



Incorporating history and deviations in forward–backward splitting

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Abstract

We propose a variation of the forward–backward splitting method for solving structured monotone inclusions. Our method integrates past iterates and two deviation vectors into the update equations. These deviation vectors bring flexibility to the algorithm and can be chosen arbitrarily as long as they together satisfy a norm condition. We present special cases where the deviation vectors, selected as predetermined linear combinations of previous iterates, always meet the norm condition. Notably, we introduce an algorithm employing a scalar parameter to interpolate between the conventional forward–backward splitting scheme and an accelerated $\mathcal{O}\left(\frac{1}{n^2}\right)$ -convergent forward–backward method that encompasses both the accelerated proximal point method and the Halpern iteration as special cases. The existing methods correspond to the two extremes of the allowed scalar parameter range. By choosing the interpolation scalar near the midpoint of the permissible range, our algorithm significantly outperforms these previously known methods when addressing a basic monotone inclusion problem stemming from minimax optimization.

Keywords Forward–backward splitting · Monotone inclusions · Inertial algorithms · Convergence rate · Halpern iteration · Deviations

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1 Introduction

We consider the problem of finding $x \in \mathcal{H}$ such that

$$0 \in Ax + Cx, \quad (1)$$

where $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is a maximally monotone operator, $C: \mathcal{H} \rightarrow \mathcal{H}$ is a cocoercive operator, \mathcal{H} is a real Hilbert space and $2^{\mathcal{H}}$ denotes its power set. This monotone inclusion has optimization problems [18, 29], convex-concave saddle-point problems [13], and variational inequalities [5, 14, 38] as special cases.

The forward–backward (FB) splitting method [11, 24, 27] has been widely used to solve the monotone inclusion problem (1). The gradient method, the proximal point algorithm [30], the proximal-gradient method [16], the Chambolle–Pock method [13], the Douglas–Rachford method [18, 24], and the Krasnosel’skiĭ–Mann iteration [9, Section 5.2] can all be considered special instances of the FB method. Various attempts have been made to improve the convergence of the FB splitting algorithm by incorporating information from previous iterations. Notable examples include the heavy-ball method [28], the inertial proximal point algorithm [1, 2], and inertial FB algorithms [3, 4, 6, 7, 10, 12, 15, 25, 26], which integrate prior information into the current iteration through a momentum term.

In this paper, we propose an extension to the conventional FB algorithm that includes momentum-like terms and two *deviation vectors*. These deviations have the same dimension as the underlying space of the problem and serve as adjustable parameters that provide the algorithm with great flexibility. This flexibility can be exploited to control the trajectory of the iterations with the aim to enhance algorithm convergence. To guarantee convergence, we require the deviations to satisfy a safeguarding condition that restricts the norm, but not the direction, of the deviation vectors. Our safeguarding approach is similar to those in [8, 32, 33]—which indeed are special instances of our algorithm—while it distinctly contrasts with the safeguarding conditions presented in [20, 34, 37, 40] that choose between a globally convergent and locally fast methods depending on the fulfillment of their respective safeguarding conditions.

We also introduce two special cases where the deviation vectors are predetermined linear combinations of prior iteration data. This construction ensures that the safeguarding condition is met in all iterations, implying that it does not require online evaluation. The two special cases incorporate different scalar parameters controlling the behaviour of their respective algorithms. In one case, the scalar parameter $\kappa \in (-1, 1)$ regulates the momentum used in the algorithm, with $\kappa = 0$ yielding the standard FB method. This algorithm converges weakly towards a solution of the inclusion problem for all κ within the permitted range. In the other case, the scalar parameter $e \in [0, 1]$ acts as an interpolator between the standard FB method ($e = 0$) and an accelerated FB method ($e = 1$) featuring the accelerated proximal point method from [21] and the Halpern iteration analyzed in [23] as special cases. The scalar parameter e regulates the convergence rate of the squared norm of the fixed-point residual, converging as $\mathcal{O}(\min(1/n^{2e}, 1/n))$, with $e = 1$ offering the accelerated FB method with an $\mathcal{O}(1/n^2)$ convergence rate, consistent with the rates in [21, 23].

We perform numerical evaluation of these two special cases on a simple skew-symmetric monotone inclusion problem arising from optimality conditions for the minimax problem $\max_{y \in \mathbb{R}} \min_{x \in \mathbb{R}} xy$. Our findings suggest that with $\kappa \in [0.8, 0.9]$, our first special case performs an order of magnitude better than the FB method ($\kappa = 0$) on this problem. Furthermore, by allowing $e \in [0.4, 0.5]$, we observe that our second special case outperforms the FB method ($e = 0$) by an order of magnitude and performs several orders of magnitude better than the accelerated FB method ($e = 1$), despite the latter’s stronger theoretical convergence guarantee.

The analysis of our base algorithm relies on a Lyapunov inequality. We derive this inequality by applying the monotonicity inequality of operator A and the cocoercivity inequality of operator C (that are referred to as *interpolation conditions* in the terminology of performance estimation (PEP), see for instance [31, 35, 36]), both between the last iterate and a solution of the problem, as well as the last two points generated by the algorithm. This is in contrast to the analysis conducted in [33], which restricts the use of these inequalities to only the last iteration and a solution. The inclusion of additional inequalities allows for deriving special cases such as the one that interpolates between FB splitting and accelerated FB splitting along with the associated convergence rate of $\mathcal{O}(1/n^{2e})$ where $e \in [0, 1]$. This result is not achievable via the algorithm proposed in [33].

The paper is organized as follows. In Section 2, we establish the basic definitions and notations used throughout the paper. Section 3 contains the formal presentation of the problem and introduces our proposed algorithm that is analyzed in Section 4. In Section 5, we present several special instances of our algorithm, two of which we examine numerically in Section 6. Proofs omitted for the sake of brevity are shared in Sections 7 and 8 concludes the paper.

2 Preliminaries

The set of real numbers is denoted by \mathbb{R} . \mathcal{H} denotes a real Hilbert space that is equipped with an inner product and an induced norm, respectively denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. $\mathcal{M}(\mathcal{H})$ denotes the set of bounded linear, self-adjoint, strongly positive operators on \mathcal{H} . For $M \in \mathcal{M}(\mathcal{H})$ and all $x, y \in \mathcal{H}$, the M -induced inner product and norm are denoted and defined by $\langle x, y \rangle_M := \langle x, My \rangle$ and $\|x\|_M = \sqrt{\langle x, Mx \rangle}$, respectively.

The *power set* of \mathcal{H} is denoted by $2^{\mathcal{H}}$. A map $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is characterized by its graph $\text{gra}(A) = \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Ax\}$. An operator $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is *monotone* if $\langle u - v, x - y \rangle \geq 0$ for all $(x, u), (y, v) \in \text{gra}(A)$. A monotone operator $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is *maximally monotone* if there exists no monotone operator $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ such that $\text{gra}(B)$ properly contains $\text{gra}(A)$.

Let $M \in \mathcal{M}(\mathcal{H})$. An operator $T: \mathcal{H} \rightarrow \mathcal{H}$ is said to be

- (i) *L-Lipschitz continuous* ($L \geq 0$) w.r.t. $\|\cdot\|_M$ if

$$\|Tx - Ty\|_{M^{-1}} \leq L\|x - y\|_M \quad \text{for all } x, y \in \mathcal{H};$$

(ii) $\frac{1}{\beta}$ -cocoercive ($\beta \geq 0$) w.r.t. $\|\cdot\|_M$ if

$$\beta \langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|_{M^{-1}}^2 \quad \text{for all } x, y \in \mathcal{H};$$

(iii) *nonexpansive* if it is 1-Lipschitz continuous w.r.t. $\|\cdot\|$.

Note that a $\frac{1}{\beta}$ -cocoercive operator is β -Lipschitz continuous. This holds trivially for $\beta = 0$ and for $\beta > 0$ it follows from the Cauchy-Schwarz inequality.

3 Problem statement and proposed algorithm

We consider structured monotone inclusion problems of the form

$$0 \in Ax + Cx, \tag{2}$$

that satisfy the following assumption.

Assumption 1 *Let $\beta \geq 0$ and $M \in \mathcal{M}(\mathcal{H})$ and assume that*

- (i) $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is maximally monotone,
- (ii) $C: \mathcal{H} \rightarrow \mathcal{H}$ is $\frac{1}{\beta}$ -cocoercive with respect to $\|\cdot\|_M$,
- (iii) the solution set $\text{zer}(A + C) := \{x \in \mathcal{H} : 0 \in Ax + Cx\}$ is nonempty.

Since operator C has a full domain and is maximally monotone as a cocoercive operator [9, Corollary 20.28], the operator $A + C$ is also maximally monotone [9, Corollary 25.5].

We propose the following variant of FB splitting which incorporates momentum terms and deviations in order to solve the inclusion problem in (2). The algorithm has many degrees of freedom that we will specify later in this section and in the special cases found in Section 5.

At the core of the method is a forward–backward type step, found in Step 7 of Algorithm 1, which reduces to a nominal forward–backward step when $z_n = y_n$. The update equations for the algorithm sequences y_n, z_n , and x_n involve linear combinations of momentum-like terms and the so-called *deviations*, u_n and v_n . These deviations are arbitrarily chosen provided they satisfy safeguarding condition in (3), where ℓ_n is defined in (4). When selecting the deviations, all other quantities involved in (3) are computable. These deviations offer a degree of flexibility that can be used to control the algorithm trajectory with the aim of improving convergence. In Section 5, we present examples of nontrivial deviations that a priori satisfy this condition, thus removing the need for online evaluation.

For the algorithm to be implementable, let alone convergent, the algorithm parameters must be constrained. For the FB step in Step 7 to be implementable, and the safeguarding step to be satisfied for some u_{n+1} and v_{n+1} , we require for all $n \in \mathbb{N}$ that $\gamma_n, \lambda_n, \theta_n, \hat{\theta}_n$, and $\tilde{\theta}_n$ are strictly positive and the parameters ζ_n, μ_n, α_n , and $\bar{\alpha}_n$ are non-negative. Fulfillment of these requirements allows for a trivial choice that satisfies the safeguarding condition (3), namely $u_{n+1} = v_{n+1} = 0$, which results in a

Algorithm 1

1: **Input:** initial point $x_0 \in \mathcal{H}$; the strictly positive sequences $(\gamma_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$; the non-negative sequences $(\zeta_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$; $\tilde{\beta} \geq \beta \geq 0$; and the metric $\|\cdot\|_M$ with $M \in \mathcal{M}(\mathcal{H})$.

2: Given the input parameters, for all $n \in \mathbb{N}$, define:

- (i) $\alpha_n := \frac{\mu_n}{\lambda_n + \mu_n}$;
- (ii) $\tilde{\alpha}_n := \frac{\gamma_n \mu_n}{\gamma_{n-1}(\lambda_n + \mu_n)}$;
- (iii) $\theta_n := (4 - \gamma_n \tilde{\beta})(\lambda_n + \mu_n) - 2\lambda_n^2$;
- (iv) $\hat{\theta}_n := 2\lambda_n + 2\mu_n - \gamma_n \tilde{\beta} \lambda_n^2$;
- (v) $\tilde{\theta}_n := \lambda_n + \mu_n - \lambda_n^2$;
- (vi) $\tilde{\theta}_n := (\lambda_n + \mu_n) \gamma_n \tilde{\beta}$.

3: **set:** $y_{-1} = p_{-1} = z_{-1} = x_0$ and $u_0 = v_0 = 0$ and $\gamma_{-1} = \gamma_0$

4: **for** $n = 0, 1, 2, \dots$ **do**

5: $y_n = x_n + \alpha_n(y_{n-1} - x_n) + u_n$

6: $z_n = x_n + \alpha_n(p_{n-1} - x_n) + \tilde{\alpha}_n(z_{n-1} - p_{n-1}) + \frac{\tilde{\theta}_n \gamma_n \tilde{\beta}}{\theta_n} u_n + v_n$

7: $p_n = (M + \gamma_n A)^{-1}(M z_n - \gamma_n C y_n)$

8: $x_{n+1} = x_n + \lambda_n(p_n - z_n) + \tilde{\alpha}_n \lambda_n(z_{n-1} - p_{n-1})$

9: choose u_{n+1} and v_{n+1} such that

$$(\lambda_{n+1} + \mu_{n+1}) \left(\frac{\tilde{\theta}_{n+1}}{\hat{\theta}_{n+1}} \|u_{n+1}\|_M^2 + \frac{\hat{\theta}_{n+1}}{\theta_{n+1}} \|v_{n+1}\|_M^2 \right) \leq \zeta_{n+1} \ell_n \tag{3}$$

is satisfied, where

$$\begin{aligned} \ell_n = & \frac{\theta_n}{2} \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \tilde{\beta} \lambda_n^2}{\theta_n} u_n - \frac{2\tilde{\theta}_n}{\theta_n} v_n \right\|_M^2 \\ & + 2\mu_n \gamma_n \left\langle \frac{z_n - p_n}{\gamma_n} - \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_n - p_{n-1} \right\rangle_M \\ & + \frac{\mu_n \gamma_n \tilde{\beta}}{2} \|p_n - y_n - (p_{n-1} - y_{n-1})\|_M^2 \end{aligned} \tag{4}$$

10: **end for**

novel momentum-type forward–backward scheme. Additional requirements on some of these parameters, that are needed for the convergence analysis, are discussed in Section 4.

Algorithm 1 can be viewed as an extension of the algorithm in [33]. The key difference arises from the inclusion of additional monotonicity and cocoercivity inequalities (interpolation conditions) in our analysis compared to the analysis of [33]. In contrast to the analysis in [33], we utilize inequalities not only between the last iteration points and a solution but also between points generated during the last two iterations of our algorithm. This approach provides our algorithm with an additional degree of freedom, embodied by the parameter μ_n , that stems from the degree to which these extra interpolation conditions are incorporated into the analysis. This addition yields momentum-like terms in the updates, a less restrictive safeguarding condition, and the potential to derive convergence rate estimates for several involved quantities up to $O(\frac{1}{n^2})$. Such rates are not achievable in [33] as setting μ_n to zero reverts our algorithm to that of [33].

When $\gamma_n = \gamma > 0$ for every $n \in \mathbb{N}$, the deviation vectors u_n and v_n can be chosen so that $y_n = z_n$. In this case, Step 7 of Algorithm 1 simplifies to a FB step of the form

$$p_n = \left((M + \gamma A)^{-1} \circ (M - \gamma C) \right) y_n.$$

It is widely recognized that given appropriate selections of $\gamma > 0$, $M \in \mathcal{M}(\mathcal{H})$, A , and C , this FB step can reduce to iterations of well-known algorithms. These include the Chambolle–Pock algorithm [13], the Condat–Vũ method [17, 39], the Douglas–Rachford method [24], the Krasnosel’skiĭ–Mann iteration [9, Section 5.2], and the proximal gradient method. Consequently, Algorithm 1 can be applied to all these special cases.

3.1 Preview of special cases

This section previews some special cases of Algorithm 1, which we will explore in depth in Sections 5 and 6. Specifically, we consider cases where the sequence $(\lambda_n)_{n \in \mathbb{N}}$ is non-decreasing and, for all $n \in \mathbb{N}$, $\gamma_n = \gamma > 0$,

$$v_n = \frac{(2 - \gamma \bar{\beta})^{(\lambda_n + \mu_n)}}{\hat{\theta}_n} u_n,$$

and u_n is parallel to the expression in the first norm in the ℓ_n expression in (4) that contributes to the upper bound in the safeguarding condition in (3). This yields $z_n = y_n$ and as demonstrated in Section 5, with a particular choice of μ_n , Algorithm 1 becomes

$$\begin{aligned} y_n &= x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (y_{n-1} - x_n) + u_n \\ p_n &= (M + \gamma A)^{-1} (M y_n - \gamma C y_n) \\ x_{n+1} &= x_n + \lambda_n (p_n - y_n) + (\lambda_n - \lambda_0) (y_{n-1} - p_{n-1}) \\ u_{n+1} &= \kappa_n \frac{4 - \gamma \bar{\beta} - 2\lambda_0}{2} (p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{2 - \gamma \bar{\beta} - 2\lambda_0}{4 - \gamma \bar{\beta} - 2\lambda_0} u_n), \end{aligned}$$

that is initialized with $y_{-1} = p_{-1} = x_0$ and $u_0 = 0$. This algorithm, under the condition $\kappa_n^2 \leq \zeta_{n+1}$, satisfies the safeguarding condition. As we will see later, $\zeta_{n+1} \in [0, 1]$ can be set arbitrarily (though stronger convergence conclusions can be drawn if $\zeta_n < 1$ for all $n \in \mathbb{N}$), indicating that as long as $\kappa_n \in [-1, 1]$, this new algorithm satisfies the safeguarding condition by design.

In Section 5, we present two special cases of this iteration that we numerically evaluate in Section 6. The first special case involves setting $\lambda_0 = 1$ and, for all $n \in \mathbb{N}$, $\lambda_n = \lambda_0$ and $\kappa_n = \kappa \in (-1, 1)$. As shown in Section 5, the resulting algorithm can be written as

$$\begin{aligned} p_n &= \left((M + \gamma A)^{-1} \circ (M - \gamma C) \right) (p_{n-1} + u_n - u_{n-1}) \\ u_{n+1} &= \kappa \left(\frac{2 - \gamma \bar{\beta}}{2} (p_n - p_{n-1} + u_{n-1}) + \frac{\gamma \bar{\beta}}{2} u_n \right) \end{aligned} \tag{5}$$

that is initialized with $u_{-1} = u_0 = 0$. This algorithm converges weakly to a solution of the inclusion problem and setting $\kappa = 0$ gives $u_n = 0$ for all $n \in \mathbb{N}$ and the algorithm reduces to the standard FB method.

The second special case is obtained by letting $\kappa_n = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}} \in [0, 1)$ resulting, as shown in Section 5, in the algorithm:

$$\begin{aligned}
 p_n &= (M + \gamma A)^{-1}(My_n - \gamma Cy_n) \\
 y_{n+1} &= y_n + \left(\frac{\lambda_0 \lambda_n}{\lambda_{n+1}} + \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} \frac{4 - \gamma \bar{\beta} - 2\lambda_0}{2} \right) (p_n - y_n) \\
 &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} \left((y_n - y_{n-1}) + \frac{4 - \gamma \bar{\beta}}{2} (y_{n-1} - p_{n-1}) \right),
 \end{aligned}
 \tag{6}$$

that is initialized with $y_{-1} = p_{-1}$. We will pay particular attention to the choice $\lambda_n = \left(1 - \frac{\gamma \bar{\beta}}{4}\right)^e (1 + n)^e$ with $e \in [0, 1]$. The choice $e = 0$ gives the standard FB method and, as shown in Section 5.2.1, the choice $e = 1$ has the Halpern iteration analyzed in [23] and the accelerated proximal point method in [21] as special cases. By choosing an e value between the allowed extremes, we can interpolate between these methods. We show that $\|p_n - y_n\|_M^2$ with this choice of $(\lambda_n)_{n \in \mathbb{N}}$ converges as $\mathcal{O}\left(\frac{1}{n^{2e}}\right)$ (and, if $\bar{\beta} > \beta$, as $\mathcal{O}\left(\frac{1}{n}\right)$ for all $e \in [0, 1]$) implying that the convergence rate can be tuned by selecting e . The requirements we will pose on the parameters in Section 4 to guarantee convergence state that $(\lambda_n)_{n \in \mathbb{N}}$ can grow at most linearly, meaning values of $e > 1$ are not viable and the best possible rate is $\mathcal{O}\left(\frac{1}{n^2}\right)$ obtained by letting $e = 1$. As shown in Section 5, this case recovers the exact $\mathcal{O}\left(\frac{1}{n^2}\right)$ rate results of the Halpern iteration in [23] and the accelerated proximal point method in [21].

In Section 6, we present numerical experiments on a simple skew-symmetric monotone inclusion problem, originating from the problem $\max_{y \in \mathbb{R}} \min_{x \in \mathbb{R}} xy$. We find that both Algorithms 5 and 6 can significantly outperform the standard FB method and the Halpern iteration when κ and e are appropriately chosen.

4 Convergence analysis

In this section, we conduct a Lyapunov-based convergence analysis for Algorithm 1. In Theorem 1, we define a *Lyapunov function*, V_n , based on the iterates generated by Algorithm 1, and present an identity that establishes a relation between V_{n+1} and V_n . In Theorem 2, we introduce additional assumptions and derive a *Lyapunov inequality* that serves as the main tool for the convergence and convergence rate analysis in Theorem 3.

The proof of our first theorem is lengthy and only based on algebraic manipulations and is therefore deferred to Section 7. The equality in the proof is validated with symbolic calculations in

<https://github.com/sbanert/incorporating-history-and-deviations>.

Theorem 1 *Suppose that Assumption 1 holds. Let x^* be an arbitrary point in $\text{zer}(A + C)$ and $V_0 = \|x_0 - x^*\|_M^2$, and based on the iterates generated by Algorithm 1, for all $n \in \mathbb{N}$, let*

$$V_{n+1} := \|x_{n+1} - x^*\|_M^2 + 2\lambda_{n+1}\gamma_{n+1}\alpha_{n+1}\phi_n + \ell_n, \tag{7}$$

where

$$\phi_n := \left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M + \frac{\bar{\beta}}{4} \|y_n - p_n\|_M^2, \tag{8}$$

and ℓ_n given by (4). Then,

$$\begin{aligned} V_{n+1} + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + \ell_{n-1} \\ = V_n + (\lambda_n + \mu_n) \left(\frac{\tilde{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right) \end{aligned}$$

holds for all $n \in \mathbb{N}$.

The identity relation presented in Theorem 1 can provides meaningful insights on convergence (as we will see in Theorem 3) when all its constituent terms are non-negative. The non-negativity of these terms is contingent on the selection of the parameter sequences $(\zeta_n)_{n \in \mathbb{N}}$, $(\gamma_n)_{n \in \mathbb{N}}$, $(\mu_n)_{n \in \mathbb{N}}$, and $(\lambda_n)_{n \in \mathbb{N}}$. To reduce the degrees of freedom and facilitate a clearer exposition, we constrain ourselves to a non-decreasing $(\lambda_n)_{n \in \mathbb{N}}$, and

$$\mu_n = \frac{1}{\lambda_0} \lambda_n^2 - \lambda_n \tag{9}$$

for all $n \in \mathbb{N}$. This implies $\mu_n \geq 0$ and offers a slightly less general algorithm, yet it encompasses all special cases in Section 5. We next state our restrictions on the parameter sequences that will give rise to a meaningful Lyapunov inequality in Theorems 2 and 3.

