag{23}$$

When $e_0 = 0, e_1 = 1, e_2 = 1$, singular soliton is

$$q(x,t) = -\frac{\alpha}{4a}\coth^2 x \exp\left[i(-\gamma + 5\alpha)t\right]. \tag{24}$$

Finally, if $e_0 = 0, e_1 = -1, e_2 = 1$, periodic solution is

$$q(x,t) = \frac{\alpha}{4a}\tan^2 x \exp\left[-i(\gamma + 5\alpha)t\right]. \tag{25}$$

Similarly, plugging (12) into (10) and utilizing (2) gives

$$q(x,t) = \left\{-\frac{\alpha}{a} + \frac{\alpha}{ae_1}\left(\frac{G'}{G}\right)^2\right\}\exp\left[-i\left(\gamma + 12\alpha e_1\right)t\right] \tag{26}$$

and then one gets the following solutions:

For $e_0 = 0, e_1 = 1, e_2 = -1$, bright soliton (Fig. 2) is

$$q(x,t) = -\frac{\alpha}{a}\operatorname{sech}^2 x \exp\left[-i(\gamma + 12\alpha)t\right]. \tag{27}$$

If $e_0 = 0, e_1 = 1, e_2 = 1$, other type of singular soliton emerges as

$$q(x,t) = \frac{\alpha}{a}\operatorname{csch}^2 x \exp\left[-i(\gamma + 12\alpha)t\right]. \tag{28}$$

When $e_0 = 0, e_1 = -1, e_2 = 1$, periodic wave is

$$q(x,t) = -\frac{\alpha}{a}\sec^2 x \exp\left[-i(\gamma - 12\alpha)t\right]. \tag{29}$$

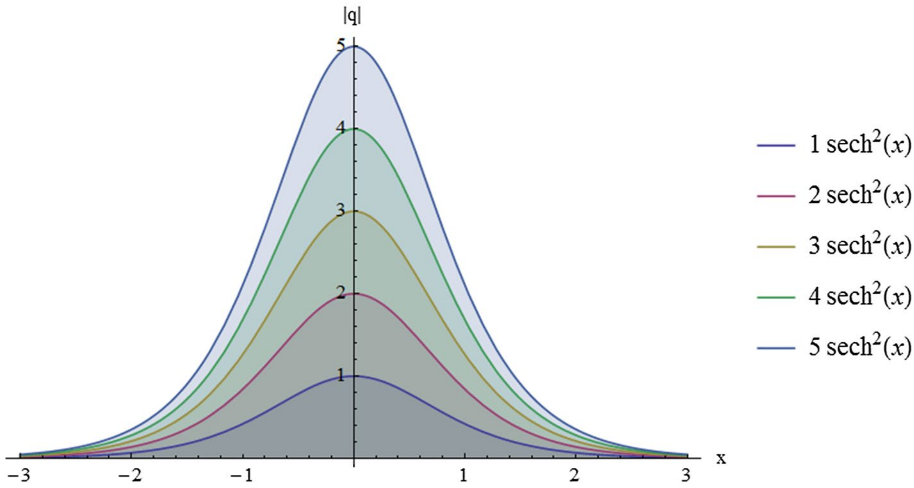

Fig. 2 Profile of bright soliton (27) for $a = -1$ and $\alpha = 1, 2, 3, 4, 5$, respectively

2.2 Power law

Power law is formulated as

$$F(s) = s^m. \tag{30}$$

Then Eq. (1) changes to

$$iq_t + a(|q|^n q)_{xx} + b|q|^{2m}q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{31}$$

When $n = m$, (31) can be integrated. Thus, (31) simplifies to

$$iq_t + a(|q|^m q)_{xx} + b|q|^{2m}q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{32}$$

and Eq. (3) changes to

$$\begin{aligned} &-(\gamma+\lambda)\phi^2 + b\phi^{2m+2} - 2(\alpha-2\beta)\left(\phi'\right)^2 + am(m+1)\phi^m\left(\phi'\right)^2 \\ &\quad - 2\alpha\phi\phi'' + a(m+1)\phi^{m+1}\phi'' = 0. \end{aligned} \tag{33}$$

Set

$$\phi = \varphi^{\frac{2}{m}} \tag{34}$$

so that Eq. (33) transforms to

$$\begin{aligned} &-m^2(\gamma+\lambda)\varphi^2 + bm^2\varphi^6 + 4(4\beta + \alpha(m-4))\left(\varphi'\right)^2 + 2a\left(m^2+3m+2\right)\varphi^2\left(\varphi'\right)^2 \\ &\quad - 4\alpha m\varphi\varphi'' + 2am(m+1)\varphi^3\varphi'' = 0. \end{aligned} \tag{35}$$

Balance principle causes $N = 1$. Then Eq. (8) becomes

$$\varphi(x)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right). \tag{36}$$

Proceeding as in the case of Kerr law, one has the solution set as:

$$\begin{aligned} b&=\frac{2a^2e_1(m+1)^3(3m+2)}{\alpha m^3},\quad \alpha_0=0,\quad \alpha_1=\frac{1}{m+1}\sqrt{-\frac{\alpha m}{ae_1}},\\ \beta&=\alpha-\frac{\alpha m}{4},\quad \gamma=\frac{2\alpha e_1^2(3m+2)+8\alpha e_0e_2(m+2)-e_1\lambda m(m+1)}{e_1m(m+1)}. \end{aligned} \tag{37}$$

Substituting (37) into (36) and employing (2) gives

$$q(x,t)=\left\{\frac{1}{m+1}\sqrt{-\frac{\alpha m}{ae_1}}\left(\frac{G'}{G}\right)\right\}^{\frac{2}{m}}e^{i\lambda t} \tag{38}$$

and thus, the solutions to (32) are derived as:

For $e_0=1, e_1=-\left(k^2+1\right), e_2=k^2$,

$$q(x,t)=\left\{\frac{1}{m+1}\sqrt{\frac{\alpha m}{a\left(k^2+1\right)}}\operatorname{cs}x\operatorname{dn}x\right\}^{\frac{2}{m}}e^{i\lambda t} \tag{39}$$

or

$$q(x,t)=\left\{\frac{k^2-1}{m+1}\sqrt{\frac{\alpha m}{a\left(k^2+1\right)}}\operatorname{sc}\ x\operatorname{nd}x\right\}^{\frac{2}{m}}e^{i\lambda t}. \tag{40}$$

If $e_0=1-k^2, e_1=2k^2-1, e_2=-k^2$,

$$q(x,t)=\left\{-\frac{1}{m+1}\sqrt{\frac{\alpha m}{a\left(1-2k^2\right)}}\operatorname{sc}\ x\operatorname{dn}x\right\}^{\frac{2}{m}}e^{i\lambda t}. \tag{41}$$

When $e_0=k^2-1, e_1=2-k^2, e_2=-1$,

$$q(x,t)=\left\{-\frac{k^2}{m+1}\sqrt{\frac{\alpha m}{a\left(k^2-2\right)}}\operatorname{sd}x\operatorname{cn}x\right\}^{\frac{2}{m}}e^{i\lambda t}. \tag{42}$$

Whenever $e_0=0, e_1=1, e_2=-1$,

$$q(x,t)=\left\{-\frac{1}{m+1}\sqrt{-\frac{\alpha m}{a}}\tanh x\right\}^{\frac{2}{m}}e^{i\lambda t}. \tag{43}$$

Finally, for $e_0 = 0, e_1 = 1, e_2 = 1$,

$$q(x,t) = \left\{ -\frac{1}{m+1}\sqrt{-\frac{\alpha m}{a}}\coth x \right\}^{\frac{2}{m}} e^{i\lambda t}. \tag{44}$$

Here, the solutions (40)–(42) stands for JE function solutions, while the solutions (43) and (44) are respectively dark and singular solitons.

