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A survey of mean-square destabilization of multidimensional linear stochastic differential systems with non-normal drift

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Abstract
Mean-square stability analysis of linear stochastic differential systems obtained perturbing ordinary systems by linear terms driven by independent Wiener processes is investigated. The so obtained stochastic regions are contractions of the asymptotic stability domain of the linear ordinary system. In this work, the mean-square stability regions exact shape is provided by means of necessary and sufficient conditions in terms of the eigenvalues of the drift and the intensities of the noises. Special attention is paid to how different structures of the perturbation affect the mean-square stability of systems with non-normal drifts. In each case, the obtained explicit stability condition reveals the role played by the parameter that controls the non-normality.

Keywords Stochastic differential systems · Linear test system · Mean square stability · Non-normal matrix

Mathematics Subject Classification 2010 60H05 · 65C30 · 65L20

1 Introduction

Stochastic differential equations (SDEs) have been used to reproduce and design the dynamics of evolutionary systems in many scientific areas (see [13] and the references therein). Recent applications in physics, medicine, epidemiology, or finance

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(see, for instance, [5, 7, 10, 14–16, 21, 22]) show a rising interest in the study of SDEs.

The linear stability theory for ordinary differential equations starts from a simple test system, usually of the form

$$d X_t = A X_t \, dt, \tag{1}$$

where A is a constant matrix with different eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{C}$. When all of them lie in the negative half-plane, the solutions tend to zero as t tends to infinity and the equilibrium position of the test system is said to be asymptotically stable. In this case, there exists a non-singular matrix T such that $T^{-1} A T = \mathrm{diag}[\lambda_1, \dots, \lambda_d]$, and the transformation $x = Tz$ uncouples the test system. So, the analysis of stability for multi-dimensional systems can be reduced to the study of scalar test equations. Unlike the deterministic case, the scalar stochastic analysis cannot be directly extended to multi-dimensional systems because in the stochastic linear test equation

$$dX_t = A X_t \, dt + \sum_{k=1}^{m} B_k X_t \, dW_t^k \tag{2}$$

with $B_1, \dots, B_m$ constant matrices, come into play at least two matrices — one for the drift and one or more for the diffusion coefficient — and hence, in general, at most one of them can be assumed to be diagonal. In other words, only in the simultaneous diagonalizable case does the analysis reduce to the study of a scalar SDE (see Komori and Mitsui [12]). Since the reduction to a scalar test equation is not justified, we have to choose test systems that reflect the essence of more general equations but simple enough to allow its analysis and interpretation.

Stability of SDEs plays an important role in the analysis of qualitative behavior of dynamical processes. Different types of stability have been proposed (see [1, 9]). In this paper, we are concerned with the concept of stability in pth mean and, more specific, the case of $p = 2$, called mean-square (MS) stability. Considering the stochastic system (2) as the result of perturbing the ordinary system (1) by means of m independent Wiener processes, we recall in Section 2 that unstable ordinary systems cannot be stabilized in the mean-square sense. And, starting from a stable ordinary system, one can consider the following questions:

(i) How strong can be the perturbation so that the solution remains stable (in the mean-square sense)? Is it possible to calculate a “stability threshold”?
(ii) Is there any linear noise perturbation that does not modify the stability condition of the ordinary system?

For the scalar case, the answers are straightforward, since the (real) ordinary equation $dX_t = \lambda X_t \, dt$ is asymptotically stable when $\lambda < 0$, and the SDE

$$dX_t = \lambda X_t \, dt + \sigma X_t \, dW_t$$

obtained introducing a multiplicative noise of intensity $\sigma \in \mathbb{R}$ is MS-stable if and only if $2\lambda + \sigma^2 < 0$.