Assumption 2 *Assume that $\varepsilon > 0$, $\epsilon_0, \epsilon_1 \geq 0$, $\lambda_0 > 0$, and that, for all $n \in \mathbb{N}$, $\mu_n = \frac{1}{\lambda_0} \lambda_n^2 - \lambda_n$ and the following hold:*

- (i) $0 \leq \zeta_n \leq 1 - \epsilon_0$;
- (ii) $\varepsilon \leq \gamma_n \bar{\beta} \leq 4 - 2\lambda_0 - \varepsilon$;
- (iii) $\lambda_{n+1} \geq \lambda_n$ and $\gamma_n \lambda_n - \gamma_{n-1} \lambda_{n-1} \leq \gamma_n \lambda_0 - \epsilon_1$;
- (iv) $\bar{\beta} \geq \beta$.

Remark 1 *Assumption 2 (i) gives an upper bound for ζ_n to be less than or equal to 1. The variable, ζ_n , multiplies ℓ_{n-1} in the right-hand side of the safeguarding condition (3), effectively contributing to limit the size of the ball from which the deviations u_n and v_n are selected. A consistent choice is $\zeta_n = 1 - \epsilon_0$. Assumption 2 (ii) sets requirements on the relation between the initial relaxation parameter λ_0 and the step size parameter γ_n . An alternative expression of the upper bound is given by*

$$4 - \gamma_n \bar{\beta} - 2\lambda_0 \geq \varepsilon, \tag{10}$$

implying that

$$\gamma_n \bar{\beta} \lambda_0 \leq (4 - \varepsilon) \lambda_0 - 2\lambda_0^2 = -(\sqrt{2}\lambda_0 - \sqrt{2})^2 + 2 - \varepsilon \lambda_0 \leq 2 - \varepsilon \lambda_0. \tag{11}$$

This inequality will be used to bound certain algorithm parameters. Note that, similarly to in [19, 22], we can allow for a $\gamma_n > \frac{2}{\bar{\beta}}$ with the trade-off of using a relaxation parameter $\lambda_0 < 1$. Assumption 2 (iii) states that $(\lambda_n)_{n \in \mathbb{N}}$ is non-decreasing, resulting in $\mu_n \geq 0$, and enforces a linear growth upper bound since γ_n is both positive and upper bounded. We will later see that our algorithm can converge as $\mathcal{O}(\frac{1}{\lambda_n^2})$, with this upper bound resulting in a best possible convergence rate of $\mathcal{O}(\frac{1}{n^2})$. Finally, Assumption 2 (iv) sets requirements on $\bar{\beta}$. While the choice $\bar{\beta} = \beta$ always works, selecting $\bar{\beta} > \beta$ guarantees convergence of certain parameter sequences. Note that when $\lambda_0 = 1$ and $\bar{\beta} = \beta$, it follows from Assumption 2 (ii) that $\gamma_n \leq \frac{2-\varepsilon}{\bar{\beta}}$, aligning with the conventional step size upper bound for forward–backward splitting.

Our convergence analysis requires that specific parameter sequences are non-negative or, in certain cases, are lower bounded by a positive number. Before showing that Assumption 2 ensures this, we provide expressions for the following sequences defined in Algorithm 1 in terms of λ_n , γ_n , and $\bar{\beta}$:

$$\begin{aligned} \alpha_n &= \frac{\mu_n}{\lambda_n + \mu_n} = \frac{\lambda_n - \lambda_0}{\lambda_n}, \\ \bar{\alpha}_n &= \frac{\gamma_n \mu_n}{\gamma_{n-1}(\lambda_n + \mu_n)} = \frac{\gamma_n}{\gamma_{n-1}} \frac{\lambda_n - \lambda_0}{\lambda_n}, \\ \theta_n &= (4 - \gamma_n \bar{\beta})(\lambda_n + \mu_n) - 2\lambda_n^2 = \frac{4 - \gamma_n \bar{\beta} - 2\lambda_0}{\lambda_0} \lambda_n^2, \\ \hat{\theta}_n &= 2\lambda_n + 2\mu_n - \gamma_n \bar{\beta} \lambda_n^2 = \frac{2 - \lambda_0 \gamma_n \bar{\beta}}{\lambda_0} \lambda_n^2, \\ \bar{\theta}_n &= \lambda_n + \mu_n - \lambda_n^2 = \frac{1 - \lambda_0}{\lambda_0} \lambda_n^2, \\ \tilde{\theta}_n &= (\lambda_n + \mu_n) \gamma_n \bar{\beta} = \frac{\gamma_n \bar{\beta}}{\lambda_0} \lambda_n^2. \end{aligned} \tag{12}$$

Note that the final four quantities are quadratic in λ_n .

Proposition 1 Consider the quantities defined in Algorithm 1 and suppose that Assumption 2 holds. The parameter sequences $(\theta_n)_{n \in \mathbb{N}}$, $(\hat{\theta}_n)_{n \in \mathbb{N}}$, $(\tilde{\theta}_n)_{n \in \mathbb{N}}$, $(\frac{\hat{\theta}_n}{\theta_n})_{n \in \mathbb{N}}$, $(\frac{\tilde{\theta}_n}{\theta_n})_{n \in \mathbb{N}}$, and $(\frac{\theta_n}{\lambda_n^2})_{n \in \mathbb{N}}$ are lower bounded by a positive constant and $(\alpha_n)_{n \in \mathbb{N}}$ and $(\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1}) - \varepsilon)_n)_{n \in \mathbb{N}}$ are non-negative.

Proof Let us first consider θ_n , $\hat{\theta}_n$, and $\tilde{\theta}_n$. Then Assumption 2, (12), (10), and (11) immediately imply that θ_n , $\hat{\theta}_n$, and $\tilde{\theta}_n$ are lower bounded by a positive constant. Moreover, since $2 - \lambda_0 \gamma_n \bar{\beta} \geq \varepsilon \lambda_0 > 0$ by (11), we have

$$\frac{\tilde{\theta}_n}{\hat{\theta}_n} = \frac{\gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} \geq \frac{\gamma_n \bar{\beta}}{2} \geq \frac{\varepsilon}{2} > 0$$

and since from (10), $4 - \gamma_n \bar{\beta} - 2\lambda_0 \geq \varepsilon > 0$, we have

$$\frac{\hat{\theta}_n}{\theta_n} = \frac{2 - \lambda_0 \gamma_n \bar{\beta}}{4 - \gamma_n \bar{\beta} - 2\lambda_0} \geq \frac{2 - \lambda_0 \gamma_n \bar{\beta}}{4} \geq \frac{\lambda_0 \varepsilon}{4} > 0$$

and

$$\frac{\theta_n}{\lambda_n^2} = \frac{4 - \gamma_n \bar{\beta} - 2\lambda_0}{\lambda_0} \geq \frac{\varepsilon}{\lambda_0} > 0.$$

That $\alpha_n \geq 0$ follows trivially from nonnegativity of μ_n and that $\lambda_n > 0$. Finally,

$$\begin{aligned} \gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1}) &= \gamma_n\lambda_n - \gamma_{n+1} \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} \lambda_{n+1} \\ &= \gamma_n\lambda_n - \gamma_{n+1}\lambda_{n+1} + \gamma_{n+1}\lambda_0 \geq \epsilon_1 \end{aligned}$$

by Assumption 2 (iii). □

In the following result, we introduce a so-called Lyapunov inequality that serves as the foundation of our main convergence results.

Theorem 2 *Suppose that Assumptions 1 and 2 hold. Let x^* be an arbitrary point in $\text{zer}(A + C)$, and the sequences $(\ell_n)_{n \in \mathbb{N}}$, $(V_n)_{n \in \mathbb{N}}$, and $(\phi_n)_{n \in \mathbb{N}}$ be constructed in terms of the iterates obtained from Algorithm 1, as per (4), (7), and (8) respectively. Then, for all $n \in \mathbb{N}$,*

(i) *the safeguarding upper bound*

$$\zeta_{n+1}\ell_n \geq \frac{\zeta_{n+1}\theta_n}{2} \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\theta_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \geq 0;$$

(ii) *the term safeguarded by $\zeta_{n+1}\ell_n$,*

$$(\lambda_{n+1} + \mu_{n+1}) \left(\frac{\tilde{\theta}_{n+1}}{\hat{\theta}_{n+1}} \|u_{n+1}\|_M^2 + \frac{\hat{\theta}_{n+1}}{\theta_{n+1}} \|v_{n+1}\|_M^2 \right) \geq 0;$$

(iii) $\phi_n \geq 0$, more specifically, if $\beta > 0$:

$$\phi_n \geq \frac{\bar{\beta}}{4} \left\| \frac{2}{\beta} (Cy_n - Cx^*) + M(p_n - y_n) \right\|_{M^{-1}}^2 + \frac{\bar{\beta} - \beta}{\bar{\beta}\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2 \geq 0,$$

and if $\beta = 0$:

$$\phi_n \geq \frac{\bar{\beta}}{4} \|p_n - y_n\|_M^2 \geq 0;$$

(iv) *the Lyapunov function $V_n \geq 0$;*

(v) the following Lyapunov inequality holds

$$V_{n+1} + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + (1 - \zeta_n)\ell_{n-1} \leq V_n.$$

Proof Theorem 2 (i). In view of (3) and since, by Assumption 2 and Proposition 1, $2\mu_n\gamma_n \geq 0$, $\theta_n > 0$, and $\zeta_{n+1} \geq 0$, the statement reduces to showing that

$$\widehat{\varphi}_n := \left\langle \frac{z_n - p_n}{\gamma_n} - \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_n - p_{n-1} \right\rangle_M + \frac{\bar{\beta}}{4} \|p_n - y_n - (p_{n-1} - y_{n-1})\|_M^2 \geq 0.$$

It follows from Step 7 of Algorithm 1 that

$$\frac{Mz_n - Mp_n}{\gamma_n} - Cy_n \in Ap_n. \tag{13}$$

Combined with monotonicity of A , this gives

$$0 \leq \left\langle \frac{Mz_n - Mp_n}{\gamma_n} - Cy_n - \frac{Mz_{n-1} - Mp_{n-1}}{\gamma_{n-1}} + Cy_{n-1}, p_n - p_{n-1} \right\rangle. \tag{14}$$

If $\beta = 0$, C is constant, implying that the right-hand side reduces to the first term defining $\widehat{\varphi}_n$. Therefore, $\widehat{\varphi}_n \geq 0$ since it is constructed by adding two non-negative terms. It remains to show $\widehat{\varphi}_n \geq 0$ when $\beta > 0$. From $\frac{1}{\beta}$ -cocoercivity of C w.r.t. $\|\cdot\|_M$, we have

$$0 \leq \langle Cy_n - Cy_{n-1}, y_n - y_{n-1} \rangle - \frac{1}{\beta} \|Cy_n - Cy_{n-1}\|_{M^{-1}}^2. \tag{15}$$

Adding (14) and (15) to form φ_n gives

$$\varphi_n := \left\langle \frac{Mz_n - Mp_n}{\gamma_n} - Cy_n - \frac{Mz_{n-1} - Mp_{n-1}}{\gamma_{n-1}} + Cy_{n-1}, p_n - p_{n-1} \right\rangle \tag{16}$$

$$+ \langle Cy_n - Cy_{n-1}, y_n - y_{n-1} \rangle - \frac{1}{\beta} \|Cy_n - Cy_{n-1}\|_{M^{-1}}^2 \tag{17}$$

$$= \left\langle \frac{z_n - p_n}{\gamma_n} - \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_n - p_{n-1} \right\rangle_M - \frac{1}{\beta} \|Cy_n - Cy_{n-1}\|_{M^{-1}}^2$$

$$+ \langle Cy_n - Cy_{n-1}, y_n - y_{n-1} - (p_n - p_{n-1}) \rangle$$

$$= \left\langle \frac{z_n - p_n}{\gamma_n} - \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_n - p_{n-1} \right\rangle_M + \frac{\bar{\beta}}{4} \|p_n - p_{n-1} - y_n + y_{n-1}\|_M^2$$

$$- \frac{\bar{\beta}}{4} \left\| \frac{2}{\beta} (Cy_n - Cy_{n-1}) + M(p_n - p_{n-1} - y_n + y_{n-1}) \right\|_{M^{-1}}^2$$

$$= \widehat{\varphi}_n - \frac{\bar{\beta}}{4} \left\| \frac{2}{\beta} (Cy_n - Cy_{n-1}) + M(p_n - p_{n-1} - y_n + y_{n-1}) \right\|_{M^{-1}}^2, \tag{18}$$

where we have used that

$$\langle s, t \rangle - \frac{1}{\delta} \|s\|_{M^{-1}}^2 = \frac{\delta}{4} \|t\|_M^2 - \frac{\delta}{4} \left\| \frac{2}{\delta} s - Mt \right\|_{M^{-1}}^2 \tag{19}$$

holds for all $t, s \in \mathcal{H}$. Since $\varphi_n \geq 0$ by construction and $\bar{\beta} \geq \beta > 0$, this implies that $\widehat{\varphi}_n \geq 0$.

Theorem 2 (ii). This follows by Assumption 2 and Proposition 1 that imply strict positiveness of $\lambda_{n+1} + \mu_{n+1}$, $\frac{\hat{\theta}_{n+1}}{\hat{\theta}_{n+1}}$, and $\frac{\hat{\theta}_{n+1}}{\hat{\theta}_{n+1}}$.

Theorem 2 (iii). Recall that

$$\phi_n = \left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M + \frac{\bar{\beta}}{4} \|y_n - p_n\|_M^2, \tag{20}$$

as defined in (8). Since $x^* \in \text{zer}(A + C)$, we have $-Cx^* \in Ax^*$, which combined with (13) and monotonicity of A gives

$$0 \leq \left\langle \frac{Mz_n - Mp_n}{\gamma_n} - Cy_n + Cx^*, p_n - x^* \right\rangle. \tag{21}$$

If $\beta = 0$, C is constant and the right hand side reduces to $\phi_n - \frac{\bar{\beta}}{4} \|y_n - p_n\|_M^2$, which is non-negative for all $n \in \mathbb{N}$ by (21). Let us now consider $\beta > 0$. From $\frac{1}{\beta}$ -cocoercivity of C w.r.t. $\|\cdot\|_M$, we have

$$0 \leq \langle Cy_n - Cx^*, y_n - x^* \rangle - \frac{1}{\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2. \tag{22}$$

Construct $\widehat{\phi}_n$ by adding (21) and (22) to get

$$\begin{aligned} \widehat{\phi}_n &:= \left\langle \frac{Mz_n - Mp_n}{\gamma_n} - Cy_n + Cx^*, p_n - x^* \right\rangle \\ &\quad + \langle Cy_n - Cx^*, y_n - x^* \rangle - \frac{1}{\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2 \\ &= \left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M + \langle Cy_n - Cx^*, y_n - p_n \rangle - \frac{1}{\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2, \\ &= \left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M + \langle Cy_n - Cx^*, y_n - p_n \rangle - \frac{1}{\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2 \\ &\quad + \left(\frac{1}{\beta} - \frac{1}{\beta} \right) \|Cy_n - Cx^*\|_{M^{-1}}^2 \\ &= \left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M + \frac{\bar{\beta}}{4} \|y_n - p_n\|_M^2 \\ &\quad - \frac{\bar{\beta}}{4} \left\| \frac{2}{\beta} (Cy_n - Cx^*) + M(p_n - y_n) \right\|_{M^{-1}}^2 - \frac{\bar{\beta} - \beta}{\beta\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2 \\ &= \phi_n - \frac{\bar{\beta}}{4} \left\| \frac{2}{\beta} (Cy_n - Cx^*) + M(p_n - y_n) \right\|_{M^{-1}}^2 - \frac{\bar{\beta} - \beta}{\beta\beta} \|Cy_n - Cx^*\|_{M^{-1}}^2, \end{aligned}$$

where (19) is used in the next to last equality. The result therefore follows since $\bar{\beta} \geq \beta > 0$ and $\widehat{\phi}_n \geq 0$ by construction.

Theorem 2 (iv). Since $\ell_n \geq 0$ by Theorem 2 (i), $\phi_n \geq 0$ by Theorem 2 (iii), and the coefficients in front of ϕ_n in the definition of V_n in (7) are non-negative by Assumption 2 and Proposition 1, we conclude that $V_n \geq 0$.

Theorem 2 (v). By Theorem 1, we have

$$\begin{aligned} V_{n+1} + \ell_{n-1} + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n \\ = V_n + (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\hat{\theta}_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\bar{\theta}_n} \|v_n\|_M^2 \right). \end{aligned}$$

Using this equality and (3) gives

$$V_{n+1} + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + \ell_{n-1} \leq V_n + \zeta_n \ell_{n-1}.$$

Moving $\zeta_n \ell_{n-1}$ to the other side gives the desired result. This concludes the proof. \square

This result demonstrates the feasibility of selecting u_{n+1} and v_{n+1} that meet the safeguarding condition. The obvious selection of $u_{n+1} = v_{n+1} = 0$ is always viable, but we will provide in Section 5 a nontrivial choice that consistently satisfies the condition and can enhance convergence. Furthermore, Theorem 2 introduces a valuable Lyapunov inequality that will underpin our conclusions on convergence. Before stating these convergence results, we show boundedness of certain coefficient sequences.

Lemma 1 Consider the quantities defined in Algorithm 1 and suppose that Assumption 2 holds. The sequences $\left(\frac{\bar{\theta}_n}{\hat{\theta}_n}\right)_{n \in \mathbb{N}}$, $\left(\frac{\lambda_n^2}{\hat{\theta}_n}\right)_{n \in \mathbb{N}}$, $\left(\frac{\bar{\theta}_n}{\hat{\theta}_n}\right)_{n \in \mathbb{N}}$, as well as $\left(\frac{(2-\gamma_n\bar{\beta})(\lambda_n+\mu_n)}{\theta_n}\right)_{n \in \mathbb{N}}$ are bounded.

Proof We have $\frac{\bar{\theta}_n}{\hat{\theta}_n} \geq 0$ by Proposition 1 and

$$\frac{\bar{\theta}_n}{\hat{\theta}_n} = \frac{\gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} \leq \frac{\gamma_n \bar{\beta}}{\lambda_0 \varepsilon} \leq \frac{4 - 2\lambda_0 - \varepsilon}{\lambda_0 \varepsilon} \leq \frac{4}{\lambda_0 \varepsilon}$$

by (11). Further, $\frac{\lambda_n^2}{\hat{\theta}_n} \geq 0$ by Proposition 1 and

$$\frac{\lambda_n^2}{\hat{\theta}_n} = \frac{\lambda_0}{(2 - \lambda_0 \gamma_n \bar{\beta})} \leq \frac{1}{\varepsilon}$$

by (11). Further, by (10),

$$\left| \frac{\bar{\theta}_n}{\theta_n} \right| = \frac{|1 - \lambda_0|}{4 - \gamma_n \bar{\beta} - 2\lambda_0} \leq \frac{|1 - \lambda_0|}{\varepsilon}$$

and, since $\gamma_n \bar{\beta} \in (0, 4)$ by Assumption 2 (ii),

$$\left| \frac{(2 - \gamma_n \bar{\beta})(\lambda_n + \mu_n)}{\theta_n} \right| \leq \frac{2(\lambda_n + \mu_n)}{\theta_n} = \frac{2}{4 - \gamma_n \bar{\beta} - 2\lambda_0} \leq \frac{2}{\varepsilon}.$$

This completes the proof. \square

Theorem 3 Suppose that Assumptions 1 and 2 hold. Let x^* be an arbitrary point in $\text{zer}(A + C)$, and the sequences $(\ell_n)_{n \in \mathbb{N}}$, $(V_n)_{n \in \mathbb{N}}$, and $(\phi_n)_{n \in \mathbb{N}}$ be constructed in terms of the iterates obtained from Algorithm 1, as per (4), (7), and (8) respectively. Then the following hold:

- (i) the sequence $(V_n)_{n \in \mathbb{N}}$ is convergent and $\ell_n \leq V_{n+1} \leq \|x_0 - x^*\|_M^2$;
- (ii) if $(\lambda_n)_{n \in \mathbb{N}}$ increasing and $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\bar{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\bar{\theta}_n} v_n \right\|_M^2 \leq \frac{2\lambda_0 \|x_0 - x^*\|_M^2}{(4 - \gamma_n \bar{\beta} - 2\lambda_0) \lambda_n^2};$$

- (iii) if $\epsilon_0 > 0$, then $(\ell_n)_{n \in \mathbb{N}}$ is summable;
- (iv) if $\epsilon_0 > 0$, then $\lambda_n u_n \rightarrow 0$, $\lambda_n v_n \rightarrow 0$, and $x_{n+1} - x_n \rightarrow 0$ as $n \rightarrow \infty$;
- (v) if $\epsilon_1 > 0$, then $(\phi_n)_{n \in \mathbb{N}}$ is summable;
- (vi) if $\epsilon_1 > 0$ and $\bar{\beta} > \beta$, then $\|y_n - p_n\|_M^2$ is summable;
- (vii) if $\epsilon_0, \epsilon_1 > 0$, $(\lambda_n)_{n \in \mathbb{N}}$ is bounded, and $p_n - x_n \rightarrow 0$, $y_n - p_n \rightarrow 0$, and $z_n - p_n \rightarrow 0$ as $n \rightarrow \infty$, then $p_n \rightarrow x^*$;
- (viii) if $\epsilon_0, \epsilon_1 > 0$ and $(\lambda_n)_{n \in \mathbb{N}}$ is constant, then $p_n \rightarrow x^*$.

Proof We base our convergence results on

$$V_{n+1} + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + (1 - \zeta_n)\ell_{n-1} \leq V_n, \tag{23}$$

from Theorem 2.

Theorem 3 (i). Recall that the sequences $(\ell_n)_{n \in \mathbb{N}}$, $(V_n)_{n \in \mathbb{N}}$, and $(\phi_n)_{n \in \mathbb{N}}$ are non-negative by Theorem 2. Additionally, by Assumption 2 (i) and Proposition 1 respectively, the quantities $1 - \zeta_n$ and $\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})$ are non-negative for all $n \in \mathbb{N}$; and thus, the quantity $2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + (1 - \zeta_n)\ell_{n-1}$ is non-negative for all $n \in \mathbb{N}$. Therefore, by [9, Lemma 5.31] the sequence $(V_n)_{n \in \mathbb{N}}$ converges and $V_{n+1} \leq V_n$ for all $n \in \mathbb{N}$ and since $\lambda_{n+1}\gamma_{n+1}\alpha_{n+1} \geq 0$ by Assumption 2 and Proposition 1,

$$\ell_n \leq V_{n+1} \leq V_n \leq \dots \leq V_0 = \|x_0 - x^*\|_M^2. \tag{24}$$

Theorem 3 (ii). Theorem 2 (i) states that

$$\frac{\theta_n}{2} \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\bar{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\bar{\theta}_n} v_n \right\|_M^2 \leq \ell_n,$$

where $\ell_n \leq \|x_0 - x^*\|_M^2$ by Theorem 3 (i). Inserting the definition of θ_n and rearranging gives the result.

