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Hermite interpolation theorems for band-limited functions of the linear canonical transforms with equidistant samples

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Abstract
We establish convergence analysis for Hermite-type interpolations for $L^2(\mathbb{R})$-entire functions of exponential type whose linear canonical transforms (LCT) are compactly supported. The results bridges the theoretical gap in implementing the derivative sampling theorems for band-limited signals in the LCT domain. Both complex analysis and real analysis techniques are established to derive the convergence analysis. The truncation error is also investigated and rigorous estimates for it are given. Nevertheless, the convergence rate is $O(1/\sqrt{N})$, which is slow. Consequently the work on regularization techniques is required.

Keywords Linear canonical transform · Sampling theory · Truncation error

1 Introduction

The linear canonical transform (LCT), extends the fractional Fourier transform (FrFT), and it plays a major role in signal and image processing, approximation theory, optics, as well as many other disciplines, see [8, 12, 15, 16, 20, 22, 28, 32]. For $f \in L^2(\mathbb{R})$

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the definition of the LCT, cf. [9, 18, 21, 25], is a four-parameter transform

$$L_f^M(x) = L_f^M[f(z)](x) = \begin{cases} \dfrac{1}{\sqrt{i2\pi b}} \displaystyle\int_{-\infty}^{\infty} f(z) e^{i\left(\frac{dz^2+ax^2-2zx}{2b}\right)} dz, & b \neq 0, \\ \sqrt{d} e^{i\frac{1}{2}cdz^2} f(dx), & b = 0, \end{cases} \tag{1.1}$$

where $a,\ b,\ c,\ d$ are real numbers satisfying $ad-cb=1$. We exclude the case $b=0$ in our study as it is trivial. When $(a,b,c,d) = (\cos(\alpha), \sin(\alpha), -\sin(\alpha), \cos(\alpha))$, the LCT becomes the FRFT. If $(a,b,c,d) = (0,1,-1,0,)$ the LCT reduces to the Fourier transform (FT). Fresnel transform is the special case when $(a,b,c,d) = (1,b,0,0)$. For more details see [26, 33, 35, 36]. The sampling theorem expansions for LCT and FRFT have been derived in many works, cf., e.g., [5, 18, 19, 27, 30–32, 34]. Let $\sigma > 0$ be fixed and let B_σ^2 be the class of bandlimited signals in the LCT domain with band-width σ, i.e., [18],

$$B_\sigma^2 := \left\{ f(z) \in L^2(\mathbb{R}) : \ \ L_f^M(x) = 0,\ |x| > \sigma \right\}.$$

If PW_σ^2 denotes the classical Paley-Wiener space of band-limited functions with band-width σ, cf. [11], then $f \in B_\sigma^2$ if and only if there is $f_o \in PW_{\sigma/b}^2$, $b > 0$, such that $f(z) = e^{-i(a/2b)z^2} f_o(z)$. The interpolation sampling theorem associated with the LCT states that, see, e.g., [18, 23, 24, 30, 32, 34], if $f \in B_\sigma^2$ then

$$f(z) = \sum_{n=-\infty}^{\infty} e^{-i\frac{a}{2b}\left(z^2 - z_n^2\right)} f(z_n) S_n(z),\ \ z \in \mathbb{R}, \tag{1.2}$$

where $z_n := \dfrac{n\pi b}{\sigma}$, $n \in \mathbb{Z}$, are equidistant samples and

$$S_n(z) := \operatorname{sinc}\left(\frac{\sigma}{b\pi}(z - z_n)\right) = \begin{cases} \dfrac{\sin\left(\frac{\sigma}{b}(z - z_n)\right)}{\frac{\sigma}{b}(z - z_n)}, & z \neq z_n, \\ 1, & z = z_n. \end{cases} \tag{1.3}$$

Series (1.2) converges absolutely on $\mathbb{C}$ and uniformly on compact subsets of $\mathbb{C}$ and on $\mathbb{R}$ as well. The error analysis for truncation, amplitude and jitter errors associated with the sampling representation (1.2) are considered in [4, 29, 32]. If we take $(a,b,d) = (0,1,0)$ in (1.2), series (1.2) turns to be the classical sampling theorem of Whittaker, Kotelnikov and Shannon (WKS),

$$f(z) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma z}{\pi} - n\right),\ \ z \in \mathbb{R}, \tag{1.4}$$

cf. [6, 11]. If $f \in PW^2_{2\sigma}$, then f can reconstructed via the following Hermite interpolation sampling theorem

$$f(z) = \sum_{n=-\infty}^{\infty} \left\{ f\left(\frac{n\pi}{\sigma}\right) + \left(z - \frac{n\pi}{\sigma}\right) f'\left(\frac{n\pi}{\sigma}\right) \right\} \operatorname{sinc}^2\left(\frac{\sigma z}{\pi} - n\right), \quad z \in \mathbb{R}. \tag{1.5}$$

Expansion (1.5) is also called the derivative sampling theorem, see [11]. Series (1.5) converges locally uniformly on $\mathbb{C}$. The Hermite sampling representation was first established by Jagerman and Fogel (1956) in [14]. Li and Fang (2006) discussed the aliasing error associated with (1.5) in [17]. In [3], truncation, amplitude and jitter errors associated with (1.5) are investigated.

The Hermite sampling theorem or derivative sampling theorem associated with the LCT is given in [18], see also [19], but nothing is said about the types of convergence of this theorem. Therefore the main outcomes of this paper are itemized as follows.

- The convergence analysis for the Hermite sampling theorem associated with the LCT is established in Section 2.
- We employ complex analysis techniques, which allow us to derive new important cases in different spaces of entire functions with prescribed growth.
- The truncation error is estimated in both local (pointwise) and global (uniform) manners in Section 3. Moreover, in Section 4 we estimate the error when the truncation error is accompanied with other types of errors.

The experimental results are presented to show the accuracy and usefulness of derived results in Section 5.

2 LCT-Hermite interpolations

Let $f \in B^2_{2\sigma}$. Define the Hermite interpolation series associated with the LCT, to be:

$$\begin{aligned} f(z) \sim \mathcal{H}[f](z) := \sum_{n=-\infty}^{\infty} e^{-i\frac{a}{2b}\left(z^2 - z_n^2\right)} \Bigg\{ &\left(1 + \frac{ia}{b} z_n (z - z_n)\right) f(z_n) \\ &+ (z - z_n) f'(z_n) \Bigg\} S_n^2(z), \end{aligned} \tag{2.1}$$

$z \in \mathbb{C}$, $z_n = \dfrac{n\pi b}{\sigma}$, $n \in \mathbb{Z}$, and $a, b \in \mathbb{R}, b > 0$ are arbitrary. We give sufficient conditions for which the $\mathcal{H}[f](z)$ converges to f. We will also derive estimates for the remainder

$$\mathcal{R}_N[f](z) := f(z) - \mathcal{H}_N[f](z), \tag{2.2}$$

where

$$\mathcal{H}_N[f](z) := \sum_{n=-N}^{N} e^{-i\frac{a}{2b}(z^2-z_n^2)} \left\{ \left(1 + \frac{ia}{b} z_n (z - z_n)\right) f(z_n) + (z - z_n) f'(z_n) \right\} S_n^2(z), \qquad N \in \mathbb{N}. \tag{2.3}$$

In the following $\mathcal{D}_N$ denotes the closed square

$$\mathcal{D}_N := \left\{ z = x + iy : x, y \in \mathbb{R}, |x|, |y| \le \left(N + \frac{1}{2}\right) \frac{\pi b}{\sigma} \right\} \subseteq \mathbb{C}, \tag{2.4}$$

$N \in \mathbb{N}$. We also denote by $\mathcal{C}_N = \partial\mathcal{D}_N$ to the boundary of $\mathcal{D}_N$. All contours are taken to be positively oriented. We denote by $\mathcal{D}_N^{o}$ to the interior of $\mathcal{D}_N$. The following lemma gives a Cauchy-type representation for the remainder $\mathcal{R}_N(f(\cdot))$.

Lemma 1 *Let $f \in B_{2\sigma}^2$. Then for any interior point z of $\mathcal{D}_N$, i.e. $z \in \mathcal{D}_N^{o}$, $N \in \mathbb{N}$,*

$$\mathcal{R}_N[f](z) = \frac{\sin^2\left(\frac{\sigma}{b} z\right)}{2\pi i} \int_{\mathcal{C}_N} \frac{e^{-i\left(\frac{a}{2b}\right)(z^2-\zeta^2)}}{(\zeta - z)\sin^2\left(\frac{\sigma}{b}\zeta\right)} f(\zeta)\, d\zeta, \tag{2.5}$$

Proof Notice that $\mathcal{R}_N[f]\left(\frac{n\pi b}{\sigma}\right) = 0$, since $\mathcal{H}_N[f]\left(\frac{n\pi b}{\sigma}\right)$ is merely $f(\frac{n\pi b}{\sigma})$, $n = -N, \cdots, N$. Therefore (2.5) holds true for $z = z_n = \frac{n\pi b}{\sigma}$, $|n| \le N$. For $z \in \mathcal{D}_N^{o}$, $z \neq z_n$, define the kernel function

$$\mathcal{K}_z(\zeta) := \frac{\sin^2\left(\frac{\sigma}{b} z\right)}{2\pi i} \times \frac{e^{-i\left(\frac{a}{2b}\right)(z^2-\zeta^2)} f(\zeta)}{(\zeta - z)\sin^2\left(\frac{\sigma}{b}\zeta\right)}. \tag{2.6}$$

As a function of ζ, $\mathcal{K}_z(\zeta)$ is a meromorphic function of ζ with a simple pole at $\zeta = z$ and with double poles at $\zeta = z_n$, $n \in \mathbb{Z}$. By the residue theorem, we obtain

$$\int_{\mathcal{C}_N} \mathcal{K}_z(\zeta) d\zeta = 2\pi i \operatorname{Res}(\mathcal{K}_z; z) + 2\pi i \sum_{n=-N}^{N} \operatorname{Res}(\mathcal{K}_z; z_n), \quad z \neq z_n, \tag{2.7}$$

where $\operatorname{Res}(\mathcal{K}_z; w)$ denotes the residue of $\mathcal{K}_z$ at w. Obviously

$$\operatorname{Res}(\mathcal{K}_z; z) = \lim_{\zeta \to z} (\zeta - z) \times \mathcal{K}_z(\zeta) = \frac{f(z)}{2\pi i}. \tag{2.8}$$

$$\begin{aligned}\operatorname{Res}(\mathcal{K}_z; z_n) &= \lim_{\zeta \to z_n} \frac{d}{d\zeta}\left[(\zeta - z_n)^2 \times \mathcal{K}_z(\zeta)\right] \\ &= \frac{-1}{2\pi i}\left\{\left(1 + \frac{ia}{b} z_n (z - z_n)\right) f(z_n) + (z - z_n) f'(z_n)\right\} \\ &\quad \times \ \operatorname{sinc}^2\left(\frac{\sigma}{b}(z - z_n)\right) e^{-i(\frac{a}{2b})(z^2 - z_n^2)}.\end{aligned} \tag{2.9}$$

Substituting from (2.8) and (2.9) into (2.7), we obtain (2.5).

