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EDITORIAL

Optimization methods in inverse problems and applications to science and engineering

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1 Introduction

Each model in Science and Engineering can be studied from a direct or an inverse point-of-view. This distinction is not well-defined. In his classic paper, Keller 1976 provides the following definition: "We call two problems inverse of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other one is newer and not so well understood. In such cases, the former is called the direct problem, while the latter is the inverse problem."

However, in general, a direct problem involves the identification of effects from causes. This is often accomplished by making predictions based on models which, in turn, may be based on well-established physical laws. Usually a direct problem analysis involves the existence and uniqueness of the solution as well as the stability with respect to initial data.

An inverse problem, on the other hand, aims to identify causes from effects. In practice, this may be done by using observed data to estimate parameters in the functional form of a model. Very often an inverse problem appears in the form of a parameter estimation problem; it can be formulated as an optimization model, and then solved using different optimization algorithms and techniques. In general an inverse problem is ill-posed and several local minima might be present.

A class of problems that has attracted a lot of attention for the variety of different applications is the one based on fixed point equations. Within this family of problems, the direct approach might involve the use of the Contraction Mapping

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Principle (Banach's Fixed Point Theorem) while the inverse problem may be based on the Collage Theorem.

The next section is devoted to recalling the inverse problem methodology for fixed point equations and its different variants.

2 Inverse problem for fixed point equations

Let (X, d) denote a complete metric space and let $T : X \to X$ be a mapping on a complete metric space (X, d). T is said to be contractive if there exists a constant $c \in [0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$. Generally, the smallest such $c \in [0, 1)$ for which the above inequality holds true is known as the contraction factor of T. If we look for solutions of the fixed point equation $Tx = x$, one can rely on the famous Banach's Fixed Point Theorem Banach 1922 which reads as follows:

If $T : X \to X$ is a contraction mapping on X with contraction factor $c \in [0, 1)$ then:

- There exists a unique element $\bar{x} \in X$, the fixed point of T, for which $T\bar{x} = \bar{x}$.
- Given any $x_0 \in X$, if we form the iteration sequence $x_{n+1} = Tx_n$, then $x_n \to \bar{x}$, i.e., $d(x_n, \bar{x}) \to 0$ as $n \to \infty$.

In other words, the fixed point $\bar{x}$ is globally attractive. Banach's Theorem guarantees the existence of a unique fixed point. Stability with respect to perturbations can also be proved Kunze et al. 2013.

A quite general class of inverse problems for fixed point equations can be formulated as follows:

> Let (X, d) be a complete metric space and a "target" element $x \in X$ that we wish to approximate. Given an $\epsilon > 0$, can we find a contraction mapping $T : X \to X$ with fixed point $\bar{x} \in X$ such that $d(\bar{x}, x) < \epsilon$?

Very briefly, the original motivation for this formulation was fractal image coding Fisher 1996; Barnsley and Hurd 1995; Lu 1997. Given the complicated nature of this problem, however, the determination of optimal mappings T by minimizing the approximation error $d(\bar{x}, x)$ is intractable as x is, in general, unknown.

This problem can be solved by means of the following simple consequence of Banach's Theorem, known in the literature as the Collage Theorem Barnsley 1989. If (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping with contraction factor $c \in [0, 1)$ then, for any $x \in X$, we have

$$d(x, \bar{x}) \leq \frac{1}{1 - c} d(x, Tx), \tag{1}$$

where $\bar{x}$ is the fixed point of T. This permits a reformulation of our original inverse problem as follows:

> Given an $\epsilon > 0$, can we find a contraction mapping $T : X \to X$ with contraction factor $c \in [0, 1)$, ideally with $c \ll 1$, such that $d(x, Tx) < \epsilon$?

In an effort to minimize the approximation error $d(\bar{x}, x)$, we now look for contraction maps T which minimize the so-called collage error $d(x, Tx)$. In other words, we look for maps T which send the target x as close as possible to itself. We refer to this approach as collage coding Kunze et al. 2013.

Barnsley and co-workers Barnsley et al. 1985; Barnsley 1989 were the first to see the potential of using the Collage Theorem for the purpose of image approximation and compression. Most, if not all fractal image coding methods are based on some kind of block-based collage coding method which follows the strategy originally presented by Jacquin Jacquin., A. 1992. A collage coding approach, however, may be applied in other, "nonfractal," situations where contractive mappings are encountered Kunze et al. 2013.