Theorem 3 (iii). That $\epsilon_0 > 0$ implies that $1 - \zeta_n > \epsilon_0 > 0$ by Assumption 2 (i) and a telescope summation of (23) gives summability of $(\ell_n)_{n \in \mathbb{N}}$.

Theorem 3 (iv). To show that $(\lambda_n u_n)_{n \in \mathbb{N}}$ and $(\lambda_n v_n)_{n \in \mathbb{N}}$ converge to 0, we note that due to (3), the summability of $(\ell_n)_{n \in \mathbb{N}}$ implies summability of

$$\left(\frac{\lambda_{n+1} + \mu_{n+1}}{\lambda_{n+1}^2} \left(\frac{\tilde{\theta}_{n+1}}{\tilde{\theta}_{n+1}} \|\lambda_{n+1} u_{n+1}\|_M^2 + \frac{\hat{\theta}_{n+1}}{\theta_{n+1}} \|\lambda_{n+1} v_{n+1}\|_M^2 \right) \right)_{n \in \mathbb{N}}. \tag{25}$$

Hence, as, for all $n \in \mathbb{N}$, by Proposition 1 and Assumption 2 the coefficients in the expression above are strictly positive, the sequences $(\lambda_n u_n)_{n \in \mathbb{N}}$ and $(\lambda_n v_n)_{n \in \mathbb{N}}$ must be convergent to zero.

Next, we show convergence to zero of $(x_{n+1} - x_n)_{n \in \mathbb{N}}$. Since $(\ell_n)_{n \in \mathbb{N}}$ is summable, Theorem 2 (i) implies that

$$\left(\frac{\theta_n}{2} \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \tilde{\beta} \lambda_n^2}{\tilde{\theta}_n} u_n - \frac{2\tilde{\theta}_n}{\theta_n} v_n \right\|_M^2 \right)_{n \in \mathbb{N}}$$

is summable. Using Lemma 2 to replace the expression inside the norm above by Lemma 2 (iii) and taking the factor $\frac{1}{\lambda_n}$ out of the norm, we get

$$\left(\frac{\theta_n}{2\lambda_n^2} \left\| x_{n+1} - x_n + \frac{\tilde{\theta}_n}{\theta_n} \lambda_n u_n + \frac{(2 - \gamma_n \tilde{\beta})(\lambda_n + \mu_n)}{\theta_n} \lambda_n v_n \right\|_M^2 \right)_{n \in \mathbb{N}},$$

which is a summable sequence too. Since, by Proposition 1, $\frac{\theta_n}{2\lambda_n^2}$ is lower bounded by a positive constant and the coefficients multiplying $\lambda_n u_n$ and $\lambda_n v_n$ are bounded by Lemma 1, we conclude, since $\lambda_n u_n \rightarrow 0$ and $\lambda_n v_n \rightarrow 0$ as $n \rightarrow \infty$, that $x_{n+1} - x_n \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 3 (v). Proposition 1 implies that $2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1}) \geq 2\epsilon_1 > 0$ and a telescope summation of (23) gives summability of $(\phi_n)_{n \in \mathbb{N}}$.

Theorem 3 (vi). Let $\beta = 0$. Then Theorem 2 (iii) immediately gives the result due to summability of $(\phi_n)_{n \in \mathbb{N}}$. Let $\beta > 0$. Then Theorem 2 (iii) and $\tilde{\beta} > \beta$ imply that

$$\left\| \frac{2}{\tilde{\beta}} (Cy_n - Cx^*) + M(p_n - y_n) \right\|_{M-1}^2 \quad \text{and} \quad \|Cy_n - Cx^*\|_{M-1}^2$$

are summable. Since

$$\begin{aligned} \|p_n - y_n\|_M^2 &= \frac{2}{\tilde{\beta}} (Cy_n - Cx^*) - \frac{2}{\tilde{\beta}} (Cy_n - Cx^*) + M(p_n - y_n) \Big\|_{M-1}^2 \\ &\leq 2 \left\| \frac{2}{\tilde{\beta}} Cy_n - Cx^* + M(p_n - y_n) \right\|_{M-1}^2 + 2 \|Cy_n - Cx^*\|_{M-1}^2, \end{aligned}$$

we conclude that $(\|p_n - y_n\|_M^2)_{n \in \mathbb{N}}$ is summable.

Theorem 3 (vii). We first show that $\|x_n - x^*\|_M^2$ converges. From Theorem 3 (i), we know that

$$V_{n+1} := \|x_{n+1} - x^*\|_M^2 + 2\lambda_{n+1}\gamma_{n+1}\alpha_{n+1}\phi_n + \ell_n,$$

converges. Since $(\lambda_n)_{n \in \mathbb{N}}$ is bounded so is $\lambda_{n+1}\gamma_{n+1}\alpha_{n+1}$ and by Theorem 3 (iii) and Theorem 3 (v) we conclude that $2\lambda_{n+1}\gamma_{n+1}\alpha_{n+1}\phi_n + \ell_n \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\|x_{n+1} - x^*\|_M^2$ converges.

Now, since $\|p_n - x_n\|_M^2 \rightarrow 0$ as $n \rightarrow \infty$ and $\|x_n - x^*\| \leq D$ for all $n \in \mathbb{N}$ and some $D \in (0, \infty)$, we conclude that

$$\begin{aligned} \left| \|p_n - x^*\|_M^2 - \|x_n - x^*\|_M^2 \right| &= \left| \|p_n - x_n\|_M^2 + 2\langle p_n - x_n, x_n - x^* \rangle \right| \\ &\leq \|p_n - x_n\|_M^2 + 2|\langle p_n - x_n, x_n - x^* \rangle| \\ &\leq \|p_n - x_n\|_M^2 + 2\|p_n - x_n\|\|x_n - x^*\| \\ &\leq \|p_n - x_n\|_M^2 + 2\|p_n - x_n\|D \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore also $\|p_n - x^*\|^2$ converges and p_n has weakly convergent subsequences. Let $(p_{n_k})_{k \in \mathbb{N}}$ be one such subsequence with weak limit point \bar{x} and construct corresponding subsequences $(y_{n_k})_{k \in \mathbb{N}}$, $(z_{n_k})_{k \in \mathbb{N}}$, and $(\gamma_{n_k})_{k \in \mathbb{N}}$. Now, Step 7 in Algorithm 1 can equivalently be written as

$$Mp_{n_k} + \gamma_{n_k}Ap_{n_k} \ni Mz_{n_k} - \gamma_{n_k}Cy_{n_k},$$

which is equivalent to that

$$Cp_{n_k} + Ap_{n_k} \ni \frac{1}{\gamma_{n_k}}M(z_{n_k} - p_{n_k}) + (Cp_{n_k} - Cy_{n_k}).$$

The right hand side converges to 0 as $k \rightarrow \infty$ since $z_{n_k} - p_{n_k} \rightarrow 0$ and $p_{n_k} - y_{n_k} \rightarrow 0$ as $k \rightarrow \infty$ and due to Lipschitz continuity of C , the uniform positive lower bound on γ_n in Assumption 2 (ii), and boundedness of $M \in \mathcal{M}(\mathcal{H})$. By weak-strong closedness of the maximal monotone operator $(A + C)$ (which is maximally monotone since C has full domain) the limit point satisfies $0 \in (A + C)\bar{x}$ by [9, Proposition 20.38]. The weak convergence result now follows from [9, Lemma 2.47].

Theorem 3 (viii). In view of Theorem 3 (vii), it is enough to show that $p_n - x_n \rightarrow 0$, $y_n - p_n \rightarrow 0$, and $z_n - p_n \rightarrow 0$ as $n \rightarrow \infty$. Since $(\lambda_n)_{n \in \mathbb{N}}$ is constant, $\mu_n = \frac{1}{\lambda_0}\lambda_n^2 - \lambda_n = 0$ and $\alpha_n = 0$. Summability of ℓ_n therefore implies through Theorem 2 (i) that

$$\left(\frac{\theta_n}{2} \left\| p_n - x_n + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\hat{\theta}_n} v_n \right\|_M^2 \right)_{n \in \mathbb{N}}$$

is summable. Since θ_n is lower bounded by a positive constant due to Proposition 1 and the coefficients in front of u_n and v_n are bounded due to Lemma 1, we conclude, since $u_n \rightarrow 0$ and $v_n \rightarrow 0$ by Theorem 3 (iv) and Assumption 2, that $p_n - x_n \rightarrow 0$ as $n \rightarrow \infty$. From the x_n update,

$$x_{k+1} = x_k + \lambda_n(p_n - z_n),$$

Theorem 3 (iv), and since $(\lambda_n)_{n \in \mathbb{N}}$ is constant, we conclude that $p_n - z_n \rightarrow 0$ as $n \rightarrow \infty$. Finally, from the y_n update,

$$y_n = x_n + u_n,$$

and since $u_n \rightarrow 0$, we conclude that $y_n - x_n \rightarrow 0$, which implies that $y_n - p_n \rightarrow 0$ as $n \rightarrow \infty$. This concludes the proof. \square

We could derive convergence properties for other quantities involved, yet we limit our discussion to these results as they are sufficient for our needs for the special cases. Notably, the conclusion in Theorem 3 (viii) aligns with a similar result presented in the authors' previous work [33]. This is due to $\mu_n = 0$, causing our algorithm to reduce to the one presented in that work.

5 A special case

The safeguarding condition in (3) typically requires the evaluation of four norms and a scalar product. However, if the vectors inside the norms are parallel, the number of norm evaluations is reduced. This section introduces an algorithm wherein we choose u_n and v_n to ensure $y_n = z_n$ for all $n \in \mathbb{N}$ and such that the safeguarding condition reduces to a scalar condition that is readily verified offline. The algorithm we propose is as follows:

$$\begin{aligned} y_n &= x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (y_{n-1} - x_n) + u_n \\ p_n &= (M + \gamma A)^{-1} (M y_n - \gamma C y_n) \\ x_{n+1} &= x_n + \lambda_n (p_n - y_n) + (\lambda_n - \lambda_0) (y_{n-1} - p_{n-1}) \\ u_{n+1} &= \kappa_n \frac{4 - \gamma \bar{\beta} - 2\lambda_0}{2} \left(p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{2 - \gamma \bar{\beta} - 2\lambda_0}{4 - \gamma \bar{\beta} - 2\lambda_0} u_n \right), \end{aligned} \tag{26}$$

where $y_{-1} = p_{-1} = x_0$, $u_0 = 0$, and a constant step size $\gamma > 0$ is used. With a constant step size, Assumption 2 (iii) reduces to

$$\lambda_n \leq \lambda_{n+1} \leq \lambda_n + \lambda_0 - \frac{\epsilon_1}{\gamma} = (2 + n)\lambda_0 - (n + 1) \frac{\epsilon_1}{\gamma}, \tag{27}$$

which gives an increasing $(\lambda_n)_{n \in \mathbb{N}}$ sequence that grows at most linearly in n .

Prior to presenting the convergence results for this algorithm, we specify a particular form for the sequence $(\lambda_n)_{n \in \mathbb{N}}$. This form separates the growth in n from the selection of λ_0 .

Assumption 3 *Let $\lambda_0 > 0$. Assume that $f : \text{dom } f \rightarrow \mathbb{R}$, with $\text{int dom } f \supseteq \{x \in \mathbb{R} : x \geq 0\}$, is differentiable (on the interior of its domain), concave, and non-decreasing, and satisfies $f(0) = 1$ and $f'(0) \in [0, 1]$. Let, for all $n \in \mathbb{N}$,*

$$\lambda_n = f(n)\lambda_0.$$

Proposition 2 *Suppose that Assumption 3 holds, then (27) and Assumption 2 (iii) hold with $\epsilon_1 = 0$. Suppose in addition that $f'(0) < 1$, then there exists $\epsilon_1 > 0$ such that (27) and Assumption 2 (iii) hold.*

Proof That f is non-decreasing trivially implies $\lambda_n \leq \lambda_{n+1}$. Concavity implies $f'(x) \leq f'(0)$ for all $x \geq 0$. Therefore

$$f(n) = 1 + \int_0^n f'(x)dx \leq 1 + \int_0^n f'(0)dx \leq 1 + n$$

and $\lambda_{n+1} \leq (2 + n)\lambda_0$. Let $a := f'(0) \in [0, 1)$, then

$$f(n) = 1 + \int_0^n f'(x)dx \leq 1 + \int_0^n f'(0)dx = 1 + an = 1 + n - (1 - a)n$$

and with $\epsilon_1 = (1 - a)\gamma\lambda_0 > 0$, we get $\lambda_{n+1} \leq (2 + n)\lambda_0 - (n + 1)\frac{\epsilon_1}{\gamma}$, as desired. \square

Example 1 *Examples of functions f that satisfy Assumption 3 for which an $\epsilon_1 > 0$ exists include functions that, for all $n \in \mathbb{N}$, satisfy $f(n) = (1 + n)^e$ with $e \in [0, 1)$, $f(n) = \frac{\log(n+2)}{\log(2)}$, and $f(n) = 1$. The choice $f(n) = (1 + n)$ requires that $\epsilon_1 = 0$.*

We will use this construction of $(\lambda_n)_{n \in \mathbb{N}}$ throughout this section and specialize Assumption 2 as follows.

Assumption 4 *Assume that $\epsilon > 0$, $\epsilon_0 \geq 0$, $\lambda_0 > 0$, and that, for all $n \in \mathbb{N}$, $\mu_n = \frac{1}{\lambda_0}\lambda_n^2 - \lambda_n$ and the following hold:*

- (i) $0 \leq \kappa_n^2 \leq 1 - \epsilon_0$;
- (ii) $\epsilon \leq \gamma\bar{\beta} \leq 4 - 2\lambda_0 - \epsilon$;
- (iii) $(\lambda_n)_{n \in \mathbb{N}}$ is given by Assumption 3;
- (iv) $\bar{\beta} \geq \beta$.

The differences to Assumption 2 are that ζ_n in Assumption 2 (i) has been replaced with κ_n^2 in Assumption 4 (i) and that Assumption 2 (iii) has been replaced with Assumption 4 (iii). If, e.g., $\kappa_n^2 \leq \zeta_{n+1}$, Proposition 2 implies that Assumption 2 holds if Assumption 4 does.

We are ready to state our convergence results for the algorithm in (26).

Proposition 3 *Suppose that Assumptions 1 and 4 hold. Then the following hold for (26):*

- (i) *if $f(n) \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$\left\| p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n}(x_n - p_{n-1}) - \frac{2 - \gamma\bar{\beta} - 2\lambda_0}{4 - \gamma\bar{\beta} - 2\lambda_0}u_n \right\|_M^2 \leq \frac{2\|y_0 - x^*\|_M^2}{(4 - \gamma\bar{\beta} - 2\lambda_0)\lambda_0 f(n)^2};$$

- (ii) *if $\bar{\beta} > \beta$ and $f'(0) < 1$, then $(\|p_n - y_n\|^2)_{n \in \mathbb{N}}$ is summable;*
- (iii) *if $(\kappa_n^2)_{n \in \mathbb{N}}$ is upper bounded by a constant less than 1 and $f'(0) = 0$ (i.e., $(\lambda_n)_{n \in \mathbb{N}}$ is constant), then $p_n \rightarrow x^* \in \text{zer}(A + C)$.*

Proof We first show that the algorithm is a special case of Algorithm 1. First note that $\gamma_n = \gamma$ implies that $\alpha_n = \bar{\alpha}_n$ for all $n \in \mathbb{N}$. Let

$$v_n = \frac{2 - \gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} u_n$$

for all $n \in \mathbb{N}$, which implies that

$$\frac{\bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} u_n + v_n = \frac{(1 - \lambda_0) \gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} u_n + \frac{2 - \gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} u_n = u_n \tag{28}$$

since $2 - \lambda_0 \gamma_n \bar{\beta} > 0$ by (11). Let us show by induction that this implies $y_n = z_n$ for all $n \in \mathbb{N}$ in Algorithm 1. Since $y_{-1} = z_{-1} = p_{-1} = x_0$ and $u_0 = 0$, we get $y_0 = z_0$. Now, assume that $y_k = z_k$ for all $k \in \{-1, \dots, n\}$, then, since $\alpha_n = \bar{\alpha}_n$ and due to (28),

$$\begin{aligned} z_{n+1} &= y_n - x_n + \alpha_n(p_{n-1} - x_n) + \bar{\alpha}_n(y_{n-1} - p_{n-1}) + \frac{\bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} u_n + v_n \\ &= y_n - x_n + \alpha_n(p_{n-1} - x_n) + u_n = y_{n+1}. \end{aligned}$$

Therefore, the z_{n+1} update of Algorithm 1 can be removed and all z_n instances replaced by y_n in (26). Moreover, the y_n and x_n updates of (26) are obtained from the corresponding sequences in Algorithm 1 by inserting $\alpha_n = \frac{\lambda_n - \lambda_0}{\lambda_n}$.

It remains to show that the u_{n+1} update satisfies the safeguarding condition. We use Theorem 2 (i), $\alpha_n = \frac{\lambda_n - \lambda_0}{\lambda_n}$, and the equality

$$\begin{aligned} \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n &= \left(\frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} - \frac{2\bar{\theta}_n}{\theta_n} \frac{2 - \gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} \right) u_n \\ &= \left(\frac{\lambda_0 \gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} - \frac{2(1 - \lambda_0)}{4 - \gamma_n \bar{\beta} - 2\lambda_0} \frac{2 - \gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} \right) u_n \\ &= \frac{\gamma_n \bar{\beta} \lambda_0 (4 - \gamma_n \bar{\beta} - 2\lambda_0) - 2(1 - \lambda_0)(2 - \gamma_n \bar{\beta})}{(2 - \lambda_0 \gamma_n \bar{\beta})(4 - \gamma_n \bar{\beta} - 2\lambda_0)} u_n \\ &= \frac{\gamma_n \bar{\beta} \lambda_0 (2 - \gamma_n \bar{\beta} - 2\lambda_0) - 2(2 - \gamma_n \bar{\beta} - 2\lambda_0)}{(2 - \lambda_0 \gamma_n \bar{\beta})(4 - \gamma_n \bar{\beta} - 2\lambda_0)} u_n \\ &= \frac{(\gamma_n \bar{\beta} \lambda_0 - 2)(2 - \gamma_n \bar{\beta} - 2\lambda_0)}{(2 - \lambda_0 \gamma_n \bar{\beta})(4 - \gamma_n \bar{\beta} - 2\lambda_0)} u_n \\ &= -\frac{(2 - \gamma_n \bar{\beta} - 2\lambda_0)}{(4 - \gamma_n \bar{\beta} - 2\lambda_0)} u_n, \end{aligned} \tag{29}$$

to conclude that

$$\ell_n \geq \frac{\theta_n}{2} \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2$$

$$= \frac{\theta_n}{2} \left\| p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{(2 - \gamma_n \bar{\beta} - 2\lambda_0)}{(4 - \gamma_n \bar{\beta} - 2\lambda_0)} u_n \right\|_M^2.$$

Now, since $\lambda_{n+1} + \mu_{n+1} = \frac{\lambda_n^2}{\lambda_0}$, we conclude that if

$$\begin{aligned} & \frac{\lambda_n^2}{\lambda_0} \left(\frac{\tilde{\theta}_{n+1}}{\hat{\theta}_{n+1}} \|u_{n+1}\|_M^2 + \frac{\hat{\theta}_{n+1}}{\tilde{\theta}_{n+1}} \|v_{n+1}\|_M^2 \right) \\ & \leq \zeta_{n+1} \frac{\theta_n}{2} \left\| p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{(2 - \gamma_n \bar{\beta} - 2\lambda_0)}{(4 - \gamma_n \bar{\beta} - 2\lambda_0)} u_n \right\|_M^2, \end{aligned}$$

the safeguarding condition in Algorithm 1 is satisfied. The vectors u_{n+1} and v_{n+1} are scalars times the quantity inside this norm. Therefore, the safeguarding condition reduces to the scalar condition

$$\frac{\lambda_n^2 \kappa_n^2 (4 - \gamma \bar{\beta} - 2\lambda_0)^2}{\lambda_0} \left(\frac{\tilde{\theta}_{n+1}}{\hat{\theta}_{n+1}} + \frac{\hat{\theta}_{n+1}}{\tilde{\theta}_{n+1}} \frac{(2 - \gamma_n \bar{\beta})^2}{(2 - \lambda_0 \gamma_n \bar{\beta})^2} \right) \leq \zeta_{n+1} \frac{\theta_n}{2}.$$

Inserting the quantities in (12) and $\mu_n = \frac{1}{\lambda_0} \lambda_n^2 - \lambda_n$ and multiplying by $\frac{2}{\theta_n} > 0$ gives

$$\kappa_n^2 \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(\frac{\gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} + \frac{2 - \lambda_0 \gamma_n \bar{\beta}}{(4 - \gamma_n \bar{\beta} - 2\lambda_0)} \frac{(2 - \gamma_n \bar{\beta})^2}{(2 - \lambda_0 \gamma_n \bar{\beta})^2} \right) \leq \zeta_{n+1}.$$

The left-hand side satisfies

$$\begin{aligned} & \kappa_n^2 \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(\frac{\gamma_n \bar{\beta}}{2 - \lambda_0 \gamma_n \bar{\beta}} + \frac{2 - \lambda_0 \gamma_n \bar{\beta}}{(4 - \gamma_n \bar{\beta} - 2\lambda_0)} \frac{(2 - \gamma_n \bar{\beta})^2}{(2 - \lambda_0 \gamma_n \bar{\beta})^2} \right) \\ & = \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \frac{\kappa_n^2}{2 - \lambda_0 \gamma_n \bar{\beta}} \frac{(\gamma_n \bar{\beta} (4 - \gamma_n \bar{\beta} - 2\lambda_0) + (2 - \gamma_n \bar{\beta})^2)}{(4 - \gamma_n \bar{\beta} - 2\lambda_0)} \\ & = \frac{\kappa_n^2}{2(2 - \lambda_0 \gamma_n \bar{\beta})} (\gamma_n \bar{\beta} (4 - \gamma_n \bar{\beta} - 2\lambda_0) + 4 - 4\gamma_n \bar{\beta} + (\gamma_n \bar{\beta})^2) \\ & = \frac{\kappa_n^2}{2(2 - \lambda_0 \gamma_n \bar{\beta})} (4 - 2\gamma_n \bar{\beta} \lambda_0) \\ & = \kappa_n^2, \end{aligned}$$

leading to the safeguarding condition

$$\kappa_n^2 \leq \zeta_{n+1}$$

which is satisfied by letting $\zeta_{n+1} = \kappa_n^2$.

Using Proposition 2 and the choice $\zeta_{n+1} = \kappa_n^2$, we conclude that Assumption 2 holds since Assumption 4 does and we can use Theorem 3 to prove convergence.

That Proposition 3 (i) holds follows from Theorem 3 (ii) since $y_0 = x_0$ and by updating the norm expression using (29).