When the matrix A in (1) is non-normal, some features of the dynamic of the system appear. The effects of non-normality in deterministic systems have been extensively studied (see [19, 20] and the references therein), but only few works analyze the stability behavior of stochastic systems obtained perturbing such non-normal systems by a multiplicative noise. To be concrete, for a two-dimensional system

$$dX_t = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} X_t \, dt \tag{3}$$

with $\lambda < 0$, $b \neq 0$, the considered stochastic perturbations, all of them driven by a single multiplicative noise term, resulted in linear SDEs of the form $dX_t = A\, X_t\, dt + B\, X_t\, dW_t$. Higham and Mao [11] showed that for $\lambda = -1$, the orthogonal perturbation given by

$$B = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$$

destabilizes the system for b sufficiently large. Later Buckwar and Kelly [3] presented an inequality relating the parameters as the necessary and sufficient condition to hold MS-stability. Finally, Tocino and Senosiaín [18] gave an equivalent but notably simpler condition. On the other hand, in [3], the diagonal perturbation given by

$$B = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

was used to carry on a similar analysis. The obtained result was slighted generalized in [18], where the terms λ in the diagonal of A were replaced by different values λ_1, λ_2. Finally, in [18], the lateral noise given by

$$B = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}$$

was considered to complete a MS-stability analysis. In the present paper (Section 3), we will use our former results in [17] to extend the MS-stability analysis to two-dimensional stochastic test systems of the form (2) obtained perturbing (3) by multiplicative noise terms driven by independent Wiener processes. We shall see how the different structures of the perturbations interact and affect to the MS-stability of the solution. The analysis is completed in Section 4, where a necessary and sufficient condition for the MS-stability of a three dimensional non-normal test system is presented.

2 Preliminaries

In this section, we will introduce some notation and basic definitions for a later use. Consider the d-dimensional Itô stochastic differential equation

$$\begin{aligned} dX_t &= f(t, X_t)\, dt + \sum_{k=1}^{m} g_k(t, X_t)\, dW_t^k, \\ X_{t_0} &= x_0, \end{aligned} \tag{4}$$

where x_0 is a constant vector; $W_t^1 \dots W_t^m$ are the components of an m-dimensional standard Wiener process, and the coefficients f and g_k, $k = 1, 2, \dots, m$, are $\mathbb{R}^d$

valued functions defined for $(t, x) \in [t_0, T] \times \mathbb{R}^d$, continuous with respect to t and satisfy the assumptions of the existence and uniqueness theorem (see Arnold [1]). Let us also assume that $f(t, 0) = 0$ and $g_k(t, 0) = 0$ for $t \geq t_0$. Notice that this implies that the process $X_t \equiv 0$, called the *equilibrium position*, is the (unique) solution of (4) with $x_0 = 0$.

Definition 1 [1, 9] The equilibrium position is said to be stable in pth mean (where $p > 0$) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{t_0 \leq t < \infty} E\|X_t(x_0)\|^p \leq \epsilon$$

for $\|x_0\| \leq \delta$.

The equilibrium position is said to be asymptotically stable in pth mean if it is stable in pth mean and if for all x_0 in a neighborhood of $x = 0$

$$\lim_{t \to \infty} E\|X_t(x_0)\|^p = 0.$$

The equilibrium position possess a stable expectation value $m_t = E[X_t(x_0)]$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $\|x_0\| \leq \delta$

$$\sup_{t_0 \leq t < \infty} \|E\, X_t(x_0)\| \leq \epsilon.$$

The equilibrium position possess a stable second moment $P(t) = E[X_t(x_0)\, X_t(x_0)']$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $\|x_0\| \leq \delta$

$$\sup_{t_0 \leq t < \infty} \|E\, X_t(x_0)\, X_t(x_0)'\| \leq \epsilon.$$

Stability in pth mean with $p = 1$ is named *stability in mean* and with $p = 2$ is called *stability in mean square* or *MS-stability*. The relations between the above stability concepts can be summarized as:

Proposition 1 [1] *(a) Stability in mean square is equivalent to stability of the second moment. (b) Stability in mean square implies stability in mean. (c) Stability in mean implies stability of the expectation value.*