Practically speaking, we consider a family of appropriate contraction mappings T_λ, where $\lambda \in \Lambda$ and Λ is a compact subset of R^n. Then we search for $\bar{\lambda}$ which minimizes the approximation error

$$d(x, \bar{x}_\lambda). \tag{2}$$

The feasible set $F \subset \Lambda$ can be defined as

$$F := \left\{ \lambda \in \mathbb{R}^n \,:\, 0 \le c_\lambda \le c < 1 \right\} \tag{3}$$

which guarantees the contractivity of T_λ for any $\lambda \in F$. A relaxed version of this problem involves the minimization of the following function

$$d(x, \bar{x}_\lambda) + \xi c_\lambda \tag{4}$$

over $\lambda \in \Lambda$ where ξ is a trade-off parameter. This formulation is quite similar to the classical Tikhonov regularization approach Tychonoff 1963; Tychonoff and Arsenin 1977.

The Collage Theorem and its variants have been used in the literature to solve a variety of inverse problems for ordinary and partial differential equations. The use of the Collage Theorem to solve inverse problems for ODEs was originally proposed in Kunze and Vrscay 1999 and developed in many subsequent works.

Related work through to 2011 is collected and expanded upon in Kunze et al. 2013. The collection includes, in particular, the introduction of a Generalized Collage Theorem for boundary value problems.

More recently, the Generalized Collage Theorem was extended to a wider class of elliptic equations problems by considering not only Hilbert spaces but also reflexive Banach spaces. Let us mention that this kind of formulation arises, for instance, when the boundary constraints are weakly imposed Berenguer et al. 2015, 2016; Kunze et al. 2015.

In Capasso et al. (2013, 2014); Kunze et al. (2013) the Collage Theorem was extended to the case of random integral and stochastic differential equations.

We included the notion of entropy and sparsity in solving inverse problems in Kunze et al. (2012). In this extended formulation, the parameter estimation minimization problem can be understood as a multiple criteria problem, with three different and conflicting criteria: The generalized collage error, the entropy associated with the unknown parameters, and the sparsity of the set of unknown parameters.

A further extension was proposed in Garralda-Guillem et al. (2020) to consider variational equations that included some kind of perturbation. One of the latest applications of the Collage Theorem is to the solution of inverse problems on perforated or porous media.

The results recalled in this section can be found with more details and applications in Kunze and La Torre (2015). Finally, the results presented in the papers Berenguer et al. 2015, 2016; Garralda-Guillem et al. 2020; Kunze and La Torre 2015; Kunze et al. 2019 are the most recent ones and they are related to extensions of the collage approach to reflexive Banach spaces, mixed variational equations, and perforated domains.

3 Contributions

This special issue aims at bringing together 15 articles that discuss recent advances of optimization methods and algorithms in inverse problems and application to science and engineering. A typical inverse problem seeks to find a mathematical model that admits given observational data as an approximate solution. This sort of question is of great interest in many application areas, including biomedical engineering and imaging, remote sensing and seismic imaging, astronomy, oceanography, atmospheric sciences and meteorology, chemical engineering and material sciences, computer vision and image processing, ecology, economics, environmental systems, and physical systems.

All papers included in this special issue consider aspects of numerical analysis, mathematical modeling, and computational methods and include topics such as: Inverse Problems Algorithms, Inverse Problems for Ordinary and Differential Equations, Inverse Problems using Nonsmooth Optimization, Inverse Problems using Multiple Criteria Optimization, Fractal-based Inverse Problems, Shape Optimization, Inverse Optimization, Inverse Problems in Image Analysis, and Regularization Techniques.

In Ramzani and Behroozifar (2020) the authors exhibit a method for numerically estimating two families of two-dimensional inverse problems. They consider inverse problems which include a time-dependent source control and their method is based on operational matrices of differential and integration and product of the shifted Legendre polynomials. They also provide illustrative examples to investigate the accuracy and applicability of the method.

In Riane and David (2021a) the authors provide a new Black-Scholes model, where the weak formulation at stake is done in the case of a general class of finite Radon measures. A numerical estimation of the parameters, by means of a gradient algorithm, shows that the estimated model is better as regards option pricing quality than the classical Black-Scholes one.

In Umer et al. (2021) the authors consider the TOPSIS approach to multiple criteria decision making and they present and extended version of it in the framework of interval type-2 trapezoidal pythagorean fuzzy numbers. They also provide an application to solar tracking system.