Before proving the main convergence results of this section, we introduce the following important estimate.

Lemma 2 *For $f \in B^2_{2\sigma}$, there exists $K > 0$, such that*

$$|f(z)| \le K \frac{e^{\frac{a}{b}xy}\, e^{\frac{2\sigma}{b}|y|}}{(1 + |y|)^{1/2}}, \quad z = x + iy \in \mathbb{C}. \tag{2.10}$$

Proof The proof is based on the fact that $f \in B^2_{2\sigma}$ iff there exists $f_0 \in PW^2_{2\sigma/b}$, such that

$$f(z) = e^{-i(a/2b)z^2} f_o(z), \quad z \in \mathbb{C}. \tag{2.11}$$

From [10, p. 158] there exist $K > 0$ such that for $z = x + iy \in \mathbb{C}$, we have

$$|f_o(z)| \le K \frac{e^{\frac{2\sigma}{b}|y|}}{(1 + |y|)^{1/2}}. \tag{2.12}$$

Moreover, $z = x + iy$,

$$\left| e^{-i(a/2b)z^2} \right| = e^{\frac{a}{b}xy}. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), we obtain (2.10).

Theorem 2.1 *Let $f \in B^2_{2\sigma}$. Then $\mathcal{H}[f](z)$ is locally uniformly convergent to f on $\mathbb{C}$.*

Proof Let $\Omega \subseteq \mathbb{C}$ be compact. We show that $\mathcal{H}[f](\cdot)$ converges uniformly to $f(\cdot)$ on Ω by showing that $\mathcal{R}_N[f](\cdot) \rightrightarrows 0$ uniformly on Ω. Indeed, let $N_o \in \mathbb{N}$ be such that $\Omega \subset \mathcal{D}^{o}_{N_o}$. Now fix $z \in \Omega$, $N \in \mathbb{N}$, for which $\mathcal{D}_{N_o} \subseteq \mathcal{D}_N$. Denote the integrals in (2.5) coming from the two horizontal legs of $\mathcal{C}_N$ by $I^{\pm}_H$, where $+$ and $-$ refer to the upper and the lower line segments, respectively, and the integrals coming from the two vertical legs of $\mathcal{C}_N$ by $I^{\pm}_V$, where $+$ and $-$ refer to the right and the left line segments, respectively. Then

$$\int_{\mathcal{C}_N} \frac{e^{-i(\frac{a}{2b})(z^2 - \zeta^2)}}{(\zeta - z)\sin^2\left(\frac{\sigma}{b}\zeta\right)} f(\zeta)\, d\zeta = I^-_H + I^+_V + I^+_H + I^-_V, \tag{2.14}$$

and direct substitutions leads immediately to

$$I_H^{\pm} = \mp \int_{-\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} \frac{e^{-i\left(\frac{a}{2b}\right)\left(z^2-\left(t\pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^2\right)} f\left(t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)}{\left(t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma} - z\right)\sin^2\left(\frac{\sigma}{b}\left(t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)\right)} dt, \tag{2.15}$$

and

$$I_V^{\pm} = \pm \int_{-\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} \frac{e^{-i\left(\frac{a}{2b}\right)\left(z^2-\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it\right)^2\right)} f\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it\right)}{\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it - z\right)\sin^2\left(\frac{\sigma}{b}\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it\right)\right)} dt. \tag{2.16}$$

Inequality (2.10), implies

$$\left| f\left(t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)\right| \leq K \frac{e^{\frac{a\pi}{\sigma}\left(N+\frac{1}{2}\right)t}\, e^{2\pi\left(N+\frac{1}{2}\right)}}{\left(1+\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^{1/2}}. \tag{2.17}$$

Also similar to (2.13), we have

$$\left| e^{i\left(\frac{a}{2b}\right)\left(t\pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^2}\right| = e^{-\frac{a\pi}{\sigma}\left(N+\frac{1}{2}\right)t}. \tag{2.18}$$

Thus, the integrals (2.15) can be estimated using (2.13), (2.17), and (2.18) to obtain

$$\begin{aligned} |I_H^{\pm}| \leq & \frac{K\, e^{\frac{a}{b}xy}\, e^{2\pi\left(N+\frac{1}{2}\right)}}{\left(1+\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^{1/2}} \\ & \times \int_{-\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} \frac{dt}{\left|\left(t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma} - z\right)\sin^2\left(\frac{\sigma}{b}\left(t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)\right)\right|}. \end{aligned} \tag{2.19}$$

Since

$$\left| t \pm i\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma} - z\right| \geq \left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma} - |y|, \quad t \in \mathbb{R}, \tag{2.20}$$

and

$$\left|\sin\left(\frac{\sigma}{b}z\right)\right| \geq \left|\sinh\left(\frac{\sigma}{b}y\right)\right| = \frac{e^{\frac{\sigma}{b}|y|}}{2}\left(1-e^{-\frac{\sigma}{b}|y|}\right), \quad z = x+iy \in \mathbb{C}, \tag{2.21}$$

then

$$\left|I_H^{\pm}\right| \leq \frac{4\,K\,e^{\frac{a}{b}xy}}{\left(1+\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^{1/2}\left(\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}-|y|\right)\left(1-e^{-\pi\left(N+\frac{1}{2}\right)}\right)^2} \times \int_{-\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} dt$$

$$= \frac{8\,K\pi\,b\,\left(N+\frac{1}{2}\right)e^{\frac{a}{b}xy}}{\sigma\left(1+\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^{1/2}\left(\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}-|y|\right)\left(1-e^{-\pi\left(N+\frac{1}{2}\right)}\right)^2}, \quad (2.22)$$

which approaches zero as $N \to \infty$ without depending on z. Now, we estimate $I_V^{\pm}$. Inequality (2.10), implies

$$\left|f\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it\right)\right| \leq K\frac{e^{\frac{a\pi}{\sigma}\left(N+\frac{1}{2}\right)t}\,e^{\frac{2\sigma}{b}|t|}}{(1+|t|)^{1/2}}. \quad (2.23)$$

Moreover,

$$\left|e^{i\left(\frac{a}{2b}\right)\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it\right)^2}\right| = e^{-\frac{a\pi}{\sigma}\left(N+\frac{1}{2}\right)t}. \quad (2.24)$$

Using (2.23), (2.24), and (2.13) leads us to the estimate

$$\left|I_V^{\pm}\right| \leq K e^{\frac{a}{b}xy}\int_{-\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} \frac{e^{\frac{2\sigma}{b}|t|}}{(1+|t|)^{1/2}} \times \frac{1}{\left|\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it-z\right)\sin^2\left(\frac{\sigma}{b}\left(\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it\right)\right)\right|}dt. \quad (2.25)$$

Again

$$\left|\pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+it-z\right| \geq \left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}-|x|, \quad t\in\mathbb{R}, \quad (2.26)$$

and, for $z = \pm\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}+iy,\ N\in\mathbb{N}$, we have

$$\left|\sin\left(\frac{\sigma}{b}z\right)\right| = \cosh\frac{\sigma}{b}(y) \geq \frac{e^{\frac{\sigma}{b}|y|}}{2}. \quad (2.27)$$

Combining (2.26), (2.27) and (2.25), we obtain

$$\begin{aligned} \left|I_V^{\pm}\right| &\le \frac{4K\, e^{\frac{a}{b}xy}}{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}-|x|}\int_{-\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} \frac{e^{\left(\frac{2\sigma}{b}-\frac{2\sigma}{b}\right)|t|}}{(1+|t|)^{1/2}}dt \\ &= \frac{8K\, e^{\frac{a}{b}xy}}{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}-|x|}\int_{0}^{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}} \frac{1}{(1+t)^{1/2}}dt \\ &= \frac{16K\, e^{\frac{a}{b}xy}\left(1+\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}\right)^{1/2}}{\left(N+\frac{1}{2}\right)\frac{\pi b}{\sigma}-|x|}, \end{aligned} \tag{2.28}$$

which tends to zero as $N\to\infty$ without depending on z. From (2.5) we conclude that $\mathcal{R}_N[f](\cdot)\rightrightarrows 0$ uniformly on Ω and the proof is complete.

As we have seen above the order of convergence of $\mathcal{H}[f](\cdot)$ to $f(\cdot)$ is $O(1/\sqrt{N})$ as $N\to\infty$. This order is very slow to establish uniform convergence on $\mathbb{R}$. However if $f(\cdot)$ satisfies a faster decay than that of $B^2_{2\sigma}$-functions, we can prove uniform convergence on $\mathbb{R}$ as in the next theorem. The proof is established using real analysis techniques.