Consider now the linear stochastic differential (2) with A, B_k real constant matrices. Recall that if X_t is the solution of (2), then its expectation $m_t = \mathrm{E}[X_t]$ is the unique solution of $\dot{m}_t = A\, m_t$. Then the stability of the expectation value for the SDE (2) reduces to the asymptotic stability of the ordinary system (1), i.e., see, e.g., [6], to the condition that all the eigenvalues of A lie in the left half-plane or, in symbols, $\nu(A) < 0$ where $\nu(A)$ stands for the spectral abscissa of A. The set of matrices A that provide asymptotically stable ordinary systems

$$\mathcal{D}_{DE} = \{A \in \mathbb{R}^{d \times d} \;:\; \nu(A) < 0\}$$

constitute the domain of stability of the (1).

Following Arnold [1], the second moment $P(t) = E[X_t X_t'] = (p_{ij}(t))$ of the solution of (2) satisfies the equation

$$\frac{dP(t)}{dt} = AP(t) + P(t)A' + \sum_{k=1}^{m} B_k P(t) B_k'. \tag{5}$$

Since $P(t)$ is symmetric, (5) has superfluous equations and can be reduced to a linear system of $d(d+1)/2$ differential equations of the form

$$\frac{dY}{dt} = \mathcal{M} Y, \tag{6}$$

where the components of the vector Y are the different $p_{ij} = E[X_t^i X_t^j]$. From Proposition 1, the MS-stability of the equilibrium position of (2) is identical to the stability of the second moment $P(t)$, i.e., to the stability of the trivial solution of the ordinary differential system (5) or its equivalent (6). We conclude that the stochastic test system (2) is asymptotically MS-stable if and only if $\nu(\mathcal{M}) < 0$. If A, B_k are the set of matrices that provide MS-asymptotically stable systems, then

$$\mathcal{D}_{SDE} = \{A, B_k \in \mathbb{R}^{d\times d} : \ \nu(\mathcal{M}) < 0\}$$

will be called the domain of MS-stability of the test SDE (2).

From Proposition 1, the asymptotic stability of the ordinary system (1), equivalent to the stability of the expectation value of (2), is a necessary condition for the MS-stability of the stochastic linear system (2):

$$\nu(\mathcal{M}) < 0 \implies \nu(A) < 0.$$

Then, only linear systems (2) with $\nu(A) < 0$ can be MS-stable.

Remark 1 Since $\mathcal{M}$ is a real matrix, using the Routh-Hurwitz criterion (see [8]), the condition $\nu(\mathcal{M}) < 0$ can be verified in terms of the coefficients of $\mathcal{M}$, without an explicit computation of its eigenvalues. For example, for a linear bi-dimensional system

$$d\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dt + \sum_{k=1}^{m} \begin{pmatrix} b_{11}^k & b_{12}^k \\ b_{21}^k & b_{22}^k \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dW_t^k ,$$

the differential equation in (6) can be written

$$\begin{pmatrix} \frac{dp_{11}(t)}{dt} \\ \frac{dp_{22}(t)}{dt} \\ \frac{dp_{12}(t)}{dt} \end{pmatrix} = \mathcal{M} \begin{pmatrix} p_{11}(t) \\ p_{22}(t) \\ p_{12}(t) \end{pmatrix},$$

where

$$\mathcal{M}=\begin{pmatrix} 2a_{11}+\sum\limits_{k=1}^{m}\left(b_{11}^{k}\right)^{2} & \sum\limits_{k=1}^{m}\left(b_{12}^{k}\right)^{2} & 2\left(a_{12}+\sum\limits_{k=1}^{m}b_{11}^{k}b_{12}^{k}\right) \\ \sum\limits_{k=1}^{m}\left(b_{21}^{k}\right)^{2} & 2a_{22}+\sum\limits_{k=1}^{m}\left(b_{22}^{k}\right)^{2} & 2\left(a_{21}+\sum\limits_{k=1}^{m}b_{21}^{k}b_{22}^{k}\right) \\ a_{21}+\sum\limits_{k=1}^{m}b_{11}^{k}b_{21}^{k} & a_{12}+\sum\limits_{k=1}^{m}b_{12}^{k}b_{22}^{k} & a_{11}+a_{22}+\sum\limits_{k=1}^{m}b_{12}^{k}b_{21}^{k}+b_{11}^{k}b_{22}^{k} \end{pmatrix}. \tag{7}$$

