



Cubic quintic Ginzburg Landau equation as a model for resonant interaction of EM field with nonlinear media

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Abstract

In this work, we investigate the possibility of approximating saturable nonlinearity, which is commonly used in complex Ginzburg–Landau equation (CGLE) for modelling resonant interaction of an electromagnetic field with nonlinear media, with cubic–quintic (CQ) nonlinearity. To validate the suggested approximations, we use variational method to estimate 2D analytical solutions of the CGLE with both saturable and CQ nonlinearity. The paper compares three ways to determine parameters of the CQ approximation and discusses the obtained results in terms of accuracy.

Keywords Cubic quintic Ginzburg Landau equation · Resonant interaction · Variational method

1 Introduction

The class of CGLEs represents a good model for describing a wide variety of phenomena in dissipative systems, such as superconductivity (Ginzburg et al. 1950), nonequilibrium phenomena (Aranson and Kramer 2002), phase transitions (Nato Advanced Study Institute 1975), binary fluid convection (Kolodner 1992), laser generated spatiotemporal dissipative structures, soliton propagation in optical fiber systems (Kodama and Hasegawa 1992), biological systems (Morales et al. 2015), etc.

The CQ CGLE, in particular, is often utilized to describe competitive effects of linear and nonlinear gain and loss. One of the reasons of high popularity of the CQ CGLE is

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the fact that there are many well developed methods for solving it both analytically and numerically. In the domain of fiber lasers, the model commonly used is the “master equation” derived by Haus (Haus et al. 1991, 1992), which, in fact, represents the stationary CGLE with cubic nonlinearity. The coefficients that appear in the Haus model were related to the physical parameters in a rather phenomenological way (Haus et al. 1994). The cubic CGLE has been investigated in sufficient details in the one-dimensional case. Analytical solutions of dissipative structures in cubic CGLE have been obtained, but all of them are unstable except the class of solutions with arbitrary amplitude in an “exotic” medium with zero linear loss/gain (Akhmediev et al. 1996).

To achieve the stabilization of the dissipative structures (solitons) in the CGLE model, higher-order nonlinear (quintic) terms have been introduced. The obtained equation is referred to as the quintic or CQ CGLE equation. The one-dimensional CQ CGLE, in the case of special relations between its parameters, has analytical solutions in the form of stable solitons with a fixed or arbitrary amplitude (Akhmediev and Afanasjev 1995; Akhmediev et al. 1997, 1998). In the general case, the one dimensional and multidimensional CQ CGLE do not have known exact solutions and have been studied numerically, as well as using different analytical approximations, including the variational method (Skarka and Aleksić 2006; Skarka et al. 2014, 2016; Aleksić et al. 2007, 2012a, b, 2014, 2015).

After the introduction of the cubic “master equation” (Haus et al. 1991, 1992), there have been derived many refinements in the form of CQ CGLE, as a model of resonant interaction of electromagnetic field with nonlinear media. These refinements have been done assuming small intensities, which allows the expansion of the Bloch equations in the series of intensities.

In this work, we further discuss the possibility of using CQ CGLE for modelling resonant interaction of an electromagnetic field. We compare three CQ approximations, including the widely used series expansion, and the CGLE with saturable nonlinearity. More precisely, the comparison is performed in terms of the parameters of stationary two-dimensional fundamental solutions, obtained using variational approach.

2 Model

We consider the response of a medium where active (amplifier) and passive atoms (saturable absorber), with almost identical transition frequencies, are embedded in the background matrix (for example, in a waveguide). Due to quick relaxation of the upper energy levels (three or four), we can reduce the model to an effective two-level scheme. When the concentration of the resonance centers is small, the polarization can be considered as an additive quantity:

$$\mathcal{P} = \mathcal{P}^{(\gamma)} + \mathcal{P}^{(\alpha)} + \mathcal{P}^{(m)} \quad (1)$$

consisting of the polarization of active ($\mathcal{P}^{(\gamma)}$), passive ($\mathcal{P}^{(\alpha)}$) atoms, and non-resonant polarization of the background matrix ($\mathcal{P}^{(m)}$).