Theorem 2.2 *If $f\in B^2_{2\sigma}$, $z^k f(z)\in L^2(\mathbb{R})$, for some $k\in\mathbb{Z}^+$, then the series (2.1) converges absolutely and uniformly on $\mathbb{R}$ to f.*

Proof Let $z\in\mathbb{R}$. Applying the triangle inequality to the convergent series (2.1), we obtain

$$\begin{aligned} |\mathcal{H}[f](z)| \le& \sum_{n=-\infty}^{\infty}\left|f(z_n)S_n^2(z)\right| + \frac{|a|}{\sigma}\left|\sin\left(\frac{\sigma}{b}z\right)\right|\sum_{n=-\infty}^{\infty}|z_n f(z_n)S_n(z)| \\ &+\frac{b}{\sigma}\left|\sin\left(\frac{\sigma}{b}z\right)\right|\sum_{n=-\infty}^{\infty}\left|f'(z_n)S_n(z)\right|. \end{aligned} \tag{2.29}$$

Using Cauchy-Schwarz inequality yields

$$\begin{aligned} |\mathcal{H}[f](z)| \le& \left\{\left(\sum_{n=-\infty}^{\infty}|f(z_n)|^2\right)^{1/2} + \frac{|a|}{\sigma}\left(\sum_{n=-\infty}^{\infty}|z_n f(z_n)|^2\right)^{1/2}\right. \\ &\left.+\frac{b}{\sigma}\left(\sum_{n=-\infty}^{\infty}\left|f'(z_n)\right|^2\right)^{1/2}\right\}\left(\sum_{n=-\infty}^{\infty}|S_n(z)|^2\right)^{1/2}, \end{aligned} \tag{2.30}$$

where we have used the fact $|S_n(z)|\le 1, z\in\mathbb{R}$. Since $f\in B^2_{2\sigma}$, then Parseval's equality takes the form

$$\int_{-\infty}^{\infty}|f(z)|^2\,dz = \frac{\pi b}{\sigma}\sum_{n=-\infty}^{\infty}\left|f\left(\frac{n\pi b}{\sigma}\right)\right|^2. \tag{2.31}$$

It is easy to see that $zf(z) \in B^2_{2\sigma}$, and hence

$$\int_{-\infty}^{\infty} |zf(z)|^2 \, dz = \frac{\pi b}{\sigma} \sum_{n=-\infty}^{\infty} \left(\frac{n\pi b}{\sigma}\right)^2 \left| f\left(\frac{n\pi b}{\sigma}\right)\right|^2. \tag{2.32}$$

For $f(z) \in B^2_{2\sigma}$, Bernstein's inequality, cf [11, p. 49],

$$\|f'\|_2 \leq \frac{2\sigma}{b} \|f\|_2, \tag{2.33}$$

implies $f'(z) \in L^2(\mathbb{R})$. Hence [7, p. 20], the Fourier transform $\mathcal{F}(f'(\cdot))$ is

$$\mathcal{F}(f'(z))(\alpha) = -i\alpha\, \mathcal{F}(f(z))(\alpha), \quad \alpha \in \mathbb{R}. \tag{2.34}$$

Then $f'(z) \in B^2_{2\sigma}$, and, [11, p. 59],

$$\int_{-\infty}^{\infty} \left|f'(z)\right|^2 dz = \frac{\pi b}{\sigma} \sum_{n=-\infty}^{\infty} \left| f'\left(\frac{n\pi b}{\sigma}\right)\right|^2. \tag{2.35}$$

Substituting from (2.31),(2.32),(2.33) and (2.35) into (2.30), we get

$$\begin{aligned} |\mathcal{H}[f](z)| &\leq \left(\sqrt{\frac{\sigma}{\pi b}} \|f\|_2 + \frac{|a|}{\sqrt{\sigma \pi b}} \|zf\|_2 + 2\sqrt{\frac{\sigma}{\pi b}} \|f\|_2\right) \left(\sum_{n=-\infty}^{\infty} |S_n(z)|^2\right)^{1/2} \\ &= \left(3\sqrt{\frac{\sigma}{\pi b}} \|f\|_2 + \frac{|a|}{\sqrt{\sigma \pi b}} \|zf\|_2\right) \left(\sum_{n=-\infty}^{\infty} |S_n(z)|^2\right)^{1/2}. \end{aligned} \tag{2.36}$$

Now let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Using the inequality, [11, pp. 114–115]

$$\left(\sum_{n=-\infty}^{\infty} |S_n(z)|^q\right)^{1/q} < p, \tag{2.37}$$

and substituting from (2.37) in (2.36) with $p = q = 2$, we obtain

$$|\mathcal{H}[f](z)| \leq 2\left(3\sqrt{\frac{\sigma}{\pi b}} \|f\|_2 + \frac{|a|}{\sqrt{\sigma \pi b}} \|zf\|_2\right). \tag{2.38}$$

Thus, series (2.1) converges absolutely on $\mathbb{R}$. Now let $N > 1$ and $\sigma_N(z)$ be the remainder

$$\begin{aligned}\sigma_N(z) &:= |\mathcal{H}[f](z) - \mathcal{H}_N[f](z)| \\ &= \Bigg| \sum_{|n|>N} e^{i(a/2b)z_n^2} \left\{ f(z_n) S_n^2(z) + \frac{ia}{\sigma} z_n f(z_n) S_n(z) \sin\left(\frac{\sigma}{b}(z - z_n)\right) \right. \\ &\quad \left. + \frac{b}{\sigma} f'(z_n) S_n(z) \sin\left(\frac{\sigma}{b}(z - z_n)\right) \right\} \Bigg|. \end{aligned} \tag{2.39}$$

Using Cauchy-Schwarz inequality yields

$$\begin{aligned}\sigma_N(z) &\leq \left\{ \left(\sum_{|n|>N} |f(z_n)|^2 \right)^{1/2} + \frac{|a|}{\sigma} \left(\sum_{|n|>N} |z_n f(z_n)|^2 \right)^{1/2} \right. \\ &\quad \left. + \frac{b}{\sigma} \left(\sum_{|n|>N} \left| f'(z_n) \right|^2 \right)^{1/2} \right\} \left(\sum_{|n|>N} |S_n(z)|^2 \right)^{1/2} \\ &\leq 2 \left\{ \left(\sum_{|n|>N} |f(z_n)|^2 \right)^{1/2} + \frac{|a|}{\sigma} \left(\sum_{|n|>N} |z_n f(z_n)|^2 \right)^{1/2} \right. \\ &\quad \left. + \frac{b}{\sigma} \left(\sum_{|n|>N} \left| f'(z_n) \right|^2 \right)^{1/2} \right\}, \end{aligned} \tag{2.40}$$

Therefore, from Bessel's inequality, $\sigma_N(z) \to 0$ as $N \to \infty$ without depending on $z \in \mathbb{R}$, which implies uniform convergence on $\mathbb{R}$.

3 Truncation error estimates

In this section, we will investigate the truncation error estimates associated with the Hermite sampling series (2.1). For $N \in \mathbb{Z}^+$, $z \in \mathbb{R}$, the truncated series of (2.1) is

$$f_N(z) := \sum_{n=-N}^{N} e^{-i\frac{a}{2b}(z^2 - z_n^2)} \left\{ \left(1 + \frac{ia}{b} z_n (z - z_n) \right) f(z_n) + (z - z_n) f'(z_n) \right\} S_n^2(z), \tag{3.1}$$

and the associated truncation error is

$$
\begin{aligned}
T_N[f](z) &:= f(z) - f_N(z) \\
&= \sum_{|n|>N} e^{-i\frac{a}{2b}(z^2-z_n^2)} \left\{ \left(1 + \frac{ia}{b} z_n (z - z_n)\right) f(z_n) + (z - z_n) f'(z_n) \right\} S_n^2(z).
\end{aligned} \tag{3.2}
$$

Theorem 3.1 *Let $f \in B_{2\sigma}^2$ and $z^k f(z) \in L^2(\mathbb{R})$, for some $k \in \mathbb{Z}^+$. Then, for $z \in \mathbb{R}$, $|z| < N\pi b/\sigma$, we have*

$$
\begin{aligned}
|T_N[f](z)| \leq \frac{\left|\sin\left(\frac{\sigma}{b}\right) z\right|^2}{(N+1)^k} &\left\{ \frac{\xi_k E_k}{\sqrt{3}} \left(\frac{1}{(N\pi - (\frac{\sigma}{b})z)^{3/2}} + \frac{1}{(N\pi + (\frac{\sigma}{b})z)^{3/2}} \right) \right. \\
&\left. + \eta_k \left(\frac{1}{\sqrt{N\pi - (\frac{\sigma}{b})z}} + \frac{1}{\sqrt{N\pi + (\frac{\sigma}{b})z}} \right) \right\},
\end{aligned} \tag{3.3}
$$

where

$$
\eta_k := \frac{\left(|a|(N+1)^k \sqrt{\int_N^\infty |zf(z)|^2 dz} + \sqrt{\pi} b\, \xi_k \left[(\sigma/b) E_k + k E_{k-1}\right]\right)}{\sigma\sqrt{\pi}}, \tag{3.4}
$$

and

$$
E_k := \sqrt{\int_{-\infty}^{\infty} |z^k f(z)|^2 dz}, \quad \xi_k := \frac{(\sigma/b)^{k+1/2}}{\pi^{k+1}\sqrt{(1-4^{-k})}}. \tag{3.5}
$$

Proof Let $N \in \mathbb{Z}^+$, $z \in \mathbb{R}$ with $|z| < N\pi b/\sigma$. Applying the triangle inequality to (3.2), we obtain

$$
\begin{aligned}
|T_N[f](z)| &\leq \sum_{|n|>N} \left| f(z_n) S_n^2(z) \right| + \frac{|a|}{\sigma} \sum_{|n|>N} \left| z_n f(z_n) \sin\left(\frac{\sigma}{b}(z - z_n)\right) S_n(z) \right| \\
&\quad + \frac{b}{\sigma} \sum_{|n|>N} \left| f'(z_n) \sin\left(\frac{\sigma}{b}(z - z_n)\right) S_n(z) \right| \\
&= \sum_{n<-N} \left| \frac{f(z_n) \sin^2(\frac{\sigma}{b})z}{\left((\frac{\sigma}{b}) z - n\pi\right)^2} \right| + \sum_{n>N} \left| \frac{f(z_n) \sin^2(\frac{\sigma}{b})z}{\left((\frac{\sigma}{b}) z - n\pi\right)^2} \right| \\
&\quad + \frac{|a|}{\sigma} \sum_{n<-N} \left| \frac{z_n f(z_n) \sin^2(\frac{\sigma}{b})z}{\left((\frac{\sigma}{b}) z - n\pi\right)} \right| + \frac{|a|}{\sigma} \sum_{n>N} \left| \frac{z_n f(z_n) \sin^2(\frac{\sigma}{b})z}{\left((\frac{\sigma}{b}) z - n\pi\right)} \right| \\
&\quad + \frac{b}{\sigma} \sum_{n<-N} \left| \frac{f'(z_n) \sin^2(\frac{\sigma}{b})z}{\left((\frac{\sigma}{b}) z - n\pi\right)} \right| + \frac{b}{\sigma} \sum_{n>N} \left| \frac{f'(z_n) \sin^2(\frac{\sigma}{b})z}{\left((\frac{\sigma}{b}) z - n\pi\right)} \right|.
\end{aligned} \tag{3.6}
$$