Proposition 3 (ii) follows from Theorem 3 (vi) due to Proposition 2 that ensures $\epsilon_1 > 0$.

Proposition 3 (iii) follows from Theorem 3 (viii) due to Proposition 2 and that $\kappa_n^2 = \zeta_{n+1}$ is upper bounded by a constant less than 1, which implies $\epsilon_0 > 0$. \square

Remark 2 *The algorithm produces points that satisfy*

$$(A + C)p_n \ni \frac{1}{\gamma}M(y_n - p_n) - C(y_n - p_n)$$

and Proposition 3 (ii) implies, if $f'(0) < 1$ and $\bar{\beta} > \beta$, that $M(y_n - p_n) - C(y_n - p_n) \rightarrow 0$ as $n \rightarrow \infty$ due to boundedness of M and Lipschitz continuity of C . Although $(p_n)_{n \in \mathbb{N}}$ may not converge, it satisfies the monotone inclusion (2) in the limit. If in addition $\|p_n - x^*\|_M$ converges, we can conclude that $p_n \rightarrow x^* \in \text{zer}(A + C)$.

A special case of (26) that we will evaluate numerically in Section 6 is found by letting $\lambda_0 = 1$, and, for all $n \in \mathbb{N}$, $\lambda_n = \lambda_0$, and $\kappa_n = \kappa \in (-1, 1)$. Then (26) reduces to

$$\begin{aligned} y_n &= x_n + u_n \\ p_n &= (M + \gamma A)^{-1}(My_n - \gamma C y_n) \\ x_{n+1} &= x_n + p_n - y_n \\ u_{n+1} &= \kappa \left(\frac{2-\gamma\bar{\beta}}{2} (p_n - x_n) + \frac{\gamma\bar{\beta}}{2} u_n \right), \end{aligned}$$

which, since $x_{n+1} = p_n - u_n$, can be written as

$$\begin{aligned} p_n &= \left((M + \gamma A)^{-1} \circ (M - \gamma C) \right) (p_{n-1} + u_n - u_{n-1}) \\ u_{n+1} &= \kappa \left(\frac{2-\gamma\bar{\beta}}{2} (p_n - p_{n-1} + u_{n-1}) + \frac{\gamma\bar{\beta}}{2} u_n \right). \end{aligned} \tag{30}$$

This algorithm is previewed in Section 3.1. Since $\kappa_n = \kappa \in (-1, 1)$ for all $n \in \mathbb{N}$ and since $(\lambda_n)_{n \in \mathbb{N}}$ is constant, Proposition 3 (iii) ensures that this algorithm produces a p_n -sequence that converges weakly to a solution.

5.1 Alternative formulation

We can eliminate the x_n sequence in (26) and express the algorithm solely in terms of y_n , p_n , and u_n . The algorithm becomes

$$\begin{aligned} p_n &= (M + \gamma A)^{-1}(My_n - \gamma C y_n) \\ y_{n+1} &= y_n + \frac{\lambda_0 \lambda_n}{\lambda_{n+1}} (p_n - y_n) + u_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} u_n \\ &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} ((y_n - y_{n-1}) + \lambda_0 (y_{n-1} - p_{n-1})) \\ u_{n+1} &= \kappa_n \left(\frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2} \left(p_n - y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} (p_{n-1} - y_{n-1}) \right) + u_n \right) \end{aligned} \tag{31}$$

with $y_{-1} = p_{-1} = x_0$ and $u_0 = 0$.

Proposition 4 *The algorithms in (26) and (31) produce the same $(y_n)_{n \in \mathbb{N}}$ and $(p_n)_{n \in \mathbb{N}}$ sequences, provided $p_{-1} = y_{-1} = y_0 = x_0$.*

Proof We remove the x_n sequence from (26) by inserting

$$x_n = \frac{\lambda_n}{\lambda_0} \left(y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} y_{n-1} - u_n \right),$$

that comes from the y_n update, into the x_{n+1} update. This gives x_{n+1} update

$$\begin{aligned} \frac{\lambda_{n+1}}{\lambda_0} \left(y_{n+1} - \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} y_n - u_{n+1} \right) &= \frac{\lambda_n}{\lambda_0} \left(y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} y_{n-1} - u_n \right) \\ &\quad + \lambda_n (p_n - y_n) + (\lambda_n - \lambda_0)(y_{n-1} - p_{n-1}). \end{aligned}$$

Multiplying by $\frac{\lambda_0}{\lambda_{n+1}}$ gives

$$\begin{aligned} y_{n+1} &= \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} y_n + u_{n+1} + \frac{\lambda_n}{\lambda_{n+1}} \left(y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} y_{n-1} - u_n \right) \\ &\quad + \frac{\lambda_0}{\lambda_{n+1}} (\lambda_n (p_n - y_n) + (\lambda_n - \lambda_0)(y_{n-1} - p_{n-1})) \\ &= y_n + u_{n+1} + \frac{\lambda_n}{\lambda_{n+1}} \left(-\frac{\lambda_0}{\lambda_n} y_n + y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} y_{n-1} - u_n \right) \\ &\quad + \frac{\lambda_0}{\lambda_{n+1}} (\lambda_n (p_n - y_n) + (\lambda_n - \lambda_0)(y_{n-1} - p_{n-1})) \\ &= y_n + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} (y_n - y_{n-1}) + u_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} u_n \\ &\quad + \frac{\lambda_0}{\lambda_{n+1}} (\lambda_n (p_n - y_n) + (\lambda_n - \lambda_0)(y_{n-1} - p_{n-1})) \\ &= y_n + \frac{\lambda_0 \lambda_n}{\lambda_{n+1}} (p_n - y_n) + u_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} u_n \\ &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} ((y_n - y_{n-1}) + \lambda_0 (y_{n-1} - p_{n-1})). \end{aligned}$$

The u_{n+1} update becomes

$$\begin{aligned} u_{n+1} &= \kappa_n \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{2 - \gamma \bar{\beta} - 2\lambda_0}{4 - \gamma \bar{\beta} - 2\lambda_0} u_n \right) \\ &= \kappa_n \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(p_n - \frac{\lambda_0}{\lambda_n} x_n - \frac{\lambda_n - \lambda_0}{\lambda_n} p_{n-1} - \frac{2 - \gamma \bar{\beta} - 2\lambda_0}{4 - \gamma \bar{\beta} - 2\lambda_0} u_n \right) \\ &= \kappa_n \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(p_n - \left(y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} y_{n-1} - u_n \right) - \frac{\lambda_n - \lambda_0}{\lambda_n} p_{n-1} - \frac{2 - \gamma \bar{\beta} - 2\lambda_0}{4 - \gamma \bar{\beta} - 2\lambda_0} u_n \right) \\ &= \kappa_n \frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(p_n - y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} (p_{n-1} - y_{n-1}) + \frac{2}{4 - \gamma \bar{\beta} - 2\lambda_0} u_n \right) \\ &= \kappa_n \left(\frac{(4 - \gamma \bar{\beta} - 2\lambda_0)}{2} \left(p_n - y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} (p_{n-1} - y_{n-1}) \right) + u_n \right). \end{aligned}$$

This concludes the proof. □

5.2 Fixed-point residual convergence rate

The convergent quantity in Proposition 3 (i) may be hard to interpret. In this section, we propose a special case of (26) and (31) such that this quantity is the fixed-point

residual, $p_n - y_n$, for the forward–backward mapping. This is achieved by letting $\kappa_n = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}}$, which implies that

$$u_{n+1} = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}} \frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2} (p_n - y_n)$$

and that the algorithm becomes

$$\begin{aligned} p_n &= (M + \gamma A)^{-1}(My_n - \gamma C y_n) \\ y_{n+1} &= y_n + \left(\frac{\lambda_0\lambda_n}{\lambda_{n+1}} + \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}} \frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2} \right) (p_n - y_n) \\ &\quad + \frac{\lambda_n-\lambda_0}{\lambda_{n+1}} \left((y_n - y_{n-1}) + \frac{(4-\gamma\bar{\beta})}{2} (y_{n-1} - p_{n-1}) \right). \end{aligned} \tag{32}$$

This algorithm converges as per the following result.

Proposition 5 *Suppose that Assumptions 1 and 4 hold. Then the following hold for (32):*

(i) *if $f(n) \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$\left\| \frac{1}{\gamma} (p_n - y_n) \right\|_M^2 \leq \frac{2\|y_0 - x^*\|_M^2}{\gamma^2(4 - \gamma\bar{\beta} - 2\lambda_0)\lambda_0 f(n)^2};$$

- (ii) *if $\bar{\beta} > \beta$ and $f'(0) < 1$, then $(\|p_n - y_n\|^2)_{n \in \mathbb{N}}$ is summable;*
- (iii) *if $(\lambda_n)_{n \in \mathbb{N}}$ is constant, the algorithm reduces to relaxed forward–backward splitting and $p_n \rightarrow x^* \in \text{zer}(A + C)$.*

Proof The claims follow from Propositions 3 and 4 by showing that, for all $n \in \mathbb{N}$: the choice $\kappa_n = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}}$ in (31) gives (32); by noting that $\kappa_n = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}}$ satisfies Assumption 4 (i) with $\epsilon_0 > 0$ if f (and consequently $(\lambda_n)_{n \in \mathbb{N}}$) is bounded and with $\epsilon_0 = 0$ if f (and consequently $(\lambda_n)_{n \in \mathbb{N}}$) is unbounded; and by showing that the expression inside the norm in Proposition 3 (i) satisfies

$$p_n - x_n + \frac{\lambda_n-\lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{2-\gamma\bar{\beta}-2\lambda_0}{4-\gamma\bar{\beta}-2\lambda_0} u_n = p_n - y_n. \tag{33}$$

We will first show that the u_{n+1} update in (31) with $\kappa_n = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}}$, i.e.,

$$u_{n+1} = \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}} \left(\frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2} (p_n - y_n - \frac{\lambda_n-\lambda_0}{\lambda_n} (p_{n-1} - y_{n-1})) + u_n \right) \tag{34}$$

implies that $u_n = \frac{\lambda_n-\lambda_0}{\lambda_n} \frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2} (p_{n-1} - y_{n-1})$ for all $n \in \mathbb{N}$. For $n = 0$, since $p_{-1} = y_{-1} = u_0 = 0$, we get

$$u_1 = \frac{\lambda_1-\lambda_0}{\lambda_1} \frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2} (p_0 - y_0).$$

For $n \geq 1$, we use induction. Assume that $u_n = \frac{\lambda_n - \lambda_0}{\lambda_n} \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} (p_{n-1} - y_{n-1})$, then

$$u_{n+1} = \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} (p_n - y_n) \tag{35}$$

since the last two terms in (34) cancel, which is what we wanted to show.

The y_{n+1} update in (31) with u_{n+1} defined in (35) inserted satisfies

$$\begin{aligned} y_{n+1} &= y_n + \frac{\lambda_0 \lambda_n}{\lambda_{n+1}} (p_n - y_n) + u_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} u_n \\ &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} ((y_n - y_{n-1}) + \lambda_0 (y_{n-1} - p_{n-1})) \\ &= y_n + \frac{\lambda_0 \lambda_n}{\lambda_{n+1}} (p_n - y_n) + \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} (p_n - y_n) \\ &\quad - \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} (p_{n-1} - y_{n-1}) \\ &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} ((y_n - y_{n-1}) + \lambda_0 (y_{n-1} - p_{n-1})) \\ &= y_n + \left(\frac{\lambda_0 \lambda_n}{\lambda_{n+1}} + \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} \right) (p_n - y_n) \\ &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} (y_n - y_{n-1}) + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} \left(\lambda_0 + \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} \right) (y_{n-1} - p_{n-1}) \\ &= y_n + \left(\frac{\lambda_0 \lambda_n}{\lambda_{n+1}} + \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}} \frac{(4 - \gamma\bar{\beta} - 2\lambda_0)}{2} \right) (p_n - y_n) \\ &\quad + \frac{\lambda_n - \lambda_0}{\lambda_{n+1}} \left((y_n - y_{n-1}) + \frac{(4 - \gamma\bar{\beta})}{2} (y_{n-1} - p_{n-1}) \right), \end{aligned}$$

which equals the y_{n+1} update in (32).

Finally, using the y_n update equation in (26), i.e. $\frac{\lambda_0}{\lambda_n} x_n = y_n - \frac{\lambda_n - \lambda_0}{\lambda_n} y_{n-1} - u_n$ and the u_{n+1} definition in (35), we conclude that

$$\begin{aligned} p_n - x_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (x_n - p_{n-1}) - \frac{2 - \gamma\beta - 2\lambda_0}{4 - \gamma\beta - 2\lambda_0} u_n \\ &= p_n - y_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (y_{n-1} - p_{n-1}) + \left(1 - \frac{2 - \gamma\beta - 2\lambda_0}{4 - \gamma\beta - 2\lambda_0} \right) u_n \\ &= p_n - y_n + \frac{\lambda_n - \lambda_0}{\lambda_n} (y_{n-1} - p_{n-1}) + \frac{2}{4 - \gamma\beta - 2\lambda_0} u_n, \\ &= p_n - y_n. \end{aligned}$$

This completes the proof. □

One of the special cases previewed in Section 3.1 is obtained from (32) by letting $\lambda_n = \left(1 - \frac{\gamma\bar{\beta}}{4}\right)^e (1 + n)^e$ for all $n \in \mathbb{N}$. The resulting algorithm is numerically evaluated in Section 6 and enjoys the following convergence properties.

Corollary 1 *Suppose that Assumptions 1 and 4 hold with $\lambda_0 = \left(1 - \frac{\gamma\bar{\beta}}{4}\right)^e$ and let $f(n) = (1 + n)^e$ with $e \in [0, 1]$. Then the following hold for (32):*

(i) *if $e > 0$, then*

$$\left\| \frac{1}{\gamma} (p_n - y_n) \right\|_M^2 \leq \frac{2 \|y_0 - x^*\|_M^2}{\gamma^2 (4 - \gamma\bar{\beta} - 2\lambda_0) \lambda_0 (1 + n)^{2e}};$$

- (ii) if $\bar{\beta} > \beta$ and $e < 1$, then $(\|p_n - y_n\|^2)_{n \in \mathbb{N}}$ is summable;
- (iii) if $e = 0$, the algorithm reduces to forward-backward splitting and $p_n \rightarrow x^* \in \text{zer}(A + C)$.

Corollary 1 (i) states that $(\|p_n - y_n\|^2)_{n \in \mathbb{N}}$ converges as $\mathcal{O}\left(\frac{1}{n^{2e}}\right)$. When $\bar{\beta} > \beta$ and $e < 1$. Corollary 1 (ii) gives that $(\|p_n - y_n\|^2)_{n \in \mathbb{N}}$ converges as $\mathcal{O}\left(\frac{1}{n}\right)$ due to its summability. This gives a combined $\mathcal{O}\left(\min\left(\frac{1}{n}, \frac{1}{n^{2e}}\right)\right)$ convergence rate and implies tunability of the convergence rate by selecting $e \in [0, 1]$. Letting $e = 1$ implies that our algorithm, as we will see in Section 5.2.1, reduces to the accelerated proximal point method and the Halpern iteration that converge as $O\left(\frac{1}{n^2}\right)$.

5.2.1 Accelerated proximal point method and Halpern iteration

Letting $f(n) = 1 + n$ and $\lambda_0 = \left(1 - \frac{\gamma\bar{\beta}}{4}\right)$ to get $\lambda_n = \left(1 - \frac{\gamma\bar{\beta}}{4}\right)(1 + n)$, we get that the y_{n+1} update of (32) satisfies

$$\begin{aligned} y_{n+1} &= y_n + \left(\frac{\lambda_0\lambda_n}{\lambda_{n+1}} + \frac{\lambda_{n+1}-\lambda_0}{\lambda_{n+1}} \frac{(4-\gamma\bar{\beta}-2\lambda_0)}{2}\right) (p_n - y_n) \\ &\quad + \frac{\lambda_n-\lambda_0}{\lambda_{n+1}} \left((y_n - y_{n-1}) + \frac{(4-\gamma\bar{\beta})}{2}(y_{n-1} - p_{n-1})\right) \\ &= y_n + \left(1 - \frac{\gamma\bar{\beta}}{4}\right) \left(\frac{n+1}{n+2} + \frac{n+1}{n+2}\right) (p_n - y_n) \\ &\quad + \frac{n}{n+2} \left((y_n - y_{n-1}) + \frac{(4-\gamma\bar{\beta})}{2}(y_{n-1} - p_{n-1})\right) \\ &= y_n + \frac{4-\gamma\bar{\beta}}{2} \frac{n+1}{n+2} (p_n - y_n) \\ &\quad + \frac{n}{n+2} \left((y_n - y_{n-1}) + \frac{(4-\gamma\bar{\beta})}{2}(y_{n-1} - p_{n-1})\right) \\ &= \frac{\gamma\bar{\beta}(1+n)}{4+2n} y_n + \frac{n(2-\gamma\bar{\beta})}{4+2n} y_{n-1} + \frac{(1+n)(4-\gamma\bar{\beta})}{4+2n} p_n - \frac{n(4-\gamma\bar{\beta})}{4+2n} p_{n-1} \end{aligned}$$

and algorithm (32) becomes

$$\begin{aligned} p_n &= (M + \gamma A)^{-1} (M - \gamma C) y_n, \\ y_{n+1} &= \frac{\gamma\bar{\beta}(1+n)}{4+2n} y_n + \frac{n(2-\gamma\bar{\beta})}{4+2n} y_{n-1} + \frac{(1+n)(4-\gamma\bar{\beta})}{4+2n} p_n - \frac{n(4-\gamma\bar{\beta})}{4+2n} p_{n-1}. \end{aligned} \tag{36}$$

From Corollary 1, we conclude since $e = 1$ and by letting $\beta = \bar{\beta}$ that this algorithm converges as

$$\left\| \frac{1}{\gamma} (p_n - y_n) \right\|_M^2 \leq \frac{16 \|y_0 - x^*\|_M^2}{\gamma^2 (4 - \gamma\beta)^2 (1 + n)^2}.$$

By letting $C = 0$ and consequently $\beta = 0$, we arrive at the accelerated proximal point method in [21] and the $O\left(\frac{1}{n^2}\right)$ convergence rate results found in [21] is recovered by Corollary 1.

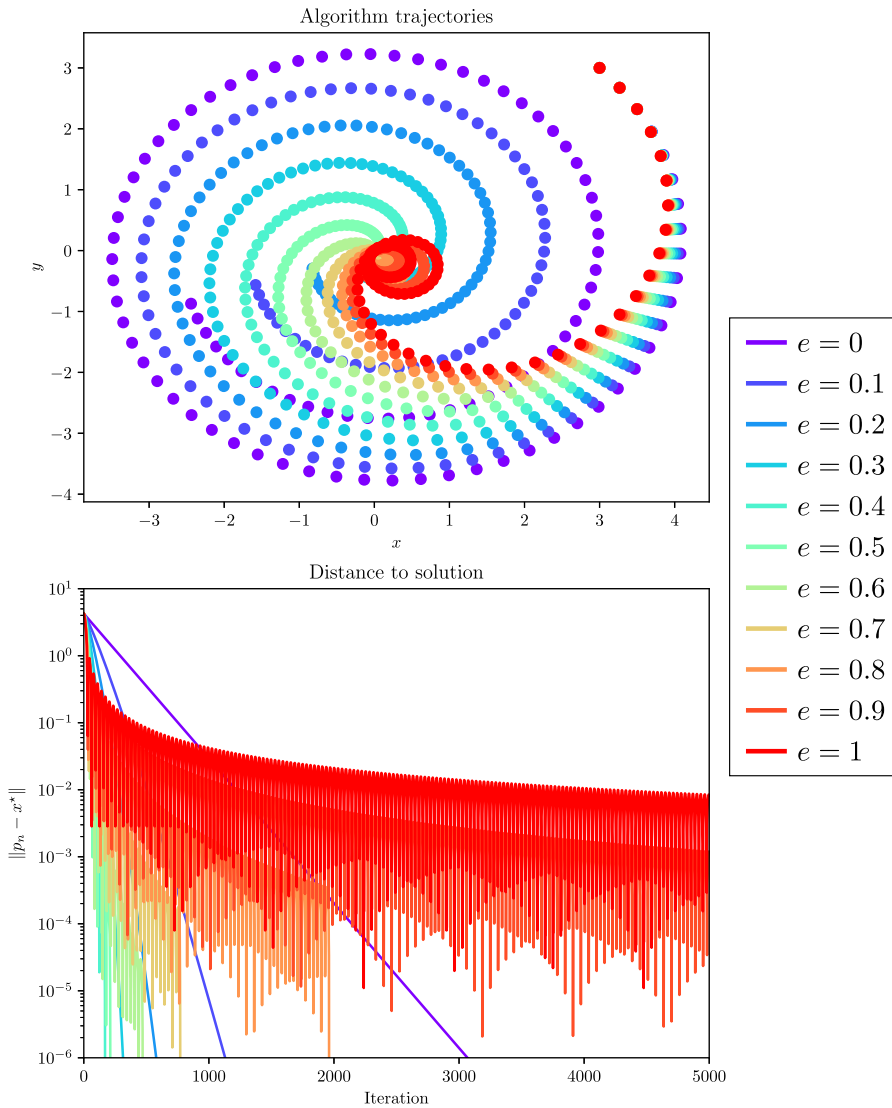


Fig. 1 We evaluate the algorithm in (32) with λ_n in (37) with the different choices of e specified in the legend to the right. The upper figure shows the 100 first p_n -iterates for the different e and the lower figure shows the distance to the unique solution. The algorithm with $e = 0$ is standard forward–backward splitting (here, only the backward part is used) and the algorithm with $e = 1$ is the accelerated proximal point in [21]. For small e , we get a strictly decreasing distance to solution, while large e gives an oscillatory behavior. Our new algorithms with e chosen around the middle of the allowed range strike a good balance and achieves superior performance compared to the previously known methods

If we let $A = 0$, $\bar{\beta} = \beta$, and $\gamma\beta = 2$, the forward–backward mapping in (36) satisfies

$$p_n = (M - \frac{2}{\beta}C)y_n$$

where $(M - \frac{2}{\beta}C) := N$ is nonexpansive in the $\|\cdot\|_M$ norm. This implies that the algorithm aims at solving the nonexpansive fixed-point equation $y = Ny$ by iterating

$$p_n = Ny_n$$

$$y_{n+1} = \frac{1+n}{2+n}y_n + \frac{1+n}{2+n}p_n - \frac{n}{2+n}p_{n-1},$$

which is the Halpern iteration studied in [23]. This is seen by recursively inserting y_n into the y_{n+1} update to get

$$y_{n+1} = \frac{1}{n+2}y_0 + \frac{n+1}{n+2}Ny_n,$$

which is the formulation used in [23]. From Corollary 1, we conclude that this iteration converges as

$$\|p_n - y_n\|_M^2 \leq \frac{4\|y_0 - x^*\|_M^2}{(1+n)^2},$$

which recovers the convergence result in [23]. Interestingly, although the convergence rate is optimized by this choice of λ_n , it does not perform very well in practice. Other choices of $(\lambda_n)_{n \in \mathbb{N}}$ with slower rate guarantees can give significantly better practical performance as demonstrated in Section 6.