2.3 Parabolic law

For this law,

$$F(s) = b_1 s + b_2 s^2 \tag{45}$$

with the constants b_1 and b_2. Then (1) changes to

$$iq_t + a(|q|^n q)_{xx} + \left(b_1|q|^2 + b_2|q|^4\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{46}$$

When $n = 2$, (46) can be integrated. Thus, (46) simplifies to

$$iq_t + a\left(|q|^2 q\right)_{xx} + \left(b_1|q|^2 + b_2|q|^4\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{47}$$

and Eq. (3) becomes

$$-(\gamma+\lambda)\phi^2 + b_1\phi^4 + b_2\phi^6 - 2(\alpha - 2\beta)\left(\phi'\right)^2 + 6a\phi^2\left(\phi'\right)^2 - 2\alpha\phi\phi'' + 3a\phi^3\phi'' = 0. \tag{48}$$

Balance principle causes $N = 1$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right). \tag{49}$$

Proceeding as in previous sections, the results procured are:

$$\begin{aligned} b_2 &= \frac{3a\left(18ae_1 - b_1\right)}{\alpha}, \quad \alpha_0 = 0, \quad \alpha_1 = \frac{2\sqrt{\alpha}}{\sqrt{b_1 - 18ae_1}}, \\ \beta &= \frac{\alpha}{2}, \quad \gamma = \frac{6a\left(8\alpha e_1^2 + 16\alpha e_0 e_2 - 3e_1\lambda\right) + b_1\left(\lambda - 4\alpha e_1\right)}{18ae_1 - b_1}. \end{aligned} \tag{50}$$

Substituting (50) into (49) and employing (2) gives

$$q(x,t) = \frac{2\sqrt{\alpha}}{\sqrt{b_1 - 18\alpha e_1}}\left(\frac{G'}{G}\right)e^{i\lambda t} \tag{51}$$

and thus, the solutions to (47) are found as:

For $e_0 = 1, e_1 = -\left(k^2+1\right), e_2 = k^2$,

$$q(x,t)=\frac{2\sqrt{\alpha}}{\sqrt{b_1+18\alpha\left(k^2+1\right)}}\,\mathrm{cs}\;\,x\,\mathrm{dn}\,x\,e^{i\lambda t} \tag{52}$$

or

$$q(x,t)=-\frac{2\left(1-k^2\right)\sqrt{\alpha}}{\sqrt{b_1+18\alpha\left(k^2+1\right)}}\,\mathrm{sc}\;\,x\,\mathrm{nd}\,x\,e^{i\lambda t}. \tag{53}$$

If $e_0=1-k^2, e_1=2k^2-1, e_2=-k^2$,

$$q(x,t)=-\frac{2\sqrt{\alpha}}{\sqrt{b_1-18\alpha\left(2k^2-1\right)}}\,\mathrm{sc}\;\,x\,\mathrm{dn}\,x\,e^{i\lambda t}. \tag{54}$$

When $e_0=k^2-1, e_1=2-k^2, e_2=-1$,

$$q(x,t)=-\frac{2k^2\sqrt{\alpha}}{\sqrt{b_1-18\alpha\left(2-k^2\right)}}\,\mathrm{sd}\;\,x\,\mathrm{cn}\,x\,e^{i\lambda t}. \tag{55}$$

Whenever $e_0=0, e_1=1, e_2=-1$,

$$q(x,t)=-\frac{2\sqrt{\alpha}}{\sqrt{b_1-18\alpha}}\tanh x\,e^{i\lambda t}. \tag{56}$$

Finally, when $e_0=0, e_1=1, e_2=1$,

$$q(x,t)=-\frac{2\sqrt{\alpha}}{\sqrt{b_1-18\alpha}}\coth x\,e^{i\lambda t}. \tag{57}$$

Here, JE function solutions are represented by Eqs. (52)–(55), while dark and singular solitons are respectively indicated in Eqs. (56) and (57).

2.4 Dual-power law

This law occurs when

$$F(s)=b_1s^m+b_2s^{2m} \tag{58}$$

with the constants b_1 and b_2. Thus, (1) changes to

$$\begin{aligned} iq_t+a(|q|^nq)_{xx}+\left(b_1|q|^{2m}+\;b_2|q|^{4m}\right)q=\frac{1}{|q|^2q^*}\Big[\alpha|q|^2\left(|q|^2\right)_{xx} \\ -\beta\left\{\left(|q|^2\right)_x\right\}^2\Big]+\gamma q. \end{aligned} \tag{59}$$

For the integration of Eq. (59), $n=2m$ is selected. Thus, (59) simplifies to

$$iq_t + a\left(|q|^{2m}q\right)_{xx} + \left(b_1|q|^{2m} + b_2|q|^{4m}\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{60}$$

and (3) reduces to

$$-(\gamma+\lambda)\phi^2 + b_1\phi^{2m+2} + b_2\phi^{4m+2} - 2(\alpha-2\beta)\left(\phi'\right)^2 + 2am(2m+1)\phi^{2m}\left(\phi'\right)^2 - 2\alpha\phi\phi'' + a(2m+1)\phi^{2m+1}\phi'' = 0. \tag{61}$$

Set

$$\phi = \varphi^{\frac{1}{m}}. \tag{62}$$

Then Eq. (61) becomes

$$-m^2(\gamma+\lambda)\varphi^2 + b_1m^2\varphi^4 + b_2m^2\varphi^6 + 2(\alpha(m-2)+2\beta)\left(\varphi'\right)^2 + a\left(2m^2+3m+1\right)\varphi^2\left(\varphi'\right)^2 - 2m\alpha\varphi\varphi'' + am(2m+1)\varphi^3\varphi'' = 0. \tag{63}$$

Balance principle causes $N = 1$. In this case, Eq. (8) reads as

$$\varphi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right). \tag{64}$$

Following the path in the previous sections yields

$$b_1 = -\frac{4b_2\alpha m}{a\left(6m^2+5m+1\right)} + \frac{2a(2m+1)^2e_1}{m^2}, \quad \alpha_0 = 0, \quad \alpha_1 = \frac{\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}, \quad \beta = \alpha - \frac{\alpha m}{2},$$
$$\gamma = \frac{m^3b_2\left(4\alpha e_1 - m\lambda\right) - (m+1)(3m+1)(2am+a)^2\left(e_1^2 - 4e_0e_2\right)}{m^4b_2}. \tag{65}$$

Putting (65) into (64) and utilizing (2) leads to

$$q(x,t) = \left\{\frac{\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\left(\frac{G'}{G}\right)\right\}^{\frac{1}{m}} e^{i\lambda t} \tag{66}$$

and as a consequence, Eq. (60) possess the following solutions:

If $e_0 = 1$, $e_1 = -\left(k^2+1\right)$, $e_2 = k^2$,

$$q(x,t) = \left\{\frac{\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\,\mathrm{cs}\,x\,\mathrm{dn}\,x\right\}^{\frac{1}{m}} e^{i\lambda t} \tag{67}$$

or

$$q(x,t)=\left\{-\frac{\left(1-k^2\right)\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\,\mathrm{sc}\;x\,\mathrm{nd}\,x\right\}^{\frac{1}{m}}e^{i\lambda t}. \tag{68}$$

For $e_0=1-k^2, e_1=2k^2-1, e_2=-k^2$,

$$q(x,t)=\left\{-\frac{\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\,\mathrm{sc}\,x\,\mathrm{dn}\,x\right\}^{\frac{1}{m}}e^{i\lambda t}. \tag{69}$$

When $e_0=k^2-1, e_1=2-k^2, e_2=-1$,

$$q(x,t)=\left\{-\frac{k^2\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\,\mathrm{sd}\,x\,\mathrm{cn}\,x\right\}^{\frac{1}{m}}e^{i\lambda t}. \tag{70}$$

Whenever $e_0=0, e_1=1, e_2=-1$,

$$q(x,t)=\left\{-\frac{\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\tanh x\right\}^{\frac{1}{m}}e^{i\lambda t}. \tag{71}$$

In the case of $e_0=0, e_1=1, e_2=1$,

$$q(x,t)=\left\{-\frac{\sqrt{-a(m(6m+5)+1)}}{m\sqrt{b_2}}\coth x\right\}^{\frac{1}{m}}e^{i\lambda t}. \tag{72}$$

Here, JE function solutions are stood for by Eqs. (67)–(70), while dark and singular solitons are respectively given in Eqs. (71) and (72).