Cauchy-Schwarz inequality implies

$$\begin{aligned}|T_N[f](z)| \le &\left(\sum_{n>N} |f(z_n)|^2\right)^{1/2} \left(\sum_{n>N} \frac{|\sin(\frac{\sigma}{b})z|^4}{\left(n\pi - \left(\frac{\sigma}{b}\right)z\right)^4}\right)^{1/2} + \left(\sum_{n<-N} |f(z_n)|^2\right)^{1/2}\\ &\times \left(\sum_{n<-N} \frac{|\sin(\frac{\sigma}{b})z|^4}{\left(n\pi - \left(\frac{\sigma}{b}\right)z\right)^4}\right)^{1/2} + \frac{|a|}{\sigma}\left(\sum_{n>N} |z_n f(z_n)|^2\right)^{1/2} \left(\sum_{n>N} \frac{|\sin(\frac{\sigma}{b})z|^4}{(n\pi - \left(\frac{\sigma}{b}\right)z)^2}\right)^{1/2}\\ &+ \frac{|a|}{\sigma}\left(\sum_{n<-N} |z_n f(z_n)|^2\right)^{1/2} \left(\sum_{n<-N} \frac{|\sin(\frac{\sigma}{b})z|^4}{(n\pi - \left(\frac{\sigma}{b}\right)z)^2}\right)^{1/2} + \frac{b}{\sigma}\left(\sum_{n>N} \left|f'(z_n)\right|^2\right)^{1/2}\\ &\times \left(\sum_{n>N} \frac{|\sin(\frac{\sigma}{b})z|^4}{(n\pi - \left(\frac{\sigma}{b}\right)z)^2}\right)^{1/2} + \frac{b}{\sigma}\left(\sum_{n<-N} \left|f'(z_n)\right|^2\right)^{1/2} \left(\sum_{n<-N} \frac{|\sin(\frac{\sigma}{b})z|^4}{(n\pi - \left(\frac{\sigma}{b}\right)z)^2}\right)^{1/2}. \end{aligned} \tag{3.7}$$

Recalling that $|z| < N\pi b/\sigma$, we have for $\alpha > 0$, cf. [2, p. 340],

$$\sum_{\pm n>N} \frac{1}{(n\pi - \left(\frac{\sigma}{b}\right)z)^{\alpha+1}} < \frac{1}{\alpha\pi(N\pi \mp \left(\frac{\sigma}{b}\right)z)^{\alpha}}. \tag{3.8}$$

Since $f(z) \in B^2_{2\sigma}$ and $z^k f(z) \in L^2(\mathbb{R})$, then we have, cf. [13, p. 716],

$$\left(\sum_{\pm n>N} |f(z_n)|^2\right)^{1/2} \le \frac{\sqrt{\pi}\xi_k E_k}{(N+1)^k}, \tag{3.9}$$

and, cf. [3, p. 1303]

$$\left(\sum_{\pm n>N} \left|f'(z_n)\right|^2\right)^{1/2} \le \frac{\sqrt{\pi}\xi_k\left[(\sigma/b)E_k + kE_{k-1}\right]}{(N+1)^k}. \tag{3.10}$$

Moreover,

$$\left(\sum_{\pm n>N} |z_n f(z_n)|^2\right)^{1/2} \le \sqrt{\int_N^{\infty} |zf(z)|^2 dz}. \tag{3.11}$$

Applying (3.8) with $\alpha = 3$ in the first two sums and $\alpha = 1$ in the last four ones in (3.7), and substituting from (3.9)-(3.11) into (3.7), we obtain (3.3).

Corollary 1 *Let $f \in B^2_{2\sigma}$ and $z^k f(z) \in L^2(\mathbb{R})$, for some $k \in \mathbb{Z}^+$. Then*

$$|T_N[f](z)| \le \frac{\sqrt{2/\pi}}{(N+1)^k}\left\{\frac{2\xi_k E_k}{\sqrt{3}\pi}\left(1 + \frac{1}{(4N-1)^{3/2}}\right) + \eta_k\left(1 + \frac{1}{\sqrt{4N-1}}\right)\right\}, \tag{3.12}$$

uniformly on $|z| < N\pi b/\sigma$.

Proof By simple calculations, we find that the left hand side in (3.3) has absolute maximum

$$\frac{\sqrt{2/\pi}}{(N+1)^k}\left\{\frac{2\xi_k\, E_k}{\sqrt{3}\,\pi}\left(1+\frac{1}{(4N-1)^{3/2}}\right)+\eta_k\left(1+\frac{1}{\sqrt{4N-1}}\right)\right\}, \tag{3.13}$$

at $z=\pm(N-1/2)b\pi/\sigma$. This completes the proof.

In the following we establish a global error estimate for the truncation error on $\mathbb{R}$.

Theorem 3.2 *Suppose that $f\in B^2_{2\sigma}$ satisfies a decay condition of the form*

$$|f(z)|\leq\frac{A}{|z|^{\beta+1}},\quad z\neq 0, \tag{3.14}$$

where $A>0$ and $1/2<\beta\leq 1$ are constants. Then for $z\in\mathbb{R}$ and $N\in\mathbb{Z}^+$, we have

$$\begin{aligned}|T_N[f](z)|\leq\frac{4b}{\sigma}&\left\{\frac{M_1\,(\rho(N)+2)}{\sqrt{N}}+\frac{M_2}{N^{\beta+1/2}}\left(\rho(2N)+\pi/\sqrt{2(2\beta+1)}+4\right)\right.\\&\left.+\frac{|a|b\pi^2\,M_2}{\sqrt{2(2\beta-1)}\,\sigma^2N^{\beta-1/2}}\right\},\end{aligned} \tag{3.15}$$

where

$$M_1:=\frac{3\sigma}{\pi b}\left(|f(0)|+A\left(\frac{\sigma}{\pi b}\right)^{\beta+1}\right),\ M_2:=A\left(\frac{\sigma}{\pi b}\right)^{\beta+2},\ \rho(z):=\gamma+\log(z), \tag{3.16}$$

and γ is the Euler-Mascheroni constant. Moreover if $N\geq 8$, then for $z\in\mathbb{R}$ we have the improved estimate

$$|T_N[f](z)|\leq\frac{2\,b\,e}{\sigma}\left\{\frac{M_1\,\rho\left(N^2\right)}{N}+\frac{M_2\varrho(N^2)}{N^{\beta+1}}+\frac{|a|\,b\,\pi^2M_2}{\sqrt{2}\,\sigma^2N^{\beta}}\right\}\ln N, \tag{3.17}$$

where

$$\varrho(x):=\left(\rho\,(2x)+\pi/\sqrt{2}+2\right). \tag{3.18}$$

Proof Let $N\in\mathbb{Z}^+$, $z\in\mathbb{R}$. Applying triangle inequality to (3.2) we obtain

$$\begin{aligned}|T_N[f](z)|\leq&\sum_{|n|>N}\left|f\,(z_n)\,S_n^2(z)\right|+\frac{|a|}{\sigma}\sum_{|n|>N}\left|z_nf\,(z_n)\sin\left(\frac{\sigma}{b}(z-z_n)\right)\,S_n(z)\right|\\&+\frac{b}{\sigma}\sum_{|n|>N}\left|f'\,(z_n)\sin\left(\frac{\sigma}{b}(z-z_n)\right)\,S_n(z)\right|.\end{aligned} \tag{3.19}$$

Using Cauchy-Schwarz inequality and since $|S_n(z)| \leq 1$, we obtain

$$|T_N[f](z)| \leq \left\{ \left(\sum_{|n|>N} |f(z_n)|^2 \right)^{1/2} + \frac{|a|}{\sigma} \left(\sum_{|n|>N} |z_n f(z_n)|^2 \right)^{1/2} + \frac{b}{\sigma} \left(\sum_{|n|>N} |f'(z_n)|^2 \right)^{1/2} \right\} \left(\sum_{|n|>N} |S_n(z)|^2 \right)^{1/2}.$$

Inequality (3.14) implies

$$\left(\sum_{|n|>N} \left| f\left(\frac{n\pi b}{\sigma}\right) \right|^2 \right)^{1/2} \leq A \left(\frac{\sigma}{\pi b}\right)^{\beta+1} \left(\sum_{|n|>N} \frac{1}{|n|^{2\beta+2}} \right)^{1/2} = A \left(\frac{\sigma}{\pi b}\right)^{\beta+1} \left(2 \sum_{n>N} \frac{1}{n^{2\beta+2}} \right)^{1/2}$$
$$\leq A \left(\frac{\sigma}{\pi b}\right)^{\beta+1} \left(2 \int_N^\infty \frac{1}{z^{2\beta+2}} dz \right)^{1/2} = M_2 \left(\frac{\pi b}{\sigma}\right) \sqrt{\frac{2}{2\beta+1}} \frac{1}{N^{\beta+1/2}}. \quad (3.20)$$

Similarly,

$$\left(\sum_{|n|>N} \left| \left(\frac{n\pi b}{\sigma}\right) f\left(\frac{n\pi b}{\sigma}\right) \right|^2 \right)^{1/2} \leq A \left(\frac{\sigma}{\pi b}\right)^{\beta} \left(\sum_{|n|>N} \frac{1}{|n|^{2\beta}} \right)^{1/2} = A \left(\frac{\sigma}{\pi b}\right)^{\beta} \left(2 \sum_{n>N} \frac{1}{n^{2\beta}} \right)^{1/2}$$
$$\leq A \left(\frac{\sigma}{\pi b}\right)^{\beta} \left(2 \int_N^\infty \frac{1}{z^{2\beta}} dz \right)^{1/2} = M_2 \left(\frac{\pi b}{\sigma}\right)^2 \sqrt{\frac{2}{2\beta-1}} \frac{1}{N^{\beta-1/2}}. \quad (3.21)$$

Since $f \in B^2_{2\sigma}$ and it decays according to (3.14), then we have, cf. [3, p. 1304]

$$\left(\sum_{\pm n>N} \left| f'\left(\frac{n\pi b}{\sigma}\right) \right|^2 \right)^{1/2} < \frac{M_1(\rho(N)+2)}{\sqrt{N}} + \frac{M_2(\rho(2N)+4)}{N^{\beta+1/2}}. \quad (3.22)$$

Hence,

$$\left(\sum_{|n|>N} \left| f'\left(\frac{n\pi b}{\sigma}\right) \right|^2 \right)^{1/2} < \frac{2M_1(\rho(N)+2)}{\sqrt{N}} + \frac{2M_2(\rho(2N)+4)}{N^{\beta+1/2}}. \quad (3.23)$$