6 Numerical examples

In this section, we apply our proposed algorithms on the problem $0 \in Az$, where $z = (x, y)$ and

$$Az = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} z = (-y, x)$$

Table 1 We report the number of iterations to reach accuracy $\|p_n - x^*\| \leq 10^{-6}$ for the algorithm in (32) with λ_n in (37) and different choices of e

| e | # iter. | e | # iter. | e | # iter. | e | # iter. |
|-----|---------|-----|---------|-----|----------|-----|---------|
| 0.0 | 3068 | 0.1 | 1131 | 0.2 | 580 | 0.3 | 314 |
| 0.4 | 170 | 0.5 | 212 | 0.6 | 471 | 0.7 | 771 |
| 0.8 | 1961 | 0.9 | 10625 | 1.0 | 21213167 | | |

The theory predicts a $O(1/n^{2e})$ convergence rate for the squared fixed point residual norm. Although $e = 1$ (which gives the accelerated proximal point method in [21]) gives the best theoretical rate, it performs the worst. Standard forward–backward splitting is obtained with $e = 0$. We find that values of e in between can perform considerably better than these previously known algorithms

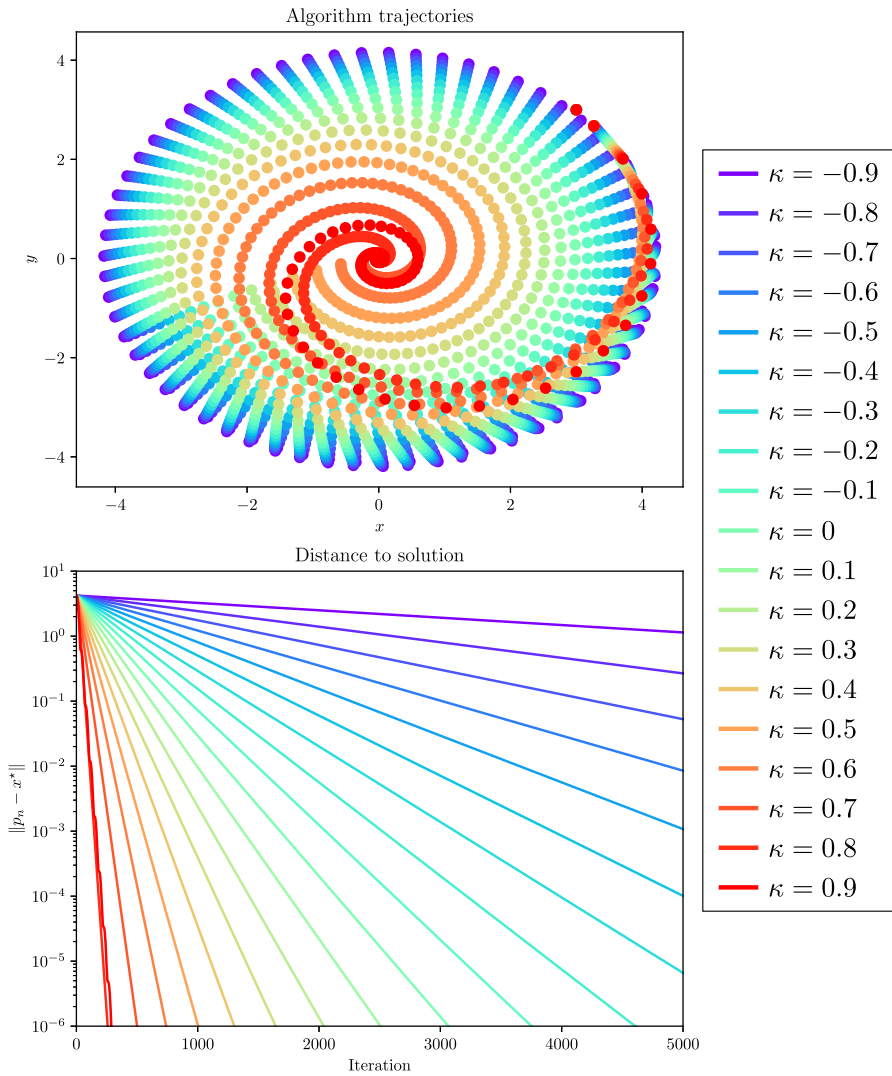


Fig. 2 We evaluate the algorithm in (30) with $\kappa \in \{-0.9, -0.8, \dots, 0.9\}$ as specified in the legend to the right. The upper figure shows the 100 first p_n -iterates for the different κ and the lower figure shows the distance to the unique solution. The performance is best for $\kappa \in \{0.8, 0.9\}$ and many choices of $\kappa \in (0, 1)$ outperform the standard forward–backward splitting method that is obtained by letting $\kappa = 0$

for all $z \in \mathbb{R}^2$. The operator $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is skew-symmetric and the monotone inclusion problem $0 \in Az$ can be interpreted as an optimality condition for the minimax problem

$$\max_{y \in \mathbb{R}} \min_{x \in \mathbb{R}} xy$$

with unique solution $x = y = 0$. We will in particular evaluate the algorithm in (30) with different choices of $\kappa \in (-1, 1)$ and the algorithm in (32) with

$$\lambda_n = \left(1 - \frac{\gamma \bar{\beta}}{4}\right)^e (1 + n)^e \tag{37}$$

for all $n \in \mathbb{N}$ and $e \in [0, 1]$. According to Propositions 3 and 5, (30) with $\kappa \in (-1, 1)$ and (32) with λ_n in (37) and $e = 0$ (corresponding to the standard FB method) converge weakly to a solution of the inclusion problem. As per Corollary 1, (32) with λ_n in (37) and $e \in (0, 1]$ converges in squared norm of the fixed point residual as $O\left(\frac{1}{n^{2e}}\right)$ and when $e < 1$ and $\bar{\beta} > \beta$, it does so with a rate of $O\left(\frac{1}{n}\right)$.

For all our experiments, parameters $\gamma = 0.1$ and $\bar{\beta} = 0.001$ (which is feasible since $C = 0$ and therefore $\beta = 0$) are used, and starting points $y_{-1} = p_{-1} = y_0 = (3, 3)$ and $x_{-1} = p_{-1} = x_0 = (3, 3)$ for (32) and (30) respectively.

In Fig. 1 and Table 1 we report numerical results for the algorithm in (32) with λ_n in (37) and $e \in \{0, 0.1, \dots, 1\}$ and $M = \text{Id}$. The choice $e = 0$ gives standard forward–backward splitting and $e = 1$ gives the accelerated proximal point method in [21]. The other choices of e gives rise to new algorithms. The figure shows that the distance to the unique solution behaves over-damped for small values of e and under-damped for large values of e . There is a sweet spot in the middle that has the right level of damping and performs significantly better than the previously known special cases with $e = 0$ and $e = 1$, at least for this problem.

In Fig. 2 and Table 2, we report numerical results for the algorithm in (30) with $\kappa \in \{-0.9, -0.8, \dots, 0.9\}$ and $M = \text{Id}$. The theory predicts sequence convergence towards a solution of the problem for all $\kappa \in (-1, 1)$. The choice $\kappa = 0$ gives rise to standard forward–backward splitting and all other values of κ define new algorithms.

Table 2 We report the number of iterations to reach accuracy $\|p_n - x^*\| \leq 10^{-6}$ for the algorithm in (30) with $\kappa \in \{-0.9, -0.8, \dots, 0.9\}$

| κ | # iter. | κ | # iter. | κ | # iter. | κ | # iter. |
|----------|---------|----------|---------|----------|---------|----------|---------|
| -0.9 | 58350 | -0.8 | 27653 | -0.7 | 17414 | -0.6 | 12292 |
| -0.5 | 9219 | -0.4 | 7170 | -0.3 | 5706 | -0.2 | 4607 |
| -0.1 | 3752 | 0.0 | 3068 | 0.1 | 2507 | 0.2 | 2040 |
| 0.3 | 1643 | 0.4 | 1302 | 0.5 | 1005 | 0.6 | 741 |
| 0.7 | 501 | 0.8 | 258 | 0.9 | 288 | | |

The theory predicts sequence for all these values of κ . The choices $\kappa = 0.8$ and $\kappa = 0.9$ significantly outperform standard forward–backward splitting that is obtained by letting $\kappa = 0$

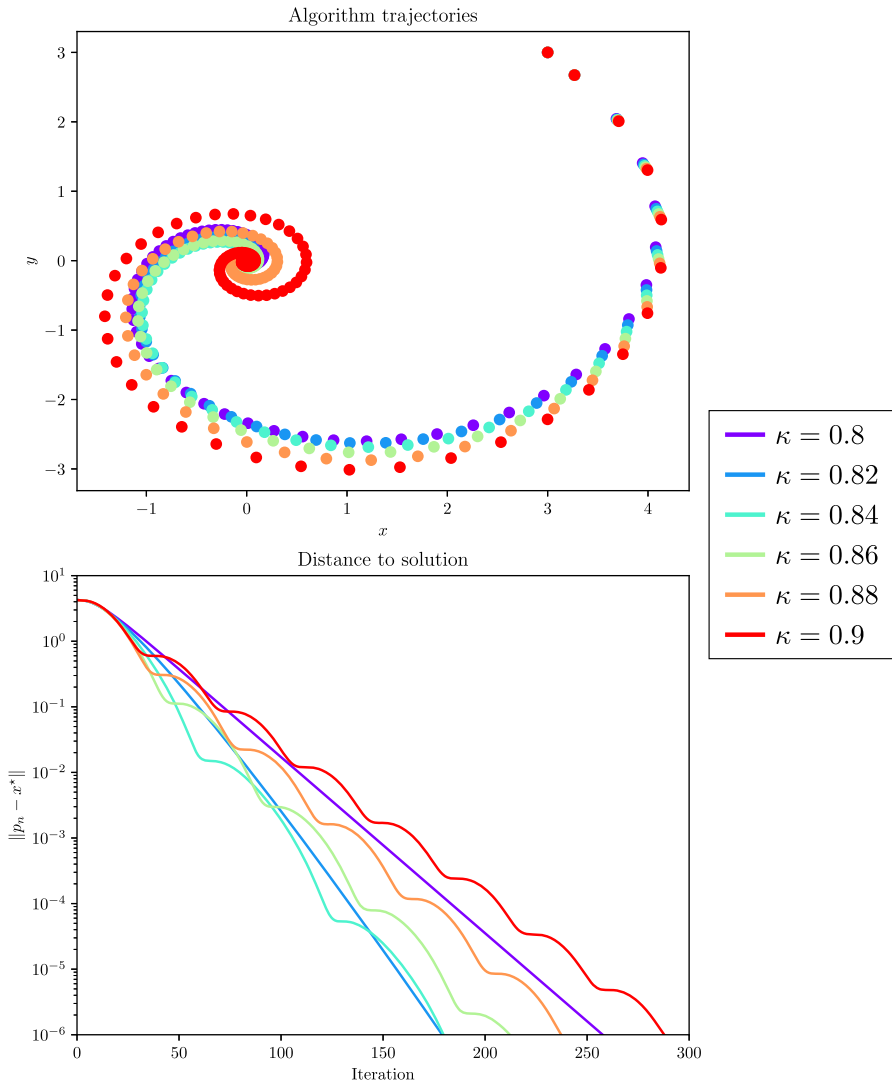


Fig. 3 We evaluate the algorithm in (30) with $\kappa \in \{0.8, 0.82, \dots, 0.9\}$ as specified in the legend to the right. The upper figure shows the 100 first p_n -iterates for the different κ and the lower figure shows the distance to the unique solution. All these choices perform well and we go from a non-oscillatory behavior to an oscillatory behaviour within this range of κ

Table 3 We report the number of iterations to reach accuracy $\|p_n - x^*\| \leq 10^{-6}$ for the algorithm in (30) with $\kappa \in \{0.8, 0.82, \dots, 0.9\}$

| κ | # iter. | κ | # iter. | κ | # iter. |
|----------|---------|----------|---------|----------|---------|
| 0.80 | 258 | 0.82 | 179 | 0.84 | 180 |
| 0.86 | 213 | 0.88 | 238 | 0.90 | 288 |

The theory predicts sequence convergence with all these κ . All choices of κ perform well

The figure reveals that the performance is best for $\kappa \in [0.8, 0.9]$, significantly better than standard FB splitting with $\kappa = 0$.

In Fig. 3 and Table 3, we provide numerical results over a finer grid of the best performing κ . We set the range to $\kappa \in [0.8, 0.9]$ and use a spacing of 0.02. We see that for $\kappa = 0.8$ and $\kappa = 0.82$, the distance to solution is non-oscillatory, while it oscillates for greater values of κ . All these choices of κ perform very well.

7 Deferred results and proofs

In what follows, we present some results that have been used in the previous sections along with the proof of Theorem 1 that was deferred to this section. Prior to that, we define the auxiliary parameter

$$\theta'_n := (2 - \gamma_n \bar{\beta})\mu_n + 2\bar{\alpha}_n \bar{\theta}_n \tag{38}$$

which frequently appears throughout this section.

We begin by establishing some identities between the parameters defined in Algorithm 1. These identities are used several times in the proof of Theorem 1.

Proposition 6 Consider the auxiliary parameters defined in Step 2 of Algorithm 1. Then, for all $n \in \mathbb{N}$, the following identities hold

- (i) $\theta_n = (2 - \gamma_n \bar{\beta})\bar{\theta}_n + \hat{\theta}_n$;
- (ii) $\theta_n = 2\bar{\theta}_n + (2 - \gamma_n \bar{\beta})(\lambda_n + \mu_n)$;
- (iii) $\lambda_n^2 \theta_n = \hat{\theta}_n (\lambda_n + \mu_n) - 2\bar{\theta}_n^2$.

Proof For Proposition 6 (i), from definition of $\bar{\theta}_n$ and $\hat{\theta}_n$, we have

$$\begin{aligned} (2 - \gamma_n \bar{\beta})\bar{\theta}_n + \hat{\theta}_n &= (2 - \gamma_n \bar{\beta})(\lambda_n + \mu_n - \lambda_n^2) + (2\lambda_n + 2\mu_n - \gamma_n \bar{\beta} \lambda_n^2) \\ &= (2 - \gamma_n \bar{\beta})(\lambda_n + \mu_n) - 2\lambda_n^2 + \gamma_n \bar{\beta} \lambda_n^2 + (2\lambda_n + 2\mu_n - \gamma_n \bar{\beta} \lambda_n^2) \\ &= (2 - \gamma_n \bar{\beta})(\lambda_n + \mu_n) - 2\lambda_n^2 + 2(\lambda_n + \mu_n) \\ &= (4 - \gamma_n \bar{\beta})(\lambda_n + \mu_n) - 2\lambda_n^2 \\ &= \theta_n, \end{aligned}$$

which holds by definition of θ_n in Algorithm 1. For Proposition 6 (ii) we have

$$\begin{aligned} 2\bar{\theta}_n + (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) &= 2(\lambda_n + \mu_n - \lambda_n^2) + (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) \\ &= -2\lambda_n^2 + (4 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) \\ &= \theta_n. \end{aligned}$$

For Proposition 6 (iii), after moving all terms to the left-hand side of the equality we get

$$\begin{aligned} \lambda_n^2\theta_n + 2\bar{\theta}_n^2 - \hat{\theta}_n(\lambda_n + \mu_n) &= \lambda_n^2\left((2 - \gamma_n\bar{\beta})\bar{\theta}_n + \hat{\theta}_n\right) + 2\bar{\theta}_n^2 - \hat{\theta}_n(\lambda_n + \mu_n) \\ &= \bar{\theta}_n\left(\lambda_n^2(2 - \gamma_n\bar{\beta}) + 2\bar{\theta}_n\right) - \hat{\theta}_n(\lambda_n + \mu_n - \lambda_n^2) \\ &= \bar{\theta}_n\left(2\lambda_n + 2\mu_n - \gamma_n\bar{\beta}\lambda_n^2\right) - \hat{\theta}_n\bar{\theta}_n = \bar{\theta}_n\hat{\theta}_n - \hat{\theta}_n\bar{\theta}_n, \end{aligned}$$

where in the first equality θ_n is substituted using Proposition 6 (i) and in the second and the third equalities, definitions of $\bar{\theta}_n$ and $\hat{\theta}_n$ are used, respectively. \square

We note from Proposition 6 (iii) that the assumption $\theta_n > 0$ in Algorithm 1 implies $\hat{\theta}_n > 0$.

The next lemma provides alternative expressions for the term inside the first norm in (4).

Lemma 2 *Suppose that Assumption 1 holds and consider the sequences generated by Algorithm 1. Then, for all $n \in \mathbb{N}$, the following*

- (i) $p_n - (1 - \alpha_n)x_n - \alpha_n p_{n-1} + \frac{\gamma_n\bar{\beta}\lambda_n^2}{\hat{\theta}_n}u_n - \frac{2\bar{\theta}_n}{\theta_n}v_n$;
- (ii) $p_n - \frac{2\bar{\theta}_n}{\theta_n}z_n + \frac{\tilde{\theta}_n}{\theta_n}y_n - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} + \frac{2\bar{\theta}_n\bar{\alpha}_n}{\theta_n}z_{n-1} - \frac{\tilde{\theta}_n\alpha_n}{\theta_n}y_{n-1}$;
- (iii) $\frac{1}{\lambda_n}(x_{n+1} - x_n) + \frac{\tilde{\theta}_n}{\hat{\theta}_n}u_n + \frac{(2-\gamma_n\bar{\beta})(\lambda_n+\mu_n)}{\theta_n}v_n$.

Proof We, first, show that Lemma 2 (ii) represents the same vector as Lemma 2 (i):

$$\begin{aligned} &p_n - \frac{2\bar{\theta}_n}{\theta_n}z_n + \frac{\tilde{\theta}_n}{\theta_n}y_n - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} + \frac{2\bar{\theta}_n\bar{\alpha}_n}{\theta_n}z_{n-1} - \frac{\tilde{\theta}_n\alpha_n}{\theta_n}y_{n-1} \\ &= p_n - \frac{2\bar{\theta}_n}{\theta_n}(z_n - \bar{\alpha}_nz_{n-1}) + \frac{\tilde{\theta}_n}{\theta_n}(y_n - \alpha_ny_{n-1}) - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} \\ &= p_n - \frac{2\bar{\theta}_n}{\theta_n}\left((1 - \alpha_n)x_n + (\alpha_n - \bar{\alpha}_n)p_{n-1} + \frac{\bar{\theta}_n\gamma_n\bar{\beta}}{\theta_n}u_n + v_n\right) \\ &\quad + \frac{\tilde{\theta}_n}{\theta_n}\left((1 - \alpha_n)x_n + u_n\right) - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} \\ &= p_n - \frac{2(1-\alpha_n)\bar{\theta}_n - (1-\alpha_n)\tilde{\theta}_n + 2\lambda_n}{\theta_n}x_n - \frac{\theta'_n + 2\bar{\theta}_n(\alpha_n - \bar{\alpha}_n)}{\theta_n}p_{n-1} \\ &\quad + \frac{\tilde{\theta}_n\bar{\theta}_n - 2\bar{\theta}_n^2\gamma_n\bar{\beta}}{\theta_n\theta_n}u_n - \frac{2\bar{\theta}_n}{\theta_n}v_n \\ &= p_n - (1 - \alpha_n)x_n - \alpha_n p_{n-1} + \frac{\gamma_n\bar{\beta}\lambda_n^2}{\hat{\theta}_n}u_n - \frac{2\bar{\theta}_n}{\theta_n}v_n \end{aligned}$$

where the coefficients of the last equality are found as follows. The numerator of the coefficient of x_n reads

$$\begin{aligned}
 & 2(1 - \alpha_n)\bar{\theta}_n - (1 - \alpha_n)\tilde{\theta} + 2\lambda_n \\
 &= (1 - \alpha_n)(\theta_n - (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)) \\
 &\quad - (1 - \alpha_n)\gamma_n\bar{\beta}(\lambda_n + \mu_n) + 2\lambda_n \tag{39} \\
 &= (1 - \alpha_n)\theta_n - 2(1 - \alpha_n)(\lambda_n + \mu_n) + 2\lambda_n \\
 &= (1 - \alpha_n)\theta_n - 2\frac{\lambda_n}{\lambda_n + \mu_n}(\lambda_n + \mu_n) + 2\lambda_n = (1 - \alpha_n)\theta_n
 \end{aligned}$$

where in the first equality, $\bar{\theta}_n$ is substituted from Proposition 6 (ii), and $\tilde{\theta}_n$ and α_n are substituted by their definitions in Algorithm 1. The numerator of the coefficient of p_{n-1} is

$$\begin{aligned}
 \theta'_n + 2\bar{\theta}_n(\alpha_n - \bar{\alpha}_n) &= (2 - \gamma_n\bar{\beta})\mu_n + 2\bar{\alpha}_n\bar{\theta}_n + 2\bar{\theta}_n(\alpha_n - \bar{\alpha}_n) \\
 &= (2 - \gamma_n\bar{\beta})\mu_n + 2\bar{\theta}_n\alpha_n \tag{40} \\
 &= (2 - \gamma_n\bar{\beta})\alpha_n(\lambda_n + \mu_n) + 2\bar{\theta}_n\alpha_n = \alpha_n\theta_n
 \end{aligned}$$

where in the first equality (38) is used, the third equality is obtained using the definition of α_n , and Proposition 6 (ii) is utilized in the last equality. For the numerator of u_n we get

$$\begin{aligned}
 \hat{\theta}\tilde{\theta}_n - 2\bar{\theta}_n^2\gamma_n\bar{\beta} &= \hat{\theta}\gamma_n\bar{\beta}(\lambda_n + \mu_n) - 2\bar{\theta}_n^2\gamma_n\bar{\beta} \\
 &= \gamma_n\bar{\beta}(\hat{\theta}_n(\lambda_n + \mu_n) - 2\bar{\theta}_n^2) = \gamma_n\bar{\beta}\lambda_n^2\theta_n \tag{41}
 \end{aligned}$$

where the first equality is obtained by substitution of the definition of $\tilde{\theta}_n$ from Algorithm 1, and in the last equality Proposition 6 (iii) is used.