2.5 Quadratic-cubic law

This nonlinear form arises when

$$F(s)=b_1\sqrt{s}+b_2 s \tag{73}$$

with the constants b_1 and b_2. Thus, the model (1) becomes

$$iq_t+a(|q|^n q)_{xx}+\left(b_1|q|+b_2|q|^2\right)q=\frac{1}{|q|^2q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx}-\beta\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q. \tag{74}$$

Picking $n=1$, Eq. (74) can be integrated. Thus, (74) modifies to

$$iq_t+a(|q|q)_{xx}+\left(b_1|q|+b_2|q|^2\right)q=\frac{1}{|q|^2q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx}-\beta\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q \tag{75}$$

and Eq. (3) reduces to

$$-(\gamma+\lambda)\phi^2+b_1\phi^3+b_2\phi^4-2(\alpha-2\beta)\left(\phi'\right)^2+2a\phi\left(\phi'\right)^2-2\alpha\phi\phi''+2a\phi^2\phi''=0. \tag{76}$$

Balance principle causes $N = 2$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2. \tag{77}$$

Proceeding as in previous sections, one secures two solution set as

$$\begin{aligned} &b_1 = -\frac{2\alpha b_2}{5a} + 32ae_1, \quad \alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{20a}{b_2}, \\ &\beta = \frac{3\alpha}{4}, \quad \lambda = \frac{b_2\left(8\alpha e_1 - \gamma\right) + 240a^2\left(4e_0e_2 - e_1^2\right)}{b_2}. \end{aligned} \tag{78}$$

$$\begin{aligned} &e_0 = 0, \quad \alpha_0 = \frac{20ae_1}{b_2}, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{20a}{b_2}, \\ &\beta = \frac{5\left(a\left(16ae_1 + b_1\right) + \alpha b_2\right)}{4b_2}, \quad \lambda = \frac{20ae_1\left(16ae_1 + b_1\right) - b_2\left(\gamma - 4\alpha e_1\right)}{b_2}. \end{aligned} \tag{79}$$

Substituting (78) into (77) and employing (2) gives

$$q(x,t) = -\frac{20a}{b_2}\left(\frac{G'}{G}\right)^2 \exp\left[i\left(\frac{b_2\left(8\alpha e_1 - \gamma\right) + 240a^2\left(4e_0e_2 - e_1^2\right)}{b_2}\right)t\right] \tag{80}$$

and consequently, the solutions for the governing model (75) are:

For $e_0 = 1, e_1 = -\left(k^2+1\right), e_2 = k^2$,

$$q(x,t) = -\frac{20a}{b_2}\,\mathrm{cs}^2x\,\mathrm{dn}^2x\ \exp\left[-i\left(\frac{b_2\left(8\alpha k^2 + 8\alpha + \gamma\right) + 240a^2\left(k^4 - 2k^2 + 1\right)}{b_2}\right)t\right] \tag{81}$$

or

$$q(x,t) = -\frac{20a\left(1-k^2\right)^2}{b_2}\,\mathrm{sc}^2x\,\mathrm{nd}^2x \exp\left[-i\left(\frac{b_2\left(8\alpha k^2 + 8\alpha + \gamma\right) + 240a^2\left(k^4 - 2k^2 + 1\right)}{b_2}\right)t\right]. \tag{82}$$

If $e_0 = 1 - k^2, e_1 = 2k^2 - 1, e_2 = -k^2$,

$$q(x,t) = -\frac{20a}{b_2}\,\mathrm{sc}^2x\,\mathrm{dn}^2x \exp\left[i\left(\frac{b_2\left(16\alpha k^2 - 8\alpha - \gamma\right) - 240a^2}{b_2}\right)t\right]. \tag{83}$$

When $e_0 = k^2 - 1, e_1 = 2 - k^2, e_2 = -1$,

$$q(x,t) = -\frac{20ak^4}{b_2}\,\mathrm{sd}^2x\,\mathrm{cn}^2x \exp\left[-i\left(\frac{b_2\left(8\alpha k^2 - 16\alpha + \gamma\right) + 240a^2k^4}{b_2}\right)t\right]. \tag{84}$$

For the case $e_0 = 0, e_1 = 1, e_2 = -1$,

$$q(x,t) = -\frac{20a}{b_2}\tanh^2 x \exp\left[i\left(\frac{b_2(8\alpha-\gamma)-240a^2}{b_2}\right)t\right]. \tag{85}$$

In the case of $e_0 = 0, e_1 = 1, e_2 = 1,$

$$q(x,t) = -\frac{20a}{b_2}\coth^2 x \exp\left[i\left(\frac{b_2(8\alpha-\gamma)-240a^2}{b_2}\right)t\right]. \tag{86}$$

Here, the solutions given by Eqs. (81)–(84) are JE function solutions, while the solutions mentioned in Eqs. (85) and (86) are dark and singular solitons, respectively.

Similarly, inserting (79) into (77) and using (2) gives rise to

$$q(x,t) = \left\{\frac{20ae_1}{b_2} - \frac{20a}{b_2}\left(\frac{G'}{G}\right)^2\right\}\exp\left[i\left(\frac{20ae_1\left(16ae_1+b_1\right)-b_2\left(\gamma-4\alpha e_1\right)}{b_2}\right)t\right] \tag{87}$$

and hence, the solutions are listed as follows:

For $e_0 = 0, e_1 = 1, e_2 = -1,$

$$q(x,t) = \frac{20a}{b_2}\operatorname{sech}^2 x \exp\left[i\left(\frac{20a\left(16a+b_1\right)-b_2(\gamma-4\alpha)}{b_2}\right)t\right]. \tag{88}$$

If $e_0 = 0, e_1 = 1, e_2 = 1,$

$$q(x,t) = -\frac{20a}{b_2}\operatorname{csch}^2 x \exp\left[i\left(\frac{20a\left(16a+b_1\right)-b_2(\gamma-4\alpha)}{b_2}\right)t\right]. \tag{89}$$

When $e_0 = 0, e_1 = -1, e_2 = 1,$

$$q(x,t) = -\frac{20a}{b_2}\sec^2 x \exp\left[i\left(\frac{20a\left(16a-b_1\right)-b_2(\gamma+4\alpha)}{b_2}\right)t\right]. \tag{90}$$

Here, bright and singular solitons are respectively given by Eqs. (88) and (89), while periodic wave is given Eq. (90).