Substituting from (3.20), (3.21), and (3.23) in (3.20) yields

$$|T_N[f](z)| \le \frac{b}{\sigma}\left\{\frac{2M_1(\rho(N)+2)}{\sqrt{N}} + \frac{M_2}{N^{\beta+1/2}}\left[\pi\sqrt{\frac{2}{2\beta+1}} + 2(\rho(2N)+4)\right]\right.$$
$$\left. + \sqrt{\frac{2}{2\beta-1}}\frac{|a|b\pi^2 M_2}{\sigma^2 N^{\beta-1/2}}\right\}\left(\sum_{|n|>N}|S_n(z)|^2\right)^{1/2}. \tag{3.24}$$

Substituting from (2.37) in (3.24) with $p = q = 2$ we obtain (3.15).
Now let $N \ge 8$. Using Hölder's inequality, (2.37), and $|S_n(z)| \le 1$, inequality (3.19) turns out to

$$|T_N[f](z)| \le \left\{\left(\sum_{|n|>N}|f(z_n)|^p\right)^{1/p} + \frac{|a|}{\sigma}\left(\sum_{|n|>N}|z_n f(z_n)|^p\right)^{1/p}\right.$$
$$\left. + \frac{b}{\sigma}\left(\sum_{|n|>N}\left|f'(z_n)\right|^p\right)^{1/p}\right\}\left(\sum_{|n|>N}|S_n(z)|^q\right)^{1/q},$$
$$\le \left\{\left(\sum_{|n|>N}|f(z_n)|^p\right)^{1/p} + \frac{|a|}{\sigma}\left(\sum_{|n|>N}|z_n f(z_n)|^p\right)^{1/p}\right.$$
$$\left. + \frac{b}{\sigma}\left(\sum_{|n|>N}\left|f'(z_n)\right|^p\right)^{1/p}\right\}p. \tag{3.25}$$

Using (3.14) and choosing $p \ge 2$ we get

$$\left(\sum_{|n|>N}\left|f\left(\frac{n\pi b}{\sigma}\right)\right|^p\right)^{1/p} \le M_2\left(\frac{\pi b}{\sigma}\right)\left(\frac{2}{p\beta+p-1}\right)^{1/p}\frac{1}{N^{\beta+1-1/p}}, \tag{3.26}$$

and

$$\left(\sum_{|n|>N}\left|\left(\frac{n\pi b}{\sigma}\right)f\left(\frac{n\pi b}{\sigma}\right)\right|^p\right)^{1/p} \le M_2\left(\frac{\pi b}{\sigma}\right)^2\left(\frac{2}{p\beta-1}\right)^{1/p}\frac{1}{N^{\beta-1/p}}. \tag{3.27}$$

Moreover, cf. [3, p. 1309],

$$\left(\sum_{\pm n>N}\left|f'\left(\frac{n\pi b}{\sigma}\right)\right|^p\right)^{1/p} < \frac{M_1\rho(N^2)}{N^{1-1/p}} + \frac{M_2\left(\rho(2N^2)+2\right)}{N^{\beta+1-1/p}}, \tag{3.28}$$

and then

$$\left(\sum_{|n|>N}\left|f'\left(\frac{n\pi b}{\sigma}\right)\right|^{p}\right)^{1/p}<\frac{2M_1\rho(N^2)}{N^{1-1/p}}+\frac{2M_2\left(\rho(2N^2)+2\right)}{N^{\beta+1-1/p}}. \tag{3.29}$$

Combining (3.25), (3.26), (3.27), and (3.29), for $p\geq 2$ yields

$$\begin{aligned}|T_N[f](z)|\leq \frac{b}{\sigma}\Bigg\{&\frac{2M_1\,\rho\left(N^2\right)}{N^{1-1/p}}+\left(\frac{2}{p\beta-1}\right)^{1/p}\frac{|a|\,b\,\pi^2M_2}{\sigma^2N^{\beta-1/p}}\\&+\frac{M_2}{N^{\beta+1-1/p}}\left[\pi\left(\frac{2}{p\beta+p-1}\right)^{1/p}+2\left(\rho\left(2N^2\right)+2\right)\right]\Bigg\}\,p.\end{aligned} \tag{3.30}$$

Letting $p:=\ln N$ and $p\geq 2$ we can see using elementary calculus that

$$\left(\frac{2}{p\,\beta+p-1}\right)^{1/p}<\sqrt{2},\quad \left(\frac{2}{p\,\beta-1}\right)^{1/p}<\sqrt{2},\quad N^{-1/p}=\frac{1}{e}. \tag{3.31}$$

Combining (3.30) and (3.31) we get (3.17).

4 Combining various types of errors

Several types of errors arise when (2.1) is used other than the truncation error, among which are amplitude and jitter errors. The amplitude error associated with (2.1) arises when the exact samples $f(z_n), f'(z_n), n\in\mathbb{Z}$ are replaced by approximate closer ones $\widetilde{f}(z_n), \widetilde{f}'(z_n)$ in the sampling series (2.1). Let $\varepsilon_n := f(z_n)-\widetilde{f}(z_n)$, $\varepsilon'_n := f'(z_n)-\widetilde{f}'(z_n)$ be uniformly bounded by ε, i.e., $|\varepsilon_n|, \left|\varepsilon'_n\right|<\varepsilon$ for a sufficiently small $\varepsilon>0$. The amplitude error is defined for $z\in\mathbb{R}$ by

$$\begin{aligned}A_\varepsilon[f](z):=\sum_{n=-\infty}^{\infty}e^{-i\left(\frac{a}{2b}\right)(z^2-z_n^2)}\Bigg[&\left(1+\frac{ia}{b}z_n(z-z_n)\right)\left\{f(z_n)-\widetilde{f}(z_n)\right\}\\&+(z-z_n)\left\{f'(z_n)-\widetilde{f}'(z_n)\right\}\Bigg]S_n^2(z).\end{aligned} \tag{4.1}$$

If $f\in B^2_{2\sigma}$ satisfies the condition (3.14) and

$$|\varepsilon_n|\leq|f(z_n)|,\quad |\varepsilon'_n|\leq\left|f'(z_n)\right|,\quad n\in\mathbb{Z}, \tag{4.2}$$

then, for $0 < \varepsilon \le \min\{\pi b/\sigma, \sigma/\pi b, 1/\sqrt{e}\}$, (4.1) is estimated in [1] by

$$\| A_\varepsilon[f](z) \|_\infty \le \frac{4\, e^{1/4}}{\sigma(\beta+1)} \left\{ \sqrt{3}\, e\,(\sigma+b) + \varrho\left(\varepsilon^{-10}\right) + \sqrt{2}\,|a| \left(\frac{\sigma}{\pi b}\right)^3 \varepsilon^{\frac{-1}{\beta+1}} \right.$$
$$\left. + \frac{2\pi b^2}{\sigma} M_1 \rho\left(\varepsilon^{-10}\right) + \sqrt{2}\,|a| \left(\frac{\pi b}{\sigma}\right)^3 M_2 \right\} \varepsilon \log(1/\varepsilon), \quad (4.3)$$

where M_1, M_2, $\rho(x)$ and $\varrho(x)$ are defined in (3.16) and (3.18).

The jitter error associated with (2.1) arises when the nodes z_n, $n \in \mathbb{Z}$ of (2.1) are deviated from the exact nodes. Let δ_n, δ_n', $n \in \mathbb{Z}$ denote the set of deviation values, such that $|\delta_n|, |\delta_n'| \le \delta$ for a sufficiently small $\delta > 0$. For $z \in \mathbb{R}$, the jitter error associated with (2.1) is defined by

$$J_\delta[f](z) := \sum_{n=-\infty}^{\infty} e^{i\left(\frac{a}{2b}\right)(z_n^2 - z^2)} \left\{ \left(1 + \frac{ia}{b} z_n (z - z_n)\right) \left\{ f(z_n) - f(z_n + \delta_n) \right\} \right.$$
$$\left. + (z - z_n) \left\{ f'(z_n) - f'\left(z_n + \delta_n'\right) \right\} \right\} S_n^2(z). \quad (4.4)$$

Let $f \in B^2_{2\sigma}$ satisfy the condition (3.14). For $0 < \delta \le \min\{\pi b/\sigma, \sigma/\pi b, 1/\sqrt{e}\}$, (4.4) is estimated in [1] by

$$\| J_\delta[f](z) \|_\infty \le \frac{4e^{1/4}}{\sigma(\beta+1)} \left\{ C \|f'\|_\infty + \sqrt{3}\, b\, e \|f''\|_\infty + \frac{4\pi b^2 M_1 \rho\left(\delta^{-10}\right)}{\sigma} \right.$$
$$\left. + 2\sqrt{2}\,|a|\, M_2 \left(\frac{\pi b}{\sigma}\right)^3 + 2\varrho\left(\delta^{-10}\right) \right\} \delta \log(1/\delta), \quad (4.5)$$

where

$$C := \sqrt{5}\, \sigma e^{3/4} + \sqrt{2} |a| \delta^{-1/\beta+1} \left(\frac{\sigma}{\pi b}\right)^3, \quad (4.6)$$

M_1, M_2, $\rho(x)$ and $\varrho(x)$ are defined in (3.16) and (3.18).

Since only a finite number of samples are available in particular applications, the amplitude and jitter errors are the truncated ones

$$\left| f(z) - \widetilde{f}_N(z) \right| = \left| f(z) - \sum_{n=-N}^{N} e^{i\frac{a}{2b}(z_n^2 - z^2)} \left\{ (1 + (ia/b) z_n (z - z_n))\, \widetilde{f}(z_n) \right. \right.$$
$$\left. \left. + (z - z_n) \widetilde{f}'(z_n) \right\} \mathbf{S}_n(z) \right|$$
$$\le \mathcal{B}(N, f; z) + \mathcal{A}(\varepsilon, f), \quad (4.7)$$

and

$$\begin{aligned}|f(z)-f_N(\delta,z)| &= \left| f(z) - \sum_{n=-N}^{N} e^{i\frac{a}{2b}(z_n^2-z^2)} \{(1+(ia/b)z_n(z-z_n))\, f(z_n+\delta_n) \right. \\ &\quad \left. + (z-z_n) f'(z_n+\delta'_n)\Big\}\, \mathbf{S}_n(z) \right| \\ &\le \mathcal{B}(N,f;z) + \mathcal{J}(\delta,f), \end{aligned} \tag{4.8}$$

where

$$\widetilde{f}_N(z) = \sum_{|n|\le N} e^{i\frac{a}{2b}(z_n^2-z^2)} \left\{(1+(ia/b)z_n(z-z_n))\, \widetilde{f}(z_n) + (z-z_n)\widetilde{f}'(z_n)\right\} \mathbf{S}_n(z), \tag{4.9}$$

$$\begin{aligned} f_N(\delta,z) = \sum_{|n|\le N} & e^{i\frac{a}{2b}(z_n^2-z^2)} \{(1+(ia/b)z_n(z-z_n))\, f(z_n+\delta_n) \\ & + (z-z_n) f'(z_n+\delta'_n)\Big\}\, \mathbf{S}_n(z), \end{aligned} \tag{4.10}$$

and $\mathcal{B}(N, f; z)$, $\mathcal{A}(\varepsilon, f)$, and $\mathcal{J}(\delta, f)$ denote to the bound of the truncation error in (3.3), the bound of the amplitude error in (4.3), and the bound of jitter error in (4.5) respectively.