Now, we show that Lemma 2 (ii) and (iii) represent the same vector. Starting from Lemma 2 (ii), we have

$$\begin{aligned}
 & p_n - \frac{2\bar{\theta}_n}{\theta_n}z_n + \frac{\tilde{\theta}_n}{\theta_n}y_n - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} + \frac{2\bar{\theta}_n\bar{\alpha}_n}{\theta_n}z_{n-1} - \frac{\bar{\theta}_n\alpha_n}{\theta_n}y_{n-1} \\
 &= p_n - \frac{2\bar{\theta}_n}{\theta_n}(z_n - \bar{\alpha}_nz_{n-1}) + \frac{\tilde{\theta}_n}{\theta_n}(y_n - \alpha_ny_{n-1}) - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} \\
 &= \frac{1}{\lambda_n}(x_{n+1} - x_n) + z_n + \bar{\alpha}_n(p_{n-1} - z_{n-1}) - \frac{2\bar{\theta}_n}{\theta_n}(z_n - \bar{\alpha}_nz_{n-1}) \\
 &\quad + \frac{\tilde{\theta}_n}{\theta_n}(y_n - \alpha_ny_{n-1}) - \frac{2\lambda_n}{\theta_n}x_n - \frac{\theta'_n}{\theta_n}p_{n-1} \\
 &= \frac{1}{\lambda_n}(x_{n+1} - x_n) + \frac{\theta_n - 2\bar{\theta}_n}{\theta_n}(z_n - \bar{\alpha}_nz_{n-1}) + \frac{\tilde{\theta}_n}{\theta_n}(y_n - \alpha_ny_{n-1}) - \frac{2\lambda_n}{\theta_n}x_n \\
 &\quad + \frac{\bar{\alpha}_n\theta_n - \theta'_n}{\theta_n}p_{n-1} \\
 &= \frac{1}{\lambda_n}(x_{n+1} - x_n) + \frac{\theta_n - 2\bar{\theta}_n}{\theta_n} \left((1 - \alpha_n)x_n + (\alpha_n - \bar{\alpha}_n)p_{n-1} + \frac{\bar{\theta}_n\gamma_n\bar{\beta}}{\theta_n}u_n + v_n \right) \\
 &\quad + \frac{\tilde{\theta}_n}{\theta_n}((1 - \alpha_n)x_n + u_n) - \frac{2\lambda_n}{\theta_n}x_n + \frac{\bar{\alpha}_n\theta_n - \theta'_n}{\theta_n}p_{n-1}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\lambda_n}(x_{n+1} - x_n) + \frac{(\theta_n - 2\bar{\theta}_n)\bar{\theta}_n\gamma_n\bar{\beta} + \tilde{\theta}_n\hat{\theta}_n}{\theta_n\hat{\theta}_n}u_n + \frac{\theta_n - 2\bar{\theta}_n}{\theta_n}v_n \\
 &\quad + \frac{(\theta_n - 2\bar{\theta}_n)(1 - \alpha_n) + \tilde{\theta}_n(1 - \alpha_n) - 2\lambda_n}{\theta_n}x_n + \frac{(\theta_n - 2\bar{\theta}_n)(\alpha_n - \bar{\alpha}_n) + \bar{\alpha}_n\theta_n - \theta'_n}{\theta_n}p_{n-1} \\
 &= \frac{1}{\lambda_n}(x_{n+1} - x_n) + \frac{\tilde{\theta}_n}{\hat{\theta}_n}u_n + \frac{(2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)}{\theta_n}v_n
 \end{aligned}$$

In the second equality, the definition of x_{n+1} in Step 8 of Algorithm 1 is used. In the fourth equality, the definition of y_n in Step 5 and the definition of z_n in Step 6 of Algorithm 1 are used. In the last equality, the coefficient of x_n is found to be $-\frac{1}{\lambda_n}$ by (39), the coefficient of p_{n-1} is zero by (40), the coefficient of v_n is found by Proposition 6 (ii), and for the coefficient of u_n we have

$$\begin{aligned}
 (\theta_n - 2\bar{\theta}_n)\bar{\theta}_n\gamma_n\bar{\beta} + \tilde{\theta}_n\hat{\theta}_n &= \theta_n\bar{\theta}_n\gamma_n\bar{\beta} - 2\bar{\theta}^2\gamma_n\bar{\beta} + \tilde{\theta}_n\hat{\theta}_n \\
 &= \theta_n\bar{\theta}_n\gamma_n\bar{\beta} + \gamma_n\bar{\beta}\lambda_n^2\theta_n \\
 &= \theta_n\gamma_n\bar{\beta}(\bar{\theta}_n + \lambda_n^2) = \theta_n\gamma_n\bar{\beta}(\lambda_n + \mu_n)
 \end{aligned}$$

where the second equality is obtained by (41), and in the last equality the definition of $\bar{\theta}_n$ is used. This concludes the proof. \square

7.1 Proof of Theorem 1

Proof The only (non-trivial) divisors that will appear the proof (as well as Algorithm 1) are θ_n and $\hat{\theta}_n$. In Algorithm 1, we assume $\theta_n > 0$ for all $n \in \mathbb{N}$, which by Proposition 6 (iii) (and since $\lambda_n > 0$ and $\mu_n \geq 0$) implies that $\hat{\theta}_n > 0$. Therefore, there are no divisions by zero.

Let us define the following quantity

$$\begin{aligned}
 \Delta_n := &V_{n+1} - V_n + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + \ell_{n-1} \\
 &- (\lambda_n + \mu_n) \left(\frac{\tilde{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right) \tag{42}
 \end{aligned}$$

and prove the result by showing that, for all $n \in \mathbb{N}$, it is identical to zero. By substituting V_{n+1} and V_n in (42), we get

$$\begin{aligned}
 \Delta_n &= \|x_{n+1} - x^*\|_M^2 + \ell_n + 2\lambda_{n+1}\gamma_{n+1}\alpha_{n+1}\phi_n \\
 &\quad - \|x_n - x^*\|_M^2 - \ell_{n-1} - 2\lambda_n\gamma_n\alpha_n\phi_{n-1} \\
 &\quad + 2\gamma_n(\lambda_n - \bar{\alpha}_{n+1}\lambda_{n+1})\phi_n + \ell_{n-1} - (\lambda_n + \mu_n) \left(\frac{\tilde{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right) \\
 &= \|x_{n+1} - x^*\|_M^2 - \|x_n - x^*\|_M^2 + \ell_n - 2\lambda_n\gamma_n\alpha_n\phi_{n-1} + 2\gamma_n\lambda_n\phi_n \\
 &\quad - (\lambda_n + \mu_n) \left(\frac{\tilde{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right),
 \end{aligned}$$

where in the last equality we used $\gamma_n \bar{\alpha}_{n+1} = \gamma_{n+1} \alpha_{n+1}$. Next, substituting ℓ_n from (4), and ϕ_{n-1} and ϕ_n from (8) on the right-hand side of the last equality above, yields

$$\begin{aligned} \Delta_n &= \|x_{n+1} - x^*\|_M^2 - \|x_n - x^*\|_M^2 \\ &\quad + \frac{1}{2} \theta_n \left\| p_n - x_n + \alpha_n (x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\ &\quad + 2\mu_n \gamma_n \left\langle \frac{z_n - p_n}{\gamma_n} - \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_n - p_{n-1} \right\rangle_M \\ &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_n - y_n - (p_{n-1} - y_{n-1})\|_M^2 \\ &\quad - 2\lambda_n \gamma_n \alpha_n \left(\left\langle \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_{n-1} - x^* \right\rangle_M + \frac{\bar{\beta}}{4} \|y_{n-1} - p_{n-1}\|_M^2 \right) \\ &\quad + 2\lambda_n \gamma_n \left(\left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M + \frac{\bar{\beta}}{4} \|y_n - p_n\|_M^2 \right) \\ &\quad - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right) \\ &= \|x_{n+1} - x^*\|_M^2 - \|x_n - x^*\|_M^2 + 2\lambda_n \gamma_n \left\langle \frac{z_n - p_n}{\gamma_n}, p_n - x^* \right\rangle_M \\ &\quad - 2\lambda_n \gamma_n \alpha_n \left\langle \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_{n-1} - p_n + p_n - x^* \right\rangle_M \\ &\quad + 2\mu_n \gamma_n \left\langle \frac{z_n - p_n}{\gamma_n} - \frac{z_{n-1} - p_{n-1}}{\gamma_{n-1}}, p_n - p_{n-1} \right\rangle_M \\ &\quad + \frac{1}{2} \theta_n \left\| p_n - x_n + \alpha_n (x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\ &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_n - y_n - (p_{n-1} - y_{n-1})\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\ &\quad + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \|y_n - p_n\|_M^2 - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right) \\ &= \|x_{n+1} - x^*\|_M^2 - \|x_n - x^*\|_M^2 \\ &\quad + 2\langle \lambda_n (z_n - p_n) - \bar{\alpha}_n \lambda_n (z_{n-1} - p_{n-1}), p_n - x^* \rangle_M \\ &\quad + 2\left\langle \mu_n (z_n - p_n) + (\bar{\alpha}_n \lambda_n - \frac{\gamma_n}{\gamma_{n-1}} \mu_n) (z_{n-1} - p_{n-1}), p_n - p_{n-1} \right\rangle_M \\ &\quad + \frac{1}{2} \theta_n \left\| p_n - x_n + \alpha_n (x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\ &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_n - y_n - (p_{n-1} - y_{n-1})\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\ &\quad + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \|y_n - p_n\|_M^2 - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\theta_n} \|u_n\|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \|v_n\|_M^2 \right). \end{aligned}$$

We define

$$\omega_n := \bar{\alpha}_n \lambda_n - \frac{\gamma_n}{\gamma_{n-1}} \mu_n \tag{43}$$

and substitute it in the last equality above; and also from Step 8 of Algorithm 1, we replace $\lambda_n (z_n - p_n) - \bar{\alpha}_n \lambda_n (z_{n-1} - p_{n-1})$ by $x_n - x_{n+1}$. Then, we get

$$\begin{aligned} \Delta_n &= \|x_{n+1} - x^*\|_M^2 - \|x_n - x^*\|_M^2 + 2\langle x_n - x_{n+1}, p_n - x^* \rangle_M \\ &\quad + 2\langle \mu_n (z_n - p_n) + \omega_n (z_{n-1} - p_{n-1}), p_n - p_{n-1} \rangle_M \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2}\theta_n \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| p_n - y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 & + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n - p_n \|_M^2 - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\hat{\theta}_n} \| u_n \|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \| v_n \|_M^2 \right) \\
 = & \| x_{n+1} - p_n \|_M^2 - \| x_n - p_n \|_M^2 \\
 & + 2 \langle \mu_n(z_n - p_n) + \omega_n(z_{n-1} - p_{n-1}), p_n - p_{n-1} \rangle_M \\
 & + \frac{1}{2}\theta_n \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| p_n - y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 & + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n - p_n \|_M^2 - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\hat{\theta}_n} \| u_n \|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \| v_n \|_M^2 \right)
 \end{aligned}$$

where in the last equality we used the identity $2\langle a - b, c - d \rangle_M + \| b - d \|_M^2 - \| a - d \|_M^2 = \| b - c \|_M^2 - \| a - c \|_M^2$ for all $a, b, c, d \in \mathcal{H}$. Now, inserting x_{n+1} from Step 8 of Algorithm 1, yields

$$\begin{aligned}
 \Delta_n = & \| x_n - p_n + \lambda_n(p_n - z_n) + \lambda_n \bar{\alpha}_n(z_{n-1} - p_{n-1}) \|_M^2 - \| x_n - p_n \|_M^2 \\
 & + 2 \langle \mu_n(z_n - p_n) + \omega_n(z_{n-1} - p_{n-1}), p_n - p_{n-1} \rangle_M \\
 & + \frac{1}{2}\theta_n \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| p_n - y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 & + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n - p_n \|_M^2 - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\hat{\theta}_n} \| u_n \|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \| v_n \|_M^2 \right) \\
 = & \| \lambda_n(p_n - z_n) + \lambda_n \bar{\alpha}_n(z_{n-1} - p_{n-1}) \|_M^2 \\
 & + 2 \langle x_n - p_n, \lambda_n(p_n - z_n) + \lambda_n \bar{\alpha}_n(z_{n-1} - p_{n-1}) \rangle_M \\
 & + 2 \langle \mu_n(z_n - p_n) + \omega_n(z_{n-1} - p_{n-1}), p_n - p_{n-1} \rangle_M \\
 & + \frac{1}{2}\theta_n \left\| p_n - x_n + \alpha_n(x_n - p_{n-1}) + \frac{\gamma_n \bar{\beta} \lambda_n^2}{\hat{\theta}_n} u_n - \frac{2\bar{\theta}_n}{\theta_n} v_n \right\|_M^2 \\
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| p_n - y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 & + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n - p_n \|_M^2 - (\lambda_n + \mu_n) \left(\frac{\bar{\theta}_n}{\hat{\theta}_n} \| u_n \|_M^2 + \frac{\hat{\theta}_n}{\theta_n} \| v_n \|_M^2 \right).
 \end{aligned}$$

Next, using Lemma 2 and Steps 5–6 of Algorithm 1, we replace the terms including u_n and v_n in terms of the iterates

$$\begin{aligned}
 \Delta_n = & \| \lambda_n(p_n - z_n) + \lambda_n \bar{\alpha}_n(z_{n-1} - p_{n-1}) \|_M^2 \\
 & + 2 \langle x_n - p_n, \lambda_n(p_n - z_n) + \lambda_n \bar{\alpha}_n(z_{n-1} - p_{n-1}) \rangle_M \\
 & + 2 \langle \mu_n(z_n - p_n) + \omega_n(z_{n-1} - p_{n-1}), p_n - p_{n-1} \rangle_M \\
 & + \frac{\theta_n}{2} \left\| p_n - \frac{2\bar{\theta}_n}{\theta_n} z_n + \frac{\bar{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_n - y_n - (p_{n-1} - y_{n-1})\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 & + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \|y_n - p_n\|_M^2 - \frac{(\lambda_n + \mu_n) \tilde{\theta}_n}{\hat{\theta}_n} \|y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1}\|_M^2 \\
 & - \frac{(\lambda_n + \mu_n) \hat{\theta}_n}{\theta_n} \left\| z_n - \frac{\tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_n - \frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n \right. \\
 & \qquad \qquad \qquad \left. + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

where we used

$$\begin{aligned}
 v_n & = z_n - (1 - \alpha_n)x_n + (\bar{\alpha}_n - \alpha_n)p_{n-1} - \bar{\alpha}_n z_{n-1} - \bar{\beta} \frac{\tilde{\theta}_n \gamma_n}{\hat{\theta}_n} u_n \\
 & = z_n - \frac{\tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_n - \frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n + (\bar{\alpha}_n - \alpha_n)p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1}
 \end{aligned}$$

which is obtained by substituting u_n from Step 5 into Step 6 of Algorithm 1. Next, we expand the terms on the right-hand side of the last equality above which include p_n . This yields

$$\begin{aligned}
 \Delta_n & = \lambda_n^2 \|p_n\|_M^2 + 2 \left\langle p_n, \lambda_n^2 (-z_n + \bar{\alpha}_n z_{n-1} - \bar{\alpha}_n p_{n-1}) \right\rangle_M \\
 & \quad + \lambda_n^2 \|z_n - \bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1}\|_M^2 \\
 & \quad - 2 \lambda_n \|p_n\|_M^2 + 2 \langle p_n, \lambda_n (z_n + x_n + \bar{\alpha}_n p_{n-1} - \bar{\alpha}_n z_{n-1}) \rangle_M \\
 & \quad + 2 \langle x_n, \lambda_n (-z_n - \bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 & \quad - 2 \mu_n \|p_n\|_M^2 + 2 \langle p_n, \mu_n z_n + (\mu_n - \omega_n) p_{n-1} + \omega_n z_{n-1} \rangle_M \\
 & \quad + 2 \langle \mu_n z_n + \omega_n (z_{n-1} - p_{n-1}), -p_{n-1} \rangle_M + \frac{1}{2} \theta_n \|p_n\|_M^2 \\
 & \quad + 2 \left\langle p_n, \frac{\theta_n}{2} \left(-\frac{2\tilde{\theta}_n}{\theta_n} z_n + \frac{\tilde{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n \right) \right\rangle_M \\
 & \quad + 2 \left\langle p_n, \frac{\theta_n}{2} \left(-\frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\tilde{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\tilde{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right) \right\rangle_M \\
 & \quad + \frac{1}{2} \theta_n \left\| -\frac{2\tilde{\theta}_n}{\theta_n} z_n + \frac{\tilde{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\tilde{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\tilde{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 & \quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_n\|_M^2 + 2 \left\langle p_n, \frac{\mu_n \gamma_n \bar{\beta}}{2} (-y_n - (p_{n-1} - y_{n-1})) \right\rangle_M \\
 & \quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|-y_n - (p_{n-1} - y_{n-1})\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 & \quad + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \|p_n\|_M^2 - 2 \left\langle p_n, \frac{\lambda_n \gamma_n \bar{\beta}}{2} y_n \right\rangle_M + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \|y_n\|_M^2 \\
 & \quad - \frac{\tilde{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \|y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1}\|_M^2 \\
 & \quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| z_n - \frac{\tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_n - \frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n \right. \\
 & \qquad \qquad \qquad \left. + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\lambda_n^2 - \frac{(4-\gamma_n\bar{\beta})(\lambda_n+\mu_n)}{2} + \frac{1}{2}\theta_n \right) \|p_n\|_M^2 \\
 &+ 2 \left\langle p_n, \left(\lambda_n + \mu_n - \lambda_n^2 - \bar{\theta}_n \right) z_n + \left(\frac{\tilde{\theta}_n}{2} - \frac{(\lambda_n+\mu_n)\gamma_n\bar{\beta}}{2} \right) y_n \right\rangle_M \\
 &+ 2 \left\langle p_n, (\lambda_n - \lambda_n)x_n + \left((1 - \lambda_n)\lambda_n\bar{\alpha}_n + \mu_n - \omega_n - \frac{1}{2}\theta'_n - \frac{\mu_n\gamma_n\bar{\beta}}{2} \right) p_{n-1} \right\rangle_M \\
 &+ 2 \left\langle p_n, \left(\lambda_n^2\bar{\alpha}_n - \lambda_n\bar{\alpha}_n + \omega_n + \bar{\theta}_n\bar{\alpha}_n \right) z_{n-1} + \left(-\frac{1}{2}\tilde{\theta}_n\alpha_n + \frac{\mu_n\gamma_n\bar{\beta}}{2} \right) y_{n-1} \right\rangle_M \\
 &+ \lambda_n^2 \|z_n - \bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1}\|_M^2 + 2 \langle x_n, \lambda_n(-z_n - \bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &+ 2 \langle \mu_n z_n + \omega_n(z_{n-1} - p_{n-1}), -p_{n-1} \rangle_M \\
 &+ \frac{1}{2}\theta_n \left\| -\frac{2\bar{\theta}_n}{\theta_n} z_n + \frac{\tilde{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n\bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\tilde{\theta}_n\alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &+ \frac{\mu_n\gamma_n\bar{\beta}}{2} \| -y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n\gamma_n\alpha_n\bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 &+ \frac{\lambda_n\gamma_n\bar{\beta}}{2} \|y_n\|_M^2 - \frac{\bar{\theta}_n(\lambda_n+\mu_n)}{\theta_n} \|y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1}\|_M^2 \\
 &- \frac{\hat{\theta}_n(\lambda_n+\mu_n)}{\theta_n} \left\| z_n - \frac{\bar{\theta}_n\gamma_n\bar{\beta}}{\theta_n} y_n - \frac{(2-\gamma_n\bar{\beta})\lambda_n}{\theta_n} x_n \right. \\
 &\quad \left. + (\bar{\alpha}_n - \alpha_n)p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n\bar{\theta}_n\gamma_n\bar{\beta}}{\theta_n} y_{n-1} \right\|_M^2.
 \end{aligned}$$

All terms involving p_n in this expression are identically zero since their coefficients become zero. This is for most terms straightforward to show by substituting $\theta_n, \bar{\theta}_n, \tilde{\theta}_n, \theta'_n, \alpha_n,$ and $\bar{\alpha}_n$ defined in Algorithm 1 into the corresponding coefficients. We show this for two coefficients for which it is less obvious. For the coefficient of $\langle p_n, p_{n-1} \rangle_M$ we have

$$\begin{aligned}
 &-\lambda_n^2\bar{\alpha}_n + \lambda_n\bar{\alpha}_n + \mu_n - \omega_n - \frac{1}{2}\theta'_n - \frac{\mu_n\gamma_n\bar{\beta}}{2} \\
 &= -\lambda_n^2\bar{\alpha}_n + \lambda_n\bar{\alpha}_n + \mu_n - \left(\lambda_n\bar{\alpha}_n - \frac{\gamma_n}{\gamma_{n-1}}\mu_n \right) - \frac{1}{2}\theta'_n - \frac{\mu_n\gamma_n\bar{\beta}}{2} \\
 &= -\lambda_n^2\bar{\alpha}_n + \frac{\gamma_n}{\gamma_{n-1}}\mu_n + \frac{(2-\gamma_n\bar{\beta})\mu_n}{2} - \frac{1}{2}\theta'_n \\
 &= -\lambda_n^2\bar{\alpha}_n + (\lambda_n + \mu_n)\bar{\alpha}_n + \frac{(2-\gamma_n\bar{\beta})\mu_n}{2} - \frac{1}{2}\theta'_n \\
 &= \bar{\theta}_n\bar{\alpha}_n + \frac{(2-\gamma_n\bar{\beta})\mu_n}{2} - \frac{1}{2}\theta'_n = \frac{1}{2}\theta'_n - \frac{1}{2}\theta'_n = 0,
 \end{aligned}$$

where in the first equality ω_n is substituted from (43) and in the third equality the definition of $\bar{\alpha}_n$ is used. For the coefficient of $\langle p_n, z_{n-1} \rangle_M$

$$\begin{aligned}
 &\lambda_n^2\bar{\alpha}_n - \lambda_n\bar{\alpha}_n + \omega_n + \bar{\theta}_n\bar{\alpha}_n \\
 &= \lambda_n^2\bar{\alpha}_n - \lambda_n\bar{\alpha}_n + \lambda_n\bar{\alpha}_n - \frac{\gamma_n}{\gamma_{n-1}}\mu_n + \bar{\theta}_n\bar{\alpha}_n \\
 &= \lambda_n^2\bar{\alpha}_n - (\lambda_n + \mu_n)\bar{\alpha}_n + \bar{\theta}_n\bar{\alpha}_n = (\bar{\theta}_n - \bar{\theta}_n)\bar{\alpha}_n = 0.
 \end{aligned}$$