2.6 Log law

In the case of this law,

$$F(s) = \ln s. \tag{91}$$

Then Eq. (1) changes to

$$iq_t + a(|q|^n q)_{xx} + 2bq\ln|q| = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{92}$$

and Eq. (3) modifies to

$$-(\gamma+\lambda)\phi^2+2b\phi^2\ln|\phi|-2(\alpha-2\beta)\left(\phi'\right)^2+an(n+1)\phi^n\left(\phi'\right)^2 \\ -2\alpha\phi\phi''+a(n+1)\phi^{n+1}\phi''=0. \tag{93}$$

Employing

$$\phi=\exp\frac{1}{\varphi} \tag{94}$$

one transforms Eq. (93) into

$$2b\varphi^3-(\gamma+\lambda)\varphi^4+\Big(a(n+1)^2e^{\frac{n}{\varphi}}-4\alpha+4\beta\Big)\left(\varphi'\right)^2-\Big(a(n+1)e^{\frac{n}{\varphi}}-2\alpha\Big)\varphi^2\varphi'' \\ +2\Big(a(n+1)e^{\frac{n}{\varphi}}-2\alpha\Big)\varphi\left(\varphi'\right)^2=0. \tag{95}$$

To carry out the integration, $n=0$ must be selected. Therefore, Eq. (92) falls in

$$iq_t+aq_{xx}+2bq\ln|q|=\frac{1}{|q|^2q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx}-\beta\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q \tag{96}$$

and Eq. (95) modifies to

$$2b\varphi^3-(\gamma+\lambda)\varphi^4+(a-4\alpha+4\beta)\left(\varphi'\right)^2+2(a-2\alpha)\varphi\left(\varphi'\right)^2-(a-2\alpha)\varphi^2\varphi''=0. \tag{97}$$

Balance principle causes $N=2$. Then Eq. (8) becomes

$$\varphi(x)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\alpha_2\left(\frac{G'}{G}\right)^2 \tag{98}$$

and then, proceeding as in previous sections, one has

$$b=2e_1(2\alpha-a),\quad e_0=0,\quad \alpha_0=-\alpha_2e_1,\quad \alpha_1=0,\quad \beta=\alpha-\frac{a}{4},\quad \lambda=\frac{2(a-2\alpha)}{\alpha_2}-\gamma. \tag{99}$$

Substituting (99) into (98) and employing (2) gives

$$q(x,t)=\exp\left[-\alpha_2e_1+\alpha_2\left(\frac{G'}{G}\right)^2\right]^{-1}\exp\left[i\left(\frac{2(a-2\alpha)}{\alpha_2}-\gamma\right)t\right] \tag{100}$$

and consequently, Gaussian solitary waves to the model adopted (96) are listed as below:

If $e_0=0, e_1=1, e_2=-1$,

$$q(x,t)=\exp\left[-\alpha_2\operatorname{sech}^2x\right]^{-1}\exp\left[i\left(\frac{2(a-2\alpha)}{\alpha_2}-\gamma\right)t\right]. \tag{101}$$

For $e_0=0, e_1=1, e_2=1$,

$$q(x,t)=\exp\left[\alpha_2\operatorname{csch}^2x\right]^{-1}\exp\left[i\left(\frac{2(a-2\alpha)}{\alpha_2}-\gamma\right)t\right]. \tag{102}$$

When $e_0=0, e_1=-1, e_2=1$,

$$q(x,t) = \exp\left[\alpha_2 \sec^2 x\right]^{-1} \exp\left[i\left(\frac{2(a-2\alpha)}{\alpha_2} - \gamma\right)t\right]. \tag{103}$$

2.7 Anti-cubic law

In the case of the type of this nonlinearity,

$$F(s) = \frac{b_1}{s^2} + b_2 s + b_3 s^2 \tag{104}$$

where b_j for $j = 1, 2, 3$ are constants. Then Eq. (1) changes to

$$iq_t + a(|q|^n q)_{xx} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{105}$$

When $n = 2$, Eq. (105) can be integrated. Thus, Eq. (105) simplifies to

$$iq_t + a\left(|q|^2 q\right)_{xx} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{106}$$

and Eq. (3) modifies to

$$b_1\phi^{-2} - (\gamma+\lambda)\phi^2 + b_2\phi^4 + b_3\phi^6 - 2(\alpha-2\beta)\left(\phi'\right)^2 + 6a\phi^2\left(\phi'\right)^2 - 2\alpha\phi\phi'' + 3a\phi^3\phi'' = 0. \tag{107}$$

Balance principle causes $N = 1$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) \tag{108}$$

and then, following the same path as in previous sections leads to

$$b_1 = 0, \quad b_3 = \frac{3a\left(18ae_1 - b_2\right)}{\alpha}, \quad \alpha_0 = 0, \quad \alpha_1 = \frac{2\sqrt{\alpha}}{\sqrt{b_2 - 18ae_1}},$$
$$\beta = \frac{\alpha}{2}, \quad \gamma = \frac{6a\left(8\alpha e_1^2 + 16\alpha e_0 e_2 - 3e_1\lambda\right) + b_2\left(\lambda - 4\alpha e_1\right)}{18ae_1 - b_2}. \tag{109}$$

Plugging (109) into (108) and utilizing (2) yields

$$q(x,t) = \frac{2\sqrt{\alpha}}{\sqrt{b_2 - 18ae_1}}\left(\frac{G'}{G}\right)e^{i\lambda t}. \tag{110}$$

Since $b_1 = 0$ from the solution set (109), this form of the nonlinearity collapses to parabolic law nonlinear media. Also, because the solution (110) is the same as that of in case of parabolic law. Hence, the solutions that will be recovered are omitted.

2.8 Polynomial law

For nonlinear form

$$F(s) = b_1 s + b_2 s^2 + b_3 s^3 \tag{111}$$

where b_j for $j = 1, 2, 3$ are constants. Therefore, (1) changes to

$$iq_t + a(|q|^n q)_{xx} + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{112}$$

In the case of $n = 4$, the integration of Eq. (112) can be performed. Then Eq. (112) condenses to:

$$iq_t + a\left(|q|^4 q\right)_{xx} + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{113}$$

and Eq. (3) simplifies to

$$-(\gamma+\lambda)\phi^2 + b_1\phi^4 + b_2\phi^6 + b_3\phi^8 - 2(\alpha - 2\beta)\left(\phi'\right)^2 + 20a\phi^4\left(\phi'\right)^2 - 2\alpha\phi\phi'' + 5a\phi^5\phi'' = 0. \tag{114}$$

Balance principle causes $N = 1$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) \tag{115}$$

and following the same path in the previous sections leads to

$$b_1 = \frac{4\left(\alpha - 5a\alpha_1^4\left(e_1^2 - 4e_0e_2\right)\right)}{\alpha_1^2}, \quad b_2 = 50ae_1, \quad b_3 = -\frac{30a}{\alpha_1^2}, \quad \alpha_0 = 0, \quad \beta = \frac{\alpha}{2}, \quad \lambda = 4\alpha e_1 - \gamma. \tag{116}$$

Substituting (116) into (115) and employing (2) gives

$$q(x,t) = \alpha_1\left(\frac{G'}{G}\right)\exp\left[i\left(4\alpha e_1 - \gamma\right)t\right] \tag{117}$$

and consequently, one possess the solutions for Eq. (113) as follows:

For $e_0 = 1, e_1 = -\left(k^2+1\right), e_2 = k^2$,

$$q(x,t) = \alpha_1 \operatorname{cs} x \operatorname{dn} x \exp\left[-i\left(4\alpha\left(k^2+1\right) + \gamma\right)t\right] \tag{118}$$

or

$$q(x,t) = \alpha_1\left(k^2-1\right) \operatorname{sc} x \operatorname{nd} x \exp\left[-i\left(4\alpha\left(k^2+1\right) + \gamma\right)t\right]. \tag{119}$$

If $e_0 = 1 - k^2, e_1 = 2k^2 - 1, e_2 = -k^2$,

$$q(x,t) = -\alpha_1 \operatorname{sc} x \operatorname{dn} x \exp\left[i\left(4\alpha\left(2k^2-1\right) - \gamma\right)t\right]. \tag{120}$$

When $e_0 = k^2 - 1, e_1 = 2 - k^2, e_2 = -1$,

$$q(x,t) = -\alpha_1 k^2 \operatorname{sd} x \operatorname{cn} x \exp\left[i\left(4\alpha\left(2-k^2\right)-\gamma\right)t\right]. \tag{121}$$

Whenever $e_0 = 0, e_1 = 1, e_2 = -1$,

$$q(x,t) = -\alpha_1 \tanh x \exp[i(4\alpha-\gamma)t]. \tag{122}$$

Finally, when $e_0 = 0, e_1 = 1, e_2 = 1$,

$$q(x,t) = -\alpha_1 \coth x \exp[i(4\alpha-\gamma)t]. \tag{123}$$

Here, JE function solutions are represented by from (118) to (121), while dark and singular solitons are introduced in Eqs. (122) and (123), respectively.