5 Numerical examples

In this section, we introduce several numerical examples and comparisons between Hermite sampling approximations with LCT which introduced in this paper and the classical sampling with LCT. We also give tables illustrating the error for some numerical values. Furthermore, the results are depicted in various figures to demonstrate the accuracy of the approximations. All of the results in the following examples confirm the correctness and accuracy of the conclusions we reached in this paper. We have used *Mathematica* to derive these examples. It is convenient to introduce the following notations. Let $f \in B_\sigma^2 \subseteq B_{2\sigma}^2$. For $z \in \mathbb{R}$, $N \in \mathbb{Z}^+$, we denote by $f_N^C(z)$ and $f_N^H(z)$ to the truncated expansions:

$$f_N^C(z) := \sum_{n=-N}^{N} e^{-i\frac{a}{2b}(z^2-Z_n^2)} f(z_n)\, \mathrm{sinc}\left[\frac{\sigma}{b\pi}(z-z_n)\right], \quad z\in\mathbb{C}, \tag{5.1}$$

$$\begin{aligned} f_N^H(z) := \sum_{n=-N}^{N} & e^{-i\frac{a}{2b}(z^2-z_n^2)} \left\{ \left(1+\frac{ia}{b}z_n(z-z_n)\right) f(z_n) \right. \\ & \left. + (z-z_n) f'(z_n) \right\} S_n^2(z), \quad z\in\mathbb{C}. \end{aligned} \tag{5.2}$$

Example 1 Consider the B_{π}^{2}-function

$$f(z)=\frac{e^{-i\frac{z^2}{2}}\left(-2\pi z\cos(\sqrt{2}\pi z)+\sqrt{2}\sin(\sqrt{2}\pi z)\right)}{12\pi^2 z^3}. \tag{5.3}$$

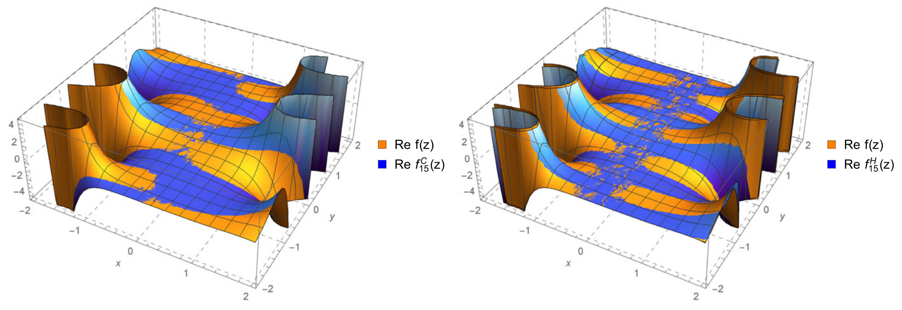
This function has both Lagrange-type and Hermite-type sampling expansions. Table 1 exhibits comparison between $f_N^C(z)$ and $f_N^H(z)$ at some points $z_i \in \mathbb{R}$, when $N = 20, 30, 50$. We apply Theorem 2.1 of [4] and (3.3) to compute the error bounds $|f(z) - f_N^C(z)|$ and $|f(z) - f_N^H(z)|$ respectively. As Table 1 indicates, the precision improves when N increases for both techniques. However, the Hermite interpolation approximations are better than the classical sampling representation of $f(z)$ in terms of the absolute errors. In addition, the error bound for the Hermite interpolations is also better than the error bound for the Shannon-type interpolations. The real and imaginary parts of the complex signal $f(z)$ and reconstructed signals $f_N^C(z)$ and $f_N^H(z)$ are plotted in Figs. 1 and 2 when $|z| \le 2 \subset \mathbb{C}$, $N = 15$. Here $a = b = d = 1/\sqrt{2}$.

Example 2 In this example we investigate the closeness between the exact error and the obtained error estimates. The function

$$f(z)=\frac{e^{-i\frac{\sqrt{3}}{2}z^2}\,\sin(2\pi z)}{2\pi^2 z\left(1-z^2\right)}, \tag{5.4}$$

Table 1 Comparison between $f_N^C(z)$ and $f_N^H(z)$ of Example 1

z_i	N	$\lvert f(z_i) - f_N^C(z_i)\rvert$	Bound	$\lvert f(z_i) - f_N^H(z_i)\rvert$	Bound
	20	8.16951×10^{-8}	4.75991×10^{-5}	2.42076×10^{-10}	2.73428×10^{-6}
0.7	30	2.48032×10^{-8}	2.63141×10^{-5}	4.94701×10^{-11}	1.53971×10^{-6}
	50	5.46396×10^{-9}	1.23863×10^{-5}	6.58691×10^{-12}	7.47049×10^{-7}
	20	3.27996×10^{-7}	9.54151×10^{-5}	9.77546×10^{-10}	1.09572×10^{-5}
1.4	30	9.93519×10^{-8}	5.26666×10^{-5}	1.98623×10^{-10}	6.16035×10^{-6}
	50	2.18601×10^{-8}	2.47712×10^{-5}	2.63668×10^{-11}	2.98655×10^{-6}
	20	7.42610×10^{-7}	1.43674×10^{-4}	2.23501×10^{-9}	2.47304×10^{-5}
2.1	30	2.24067×10^{-7}	7.90969×10^{-5}	4.49716×10^{-10}	1.38666×10^{-5}
	50	4.92011×10^{-8}	3.71533×10^{-5}	5.93981×10^{-11}	6.71355×10^{-6}
	20	1.33194×10^{-6}	1.92619×10^{-4}	4.06516×10^{-9}	4.41618×10^{-5}
2.8	30	3.99660×10^{-7}	1.05646×10^{-4}	8.06619×10^{-10}	2.46669×10^{-5}
	50	8.75084×10^{-8}	4.95313×10^{-5}	1.05779×10^{-10}	1.19199×10^{-5}
	20	2.10542×10^{-6}	2.42525×10^{-4}	6.54592×10^{-9}	6.94136×10^{-5}
3.5	30	6.27153×10^{-7}	1.32357×10^{-4}	1.27498×10^{-9}	3.85737×10^{-5}
	50	1.36813×10^{-7}	6.19041×10^{-5}	1.65649×10^{-10}	1.85941×10^{-5}



(a) (b)

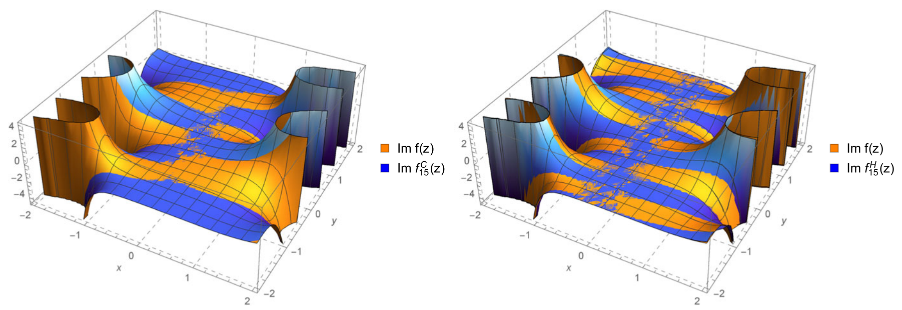
Fig. 1 Illustrations for Example 1 when $|z| \leq 2$, $z \in \mathbb{C}$. The orange surfaces in (**a**) and (**b**) are real part of $f(z)$, while the blue surfaces in (**a**) and (**b**) are real parts of $f_{15}^{C}(z)$ and $f_{15}^{H}(z)$ respectively

is a B_{π}^{2}-function, where $a = d = \sqrt{3}/2, b = 1/2$. Table 2 demonstrates comparison between the pointwise truncation error $T(N, f; z) := T_N[f](z)$ and its bound, denoted above by $\mathcal{B}(N, f; z)$. Figure 3 exhibits the absolute error $|T(N, f; z)|$ and the bound $\mathcal{B}(N, f; z)$ on the interval $[-2, 2]$. It is noted that the error bounds are quite realistic. Here we take $N = 15; 20; 25$ respectively.

Example 3 In this example, we consider the function

$$f(z) = \frac{\left(1 - \cos\left(\frac{\pi z}{4}\right)\right)^2}{\pi^4 z^4} \in B_{\pi/4}^{2}. \tag{5.5}$$

Here we compare between the uniform bound truncation error in Theorem 3.1, which is denoted by $\mathcal{U}_1$, and the uniform bound truncation error in Theorem 3.2, which is denoted by $\mathcal{U}_2$. Table 3 exhibits the truncation error $T(N, f; z)$, bound truncation error $\mathcal{B}(N, f; z)$, $\mathcal{U}_1$, and $\mathcal{U}_2$ on the interval $[-15, 15]$, where $N = 15; 20; 25$ respectively. The uniform bound $\mathcal{U}_1$ in (3.12) is smaller than the uniform bound $\mathcal{U}_2$ in (3.15).