In this case, if

$$P(x)=\det\ (xI-\mathcal{M})=x^{3}+a_{1}x^{2}+a_{2}x+a_{3} \tag{8}$$

is the characteristic polynomial of the matrix $\mathcal{M}$, the Routh-Hurwitz criterion reduces to the fulfilling of the conditions

$$a_{1}a_{2}-a_{3}>0,\ \ a_{1}>0,\ \ a_{3}>0. \tag{9}$$

3 MS-stability analysis of two-dimensional test equations

We start with the two-dimensional linear system (3), asymptotically stable when $\lambda<0$. The parameter b determines the non-normality degree of the system. In this section, we shall add to the system stochastic perturbations with different structures by means of independent Wiener processes. The analysis of each considered SDE leads to a single inequality (in terms of λ, b and the intensity of the noises) as the necessary and sufficient condition for its MS-stability.

3.1 Independent diagonal noises

Consider the test system (3) perturbed by m independent linear terms determined by diagonal matrices with different intensities:

$$d\begin{pmatrix}X^{1}(t)\\X^{2}(t)\end{pmatrix}=\begin{pmatrix}\lambda & b\\0 & \lambda\end{pmatrix}\begin{pmatrix}X^{1}(t)\\X^{2}(t)\end{pmatrix}dt+\sum_{k=1}^{m}\begin{pmatrix}\sigma_{k} & 0\\0 & \sigma_{k}\end{pmatrix}\begin{pmatrix}X^{1}(t)\\X^{2}(t)\end{pmatrix}dW_{t}^{k}\quad t>0, \tag{10}$$

with $\lambda,\sigma_{k},b\in\mathbb{R}$. In this case, the matrix in (7) is

$$\mathcal{M}_{d}=\begin{pmatrix} 2\lambda+\sum\limits_{k=0}^{m}\sigma_{k}^{2} & 0 & 2b \\ 0 & 2\lambda+\sum\limits_{k=0}^{m}\sigma_{k}^{2} & 0 \\ 0 & b & 2\lambda+\sum\limits_{k=0}^{m}\sigma_{k}^{2} \end{pmatrix}$$

Since the spectrum of $\mathcal{M}_{d}$ reduces to the point $2\lambda+\sum\limits_{k=0}^{m}\sigma_{k}^{2}$, we have

Proposition 2 *The two-dimensional test system* (10) *is asymptotically MS-stable if and only if* $2\lambda + \sigma^2 < 0$, *where* $\sigma^2 = \sum_{k=0}^{m} \sigma_k^2$.

Remark 2 This statement generalizes some results in the literature: the particular selection $\sigma_k = \pm\sigma/\sqrt{m},\ \ k = 1, 2, \ldots, m$, for a given σ was considered in [2] for diagonal drift coefficient ($b = 0$) and in [3] for a single noise ($m = 1$).

3.2 Independent orthogonal noises

Consider now the stochastic test system obtained adding to (3) independent linear noises with different intensities determined by matrices whose structure is orthogonal to the flow:

$$d\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dt + \sum_{k=1}^{m} \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dW_t^k \quad t > 0, \tag{11}$$

with $\lambda, \sigma_k, b \in \mathbb{R}$. In this case, the matrix in (7) is

$$\mathcal{M}_o = \begin{pmatrix} 2\lambda & \sum\limits_{k=0}^{m} \sigma_k^2 & 2b \\ \sum\limits_{k=0}^{m} \sigma_k^2 & 2\lambda & 0 \\ 0 & b & 2\lambda - \sum\limits_{k=0}^{m} \sigma_k^2 \end{pmatrix}$$