2.9 Triple power law

For this media,

$$F(s) = b_1 s^m + b_2 s^{2m} + b_3 s^{3m} \tag{124}$$

where b_j for $j = 1, 2, 3$ are constants. Thus, (1) changes to

$$\begin{aligned} &iq_t + a(|q|^n q)_{xx} + \left(b_1|q|^{2m} + b_2|q|^{4m} + b_3|q|^{6m}\right)q \\ &= \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \end{aligned} \tag{125}$$

By taking $n = 4m$, one can integrate Eq. (125). Then, Eq. (125) reads as

$$\begin{aligned} &iq_t + a\left(|q|^{4m} q\right)_{xx} + \left(b_1|q|^{2m} + b_2|q|^{4m} + b_3|q|^{6m}\right)q \\ &= \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \end{aligned} \tag{126}$$

and (3) reduces to

$$\begin{aligned} &-(\gamma+\lambda)\phi^2 + b_1\phi^{2m+2} + b_2\phi^{4m+2} + b_3\phi^{6m+2} - 2(\alpha-2\beta)\left(\phi'\right)^2 \\ &+ 4am(4m+1)\phi^{4m}\left(\phi'\right)^2 - 2\alpha\phi\phi'' + a(4m+1)\phi^{4m+1}\phi'' = 0. \end{aligned} \tag{127}$$

Setting

$$\phi = \varphi^{\frac{1}{m}} \tag{128}$$

Eq. (127) can be turned into

$$\begin{aligned} &-m^2(\gamma+\lambda)\varphi^2 + b_1 m^2\varphi^4 + b_2 m^2\varphi^6 + b_3 m^2\varphi^8 + 2(\alpha(m-2)+2\beta)\left(\varphi'\right)^2 \\ &+ a\left(12m^2+7m+1\right)\varphi^4\left(\varphi'\right)^2 - 2m\alpha\varphi\varphi'' + am(4m+1)\varphi^5\varphi'' = 0. \end{aligned} \tag{129}$$

Balance principle causes $N = 1$. Then Eq. (8) becomes

$$\varphi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) \tag{130}$$

and proceeding as in previous sections brings about

$$b_1 = \frac{4\alpha m - a\alpha_1^4\left(e_1^2 - 4e_0e_2\right)(3m+1)(4m+1)}{\alpha_1^2 m^2}, \quad b_2 = \frac{2ae_1(4m+1)^2}{m^2},$$
$$b_3 = -\frac{a(4m+1)(5m+1)}{\alpha_1^2 m^2}, \tag{131}$$

$$\alpha_0 = 0, \quad \beta = \alpha - \frac{\alpha m}{2}, \quad \lambda = \frac{4\alpha e_1}{m} - \gamma.$$

Putting (131) into (130) and utilizing (2) yields

$$q(x,t) = \left\{\alpha_1\left(\frac{G'}{G}\right)\right\}^{\frac{1}{m}} \exp\left[i\left(\frac{4\alpha e_1}{m} - \gamma\right)t\right] \tag{132}$$

and so, the solutions for the model under consideration (126) are extracted as the following:

For $e_0 = 1, e_1 = -\left(k^2+1\right), e_2 = k^2$, JE function solutions are

$$q(x,t) = \left\{\alpha_1 \operatorname{cs} x \operatorname{dn} x\right\}^{\frac{1}{m}} \exp\left[-i\left(\frac{4\alpha\left(k^2+1\right)}{m} + \gamma\right)t\right] \tag{133}$$

or

$$q(x,t) = \left\{\alpha_1\left(k^2-1\right) \operatorname{sc}\ x \operatorname{nd} x\right\}^{\frac{1}{m}} \exp\left[-i\left(\frac{4\alpha\left(k^2+1\right)}{m} + \gamma\right)t\right]. \tag{134}$$

If $e_0 = 1 - k^2, e_1 = 2k^2 - 1, e_2 = -k^2$, JE function solution is

$$q(x,t) = \left\{-\alpha_1 \operatorname{sc}\ x \operatorname{dn} x\right\}^{\frac{1}{m}} \exp\left[i\left(\frac{4\alpha\left(2k^2-1\right)}{m} - \gamma\right)t\right]. \tag{135}$$

When $e_0 = k^2 - 1, e_1 = 2 - k^2, e_2 = -1$, JE function solution is

$$q(x,t) = \left\{-\alpha_1 k^2 \operatorname{sd} x \operatorname{cn} x\right\}^{\frac{1}{m}} \exp\left[i\left(\frac{4\alpha\left(2-k^2\right)}{m} - \gamma\right)t\right]. \tag{136}$$

In the case of $e_0 = 0, e_1 = 1, e_2 = -1$, dark soliton is

$$q(x,t) = \left\{-\alpha_1 \tanh x\right\}^{\frac{1}{m}} \exp\left[i\left(\frac{4\alpha}{m} - \gamma\right)t\right]. \tag{137}$$

Finally, if $e_0 = 0, e_1 = 1, e_2 = 1$, singular soliton is

$$q(x,t) = \left\{-\alpha_1 \coth x\right\}^{\frac{1}{m}} \exp\left[i\left(\frac{4\alpha}{m} - \gamma\right)t\right]. \tag{138}$$

2.10 Parabolic law medium with weak non-local nonlinearity

This nonlinear media arises when

$$F(s) = b_1 s + b_2 s^2 + b_3 s_{xx} \tag{139}$$

where b_j for $j = 1, 2, 3$ are constants. Therefore Eq. (1) changes to

$$\begin{aligned} &iq_t + a(|q|^n q)_{xx} + \left(b_1|q|^2 + b_2|q|^4 + b_3\left(|q|^2\right)_{xx}\right)q \\ &= \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \end{aligned} \tag{140}$$

For the integration of Eq. (140), $n = 2$ is picked. Then Eq. (140) simplifies to

$$\begin{aligned} &iq_t + a\left(|q|^2 q\right)_{xx} + \left(b_1|q|^2 + b_2|q|^4 + b_3\left(|q|^2\right)_{xx}\right)q \\ &= \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \end{aligned} \tag{141}$$

and Eq. (3) changes to

$$-(\lambda + \gamma)\phi^2 + b_1\phi^4 + b_2\phi^6 - 2(\alpha - 2\beta)\left(\phi'\right)^2 + 6a\phi^2\left(\phi'\right)^2 - 2\alpha\phi\phi'' + 3a\phi^3\phi'' = 0. \tag{142}$$

Balance principle causes $N = 1$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) \tag{143}$$

and then one recovers the following results:

$$\begin{aligned} &b_1 = e_1\left(2a - \frac{4\alpha_1^2 b_2}{3}\right) + \frac{4\alpha}{\alpha_1^2}, \quad b_3 = -2a - \frac{\alpha_1^2 b_2}{6}, \quad \alpha_0 = 0, \\ &\beta = \frac{\alpha}{2}, \quad \lambda = 2a\alpha_1^2\left(e_1^2 - 4e_0e_2\right) - \frac{\alpha_1^4 b_2\left(e_1^2 - 4e_0e_2\right)}{3} - \gamma + 4\alpha e_1. \end{aligned} \tag{144}$$

$$\begin{aligned} b_1 = \frac{\alpha}{\alpha_1^2} - \frac{12ae_1}{5}, \quad b_2 = \frac{3a}{5\alpha_1^2}, \quad b_3 = -\frac{21a}{10}, \quad e_0 = 0, \quad \alpha_0 = \alpha_1\sqrt{e_1}, \\ \beta = \tfrac{5\alpha}{4}, \quad \lambda = 4\alpha e_1 - \gamma. \end{aligned} \tag{145}$$