(a) (b)

Fig. 2 The orange surfaces in (**a**) and (**b**) are imaginary part of $f(z)$, while the blue surfaces in (**a**) and (**b**) are imaginary parts of $f_{15}^{C}(z)$ and $f_{15}^{H}(z)$ respectively. Here $|z| \leq 2$, $z \in \mathbb{C}$

Table 2 Exact and bound truncation errors of Example 2

z_i	$N=15$		$N=20$		$N=25$	
	$\lvert T(N, f; z_i)\rvert$	$\mathcal{B}(N, f; z_i)$	$\lvert T(N, f; z_i)\rvert$	$\mathcal{B}(N, f; z_i)$	$\lvert T(N, f; z_i)\rvert$	$\mathcal{B}(N, f; z_i)$
0.25	2.25153×10^{-6}	6.89675×10^{-3}	7.33523×10^{-7}	4.53742×10^{-3}	3.0593×10^{-7}	3.27222×10^{-3}
0.75	2.27024×10^{-6}	6.9208×10^{-3}	7.3701×10^{-7}	4.54621×10^{-3}	3.06871×10^{-7}	3.27624×10^{-3}
1.25	2.3086×10^{-6}	6.96991×10^{-3}	7.44084×10^{-7}	4.56399×10^{-3}	3.08769×10^{-7}	3.28436×10^{-3}
1.75	2.36862×10^{-6}	7.04616×10^{-3}	7.54952×10^{-7}	4.59118×10^{-3}	3.1166×10^{-7}	3.29668×10^{-3}
2.25	2.45365×10^{-6}	7.15307×10^{-3}	7.69946×10^{-7}	4.62842×10^{-3}	3.1560×10^{-7}	3.31339×10^{-3}

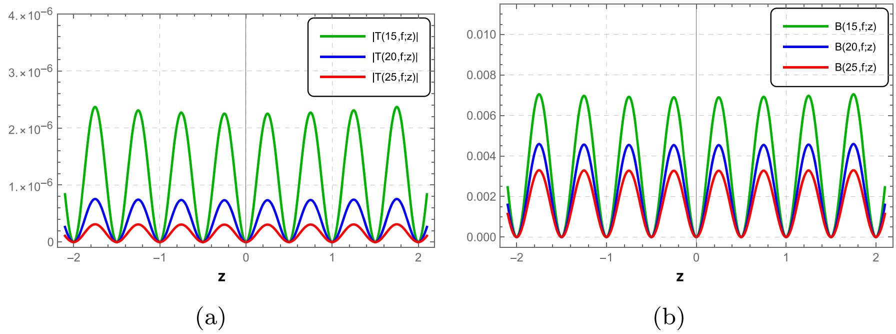


Fig. 3 Figures for $|T(N, f; z)|$ (**a**), $\mathcal{B}(N, f; z)$ (**b**), f is given in (5.4), where $z \in [-2, 2]$ and $N = 15, 20, 25$. Notice that due to the difference both figures cannot be gathered in one graph

Figure 4 illustrates the absolute error $|T(N, f; z)|$ and the bound $\mathcal{B}(N, f; z)$ on the interval $[-15, 15]$, where $N = 15; 20; 25$ respectively. Here we take $a = d = 0, b = 1, \sigma = \pi/4$.

Example 4 This example is devoted to comparison between the absolute amplitude error $\left|f(z) - \widetilde{f}_N(z)\right|$ and its bound $\mathcal{B}(N, f; z) + \mathcal{A}(\varepsilon, f)$ for the $\mathrm{B}^2_{\pi/4}$-function

$$f(z) = \frac{e^{-i\frac{z^2}{2\sqrt{3}}}\left(\sqrt{3}\pi z\cos\left(\frac{\pi z}{2\sqrt{3}}\right) - 6\sin\left(\frac{\pi z}{2\sqrt{3}}\right)\right)}{24\pi^2 z^3}, \tag{5.6}$$

where $a = d = 1/2, b = \sqrt{3}/2$, and $N = 20$. Table 4 demonstrates numerical results when $\varepsilon = 6 \times 10^{-7}, 4 \times 10^{-7}, 2 \times 10^{-7}$ respectively. It indicates the effect of ε on the amplitude error. Graphs of $\left|f(z) - \widetilde{f}_N(z)\right|$ and its bound $\mathcal{B}(N, f; z) + \mathcal{A}(\varepsilon, f)$ where $z \in [-10, 10]$ and $\varepsilon = 6 \times 10^{-7}, 4 \times 10^{-7}, 2 \times 10^{-7}$ are exhibited in Figure 5. As Table 5 indicates, the error bounds are quite realistic. Moreover, the accuracy improves when ε decreases.

Table 3 Exact, bound and uniform bound errors of Example 3

N	$\max\limits_{z\in[-15,15]} \lvert T(N, f; z)\rvert$	$\max\limits_{z\in[-15,15]} \{\mathcal{B}(N, f; z)\}$	$\mathcal{U}_1$	$\mathcal{U}_2$
15	1.59553×10^{-11}	2.59883×10^{-7}	9.34249×10^{-7}	5.42094×10^{-3}
20	5.17309×10^{-12}	1.29148×10^{-7}	5.35165×10^{-7}	4.94719×10^{-3}
25	1.38273×10^{-12}	7.49456×10^{-8}	3.45957×10^{-7}	4.60044×10^{-3}

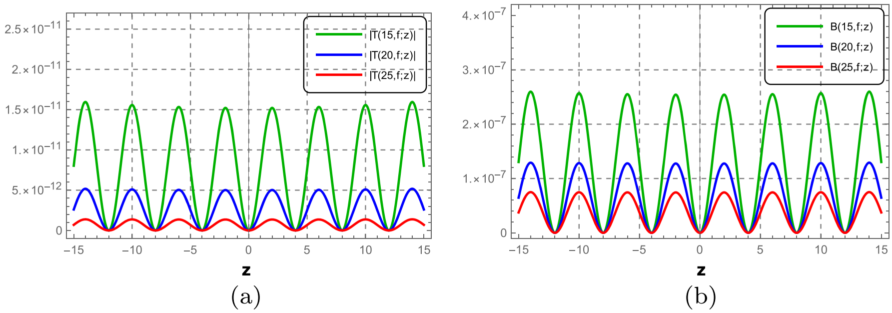


Fig. 4 Illustrations associated with Example 3. (**a**) $|T(N, f; z)|$. (**b**) $\mathcal{B}(N, f; z)$. Here $z \in [-15, 15]$ and $N = 15, 20, 25$

Example 5 Consider the following $B^2_{\pi/2}$-function

$$f(z) = \frac{e^{-i\frac{z^2}{2}} \cos\left(\frac{\pi z}{\sqrt{2}}\right)}{12\pi^2\left(2z^2 - 1\right)}. \tag{5.7}$$

Here we take $a = b = d = 1/\sqrt{2}$. In this example, we compare between the absolute jitter error $|f(z) - f_N(\delta, z)|$ and its associated bound $\mathcal{B}(N, f; z) + \mathcal{J}(\delta, f)$. Table 5 and Fig. 6 show the results when $N = 20$ and $\delta = 6 \times 10^{-6}, 4 \times 10^{-6}, 2 \times 10^{-6}$ respectively. We notice that the precision increases when δ decreases.

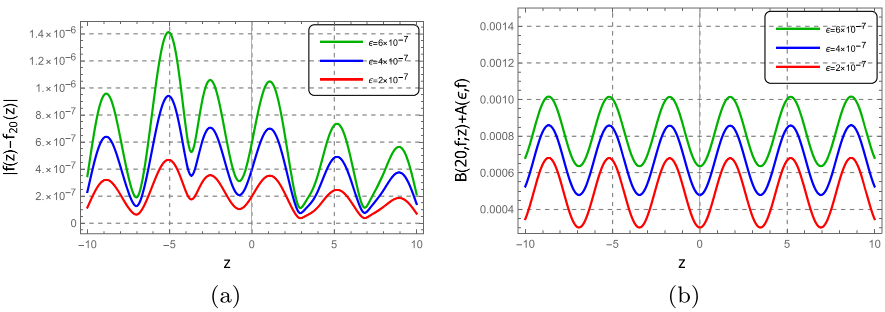


Fig. 5 Illustrative figures for $\left|f(z) - \widetilde{f}_{20}(z)\right|$ (**a**), $\mathcal{B}(20, f; z) + \mathcal{A}(\varepsilon, f)$ (**b**), where $z \in [-10, 10]$ and $\varepsilon = 6 \times 10^{-7}, 4 \times 10^{-7}, 2 \times 10^{-7}$ respectively

Table 4 Exact error $\left|f(z) - \widetilde{f}_{20}(z)\right|$ and its bound $\mathcal{B}(20, f; z) + \mathcal{A}(\varepsilon, f)$ of Example 4

z_i	$\varepsilon = 6 \times 10^{-7}$		$\varepsilon = 4 \times 10^{-7}$		$\varepsilon = 2 \times 10^{-7}$	
	Absolute error	Bound	Absolute error	Bound	Absolute error	Bound
0.5	8.90998×10^{-7}	7.08149×10^{-4}	5.94278×10^{-7}	5.51429×10^{-4}	2.97558×10^{-7}	3.73393×10^{-4}
1.0	1.04496×10^{-6}	8.70263×10^{-4}	6.97544×10^{-7}	7.13543×10^{-4}	3.50127×10^{-7}	5.35508×10^{-4}
1.5	9.54058×10^{-7}	9.97463×10^{-4}	6.37431×10^{-7}	8.40743×10^{-4}	3.20805×10^{-7}	6.62707×10^{-4}
2.0	6.49078×10^{-7}	9.92115×10^{-4}	4.34079×10^{-7}	8.35395×10^{-4}	2.19080×10^{-7}	6.57359×10^{-4}
2.5	2.83482×10^{-7}	8.58309×10^{-4}	1.89736×10^{-7}	7.01589×10^{-4}	9.59923×10^{-8}	5.23554×10^{-4}

Table 5 Exact error $|f(z) - f_{20}(\delta, z)|$ and its bound $\mathcal{B}(20, f; z) + \mathcal{J}(\delta, f)$ of Example 5

z_i	$\delta = 6 \times 10^{-6}$ Absolute error	Bound	$\delta = 4 \times 10^{-6}$ Absolute error	Bound	$\delta = 2 \times 10^{-6}$ Absolute error	Bound
0.6	3.54999×10^{-8}	2.02077×10^{-3}	2.45111×10^{-8}	1.48372×10^{-4}	1.35869×10^{-8}	8.89849×10^{-4}
1.2	1.79674×10^{-8}	1.94509×10^{-3}	1.21733×10^{-8}	1.40805×10^{-3}	6.38538×10^{-9}	8.14174×10^{-4}
1.8	1.51792×10^{-8}	1.98241×10^{-3}	1.07248×10^{-8}	1.44536×10^{-3}	6.29761×10^{-9}	8.51489×10^{-4}
2.4	7.56112×10^{-9}	1.99197×10^{-3}	5.81799×10^{-9}	1.45493×10^{-3}	4.09304×10^{-9}	8.61055×10^{-4}
3.0	1.09067×10^{-9}	1.93777×10^{-3}	7.74911×10^{-10}	1.40073×10^{-3}	5.38888×10^{-10}	8.06855×10^{-4}