with characteristic polynomial as in (8) with

$$\begin{aligned} a_1 &= \sum_{k=1}^{m} \sigma_k^2 - 6\lambda, \\ a_2 &= \left(2\lambda - \sum_{k=1}^{m} \sigma_k^2\right)\left(6\lambda + \sum_{k=1}^{m} \sigma_k^2\right), \\ a_3 &= -2b^2 \sum_{k=1}^{m} \sigma_k^2 - \left(\sum_{k=1}^{m} \sigma_k^2 - 2\lambda\right)^2 \left(\sum_{k=1}^{m} \sigma_k^2 + 2\lambda\right), \end{aligned}$$

and the Routh-Hurwitz conditions (9) for its stability become

$$32\lambda^3 - (b^2 + 16\lambda^2) \sum_{k=1}^{m} \sigma_k^2 < 0, \tag{12}$$

$$\sum_{k=1}^{m} \sigma_k^2 - 6\lambda > 0, \tag{13}$$

$$2b^2 \sum_{k=1}^{m} \sigma_k^2 + \left(\sum_{k=1}^{m} \sigma_k^2 - 2\lambda\right)^2 \left(\sum_{k=1}^{m} \sigma_k^2 + 2\lambda\right) < 0. \tag{14}$$

Notice that (14) implies $\lambda < 0$ and from here (12) and (13) hold. We have:

Theorem 3 *The two-dimensional test system* (11) *is asymptotically MS-stable if and only if*

$$2b^2\sigma^2 + (\sigma^2 - 2\lambda)^2(\sigma^2 + 2\lambda) < 0$$

where $\sigma^2 = \sum_{k=1}^m \sigma_k^2$.

Remark 3 This statement generalizes some results in the literature: If $b = 0$, the stability condition reduces to $\sigma^2 + 2\lambda < 0$ (see [2] for the particular case $\sigma_k = \pm\sigma/\sqrt{m},\ k = 1, 2, \ldots, m$). On the other hand, for $m = 1$, the condition of the theorem was obtained in [17] (see also [3]).

Using the condition in Theorem 3, a graphical representation of the domain of MS-stability of the SDE (11) can be given in $\mathbb{R}^3_{(\lambda,b,\sigma^2)}$ (see Fig. 1).

3.3 Diagonal and orthogonal noises

Consider now a linear test system with two independent noises with structures orthogonal to each other (see [4]) with respective intensities σ and ϵ,

$$d\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dW_t^1 + \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dW_t^2 \tag{15}$$