Substituting (144) into (143) and using (2) yields

$$q(x,t) = \alpha_1\left(\frac{G'}{G}\right)\exp\left[i\left(2a\alpha_1^2\left(e_1^2 - 4e_0e_2\right) - \frac{\alpha_1^4 b_2\left(e_1^2 - 4e_0e_2\right)}{3} - \gamma + 4\alpha e_1\right)t\right] \tag{146}$$

and hence, the solutions to the governing equation (141) are:

If $e_0 = 1$, $e_1 = -\left(k^2 + 1\right)$, $e_2 = k^2$,

$$q(x,t) = \alpha_1 \operatorname{cs} x \operatorname{dn} x \exp\left[i\left(2a\alpha_1^2\left(k^2 - 1\right)^2 - \frac{\alpha_1^4 b_2\left(k^2 - 1\right)^2}{3} - \gamma - 4\alpha\left(k^2 + 1\right)\right)t\right] \tag{147}$$

or

$$q(x,t)=\alpha_1\left(k^2-1\right)\,\mathrm{sc}\;x\,\mathrm{nd}\,x\exp\left[i\left(2a\alpha_1^2\left(k^2-1\right)^2-\frac{\alpha_1^4b_2\left(k^2-1\right)^2}{3}-\gamma-4\alpha\left(k^2+1\right)\right)t\right]. \tag{148}$$

For $e_0=1-k^2, e_1=2k^2-1, e_2=-k^2$,

$$q(x,t)=-\alpha_1\,\mathrm{sc}\;x\,\mathrm{dn}\,x\exp\left[i\left(2a\alpha_1^2-\frac{\alpha_1^4b_2}{3}-\gamma+4\alpha\left(2k^2-1\right)\right)t\right]. \tag{149}$$

When $e_0=k^2-1, e_1=2-k^2, e_2=-1$,

$$q(x,t)=-\alpha_1k^2\,\mathrm{sd}\,x\,\mathrm{cn}\,x\exp\left[i\left(2a\alpha_1^2k^4-\frac{\alpha_1^4b_2k^4}{3}-\gamma-4\alpha\left(k^2-2\right)\right)t\right]. \tag{150}$$

Here, the solutions (147)–(150) are JE function solutions.

Next, whenever $e_0=0, e_1=1, e_2=-1$, dark soliton is

$$q(x,t)=-\alpha_1\tanh x\exp\left[i\left(2a\alpha_1^2-\frac{\alpha_1^4b_2}{3}-\gamma+4\alpha\right)t\right]. \tag{151}$$

Finally, for $e_0=0, e_1=1, e_2=1$, singular soliton is

$$q(x,t)=-\alpha_1\coth x\exp\left[i\left(2a\alpha_1^2-\frac{\alpha_1^4b_2}{3}-\gamma+4\alpha\right)t\right]. \tag{152}$$

Similarly, plugging (145) into (143) and utilizing (2) gives rise to

$$q(x,t)=\alpha_1\left\{\sqrt{e_1}+\left(\frac{G'}{G}\right)\right\}\exp\left[i\left(4\alpha e_1-\gamma\right)t\right] \tag{153}$$

and then one has the solution as:

For $e_0=0, e_1=1, e_2=-1$,

$$q(x,t)=\alpha_1(1-\tanh x)\exp\left[i(4\alpha-\gamma)t\right]. \tag{154}$$

When $e_0=0, e_1=1, e_2=1$,

$$q(x,t)=\alpha_1(1-\coth x)\exp\left[i(4\alpha-\gamma)t\right]. \tag{155}$$

If $e_0=0, e_1=-1, e_2=1$,

$$q(x,t)=\alpha_1(i+\tan x)\exp\left[-i(4\alpha+\gamma)t\right]. \tag{156}$$

2.11 Generalized anti-cubic law

For this nonlinear media,

$$F(s)=\frac{b_1}{s^{m+1}}+b_2s^m+b_3s^{m+1} \tag{157}$$

where b_j for $j = 1, 2, 3$ are constants. Then Eq. (1) becomes to

$$iq_t + a(|q|^n q)_{xx} + \left(\frac{b_1}{|q|^{2m+2}} + b_2|q|^{2m} + b_3|q|^{2m+2}\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{158}$$

To integrate Eq. (158), $n = m + 1$ is chosen. Thus, Eq. (158) simplifies to

$$iq_t + a\left(|q|^{m+1}q\right)_{xx} + \left(\frac{b_1}{|q|^{2m+2}} + b_2|q|^{2m} + b_3|q|^{2m+2}\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{159}$$

and Eq. (3) changes to

$$-(\gamma+\lambda)\phi^2 + b_1\phi^{-2m} + b_2\phi^{2m+2} + b_3\phi^{2m+4} - 2(\alpha - 2\beta)\left(\phi'\right)^2 + a(m+1)(m+2)\phi^{m+1}\left(\phi'\right)^2 - 2\alpha\phi\phi'' + a(m+2)\phi^{m+2}\phi'' = 0. \tag{160}$$

Considering

$$\phi = \varphi^{\frac{1}{m+1}} \tag{161}$$

one turns Eq. (160) into

$$b_1(m+1)^2 - (m+1)^2(\gamma+\lambda)\varphi^2 + b_3(m+1)^2\varphi^4 + 2(\alpha(m-1) + 2\beta)\left(\varphi'\right)^2 + a(m+2)\varphi\left(\varphi'\right)^2 - 2\alpha(m+1)\varphi\varphi'' + a(m+1)(m+2)\varphi^2\varphi'' + b_2(m+1)^2\varphi(x)^{\frac{2m}{m+1}+2} = 0. \tag{162}$$

To proceed further, it is assume that $b_2 = 0$. Then, Eq. (159) changes to

$$iq_t + a\left(|q|^{m+1}q\right)_{xx} + \left(\frac{b_1}{|q|^{2m+2}} + b_3|q|^{2m+2}\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{163}$$

and Eq. (162) reads as

$$b_1(m+1)^2 - (m+1)^2(\gamma+\lambda)\varphi^2 + b_3(m+1)^2\varphi^4 + 2(\alpha(m-1) + 2\beta)\left(\varphi'\right)^2 + a(m+2)\varphi\left(\varphi'\right)^2 - 2\alpha(m+1)\varphi\varphi'' + a(m+1)(m+2)\varphi^2\varphi'' = 0. \tag{164}$$