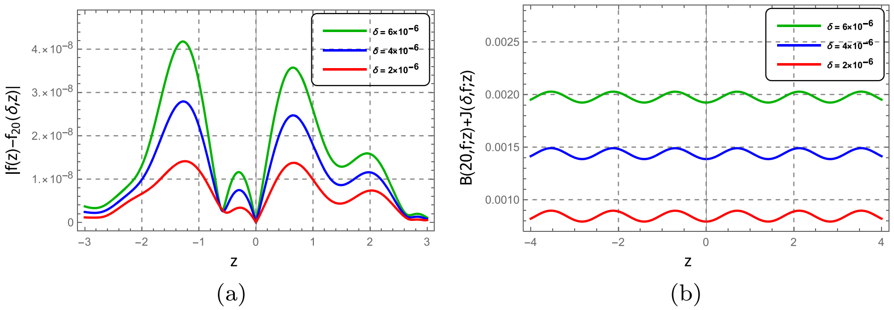


Fig. 6 Illustrations associated with Example 5. $|f(z) - f_{20}(\delta, z)|$ (**a**), $\mathcal{B}(20, f; z) + \mathcal{J}(\delta, f; z)$ (**b**), where $z \in [-3, 3]$ and $\delta = 6 \times 10^{-6}, 4 \times 10^{-6}, 2 \times 10^{-6}$ respectively

6 Conclusions

In this paper we investigated the convergence and the error analysis of the Hermite sampling theorem associated with the LCT. We first established a rigorous convergence analysis on $\mathbb{C}$ and $\mathbb{R}$. Both harmonic analysis and complex analysis approaches are utilized. The estimation of the remainder is rigorously derived as well, in both pointwise and uniform settings. It is worthwhile to emphasis that the convergence rate is as slow as $O(1/\sqrt{N})$ as $N \to \infty$ for band-limited functions. However, as in the classical case it will be improved if the band-limited function satisfies better smoothness conditions. Therefore there is a theoretical demand to derive regularized techniques to fasten the convergence rate. This can be established by incorporating the sampling representation with smoothing convergence kernels. The paper closes a gap in the convergence analysis associated with Hermite sampling representation for entire functions of exponential type.

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Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.

References

1. Al-Abdi, I.A.: Approximate hermite interpolations for compactly supported linear canonical transforms. Comput. Math. Methods, **2022**, Article ID 5243466, (2022). https://doi.org/10.1155/2022/5243466
2. Annaby, M.H., Asharabi, R.M.: Truncation, amplitude and jitter errors on ℝ for sampling series derivatives. J. Approx. Theory **163**, 336–362 (2011). https://doi.org/10.1016/j.jat.2010.10.001
3. Annaby, M.H., Asharabi, R.M.: Error analysis associated with uniform Hermite interpolations of bandlimited functions. J. Korean Math. Soc. **47**(6), 1299–1316 (2010). https://doi.org/10.4134/JKMS.2010.47.6.1299
4. Annaby, M.H., Asharabi, R.M.: Error estimates associated with sampling series of the linear canonical transforms. IMA J. Numer. Anal. **35**, 931–946 (2015). https://doi.org/10.1093/imanum/dru020
5. Annaby, M.H., Asharabi, R.M.: Derivative sampling expansions for the linear canonical transform: convergence and error analysis. J. Comput. Math. **37**(3), 431–446 (2019). https://doi.org/10.4208/jcm.1806-m2017-0215
6. Butzer, P.L., Higgins, J.R., Stens, R.L.: Sampling Theory of Signal Analysis. In: Development of Mathematics 1950–2000, Birkhäuser, Basel, 193–234 (2000)
7. Chandrasekharan, K.: Classical Fourier Transforms. Springer-Verlag, Berlin (1989)
8. Guo, Y., Li, B.-Z.: Blind image watermarking method based on linear canonical wavelet transform and QR decomposition. IET Image Process. **10**(10), 773–786 (2016). https://doi.org/10.1049/iet-ipr.2015.0818
9. Healy, J.J., Kutay, M.A., Ozaktas, H.M., Sheridan, J.T.: Linear Canonical Transforms: Theory and Applications, 3rd edn. Springer-Verlag, New York, NY, USA (2016)
10. Higgins, J.R.: Sampling theorems and the contour integral method. Appl. Anal. **41**, 155–169 (1991). https://doi.org/10.1080/00036819108840021
11. Higgins, J.R.: Sampling Theory in Fourier and Signal Analysis Foundations. Oxford University Press, Oxford (1996)
12. Huang, Z.-J., Cheng, S., Gonga, L.-H., Zhou, N.-R.: Nonlinear optical multi-image encryption scheme with two-dimensional linear canonical transform. Opt. Lasers Eng. **124**, (2020). https://doi.org/10.1016/j.optlaseng.2019.105821
13. Jagerman, D.: Bounds for truncation error of the sampling expansion. SIAM J. Appl. Math. **14**, 714–723 (1966). https://doi.org/10.1137/0114060
14. Jagerman, D., Fogel, L.: Some general aspects of the sampling theorem. IRE Trans. Inform. Theory **2**, 139–146 (1956). https://doi.org/10.1109/TIT.1956.1056821
15. Kutay, M.A., Ozaktas, H.M., Rodrigo, J.A.: Optical Implementation of Linear Canonical Transforms. In: Healy, J.J., et al. (eds.) Linear Canonical Transforms, pp. 179–194. Springer Series in Optical Sciences, Springer, New York (2016)
16. Lacaze, B.: About sampling for band-limited linear canonical. Signal Process. **91**, 1076–1078 (2011). https://doi.org/10.1016/j.sigpro.2010.09.020
17. Li, H.A., Fang, G.S.: Sampling theorem of Hermite type and aliasing error on the Sobolev class of functions. Front. Math. China **1**, 252–271 (2006). https://doi.org/10.1007/s11464-006-0006-x
18. Li, B.-Z., Tao, R., Wang, Y.: New sampling formulae related to linear canonical transform. Signal Process. **87**(5), 983–990 (2007). https://doi.org/10.1016/j.sigpro.2006.09.008
19. Liu, Y., Kou, K., Ho, I.: New sampling formulae for non-bandlimited signals associated with linear canonical transform and non linear Fourier atoms. Signal Process. **90**(5), 933–945 (2010). https://doi.org/10.1016/j.sigpro.2009.09.030
20. McBride, A., Kerr, F.: On Namias's fractional Fourier transforms. IMA J. Appl. Math. **39**, 159–175 (1987). https://doi.org/10.1093/imamat/39.2.159
21. Moshinsky, M., Quesne, C.: Linear canonical transformations and their unitary representations. J. Math. Phys. **12**(8), 1772–1783 (1971). https://doi.org/10.1063/1.1665805
22. Namias, V.: The fractional order Fourier and its application to quantum mechanics. J. Inst. Math. Appl. **25**, 180–191 (1980). https://doi.org/10.1093/imamat/25.3.241
23. Ozaktas, H.M., Mendlovic, D.: Fractional Fourier optics. J. Opt. Soc. Amer. **A** (12), 743–751 (1995). https://doi.org/10.1364/JOSAA.12.000743
24. Ozaktas, H.M., Mendlovic, D.: Fourier transforms of fractional order and their optical interpretation. Opt. Commun. **101**, 163–169 (1993). https://doi.org/10.1016/0030-4018(93)90359-D

25. Ozaktas, H.M., Zalevsky, Z., Kutay, M.A.: The Fractional Fourier Transform with Applications in Optics and Signal Processing. New York: Wiley (2001)
26. Pei, S.C., Ding, J.J.: Eigenfunctions of linear canonical transform. IEEE Trans. Acoust. Speech Signal Process. **50**, 11–26 (2002). https://doi.org/10.1109/78.972478
27. Ran, Q.W., Tan, L.-Y., Ma, J.: Sampling of bandlimited signals in fractional Fourier transform domain. Circuits Syst. Signal Process. **29**, 459–467 (2010). https://doi.org/10.1007/s00034-010-9155-y
28. Shi, J., Liu, X., Fang, X., Sha, X., Xiang, W., Zhang, Q.: Linear canonical matched filter: theory, design, and applications. IEEE Trans. Signal Process. **66**(24), 6404–6417 (2018). https://doi.org/10.1109/TSP.2018.2877193
29. Shi, J., Liu, X., Yan, F.-G., Song, W.: Error analysis of reconstruction from linear canonical transform-based sampling. IEEE Trans. Signal Process. **66**(7), 1748–1760 (2018). https://doi.org/10.1109/TSP.2018.2793866
30. Stern, A.: Sampling of linear canonical transformed signals. Signal Process. **86**(7), 1421–1425 (2006). https://doi.org/10.1016/j.sigpro.2005.07.031
31. Stern, A.: Sampling of compact signals in offset linear canonical transform domains. Signal, Image, Video Process. **1**(4), 359–367 (2007). https://doi.org/10.1007/s11760-007-0029-0
32. Tao, R., Li, B.-Z., Wang, Y., Aggrey, G.K.: On sampling of band-limited signals associated with the linear canonical transform. IEEE Trans. Signal Process. **56**, 5454–5464 (2008). https://doi.org/10.1109/TSP.2008.929333
33. Tao, R., Qi, L., Wang, Y.: Theory and Applications of the Fractional Fourier Transform, Beijing, China: Tsinghua Univ. Press (2004)
34. Xia, X.: On bandlimited signals with fractional Fourier transform. IEEE Signal Process. Lett. **3**(3), 72–74 (1996). https://doi.org/10.1109/97.481159
35. Zhao, H., Ran, Q.-W., Ma, J., Tan, L.-Y.: On bandlimited signals associated with linear cannonical transform. IEEE Signal Process. Lett. **16**, 343–345 (2009). https://doi.org/10.1109/LSP.2009.2014290
36. Zhao, H., Ran, Q.-W., Tan, L.-T., Ma, J.: Reconstruction of bandlimited signals in linear cannonical transform domain from finite nonuniformly spaced samples. IEEE Signal Process. Lett. **16**, 1047–1050 (2009). https://doi.org/10.1109/LSP.2009.2030098