From the Maxwell–Bloch equations, when the electromagnetic pulse duration is much greater than the relaxation time of the upper working level, we obtain the dimensionless CGLE for the electrical field:

$$i \frac{\partial \mathcal{E}}{\partial z} + \sigma \frac{\partial^2 \mathcal{E}}{\partial t^2} + \Delta \mathcal{E} + N(|\mathcal{E}|^2)\mathcal{E} = iG(|\mathcal{E}|^2)\mathcal{E} + i\beta_1 \frac{\partial^2 \mathcal{E}}{\partial t^2} + i\beta \Delta \mathcal{E} \tag{2}$$

where

$$N(|\mathcal{E}|^2) = -\frac{\bar{\gamma}|\mathcal{E}|^2}{(1 + |\mathcal{E}|^2)} + \frac{\bar{\alpha}b|\mathcal{E}|^2}{(1 + b|\mathcal{E}|^2)}, \tag{3}$$

$$G(|\mathcal{E}|^2) = -1 + \gamma - \alpha - \frac{\gamma|\mathcal{E}|^2}{(1 + |\mathcal{E}|^2)} + \frac{\alpha b|\mathcal{E}|^2}{(1 + b|\mathcal{E}|^2)}, \tag{4}$$

and $\bar{\gamma} = \gamma\delta_\gamma$, $\bar{\alpha} = \alpha\delta_\alpha$. The constants δ_γ , δ_α are dimensionless detunings between the gain (amplification) and loss (absorption), with the respect to the spectral line center and the frequency of empty cavity mode. The coefficients γ and α represent the linear amplification of active atoms and the absorption of passive atoms, (normalized to the absorption coefficient of the background medium), respectively (Fedorov et al. 2000). The linear part of the polarization determines the linear losses, temporal dispersion of the media, and spectral filtering (β_1). Under conditions of resonant interactions, when $\delta_\gamma \rightarrow 0$ and $\delta_\alpha \rightarrow 0$, term $N(I)$ in Eq. (2) can be neglected.

The function $G(I)$ has a maximum

$$G_m = -1 - \alpha + \gamma + \frac{\alpha b}{b-1} \left(1 - \sqrt{\frac{\gamma}{\alpha b}}\right)^2 \tag{5}$$

for

$$I = I_m = \left(1 - \sqrt{\frac{\gamma}{\alpha b}}\right) \left(b\sqrt{\frac{\gamma}{\alpha b}} - 1\right)^{-1}, \tag{6}$$

if the parameters satisfy the following relation:

$$\frac{\alpha}{b} < \gamma < \alpha b. \tag{7}$$

Outside this region the function G monotonically either decreases, when $\gamma > \alpha b$, or increases, when $\gamma < \alpha/b$. To establish the energy balance and the possibility of the existence of localized structures, the function $G(I)$ should be positive in a finite interval of intensity, i.e. $G(I_m) > 0$, and $G(I = 0) < 0$, in order to have a stable background. These two conditions in the parameter space become:

$$\frac{1}{b} \left(\sqrt{b-1} + \sqrt{\alpha}\right)^2 = \gamma_{min} < \gamma < \gamma_{max} = 1 + \alpha, \quad \alpha > \frac{1}{b-1}. \tag{8}$$

Under the previous conditions, function $G(I)$ has two zeros:

$$I_{0\pm} = \frac{b\gamma - b - 1 - \alpha \pm \sqrt{(b\gamma - b - 1 - \alpha)^2 + 4b(\gamma - \alpha - 1)}}{2b}, \tag{9}$$

as shown by the middle curve in Fig. 1a. The lower curve ($\alpha/b < \gamma < \gamma_{min}$) does not have positive regions, thus the function G provides only loss. The fact that in the upper curve ($\gamma_{max} < \gamma < \alpha b$) we have $G(0) > 0$ means the linear gain is bigger than the linear loss,

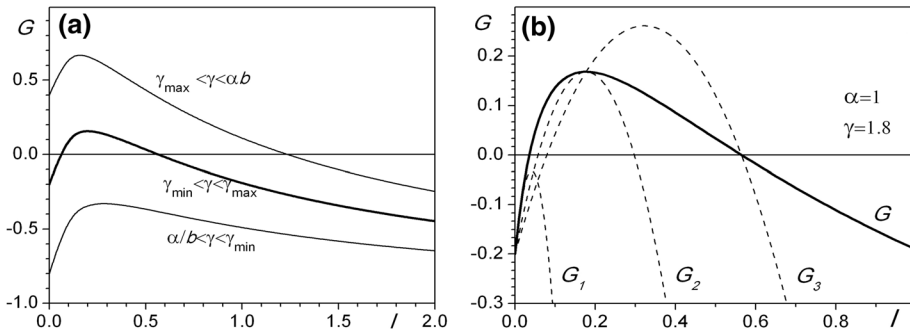


Fig. 1 **a** G as a function of intensity I for different regions of parameter γ . **b** $G(I)$ and its three approximations for $\alpha = 1$ and $\gamma = 1.8$

which causes an unstable background. The aforementioned properties of the lower and upper curve prevent formation of stable localized structures.