Next, for the terms containing z_n , we do a similar procedure of expanding, reordering, and recollecting the terms as we did for p_n . This gives

$$\begin{aligned}
 \Delta_n &= \lambda_n^2 \|z_n - \bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1}\|_M^2 + 2\langle x_n, \lambda_n(-z_n - \bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &\quad + 2\langle \mu_n z_n + \omega_n(z_{n-1} - p_{n-1}), -p_{n-1} \rangle_M \\
 &\quad + \frac{1}{2}\theta_n \left\| -\frac{2\bar{\theta}_n}{\theta_n} z_n + \frac{\bar{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| -y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 &\quad + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n \|_M^2 - \frac{\bar{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \| y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1} \|_M^2 \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| z_n - \frac{\bar{\theta}_n \gamma_n \bar{\beta}}{\theta_n} y_n - \frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n \right. \\
 &\quad \quad \left. + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2 \\
 &= \lambda_n^2 \|z_n\|_M^2 + 2\left\langle z_n, \lambda_n^2(-\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1}) \right\rangle_M + \lambda_n^2 \|-\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1}\|_M^2 \\
 &\quad + 2\langle z_n, -\lambda_n x_n \rangle_M + 2\langle x_n, \lambda_n(-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &\quad + 2\langle z_n, -\mu_n p_{n-1} \rangle_M + 2\langle p_{n-1}, -\omega_n(z_{n-1} - p_{n-1}) \rangle_M + \frac{2\bar{\theta}_n^2}{\theta_n} \|z_n\|_M^2 \\
 &\quad + 2\left\langle z_n, -\bar{\theta}_n \left(\frac{\bar{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right) \right\rangle_M \\
 &\quad + \frac{1}{2}\theta_n \left\| \frac{\bar{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| -y_n - (p_{n-1} - y_{n-1}) \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 &\quad + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n \|_M^2 - \frac{\bar{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \| y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1} \|_M^2 \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \| z_n \|_M^2 + 2\left\langle z_n, -\frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left(-\frac{\bar{\theta}_n \gamma_n \bar{\beta}}{\theta_n} y_n - \frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n \right) \right\rangle_M \\
 &\quad + 2\left\langle z_n, -\frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left((\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right) \right\rangle_M \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| -\frac{\bar{\theta}_n \gamma_n \bar{\beta}}{\theta_n} y_n - \frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n \right. \\
 &\quad \quad \left. + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2 \\
 &= \frac{\lambda_n^2 \theta_n + 2\bar{\theta}_n^2 - \hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \|z_n\|_M^2 + 2\left\langle z_n, \frac{\bar{\theta}_n (\gamma_n \bar{\beta} (\lambda_n + \mu_n) - \hat{\theta}_n)}{\theta_n} y_n \right\rangle_M \\
 &\quad + 2\left\langle z_n, \frac{2\bar{\theta}_n \lambda_n - \lambda_n \theta_n + (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n) \lambda_n}{\theta_n} x_n \right\rangle_M \\
 &\quad + 2\left\langle z_n, \frac{\lambda_n^2 \bar{\alpha}_n \theta_n - \mu_n \theta_n + \bar{\theta}_n \theta'_n - \hat{\theta}_n (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n)}{\theta_n} p_{n-1} \right\rangle_M \\
 &\quad + 2\left\langle z_n, \frac{\bar{\alpha}_n (\hat{\theta}_n (\lambda_n + \mu_n) - 2\bar{\theta}_n^2 - \lambda_n^2 \theta_n)}{\theta_n} z_{n-1} + \frac{\alpha_n \bar{\theta}_n (\hat{\theta}_n - \gamma_n \bar{\beta} (\lambda_n + \mu_n))}{\theta_n} y_{n-1} \right\rangle_M \\
 &\quad + \lambda_n^2 \|-\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1}\|_M^2 + 2\langle x_n, \lambda_n(-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M
 \end{aligned}$$

$$\begin{aligned}
 &+ 2\langle p_{n-1}, -\omega_n(z_{n-1} - p_{n-1}) \rangle_M \\
 &+ \frac{1}{2}\theta_n \left\| \frac{\bar{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n\bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n\alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &+ \frac{\mu_n\gamma_n\bar{\beta}}{2} \left\| -y_n - p_{n-1} + y_{n-1} \right\|_M^2 - \frac{\lambda_n\gamma_n\alpha_n\bar{\beta}}{2} \left\| y_{n-1} - p_{n-1} \right\|_M^2 \\
 &+ \frac{\lambda_n\gamma_n\bar{\beta}}{2} \left\| y_n \right\|_M^2 - \frac{\bar{\theta}_n(\lambda_n+\mu_n)}{\hat{\theta}_n} \left\| y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1} \right\|_M^2 \\
 &- \frac{\hat{\theta}_n(\lambda_n+\mu_n)}{\theta_n} \left\| -\frac{\bar{\theta}_n\gamma_n\bar{\beta}}{\hat{\theta}_n} y_n - \frac{(2-\gamma_n\bar{\beta})\lambda_n}{\hat{\theta}_n} x_n \right. \\
 &\quad \left. + (\bar{\alpha}_n - \alpha_n)p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n\bar{\theta}_n\gamma_n\bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

Now, we show that all the coefficients of the terms containing z_n are identical to zero. The coefficients of $\|z_n\|_M^2$ and $\langle z_n, z_{n-1} \rangle_M$ are zero by Proposition 6 (iii). For the coefficient of $\langle z_n, x_n \rangle_M$ we have

$$2\bar{\theta}_n\lambda_n - \lambda_n\theta_n + (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)\lambda_n = \lambda_n(2\bar{\theta}_n + (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) - \theta_n)$$

which is identical to zero by Proposition 6 (ii). For the coefficient of $\langle z_n, p_{n-1} \rangle_M$ we have

$$\begin{aligned}
 &\lambda_n^2\bar{\alpha}_n\theta_n - \mu_n\theta_n + \bar{\theta}_n\theta'_n - \hat{\theta}_n(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n) \\
 &= \lambda_n^2\bar{\alpha}_n\theta_n - \mu_n\theta_n + \bar{\theta}_n((2 - \gamma_n\bar{\beta})\mu_n + 2\bar{\alpha}_n\bar{\theta}_n) - \hat{\theta}_n(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n) \\
 &= \bar{\alpha}_n(\lambda_n^2\theta_n + 2\bar{\theta}_n^2 - (\lambda_n + \mu_n)\hat{\theta}_n) - \mu_n\theta_n \\
 &\quad + (2 - \gamma_n\bar{\beta})\mu_n\bar{\theta}_n + (\lambda_n + \mu_n)\alpha_n\hat{\theta}_n \\
 &= (-\theta_n + (2 - \gamma_n\bar{\beta})\bar{\theta}_n + \hat{\theta}_n)\mu_n = 0
 \end{aligned}$$

where in the first equality, θ'_n is substituted and in the third equality Proposition 6 (iii) is used and in the last equality Proposition 6 (i) is used. Therefore, all terms containing z_n can be eliminated from Δ_n and we are left with

$$\begin{aligned}
 \Delta_n = & \lambda_n^2 \left\| -\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1} \right\|_M^2 + 2\langle x_n, \lambda_n(-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 & + 2\langle p_{n-1}, -\omega_n(z_{n-1} - p_{n-1}) \rangle_M \\
 & + \frac{1}{2}\theta_n \left\| \frac{\bar{\theta}_n}{\theta_n} y_n - \frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n\bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n\alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 & + \frac{\mu_n\gamma_n\bar{\beta}}{2} \left\| -y_n - p_{n-1} + y_{n-1} \right\|_M^2 - \frac{\lambda_n\gamma_n\alpha_n\bar{\beta}}{2} \left\| y_{n-1} - p_{n-1} \right\|_M^2 \\
 & + \frac{\lambda_n\gamma_n\bar{\beta}}{2} \left\| y_n \right\|_M^2 - \frac{\bar{\theta}_n(\lambda_n+\mu_n)}{\hat{\theta}_n} \left\| y_n - (1 - \alpha_n)x_n - \alpha_n y_{n-1} \right\|_M^2 \\
 & - \frac{\hat{\theta}_n(\lambda_n+\mu_n)}{\theta_n} \left\| -\frac{\bar{\theta}_n\gamma_n\bar{\beta}}{\hat{\theta}_n} y_n - \frac{(2-\gamma_n\bar{\beta})\lambda_n}{\hat{\theta}_n} x_n \right. \\
 & \quad \left. + (\bar{\alpha}_n - \alpha_n)p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n\bar{\theta}_n\gamma_n\bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

$$\begin{aligned}
 &= \lambda_n^2 \| -\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1} \|_M^2 + 2 \langle x_n, \lambda_n (-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &\quad + 2 \langle p_{n-1}, -\omega_n (z_{n-1} - p_{n-1}) \rangle_M + \frac{\tilde{\theta}_n^2}{2\theta_n} \| y_n \|_M^2 \\
 &\quad + 2 \left\langle y_n, \frac{\tilde{\theta}_n}{2} \left(-\frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\tilde{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\tilde{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right) \right\rangle_M \\
 &\quad + \frac{1}{2} \theta_n \left\| -\frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\tilde{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\tilde{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| y_n \|_M^2 + 2 \left\langle y_n, \frac{\mu_n \gamma_n \bar{\beta}}{2} (p_{n-1} - y_{n-1}) \right\rangle_M + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| p_{n-1} - y_{n-1} \|_M^2 \\
 &\quad + \frac{\lambda_n \gamma_n \bar{\beta}}{2} \| y_n \|_M^2 - \frac{\tilde{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \| y_n \|_M^2 \\
 &\quad - 2 \left\langle y_n, \frac{\tilde{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} ((\alpha_n - 1)x_n - \alpha_n y_{n-1}) \right\rangle_M \\
 &\quad - \frac{\tilde{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \| (\alpha_n - 1)x_n - \alpha_n y_{n-1} \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 &\quad - \frac{\tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n)}{\hat{\theta}_n \theta_n} \| y_n \|_M^2 + 2 \left\langle y_n, \frac{\tilde{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n)}{\theta_n} \left(-\frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n \right) \right\rangle_M \\
 &\quad + 2 \left\langle y_n, \frac{\tilde{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n)}{\theta_n} \left((\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right) \right\rangle_M \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| -\frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2 \\
 &= \left(\frac{\tilde{\theta}_n^2}{2\theta_n} + \frac{\mu_n \gamma_n \bar{\beta}}{2} + \frac{\lambda_n \gamma_n \bar{\beta}}{2} - \frac{\tilde{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} - \frac{\tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n)}{\hat{\theta}_n \theta_n} \right) \| y_n \|_M^2 \\
 &\quad + 2 \left\langle y_n, \left(-\frac{\lambda_n \tilde{\theta}_n}{\theta_n} + \frac{\tilde{\theta}_n (\lambda_n + \mu_n) (1 - \alpha_n)}{\hat{\theta}_n} - \frac{\tilde{\theta}_n \lambda_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) (2 - \gamma_n \bar{\beta})}{\theta_n \hat{\theta}_n} \right) x_n \right\rangle_M \\
 &\quad + 2 \left\langle y_n, \left(-\frac{\tilde{\theta}_n \theta'_n}{2\theta_n} + \frac{\mu_n \gamma_n \bar{\beta}}{2} + \frac{\tilde{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n)}{\theta_n} \right) p_{n-1} \right\rangle_M \\
 &\quad + 2 \left\langle y_n, \left(\frac{\tilde{\theta}_n \tilde{\theta}_n \bar{\alpha}_n}{\theta_n} - \frac{\tilde{\theta}_n \bar{\alpha}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n)}{\theta_n} \right) z_{n-1} \right\rangle_M \\
 &\quad + 2 \left\langle y_n, \left(-\frac{\tilde{\theta}_n^2 \alpha_n}{2\theta_n} - \frac{\mu_n \gamma_n \bar{\beta}}{2} + \frac{\tilde{\theta}_n \alpha_n (\lambda_n + \mu_n)}{\hat{\theta}_n} + \frac{\alpha_n \tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n)}{\theta_n \hat{\theta}_n} \right) y_{n-1} \right\rangle_M \\
 &\quad + \lambda_n^2 \| -\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1} \|_M^2 + 2 \langle x_n, \lambda_n (-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \| p_{n-1} - y_{n-1} \|_M^2 + 2 \langle p_{n-1}, -\omega_n (z_{n-1} - p_{n-1}) \rangle_M \\
 &\quad + \frac{1}{2} \theta_n \left\| -\frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\tilde{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\tilde{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &\quad - \frac{\tilde{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \| (\alpha_n - 1)x_n - \alpha_n y_{n-1} \|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \| y_{n-1} - p_{n-1} \|_M^2 \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| -\frac{(2 - \gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \tilde{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

Now, we show that all the coefficients of the terms containing y_n are identically zero. For the coefficient of $\|y_n\|_M^2$ we have

$$\begin{aligned}
 & \left(\theta_n \hat{\theta}_n \gamma_n \bar{\beta} - 2\theta_n \tilde{\theta}_n - 2\tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 \right) (\lambda_n + \mu_n) + \tilde{\theta}_n^2 \hat{\theta}_n \\
 &= \left(\theta_n \gamma_n \bar{\beta} (2\lambda_n + 2\mu_n - \lambda_n^2 \gamma_n \bar{\beta}) - 2\theta_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) - 2\tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 \right) (\lambda_n + \mu_n)
 \end{aligned}$$

$$\begin{aligned}
 & + \tilde{\theta}_n^2 \hat{\theta}_n \\
 & = -\left(\theta_n \lambda_n^2 + 2\tilde{\theta}_n^2\right) (\lambda_n + \mu_n) \gamma_n^2 \bar{\beta}^2 + (\lambda_n + \mu_n)^2 \gamma_n^2 \bar{\beta}^2 \hat{\theta}_n \\
 & = \left((\lambda_n + \mu_n) \hat{\theta}_n - \theta_n \lambda_n^2 - 2\tilde{\theta}_n^2\right) (\lambda_n + \mu_n) \gamma_n^2 \bar{\beta}^2,
 \end{aligned}$$

which, by Proposition 6 (iii), is identical to zero. Now, for the coefficient of $\langle y_n, x_n \rangle_M$ we have

$$\begin{aligned}
 & \tilde{\theta}_n \theta_n (\lambda_n + \mu_n) (1 - \alpha_n) - \lambda_n \tilde{\theta}_n \hat{\theta}_n - \tilde{\theta}_n \lambda_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) (2 - \gamma_n \bar{\beta}) \\
 & = \left(\theta_n (\lambda_n + \mu_n) (1 - \alpha_n) - \lambda_n \hat{\theta}_n - \tilde{\theta}_n \lambda_n (2 - \gamma_n \bar{\beta})\right) \tilde{\theta}_n \\
 & = \left(\theta_n - \hat{\theta}_n - \tilde{\theta}_n (2 - \gamma_n \bar{\beta})\right) \lambda_n \tilde{\theta}_n
 \end{aligned}$$

which is identically zero by Proposition 6 (i). For the coefficient of $\langle y_n, p_{n-1} \rangle_M$ we have

$$\begin{aligned}
 & \mu_n \gamma_n \bar{\beta} \theta_n + 2\tilde{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n) - \tilde{\theta}_n \theta'_n \\
 & = \mu_n \gamma_n \bar{\beta} \theta_n + 2\tilde{\theta}_n \tilde{\theta}_n (\bar{\alpha}_n - \alpha_n) - \tilde{\theta}_n ((2 - \gamma_n \bar{\beta}) \mu_n + 2\bar{\alpha}_n \tilde{\theta}_n) \\
 & = \mu_n \gamma_n \bar{\beta} \theta_n + 2\tilde{\theta}_n \tilde{\theta}_n \bar{\alpha}_n - 2\tilde{\theta}_n \tilde{\theta}_n \alpha_n - (2 - \gamma_n \bar{\beta}) \mu_n \tilde{\theta}_n - 2\bar{\alpha}_n \tilde{\theta}_n \tilde{\theta}_n \\
 & = \mu_n \gamma_n \bar{\beta} \theta_n - 2\tilde{\theta}_n \tilde{\theta}_n \alpha_n - (2 - \gamma_n \bar{\beta}) \mu_n \tilde{\theta}_n \\
 & = \mu_n \gamma_n \bar{\beta} \theta_n - 2\tilde{\theta}_n \mu_n \gamma_n \bar{\beta} - (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n) \mu_n \gamma_n \bar{\beta} \\
 & = \mu_n \gamma_n \bar{\beta} (\theta_n - 2\tilde{\theta}_n - (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n)) \\
 & = \mu_n \gamma_n \bar{\beta} \left(\theta_n - 2(\lambda_n + \mu_n - \lambda_n^2) - (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n)\right) \\
 & = \mu_n \gamma_n \bar{\beta} \left(\theta_n + 2\lambda_n^2 - (4 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n)\right)
 \end{aligned}$$

which is equal to zero by the definition of θ_n . The equivalence of the coefficient of $\langle y_n, z_{n-1} \rangle_M$ to zero follows from the definition of $\tilde{\theta}_n$. For the coefficient of $\langle y_n, y_{n-1} \rangle_M$ we have

$$\begin{aligned}
 & 2\alpha_n \tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n) - \tilde{\theta}_n^2 \hat{\theta}_n \alpha_n + 2\theta_n \tilde{\theta}_n \alpha_n (\lambda_n + \mu_n) - \mu_n \gamma_n \bar{\beta} \theta_n \hat{\theta}_n \\
 & = 2\alpha_n \tilde{\theta}_n^2 \gamma_n \bar{\beta} \tilde{\theta}_n - \tilde{\theta}_n^2 \hat{\theta}_n \alpha_n + 2\theta_n \tilde{\theta}_n \alpha_n (\lambda_n + \mu_n) - \alpha_n \tilde{\theta}_n \theta_n \hat{\theta}_n \\
 & = \alpha_n \tilde{\theta}_n \left(2\tilde{\theta}_n^2 \gamma_n \bar{\beta} - \tilde{\theta}_n \hat{\theta}_n + 2\theta_n (\lambda_n + \mu_n) - \theta_n \hat{\theta}_n\right) \\
 & = \alpha_n \tilde{\theta}_n \left(2\tilde{\theta}_n^2 \gamma_n \bar{\beta} - \tilde{\theta}_n \hat{\theta}_n + \theta_n \left(2(\lambda_n + \mu_n) - 2(\lambda_n + \mu_n) + \lambda_n^2 \gamma_n \bar{\beta}\right)\right) \\
 & = \alpha_n \tilde{\theta}_n \left(2\tilde{\theta}_n^2 \gamma_n \bar{\beta} - \tilde{\theta}_n \hat{\theta}_n + \theta_n \lambda_n^2 \gamma_n \bar{\beta}\right) \\
 & = \alpha_n \tilde{\theta}_n \left(2\tilde{\theta}_n^2 \gamma_n \bar{\beta} - \hat{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) + \theta_n \lambda_n^2 \gamma_n \bar{\beta}\right) \\
 & = \alpha_n \tilde{\theta}_n \gamma_n \bar{\beta} \left(2\tilde{\theta}_n^2 - \hat{\theta}_n (\lambda_n + \mu_n) + \theta_n \lambda_n^2\right)
 \end{aligned}$$

which by Proposition 6 (iii) is identical to zero. Therefore, all the coefficients of the terms containing y_n are zero and we can eliminate those terms. The remaining terms are

$$\begin{aligned}
 \Delta_n &= \\
 &\lambda_n^2 \left\| -\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1} \right\|_M^2 + 2 \langle x_n, \lambda_n (-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_{n-1} - y_{n-1}\|_M^2 + 2 \langle p_{n-1}, -\omega_n (z_{n-1} - p_{n-1}) \rangle_M \\
 &\quad + \frac{1}{2} \theta_n \left\| -\frac{2\lambda_n}{\theta_n} x_n - \frac{\theta'_n}{\theta_n} p_{n-1} + \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} - \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 \\
 &\quad - \frac{\bar{\theta}_n (\lambda_n + \mu_n)}{\hat{\theta}_n} \|(\alpha_n - 1)x_n - \alpha_n y_{n-1}\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| -\frac{(2-\gamma_n \bar{\beta}) \lambda_n}{\hat{\theta}_n} x_n + (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2 \\
 &= \lambda_n^2 \left\| -\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1} \right\|_M^2 + 2 \langle x_n, \lambda_n (-\bar{\alpha}_n p_{n-1} + \bar{\alpha}_n z_{n-1}) \rangle_M \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_{n-1} - y_{n-1}\|_M^2 + 2 \langle p_{n-1}, -\omega_n (z_{n-1} - p_{n-1}) \rangle_M \\
 &\quad + \frac{2\lambda_n^2}{\theta_n} \|x_n\|_M^2 + 2 \left\langle x_n, \lambda_n \left(\frac{\theta'_n}{\theta_n} p_{n-1} - \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} + \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right) \right\rangle_M \\
 &\quad + \frac{1}{2} \theta_n \left\| \frac{\theta'_n}{\theta_n} p_{n-1} - \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} + \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 &\quad - \frac{\bar{\theta}_n (\lambda_n + \mu_n) (1 - \alpha_n)^2}{\hat{\theta}_n} \|x_n\|_M^2 + 2 \left\langle x_n, \frac{\bar{\theta}_n \alpha_n (\lambda_n + \mu_n) (\alpha_n - 1)}{\hat{\theta}_n} y_{n-1} \right\rangle_M \\
 &\quad - \frac{(\lambda_n + \mu_n) (2 - \gamma_n \bar{\beta})^2 \lambda_n^2}{\theta_n \hat{\theta}_n} \|x_n\|_M^2 - \frac{\bar{\theta}_n \alpha_n^2 (\lambda_n + \mu_n)}{\hat{\theta}_n} \|y_{n-1}\|_M^2 \\
 &\quad + 2 \left\langle x_n, \frac{\lambda_n (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n)}{\theta_n} \left((\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right) \right\rangle_M \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2 \\
 &= \left(\frac{2\lambda_n^2}{\theta_n} - \frac{\bar{\theta}_n (\lambda_n + \mu_n) (1 - \alpha_n)^2}{\hat{\theta}_n} - \frac{(\lambda_n + \mu_n) (2 - \gamma_n \bar{\beta})^2 \lambda_n^2}{\theta_n \hat{\theta}_n} \right) \|x_n\|_M^2 \\
 &\quad + 2 \left\langle x_n, \left(\frac{\lambda_n \theta'_n}{\theta_n} + \frac{\lambda_n (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n)}{\theta_n} - \lambda_n \bar{\alpha}_n \right) p_{n-1} \right\rangle_M \\
 &\quad + 2 \left\langle x_n, \left(-\frac{2\lambda_n \bar{\theta}_n \bar{\alpha}_n}{\theta_n} - \frac{\lambda_n \bar{\alpha}_n (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n)}{\theta_n} + \lambda_n \bar{\alpha}_n \right) z_{n-1} \right\rangle_M \\
 &\quad + 2 \left\langle x_n, \left(\frac{\lambda_n \bar{\theta}_n \alpha_n}{\theta_n} + \frac{\bar{\theta}_n \alpha_n (\lambda_n + \mu_n) (\alpha_n - 1)}{\hat{\theta}_n} + \frac{\bar{\theta}_n \alpha_n \lambda_n \gamma_n \bar{\beta} (2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n)}{\theta_n \hat{\theta}_n} \right) y_{n-1} \right\rangle_M \\
 &\quad + \lambda_n^2 \left\| -\bar{\alpha}_n z_{n-1} + \bar{\alpha}_n p_{n-1} \right\|_M^2 - \frac{\bar{\theta}_n \alpha_n^2 (\lambda_n + \mu_n)}{\hat{\theta}_n} \|y_{n-1}\|_M^2 \\
 &\quad + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_{n-1} - y_{n-1}\|_M^2 + 2 \langle p_{n-1}, -\omega_n (z_{n-1} - p_{n-1}) \rangle_M \\
 &\quad + \frac{1}{2} \theta_n \left\| \frac{\theta'_n}{\theta_n} p_{n-1} - \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} + \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 &\quad - \frac{\hat{\theta}_n (\lambda_n + \mu_n)}{\theta_n} \left\| (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2
 \end{aligned}$$