Interval-valued fuzzy sets have widely been acknowledged as proficient in modeling suspicions and practical in assigning an interval of values where allotting an accurate and precise number to an expert's outlook is too restrictive. In Touqeer et al. (2021) the authors establish a technique to solve linear programming network problems with constraints concerning interval-valued neutrosophic numbers. They also present an application to energy scheduling problem with constraints represented as interval-valued trapezoidal neutrosophic numbers.

In Riane and David (2021b) the authors explore how to control the solutions of PDEs on fractal sets. They discuss the extension of classical results of control theory to self-similar sets, and apply them to the benchmark case of the Sierpiński Gasket.

Phaseless inverse scattering problems appear often in practical applications since phaseless data are relatively easier to measure than the phased data, but they are also numerically more difficult to solve due to the translation invariance property. In Jiang and Liu (2021) the authors numerically compare several source localization algorithms based on different norm formulations in the context of inverse scattering. They propose an improved phase retrieval algorithm. They also discuss a simple criterion of minimizing the condition number of the underlying linear least square system for optimizing the choices of scattering strengths.

In Li et al. (2021) and Urbaniak et al. (2021) the authors consider the problem of modifying L^2-based approximations so that they "conform" in a better way to Weber's model of perception: Given a greyscale background intensity $I > 0$, the minimum change in intensity ΔI perceived by the human visual system is $\frac{\Delta I}{Ia} = C$, where $a > 0$ and $C > 0$ are constants. In the first paper Li et al. 2021, the authors modify the usual integral formulas used to define L^2 distances between functions. The pointwise differences $|u(x) - v(x)|$ which comprise the L^2 (or L^p) integrands are replaced with measures of the appropriate greyscale intervals. These measures are defined in terms of density functions which decrease at rates that conform with Weber's model of perception. The existence of such measures is proved in the paper. We also define the "best Weberized approximation" of a function in terms of these metrics and also prove the existence and uniqueness of such an approximation. In the second paper Urbaniak et al. 2021 the authors "Weberize" the L^2 metric by inserting an intensity-dependent weight function into its integral. The weight function will depend on the exponent a so that Weber's model is accommodated for all $a > 0$. They also define the "best Weberized approximation" of a function and also prove the existence and uniqueness of such an approximation.

In Otero et al. (2020) the authors introduce a general framework that encompasses a wide range of imaging applications in which the Structural Similariy Index SSIM can be employed as a fidelity measure. Subsequently, the authors show how the framework can be used to cast some standard as well as original imaging tasks into optimization problems, followed by a discussion of a number of novel numerical strategies for their solution.

In Yan et al. (2021) the authors transform the 2-dimension uncertain linguistic variable into a cloud model which converses the qualitative information to the quantitative information. The model is formulated and solved by means of multiple attribute group decision making techniques.

In Tadi and Radenkovic (2021) the authors are concerned with inverse wave scattering in one and two dimensional domains, that is the problem of recovering an unknown function based on measurements collected at the boundary of the domain. They develop two numerical algorithms: For one-dimensional problem they assume only one point of the domain is accessible. For the two-dimensional domain, the outer boundary is assumed to be accessible. A number of numerical examples are used to show their applicability and robustness to noise.

In Atanasov et al. (2021) the authors analyze a new model constituted by a system of three ordinary differential equations that account for the change in time of the population size of the hive bees, forage workers and infected foragers. It models the condition as a contagion, transmitted by both bee-to-bee and bee-to-plant interaction. The authors solve a parameter identification inverse problem to reconstruct the values which are directly unobservable in honeybee management.

Computing sparse solutions to overdetermined linear systems is a ubiquitous problem in several fields such as regression analysis, signal and image processing, information theory and machine learning. In Aktas et al. (2021) the authors report on an efficient and modular implicit enumeration algorithm to find provably optimal solutions to the NP-hard problem of sparsity-constrained non-negative least squares. They also report numerical results with real test data as well as comparisons of competing methods and an application to hyperspectral imaging.

In Garralda-Guillem and Lopez (2021) the authors use a minimax equality to prove the existence of a solution of certain system of variational equations and then they provide a numerical approximation of such a solution. They also propose a numerical method to solve a collage-type inverse problem associated with the corresponding system.

In Chaofan Huang et al. (2021) the authors, motivated by the parameter identification problem of a reaction-diffusion transport model in a vapor phase infiltration processes, propose a Bayesian optimization procedure for solving the inverse problem that aims to find an input setting that achieves a desired functional output.

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