In order to simplify analytical estimation of Eq. (2), we consider three approximations of G , by using a second-order polynomial:

$$G_a(I) = -\delta + \epsilon_a I - \mu_a I^2, \quad a = 1, 2, 3 \tag{10}$$

which satisfy the condition $G_a(0) = G(0)$ i.e.

$$\delta = 1 + \alpha - \gamma. \tag{11}$$

The first approximation (G_1) is a series expansion, the second (G_2) reaches maximum value at I_m , and the third (G_3) has the value of zero at I_{0+} , and $G(I_m)$ at I_m . Thus:

1. Approximation G_1 : the series expansion of Eq. (4)

$$\epsilon_1 = b\alpha - \gamma, \quad \mu_1 = b^2\alpha - \gamma \tag{12}$$

2. Approximation G_2 : $dG_2(I_m)/dI_m = 0$ and $G_2(I_m) = G(I_m)$

$$\epsilon_2 = \frac{2\alpha b}{b-1} \left(1 - \sqrt{\frac{\gamma}{\alpha b}} \right) \left(b\sqrt{\frac{\gamma}{\alpha b}} - 1 \right), \quad \mu_2 = \frac{\alpha b}{b-1} \left(b\sqrt{\frac{\gamma}{\alpha b}} - 1 \right)^2 \tag{13}$$

3. Approximation G_3 : $G_3(I_{0+}) = G(I_{0+}) = 0$ and $G_3(I_m) = G(I_m)$

$$\epsilon_3 = \frac{G(I_m)I_{0+}}{I_m(I_{0+} - I_m)} + \frac{(I_{0+} + I_m)}{I_m I_{0+}} \delta, \quad \mu_3 = \frac{G(I_m)}{I_m(I_{0+} - I_m)} + \frac{\delta}{I_m I_{0+}} \tag{14}$$

Figure 1b shows the saturable nonlinearity (G) and its three approximations (G_1, G_2, G_3) as functions of intensity I , for parameters $\alpha = 1$ and $\gamma = 1.8$. Clearly, the most widely used approximation G_1 deviates the most from G , while G_2 and G_3 better resemble G . In the next section we analyze each approximation in more details.

The coefficients δ , ϵ_a and μ_a account for linear loss, cubic gain, and quintic loss, respectively. If the loss saturates faster than the gain ($b > 1$), then coefficients ϵ_a and μ_a are positive.

3 Variational approach

In order to validate the approximations, we use the variational method to solve the two-dimensional CGLE:

$$i \frac{\partial \mathcal{E}}{\partial z} + \Delta \mathcal{E} = iG(|\mathcal{E}|^2)\mathcal{E} + i\beta \Delta \mathcal{E} \tag{15}$$

with saturable nonlinearity and its CQ approximations, in the form of localized fundamental solitons. We adopt the trial function in the form of a Gaussian:

$$\mathcal{E} = A \exp \left[-\frac{r^2}{2R^2} + iCr^2 + i\psi \right] \tag{16}$$

where the real variational parameters are the amplitude $A = A(z)$, width $R = R(z)$, wave-front curvature $C(z)$, and phase $\psi = \psi(z)$.

Skipping technical details, along the lines of Skarka and Aleksić (2006), for the saturable nonlinearity of Eq. (4) we obtain the following system of the first-order evolution equations for the parameters of ansatz in Eq. (16):

$$\begin{aligned} \frac{dA}{dz} = & -4Ac - \delta A - \frac{2A\beta}{R^2} \\ & + \gamma \frac{\text{Li}_2(-A^2) + 2 \log(A^2 + 1)}{A} - \alpha \frac{\text{Li}_2(-A^2b) + 2 \log(A^2b + 1)}{Ab}, \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{dR}{dz} = & 4cR - 4\beta c^2 R^3 + \frac{\beta}{R} \\ & + R \left(\alpha \frac{\text{Li}_2(-A^2b) + \log(A^2b + 1)}{A^2b} - \gamma \frac{\text{Li}_2(-A^2) + \log(A^2 + 1)}{A^2} \right), \end{aligned} \tag{18}$$

$$\frac{dC}{dz} = -4C^2 - \frac{4C\beta}{R^2} + \frac{1}{R^4}, \tag{19}$$

$$\frac{d\psi}{dz} = 4C\beta - \frac{2}{R^2}. \tag{20}$$

From the fixed points of Eqs.(17)–(19):

$$\frac{dA}{dz} = \frac{dR}{dz} = \frac{dC}{dz} = 0 \tag{21}$$

we obtain the steady-state solutions, which are determined by the following system of equations:

$$R = \frac{-\beta C + |C| \sqrt{1 + \beta^2}}{C^2}, \tag{22}$$

$$C = \gamma \frac{(\text{Li}_2(-A^2) + \log(A^2 + 1))}{4A^2(\beta^2 + 1)} - \alpha \frac{(\text{Li}_2(-A^2b) + \log(A^2b + 1))}{4A^2b(\beta^2 + 1)}, \tag{23}$$

$$\alpha \frac{\log(A^2b + 1)}{A^2b} - \gamma \frac{\log(A^2 + 1)}{A^2} + 4 \frac{\beta_e}{1 - \beta_e^2} |C| + \delta = 0, \tag{24}$$

where $\beta_e = \beta/\sqrt{1 + \beta^2}$ and $\text{Li}_2(x)$ is the polylogarithm function.

In the case of CQ approximations Eq. (10), the stationary values of the trial function parameters are given by the following system of equations ($a = 1, 2, 3$):

$$R_a = \frac{-\beta C_a + |C_a| \sqrt{1 + \beta^2}}{C_a^2}, \tag{25}$$

$$C_a = \frac{A_a^2(9\varepsilon_a - 8A_a^2\mu_a)}{144(\beta^2 + 1)}, \tag{26}$$

$$\delta + \frac{A_a^4\mu_a}{3} - \frac{A_a^2\varepsilon_a}{2} + \frac{A_a^2\beta_e|9\varepsilon_a - 8A_a^2\mu_a|}{36} = 0. \tag{27}$$

Figure 2 represents the solutions of Eqs. (24) and (27), i.e. dependence of the amplitude of the stationary solution as a function of parameter γ , for two values of the parameters $\beta_e = 0; 0.5$, and $\alpha = 1$, shown in Fig. 2a and $\alpha = 3$ in Fig. 2b. The solution of Eq. (24) is denoted as G while the solutions of Eq. (27) are denoted by G_1, G_2 and G_3 . The series expansion G_1 , which is widely used in many research papers, represents a good approximation only for small values of the intensity. Approximation G_2 shows better accuracy for slightly larger values of the intensity, while G_3 gives values fairly close to the saturable nonlinearity over the whole intensity domain.

From Eqs. (22), (25) and (26) we can see that in the case when the wave front curvature (C and C_a) approaches zero, the radius (R and R_a) approaches infinity. Therefore, the nonzero value of the wave front curvature is the essential condition in order to have a finite radius, i.e. the existence of localized solutions. The zero values of C and C_a are marked with circles in Fig. 2.

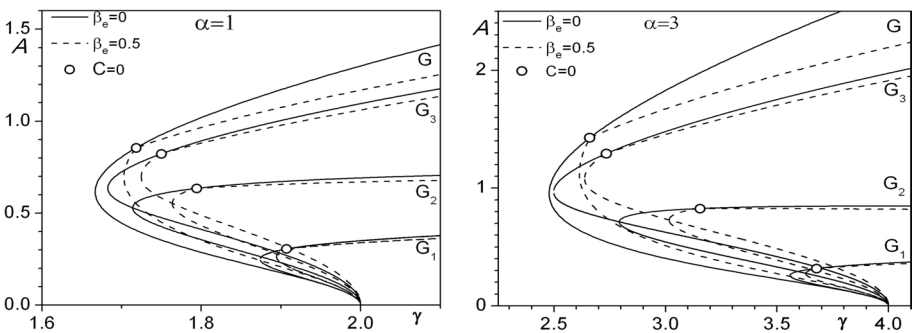


Fig. 2 The amplitude of the stationary solution as a function of the γ for saturable nonlinearity (denoted by G) and three CQ approximations (G_1, G_2 and G_3). Parameter $\alpha = 1$ —left and $\alpha = 3$ —right

4 Conclusion

A common model for resonant interaction of electromagnetic waves with nonlinear media is the CGLE with saturable nonlinearity. Due to limitations in the analytical analysis of that kind of a model, we approximate the saturable nonlinearity with CQ nonlinearity, whose analytical and numerical aspects have been extensively studied.

Using the variational method, we analyze three different second-order polynomial approximations of the saturable nonlinearity. The obtained results show the differences of the steady-state solutions between different approximations and the saturable nonlinearity, and depending on the application, suggest the most accurate approach. The most adequate approximation is the second-order polynomial, which has the same value at the linear limit and the common distant zero with saturable nonlinearity.

In the parameter space of the resonant medium, the region of possible approximations is determined. The parameters of the polynomial approximation are related to the physical parameters of the resonant medium.

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