Balance principle leads to $N = 2$. This means that Eq. (8) becomes

$$\varphi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2 \tag{165}$$

and then two solution set are constructed as below:

$$b_1 = -\frac{8a\alpha_2^3\left(e_1^3 - 36e_0e_1e_2\right)^2(m+2)}{81\left(e_1^2 + 12e_0e_2\right)(m+1)^2}, \quad b_3 = -\frac{2a(m+2)(3m+5)}{\alpha_2(m+1)^2}, \quad \alpha_0 = -\frac{2\alpha_2 e_1}{3}, \quad \alpha_1 = 0,$$

$$\alpha = \frac{a\alpha_2 e_1\left(e_1^2 - 36e_0e_2\right)(m+2)}{9\left(e_1^2 + 12e_0e_2\right)(m+1)}, \quad \beta = \frac{a\alpha_2 e_1\left(e_1^2 - 36e_0e_2\right)(m+2)(m+5)}{36\left(e_1^2 + 12e_0e_2\right)(m+1)},$$

$$\gamma = -\lambda - \frac{2a\alpha_2\left(e_1^2 + 12e_0e_2\right)(m+2)(m+3)}{3(m+1)^2}. \tag{166}$$

$$b_1 = 0, \quad b_3 = -\frac{2a(m+2)(3m+5)}{\alpha_2(m+1)^2}, \quad e_0 = 0, \quad \alpha_0 = -\alpha_2 e_1, \quad \alpha_1 = 0,$$

$$\alpha = \frac{a\alpha_2 e_1(m+2)^2}{m+1} + \frac{(m+1)(\gamma+\lambda)}{4e_1}, \quad \beta = \frac{a\alpha_2 e_1(m+2)^2}{m+1} + \frac{(m+1)(m+5)(\gamma+\lambda)}{16e_1}. \tag{167}$$

Substituting (166) into (165) and employing (2) gives

$$q(x,t) = \left\{-\frac{2\alpha_2 e_1}{3} + \alpha_2\left(\frac{G'}{G}\right)^2\right\}^{\frac{1}{m+1}} e^{i\lambda t} \tag{168}$$

and thus, the solutions to (163) are listed as:

If $e_0 = 1$, $e_1 = -\left(k^2+1\right)$, $e_2 = k^2$,

$$q(x,t) = \left\{\frac{2\alpha_2\left(k^2+1\right)}{3} + \alpha_2 \operatorname{cs}^2 x \operatorname{dn}^2 x\right\}^{\frac{1}{m+1}} e^{i\lambda t} \tag{169}$$

or

$$q(x,t) = \left\{\frac{2\alpha_2\left(k^2+1\right)}{3} + \alpha_2\left(1-k^2\right)^2 \operatorname{sc}^2 x \operatorname{nd}^2 x\right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{170}$$

For $e_0 = 1 - k^2$, $e_1 = 2k^2 - 1$, $e_2 = -k^2$,

$$q(x,t) = \left\{-\frac{2\alpha_2\left(2k^2-1\right)}{3} + \alpha_2 \operatorname{sc}^2 x \operatorname{dn}^2 x\right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{171}$$

When $e_0 = k^2 - 1$, $e_1 = 2 - k^2$, $e_2 = -1$,

$$q(x,t) = \left\{-\frac{2\alpha_2\left(2-k^2\right)}{3} + \alpha_2 k^4 \operatorname{sd}^2 x \operatorname{cn}^2 x\right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{172}$$

Whenever $e_0 = 0$, $e_1 = 1$, $e_2 = -1$,

$$q(x,t) = \left\{-\frac{2\alpha_2}{3} + \alpha_2 \tanh^2 x\right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{173}$$

Finally, if $e_0 = 0, e_1 = 1, e_2 = 1,$

$$q(x,t) = \left\{ -\frac{2\alpha_2}{3} + \alpha_2 \coth^2 x \right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{174}$$

Here, JE function solutions are represented by Eqs. (169)–(172), while dark and singular solitons are respectively indicated in Eqs. (173) and (174).

Similarly, putting (167) into (165) and using (2) leads to

$$q(x,t) = \left\{ -\alpha_2 e_1 + \alpha_2 \left(\frac{G'}{G} \right)^2 \right\}^{\frac{1}{m+1}} e^{i\lambda t} \tag{175}$$

and thus, one acquires bright and singular solitons and also periodic wave solution, respectively as:

If $e_0 = 0, e_1 = 1, e_2 = -1,$

$$q(x,t) = \left\{ -\alpha_2 \operatorname{sech}^2 x \right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{176}$$

For $e_0 = 0, e_1 = 1, e_2 = 1,$

$$q(x,t) = \left\{ \alpha_2 \operatorname{csch}^2 x \right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{177}$$

When $e_0 = 0, e_1 = -1, e_2 = 1,$

$$q(x,t) = \left\{ \alpha_2 \sec^2 x \right\}^{\frac{1}{m+1}} e^{i\lambda t}. \tag{178}$$

2.12 Cubic–quartic law

For CQ nonlinearity,

$$F(s) = b_1 s + b_2 s^{\frac{3}{2}} \tag{179}$$

with the constants b_1 and b_2. Thus Eq. (1) becomes

$$iq_t + a(|q|^n q)_{xx} + \left(b_1|q|^2 + b_2|q|^3\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{180}$$

By selecting $n = 2$, one can perform the integration of Eq. (180). Therefore Eq. (180) condenses to:

$$iq_t + a\left(|q|^2 q\right)_{xx} + \left(b_1|q|^2 + b_2|q|^3\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{181}$$

and Eq. (3) simplifies to

$$-(\gamma + \lambda)\phi^2 + b_1\phi^4 + b_2\phi^5 - 2(\alpha - 2\beta)\left(\phi'\right)^2 + 6a\phi^2\left(\phi'\right)^2 - 2\alpha\phi\phi'' + 3a\phi^3\phi'' = 0. \tag{182}$$

Balance principle causes $N = 2$. Then Eq. (8) becomes

$$\phi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2 \tag{183}$$

and then the results given below are derived:

$$b_1 = 72ae_1, \quad b_2 = -\frac{42a}{\alpha_2}, \quad \alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha = \frac{15\alpha_2^2 a\left(e_1^2 - 4e_0e_2\right)}{4}, \\ \beta = \frac{45\alpha_2^2 a\left(e_1^2 - 4e_0e_2\right)}{16}, \quad \lambda = 30a\alpha_2^2 e_1\left(e_1^2 - 4e_0e_2\right) - \gamma. \tag{184}$$

$$b_1 = -36ae_1, \quad b_2 = -\frac{42a}{\alpha_2}, \quad e_0 = 0, \quad \alpha_0 = -\alpha_2 e_1, \quad \alpha_1 = 0, \quad \beta = \frac{5\alpha}{4}, \quad \lambda = 4\alpha e_1 - \gamma. \tag{185}$$

Substituting (184) into (183) and employing (2) gives

$$q(x,t) = \alpha_2\left(\frac{G'}{G}\right)^2 \exp\left[i\left(30a\alpha_2^2 e_1\left(e_1^2 - 4e_0e_2\right) - \gamma\right)t\right]. \tag{186}$$

As a results, JE function solutions, dark and singular solitons to the model (181) are written down as:

If $e_0 = 1, e_1 = -\left(k^2+1\right), e_2 = k^2$,

$$q(x,t) = \alpha_2 \,\mathrm{cs}^2 x \,\mathrm{dn}^2 x \exp\left[-i\left(30a\alpha_2^2\left(k^2+1\right)\left(k^2-1\right)^2 + \gamma\right)t\right] \tag{187}$$

or

$$q(x,t) = \alpha_2\left(1-k^2\right)^2 \,\mathrm{sc}^2 x \,\mathrm{nd}^2 x \exp\left[-i\left(30a\alpha_2^2\left(k^2+1\right)\left(k^2-1\right)^2 + \gamma\right)t\right]. \tag{188}$$

For $e_0 = 1-k^2, e_1 = 2k^2-1, e_2 = -k^2$,

$$q(x,t) = \alpha_2 \,\mathrm{sc}^2 x \,\mathrm{dn}^2 x \exp\left[i\left(30a\alpha_2^2\left(2k^2-1\right) - \gamma\right)t\right]. \tag{189}$$

When $e_0 = k^2-1, e_1 = 2-k^2, e_2 = -1$,

$$q(x,t) = \alpha_2 k^4 \,\mathrm{sd}^2 x \,\mathrm{cn}^2 x \exp\left[i\left(30a\alpha_2^2 k^4\left(2-k^2\right) - \gamma\right)t\right]. \tag{190}$$