We want to show that all the coefficients of the terms containing x_n are zero. For the coefficient of $\|x_n\|_M^2$ we have

$$\begin{aligned}
 & 2\lambda_n^2\hat{\theta}_n - (\lambda_n + \mu_n)(2 - \gamma_n\bar{\beta})^2\lambda_n^2 - \theta_n\tilde{\theta}_n(\lambda_n + \mu_n)(1 - \alpha_n)^2 \\
 &= 2\lambda_n^2\hat{\theta}_n - (\lambda_n + \mu_n)(2 - \gamma_n\bar{\beta})^2\lambda_n^2 - \theta_n\lambda_n^2\gamma_n\bar{\beta} \\
 &= \lambda_n^2\left(2\hat{\theta}_n - (\lambda_n + \mu_n)(2 - \gamma_n\bar{\beta})^2 - \theta_n\gamma_n\bar{\beta}\right) \\
 &= \lambda_n^2\left(2\left(2\lambda_n + 2\mu_n - \gamma_n\bar{\beta}\lambda_n^2\right) - (\lambda_n + \mu_n)\left(4 - 4\gamma_n\bar{\beta} + \gamma_n^2\bar{\beta}^2\right) - \theta_n\gamma_n\bar{\beta}\right) \\
 &= \lambda_n^2\left(-2\gamma_n\bar{\beta}\lambda_n^2 - (\lambda_n + \mu_n)\left(-4\gamma_n\bar{\beta} + \gamma_n^2\bar{\beta}^2\right) \right. \\
 &\quad \left. - \left((4 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) - 2\lambda_n^2\right)\gamma_n\bar{\beta}\right) \\
 &= \lambda_n^2\left(-2\gamma_n\bar{\beta}\lambda_n^2 + 2\lambda_n^2\gamma_n\bar{\beta}\right) = 0
 \end{aligned} \tag{44}$$

where in the first equality $\tilde{\theta}_n$ and α_n and in the third equality $\hat{\theta}_n$ are substituted by their definition from Algorithm 1. For the coefficient of $\langle x_n, p_{n-1} \rangle$ we have

$$\begin{aligned}
 & \lambda_n\theta'_n + \lambda_n(2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n) - \lambda_n\bar{\alpha}_n\theta_n \\
 &= \lambda_n\left((2 - \gamma_n\bar{\beta})\mu_n + 2\bar{\alpha}_n\tilde{\theta}_n\right) + \lambda_n(2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n) - \lambda_n\bar{\alpha}_n\theta_n \\
 &= 2\lambda_n\bar{\alpha}_n\tilde{\theta}_n + (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)\lambda_n\bar{\alpha}_n - \lambda_n\bar{\alpha}_n\theta_n \\
 &\quad - (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n)\lambda_n\alpha_n + (2 - \gamma_n\bar{\beta})\lambda_n\mu_n \\
 &= \lambda_n\bar{\alpha}_n(2\tilde{\theta}_n + (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) - \theta_n) \\
 &\quad - (2 - \gamma_n\bar{\beta})\lambda_n\mu_n + (2 - \gamma_n\bar{\beta})\lambda_n\mu_n
 \end{aligned}$$

which by Proposition 6 (ii) is zero. For the coefficient of $\langle x_n, z_{n-1} \rangle$ we have

$$\begin{aligned}
 & \lambda_n\bar{\alpha}_n\theta_n - 2\lambda_n\tilde{\theta}_n\bar{\alpha}_n - \lambda_n\bar{\alpha}_n(2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) \\
 &= \lambda_n\bar{\alpha}_n(\theta_n - 2\tilde{\theta}_n - (2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n))
 \end{aligned}$$

which is identically zero by Proposition 6 (ii). For the coefficient of $\langle x_n, y_{n-1} \rangle$ we have

$$\begin{aligned}
 & \lambda_n\tilde{\theta}_n\alpha_n\hat{\theta}_n + \tilde{\theta}_n\alpha_n\lambda_n\gamma_n\bar{\beta}(2 - \gamma_n\bar{\beta})(\lambda_n + \mu_n) - \theta_n\tilde{\theta}_n\alpha_n(\lambda_n + \mu_n)(1 - \alpha_n) \\
 &= \lambda_n\tilde{\theta}_n\alpha_n\hat{\theta}_n + \tilde{\theta}_n\alpha_n\lambda_n\tilde{\theta}_n(2 - \gamma_n\bar{\beta}) - \theta_n\tilde{\theta}_n\alpha_n\lambda_n \\
 &= \lambda_n\tilde{\theta}_n\alpha_n\left(\hat{\theta}_n + \tilde{\theta}_n(2 - \gamma_n\bar{\beta}) - \theta_n\right)
 \end{aligned}$$

which by Proposition 6 (i) is identically zero. Now, expanding all the remaining terms, reordering and recollecting them give

$$\begin{aligned}
 \Delta_n &= \\
 & \lambda_n^2\|-\bar{\alpha}_nz_{n-1} + \bar{\alpha}_np_{n-1}\|_M^2 - \frac{\tilde{\theta}_n\alpha_n^2(\lambda_n + \mu_n)}{\hat{\theta}_n}\|y_{n-1}\|_M^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_{n-1} - y_{n-1}\|_M^2 + 2\langle p_{n-1}, -\omega_n(z_{n-1} - p_{n-1}) \rangle_M \\
 & + \frac{1}{2} \theta_n \left\| \frac{\theta'_n}{\theta_n} p_{n-1} - \frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} + \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right\|_M^2 - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1} - p_{n-1}\|_M^2 \\
 & - \frac{\hat{\theta}_n(\lambda_n + \mu_n)}{\theta_n} \left\| (\bar{\alpha}_n - \alpha_n) p_{n-1} - \bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right\|_M^2 \\
 = & \lambda_n^2 \bar{\alpha}_n^2 \|p_{n-1}\|_M^2 + 2\langle p_{n-1}, -\lambda_n^2 \bar{\alpha}_n^2 z_{n-1} \rangle_M + \lambda_n^2 \bar{\alpha}_n^2 \|z_{n-1}\|_M^2 \\
 & + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|p_{n-1}\|_M^2 + 2\langle p_{n-1}, -\frac{\mu_n \gamma_n \bar{\beta}}{2} y_{n-1} \rangle_M + \frac{\mu_n \gamma_n \bar{\beta}}{2} \|y_{n-1}\|_M^2 \\
 & - \frac{\bar{\theta}_n \alpha_n^2 (\lambda_n + \mu_n)}{\hat{\theta}_n} \|y_{n-1}\|_M^2 + 2\omega_n \|p_{n-1}\|_M^2 + 2\langle p_{n-1}, -\omega_n z_{n-1} \rangle_M \\
 & + \frac{\theta_n'^2}{2\theta_n} \|p_{n-1}\|_M^2 + 2\langle p_{n-1}, \frac{1}{2} \theta_n' \left(-\frac{2\bar{\theta}_n \bar{\alpha}_n}{\theta_n} z_{n-1} + \frac{\bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \right) \rangle_M \\
 & + \frac{2\bar{\theta}_n^2 \bar{\alpha}_n^2}{\theta_n} \|z_{n-1}\|_M^2 + 2\langle z_{n-1}, -\frac{\bar{\theta}_n \bar{\alpha}_n \bar{\theta}_n \alpha_n}{\theta_n} y_{n-1} \rangle_M + \frac{\bar{\theta}_n^2 \alpha_n^2}{2\theta_n} \|y_{n-1}\|_M^2 \\
 & - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|p_{n-1}\|_M^2 + 2\langle p_{n-1}, \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} y_{n-1} \rangle_M - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} \|y_{n-1}\|_M^2 \\
 & - \frac{\hat{\theta}_n(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n)^2}{\theta_n} \|p_{n-1}\|_M^2 - \frac{\alpha_n^2 \bar{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n)}{\hat{\theta}_n \theta_n} \|y_{n-1}\|_M^2 \\
 & - \frac{\hat{\theta}_n \bar{\alpha}_n^2 (\lambda_n + \mu_n)}{\theta_n} \|z_{n-1}\|_M^2 + 2\langle z_{n-1}, \frac{\bar{\alpha}_n \alpha_n \bar{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n)}{\theta_n} y_{n-1} \rangle_M \\
 & + 2\langle p_{n-1}, -\frac{\hat{\theta}_n(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n)}{\theta_n} \left(-\bar{\alpha}_n z_{n-1} + \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta}}{\hat{\theta}_n} y_{n-1} \right) \rangle_M \\
 = & \left(\lambda_n^2 \bar{\alpha}_n^2 + \frac{\mu_n \gamma_n \bar{\beta}}{2} + 2\omega_n + \frac{\theta_n'^2}{2\theta_n} - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} - \frac{\hat{\theta}_n(\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n)^2}{\theta_n} \right) \|p_{n-1}\|_M^2 \\
 & + 2\langle p_{n-1}, \left(-\lambda_n^2 \bar{\alpha}_n^2 - \omega_n - \frac{\bar{\theta}_n \theta_n' \bar{\alpha}_n}{\theta_n} + \frac{\hat{\theta}_n \bar{\alpha}_n (\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n)}{\theta_n} \right) z_{n-1} \rangle_M \\
 & + 2\langle p_{n-1}, \left(-\frac{\mu_n \gamma_n \bar{\beta}}{2} + \frac{\theta_n' \bar{\theta}_n \alpha_n}{2\theta_n} + \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} - \frac{\alpha_n \bar{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n)(\bar{\alpha}_n - \alpha_n)}{\theta_n} \right) y_{n-1} \rangle_M \\
 & + \left(\lambda_n^2 \bar{\alpha}_n^2 + \frac{2\bar{\theta}_n^2 \bar{\alpha}_n^2}{\theta_n} - \frac{\hat{\theta}_n \bar{\alpha}_n^2 (\lambda_n + \mu_n)}{\theta_n} \right) \|z_{n-1}\|_M^2 \\
 & + 2\langle z_{n-1}, \left(-\frac{\bar{\theta}_n \bar{\alpha}_n \bar{\theta}_n \alpha_n}{\theta_n} + \frac{\bar{\alpha}_n \alpha_n \bar{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n)}{\theta_n} \right) y_{n-1} \rangle_M \\
 & + \left(\frac{\mu_n \gamma_n \bar{\beta}}{2} - \frac{\bar{\theta}_n \alpha_n^2 (\lambda_n + \mu_n)}{\hat{\theta}_n} + \frac{\bar{\theta}_n^2 \alpha_n^2}{2\theta_n} - \frac{\lambda_n \gamma_n \alpha_n \bar{\beta}}{2} - \frac{\alpha_n^2 \bar{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n)}{\hat{\theta}_n \theta_n} \right) \|y_{n-1}\|_M^2
 \end{aligned}$$

We show that all the coefficients in the expression above are identically zero. Starting by the coefficient of $\|p_{n-1}\|_M^2$, we have

$$\begin{aligned}
 & 2\theta_n \lambda_n^2 \bar{\alpha}_n^2 + \theta_n \mu_n \gamma_n \bar{\beta} + 4\theta_n \omega_n + \theta_n'^2 - \theta_n \lambda_n \gamma_n \alpha_n \bar{\beta} - 2\hat{\theta}_n (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n)^2 \\
 & = 2\theta_n \lambda_n^2 \bar{\alpha}_n^2 + \theta_n \mu_n \gamma_n \bar{\beta} - 4\theta_n \bar{\alpha}_n \mu_n + ((2 - \gamma_n \bar{\beta}) \mu_n + 2\bar{\alpha}_n \bar{\theta}_n)^2 \\
 & \quad - \theta_n \lambda_n \gamma_n \alpha_n \bar{\beta} - 2\hat{\theta}_n (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n)^2 \\
 & = 2\bar{\alpha}_n^2 (\theta_n \lambda_n^2 + 2\bar{\theta}_n^2 - \hat{\theta}_n (\lambda_n + \mu_n)) + 4\mu_n \bar{\alpha}_n (-\theta_n + \bar{\theta}_n (2 - \gamma_n \bar{\beta}) + \hat{\theta}_n) \\
 & \quad + \theta_n \mu_n \gamma_n \bar{\beta} + (2 - \gamma_n \bar{\beta})^2 \mu_n^2 - \theta_n \lambda_n \gamma_n \alpha_n \bar{\beta} - 2\hat{\theta}_n (\lambda_n + \mu_n) \alpha_n^2
 \end{aligned}$$

$$\begin{aligned}
 &= \theta_n \mu_n \gamma_n \bar{\beta} + (2 - \gamma_n \bar{\beta})^2 \mu_n^2 - \theta_n \lambda_n \gamma_n \alpha_n \bar{\beta} - 2 \hat{\theta}_n \mu_n \alpha_n \\
 &= \theta_n \gamma_n \bar{\beta} (\mu_n - \lambda_n \alpha_n) + (2 - \gamma_n \bar{\beta})^2 \mu_n^2 - 2 \hat{\theta}_n \mu_n \alpha_n \\
 &= \theta_n \gamma_n \bar{\beta} \mu_n \alpha_n + (2 - \gamma_n \bar{\beta})^2 (\lambda_n + \mu_n) \alpha_n \mu_n - 2 \hat{\theta}_n \mu_n \alpha_n \\
 &= \mu_n \alpha_n \left(\theta_n \gamma_n \bar{\beta} + (2 - \gamma_n \bar{\beta})^2 (\lambda_n + \mu_n) - 2 \hat{\theta}_n \right)
 \end{aligned}$$

where in the first equality $\omega_n = -\bar{\alpha}_n \mu_n$ is used and θ'_n is substituted from 38, the third equality is attained from Proposition 6 (i) and (iii), and the expression to the right-hand side of the last equality is identically zero by (44). For the coefficient of $\langle p_{n-1}, z_{n-1} \rangle_M$ we have

$$\begin{aligned}
 &-\lambda_n^2 \bar{\alpha}_n^2 \theta_n - \omega_n \theta_n - \bar{\theta}_n \theta'_n \bar{\alpha}_n + \hat{\theta}_n \bar{\alpha}_n (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n) \\
 &= -\lambda_n^2 \bar{\alpha}_n^2 \theta_n - \omega_n \theta_n - \bar{\theta}_n \bar{\alpha}_n ((2 - \gamma_n \bar{\beta}) \mu_n + 2 \bar{\alpha}_n \bar{\theta}_n) \\
 &\quad + \hat{\theta}_n \bar{\alpha}_n (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n) \\
 &= \bar{\alpha}_n^2 \left(-\lambda_n^2 \theta_n - 2 \bar{\theta}_n^2 + \hat{\theta}_n (\lambda_n + \mu_n) \right) - \omega_n \theta_n - \bar{\theta}_n \bar{\alpha}_n \mu_n (2 - \gamma_n \bar{\beta}) \\
 &\quad - \hat{\theta}_n \bar{\alpha}_n (\lambda_n + \mu_n) \alpha_n \\
 &= \bar{\alpha}_n \mu_n \theta_n - \bar{\theta}_n \bar{\alpha}_n \mu_n (2 - \gamma_n \bar{\beta}) - \hat{\theta}_n \bar{\alpha}_n \mu_n \\
 &= \bar{\alpha}_n \mu_n \left(\theta_n - \bar{\theta}_n (2 - \gamma_n \bar{\beta}) - \hat{\theta}_n \right)
 \end{aligned}$$

which by Proposition 6 (i) is equal to zero. The third equality above is attained by using Proposition 6 (iii). For the coefficient of $\langle p_{n-1}, y_{n-1} \rangle_M$ we have

$$\begin{aligned}
 &\theta'_n \tilde{\theta}_n \alpha_n - \mu_n \gamma_n \bar{\beta} \theta_n + \lambda_n \gamma_n \alpha_n \bar{\beta} \theta_n - 2 \alpha_n \bar{\theta}_n \gamma_n \bar{\beta} (\lambda_n + \mu_n) (\bar{\alpha}_n - \alpha_n) \\
 &= ((2 - \gamma_n \bar{\beta}) \mu_n + 2 \bar{\alpha}_n \bar{\theta}_n) \tilde{\theta}_n \alpha_n - \mu_n \gamma_n \bar{\beta} \theta_n + \lambda_n \gamma_n \alpha_n \bar{\beta} \theta_n \\
 &\quad - 2 \alpha_n \bar{\theta}_n \tilde{\theta}_n (\bar{\alpha}_n - \alpha_n) \\
 &= (2 - \gamma_n \bar{\beta}) \mu_n \tilde{\theta}_n \alpha_n - \mu_n \gamma_n \bar{\beta} \theta_n + \lambda_n \gamma_n \bar{\beta} \theta_n \alpha_n + 2 \alpha_n \bar{\theta}_n \tilde{\theta}_n \alpha_n \\
 &= (2 - \gamma_n \bar{\beta}) \mu_n (\lambda_n + \mu_n) \gamma_n \bar{\beta} \alpha_n - (\lambda_n + \mu_n) \alpha_n \gamma_n \bar{\beta} \theta_n \\
 &\quad + \lambda_n \gamma_n \bar{\beta} \theta_n \alpha_n + 2 \alpha_n \bar{\theta}_n \mu_n \gamma_n \bar{\beta} \\
 &= (2 - \gamma_n \bar{\beta}) \mu_n (\lambda_n + \mu_n) \gamma_n \bar{\beta} \alpha_n - \mu_n \alpha_n \gamma_n \bar{\beta} \theta_n + 2 \alpha_n \bar{\theta}_n \mu_n \gamma_n \bar{\beta} \\
 &= \mu_n \gamma_n \bar{\beta} \alpha_n ((2 - \gamma_n \bar{\beta}) (\lambda_n + \mu_n) - \theta_n + 2 \bar{\theta}_n)
 \end{aligned}$$

which by Proposition 6 (ii) is identical to zero. For the coefficient of $\|z_{n-1}\|_M^2$, it is straightforward to see its equivalence to zero by Proposition 6 (iii). Likewise, the coefficient of $\langle z_{n-1}, y_{n-1} \rangle_M$ is identically zero by definition of $\tilde{\theta}_n$. The coefficient of $\|y_{n-1}\|_M^2$ is

$$\begin{aligned}
 &\mu_n \gamma_n \bar{\beta} \hat{\theta}_n \theta_n - 2 \theta_n \tilde{\theta}_n \alpha_n^2 (\lambda_n + \mu_n) + \hat{\theta}_n \tilde{\theta}_n^2 \alpha_n^2 - \lambda_n \gamma_n \alpha_n \bar{\beta} \hat{\theta}_n \theta_n \\
 &\quad - 2 \alpha_n^2 \tilde{\theta}_n^2 \gamma_n^2 \bar{\beta}^2 (\lambda_n + \mu_n)
 \end{aligned}$$

$$\begin{aligned}
 &= (\lambda_n + \mu_n)\alpha_n\gamma_n\bar{\beta}\hat{\theta}_n\theta_n - 2\theta_n\tilde{\theta}_n\alpha_n^2(\lambda_n + \mu_n) + \hat{\theta}_n\tilde{\theta}_n^2\alpha_n^2 \\
 &\quad - \lambda_n\gamma_n\alpha_n\bar{\beta}\hat{\theta}_n\theta_n - 2\alpha_n^2\tilde{\theta}_n^2\gamma_n^2\bar{\beta}^2(\lambda_n + \mu_n) \\
 &= \mu_n\alpha_n\gamma_n\bar{\beta}\hat{\theta}_n\theta_n - 2\theta_n\tilde{\theta}_n\alpha_n^2(\lambda_n + \mu_n) + \hat{\theta}_n\tilde{\theta}_n^2\alpha_n^2 - 2\alpha_n^2\tilde{\theta}_n^2\gamma_n^2\bar{\beta}^2(\lambda_n + \mu_n) \\
 &= \mu_n\alpha_n\gamma_n\bar{\beta}\hat{\theta}_n\theta_n - 2\theta_n\gamma_n\bar{\beta}\mu_n\alpha_n(\lambda_n + \mu_n) + \hat{\theta}_n\tilde{\theta}_n\mu_n\gamma_n\bar{\beta}\alpha_n - 2\alpha_n\mu_n\tilde{\theta}_n^2\gamma_n^2\bar{\beta}^2 \\
 &= \mu_n\alpha_n\gamma_n\bar{\beta}\left(\hat{\theta}_n\theta_n - 2\theta_n(\lambda_n + \mu_n) + \hat{\theta}_n\tilde{\theta}_n - 2\tilde{\theta}_n^2\gamma_n\bar{\beta}\right) \\
 &= \mu_n\alpha_n\gamma_n\bar{\beta}\left(\theta_n\left(\hat{\theta}_n - 2\lambda_n - 2\mu_n\right) + \hat{\theta}_n\gamma_n\bar{\beta}(\lambda_n + \mu_n) - 2\tilde{\theta}_n^2\gamma_n\bar{\beta}\right) \\
 &= \mu_n\alpha_n\gamma_n\bar{\beta}\left(-\theta_n\lambda_n^2\gamma_n\bar{\beta} + \hat{\theta}_n\gamma_n\bar{\beta}(\lambda_n + \mu_n) - 2\tilde{\theta}_n^2\gamma_n\bar{\beta}\right) \\
 &= \mu_n\alpha_n\gamma_n^2\bar{\beta}^2\left(-\theta_n\lambda_n^2 + \hat{\theta}_n(\lambda_n + \mu_n) - 2\tilde{\theta}_n^2\right)
 \end{aligned}$$

which by Proposition 6 (iii) is equivalent to zero. This concludes the proof. □

8 Conclusions

We have presented a variant of the well-known forward–backward algorithm. Our method incorporates momentum-like terms in the algorithm updates as well as deviation vectors. These deviation vectors can be chosen arbitrarily as long as a safeguarding condition that limits their size is satisfied. We propose special instances of our method that fulfill the safeguarding condition by design. Numerical evaluations reveal that these novel methods can significantly outperform the traditional forward–backward method as well as the accelerated proximal point method and the Halpern iteration, all of which are encompassed within our framework. This demonstrates the potential of our proposed methods for efficiently solving structured monotone inclusions.

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