While $e_0 = 0, e_1 = 1, e_2 = -1$,

$$q(x,t) = \alpha_2 \tanh^2 x \exp\left[i\left(30a\alpha_2^2 - \gamma\right)t\right]. \tag{191}$$

Finally if $e_0 = 0, e_1 = 1, e_2 = 1$,

$$q(x,t) = \alpha_2 \coth^2 x \exp\left[i\left(30a\alpha_2^2 - \gamma\right)t\right]. \tag{192}$$

Similarly, inserting (185) into (183) and employing (2) brings about

$$q(x,t) = \left\{-\alpha_2 e_1 + \alpha_2\left(\frac{G'}{G}\right)^2\right\} \exp\left[i\left(4\alpha e_1 - \gamma\right)t\right] \tag{193}$$

and then one obtains bright and singular solitons and also periodic wave, respectively as:
For $e_0 = 0, e_1 = 1, e_2 = -1$,

$$q(x,t) = -\alpha_2 \operatorname{sech}^2 x \exp[i(4\alpha - \gamma)t]. \tag{194}$$

If $e_0 = 0, e_1 = 1, e_2 = 1$,

$$q(x,t) = \alpha_2 \operatorname{csch}^2 x \exp[i(4\alpha - \gamma)t]. \tag{195}$$

When $e_0 = 0, e_1 = -1, e_2 = 1$,

$$q(x,t) = \alpha_2 \sec^2 x \exp[-i(4\alpha + \gamma)t]. \tag{196}$$

2.13 Generalized CQ law

For generalized CQ nonlinearity,

$$F(s) = b_1 s^m + b_2 s^{\frac{3m}{2}} \tag{197}$$

with the constants b_1 and b_2. Thus, Eq. (1) becomes to

$$iq_t + a(|q|^n q)_{xx} + \left(b_1|q|^{2m} + b_2|q|^{3m}\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q. \tag{198}$$

To integrate Eq. (198), it should be $n = 2m$. Then, Eq. (198) modifies to

$$iq_t + a\left(|q|^{2m} q\right)_{xx} + \left(b_1|q|^{2m} + b_2|q|^{3m}\right)q = \frac{1}{|q|^2 q^*}\left[\alpha|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \tag{199}$$

and Eq. (3) changes to

$$\begin{aligned} &-(\gamma + \lambda)\phi^2 + b_1\phi^{2m+2} + b_2\phi^{3m+2} - 2(\alpha - 2\beta)\left(\phi'\right)^2 + 2am(2m+1)\phi^{2m}\left(\phi'\right)^2 \\ &\quad - 2\alpha\phi\phi'' + a(2m+1)\phi^{2m+1}\phi'' = 0. \end{aligned} \tag{200}$$

Applying the transformation given by

$$\phi = \varphi^{\frac{2}{m}} \tag{201}$$

one transforms Eq. (200) to

$$\begin{aligned} &-m^2(\gamma + \lambda)\varphi^2 + b_1 m^2\varphi^6 + b_2 m^2\varphi^8 + 4(\alpha(m-4) + 4\beta)\left(\varphi'\right)^2 \\ &+ 2a\left(6m^2 + 7m + 2\right)\varphi^4\left(\varphi'\right)^2 - 4m\alpha\varphi\varphi'' + 2am(2m+1)\varphi^5\varphi'' = 0. \end{aligned} \tag{202}$$

Balance principle yields $N = 1$. Then Eq. (8) becomes

$$\varphi(x) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) \tag{203}$$

and then the following results fall out:

$$b_1 = \frac{8ae_1(2m+1)^2}{m^2}, \quad b_2 = -\frac{2a(2m+1)(5m+2)}{\alpha_1^2 m^2},$$
$$\alpha_0 = 0, \quad \alpha = \frac{a\alpha_1^4\left(e_1^2-4e_0e_2\right)(m(6m+7)+2)}{4m},$$
$$\beta = -\frac{a\alpha_1^4\left(e_1^2-4e_0e_2\right)(m-4)(2m+1)(3m+2)}{16m},$$
$$\lambda = \frac{2a\alpha_1^4 e_1\left(e_1^2-4e_0e_2\right)(6m^2+7m+2)}{m^2} - \gamma. \tag{204}$$

Utilizing (204) into (203) and using (2) leads to

$$q(x,t) = \left\{\alpha_1\left(\frac{G'}{G}\right)\right\}^{\frac{2}{m}} \exp\left[i\left(\frac{2a\alpha_1^4 e_1\left(e_1^2-4e_0e_2\right)(6m^2+7m+2)}{m^2} - \gamma\right)t\right] \tag{205}$$

and thus, JE function solutions, dark and singular solitons to the model equation (199) are reported as:

If $e_0 = 1, e_1 = -\left(k^2+1\right), e_2 = k^2$,

$$q(x,t) = \left\{\alpha_1 \,\mathrm{cs}\, x\, \mathrm{dn}\, x\right\}^{\frac{2}{m}} \exp\left[-i\left(\frac{2a\alpha_1^4\left(k^2+1\right)\left(k^2-1\right)^2(6m^2+7m+2)}{m^2} + \gamma\right)t\right] \tag{206}$$

or

$$q(x,t) = \left\{\alpha_1\left(k^2-1\right) \,\mathrm{sc}\, \; x\, \mathrm{nd}\, x\right\}^{\frac{2}{m}} \exp\left[-i\left(\frac{2a\alpha_1^4\left(k^2+1\right)\left(k^2-1\right)^2(6m^2+7m+2)}{m^2} + \gamma\right)t\right]. \tag{207}$$

For $e_0 = 1-k^2, e_1 = 2k^2-1, e_2 = -k^2$,

$$q(x,t) = \left\{-\alpha_1 \,\mathrm{sc}\, \; x\, \mathrm{dn}\, x\right\}^{\frac{2}{m}} \exp\left[i\left(\frac{2a\alpha_1^4\left(2k^2-1\right)(6m^2+7m+2)}{m^2} - \gamma\right)t\right]. \tag{208}$$

When $e_0 = k^2-1, e_1 = 2-k^2, e_2 = -1$,

$$q(x,t) = \left\{-\alpha_1 k^2 \,\mathrm{sd}\, x\, \mathrm{cn}\, x\right\}^{\frac{2}{m}} \exp\left[i\left(\frac{2a\alpha_1^4 k^4\left(2-k^2\right)(6m^2+7m+2)}{m^2} - \gamma\right)t\right]. \tag{209}$$

Whenever $e_0 = 0, e_1 = 1, e_2 = -1$,

$$q(x,t) = \left\{-\alpha_1 \tanh x\right\}^{\frac{2}{m}} \exp\left[i\left(\frac{2a\alpha_1^4(6m^2+7m+2)}{m^2} - \gamma\right)t\right]. \tag{210}$$

Finally, for $e_0 = 0, e_1 = 1, e_2 = 1$,

$$q(x,t) = \left\{-\alpha_1 \coth x\right\}^{\frac{2}{m}} \exp\left[i\left(\frac{2a\alpha_1^4(6m^2+7m+2)}{m^2} - \gamma\right)t\right]. \tag{211}$$

3 Conclusions

This work is on the derivation and exhibition of stationary solitons that emerged from CGLE that is with nonlinear CD and having several forms of SPM structures. Jacobi's elliptic functions approach has made this retrieval possible. The results are exhibited for linear temporal evolution. This paper has immediate follow-ups from several avenues. An instantaneous consequence of this paper would be to study the same model with generalized temporal evolution. This would give a generalized perspective to the model handled and studied here. Later the model would be handled numerically such as with the usage of variational iteration method, Adomian decomposition scheme and several others. Such results are yet to be reported and are currently awaited.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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