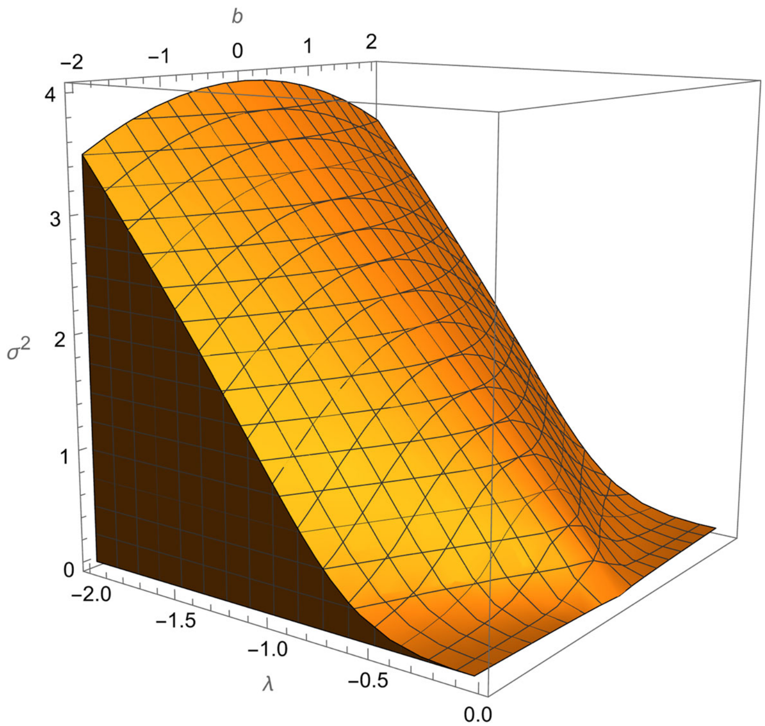

Fig. 1 MS-stability region of the system (11) with $\sigma_k = \sigma/\sqrt{m}$

where $\lambda, \sigma, \varepsilon, b \in \mathbb{R}$. In this case, the matrix in (7) is

$$\mathcal{M}_{do} = \begin{pmatrix} 2\lambda + \sigma^2 & \varepsilon^2 & 2b \\ \varepsilon^2 & 2\lambda + \varepsilon^2 & 0 \\ 0 & b & 2\lambda + \sigma^2 + \varepsilon^2 \end{pmatrix},$$

with characteristic polynomial (8) with

$$\begin{aligned} a_1 &= \varepsilon^2 - 3(2\lambda + \sigma^2), \\ a_2 &= \left(\sigma^2 - \varepsilon^2 + 2\lambda\right)\left(3\sigma^2 + \varepsilon^2 + 6\lambda\right), \\ a_3 &= -2b^2\varepsilon^2 - \left(\sigma^2 + \varepsilon^2 + 2\lambda\right)\left(\sigma^2 - \varepsilon^2 + 2\lambda\right)^2. \end{aligned}$$

Then the Routh-Hurwitz conditions (9) for the stability of $\mathcal{M}_{do}$ become

$$b^2\varepsilon^2 + 4(\varepsilon^2 - 2\lambda - \sigma^2)(2\lambda + \sigma^2)^2 > 0, \tag{16}$$

$$3(2\lambda + \sigma^2) - \varepsilon^2 < 0, \tag{17}$$

$$2b^2\varepsilon^2 + \left(\sigma^2 + \varepsilon^2 + 2\lambda\right)\left(\sigma^2 - \varepsilon^2 + 2\lambda\right)^2 < 0. \tag{18}$$

Notice that (18) implies $\sigma^2 + \varepsilon^2 + 2\lambda < 0$. From here, a straightforward computation leads to $\varepsilon^2 - 3(2\lambda + \sigma^2) > 0$ and $\varepsilon^2 - 2\lambda - \sigma^2 > 0$. Then condition (18) implies (16)–(17). This proves

Theorem 4 *The two-dimensional test system* (15) *is asymptotically MS-stable if and only if*

$$2b^2\varepsilon^2 + \left(\sigma^2 + \varepsilon^2 + 2\lambda\right)\left(\sigma^2 - \varepsilon^2 + 2\lambda\right)^2 < 0.$$

Remark 4 This result is new. Notice that $\lambda < 0$, $2\lambda + \sigma^2 < 0$, and $2\lambda + \varepsilon^2 < 0$ are necessary conditions for the MS-stability of (15). In [2], the particular test system (15) with $b = 0$ was discussed and the condition $2\lambda + \sigma^2 + \varepsilon^2 < 0$ was shown. Notice that in this case for equal normalized intensities $\sigma/\sqrt{2}$ for both noises, the condition becomes $2\lambda + \sigma^2 < 0$.

Using the condition obtained in Theorem 4, a geometrical representation of the MS-stability domain of the system (15) with $\lambda = -1$ can be plotted in $\mathbb{R}^3_{(b,\sigma^2,\varepsilon^2)}$ (see Fig. 2).

3.4 Two lateral noises

Consider the stochastic test system obtained adding two non commutative lateral noise terms

$$d\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dW_t^1 + \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} dW_t^2 \tag{19}$$

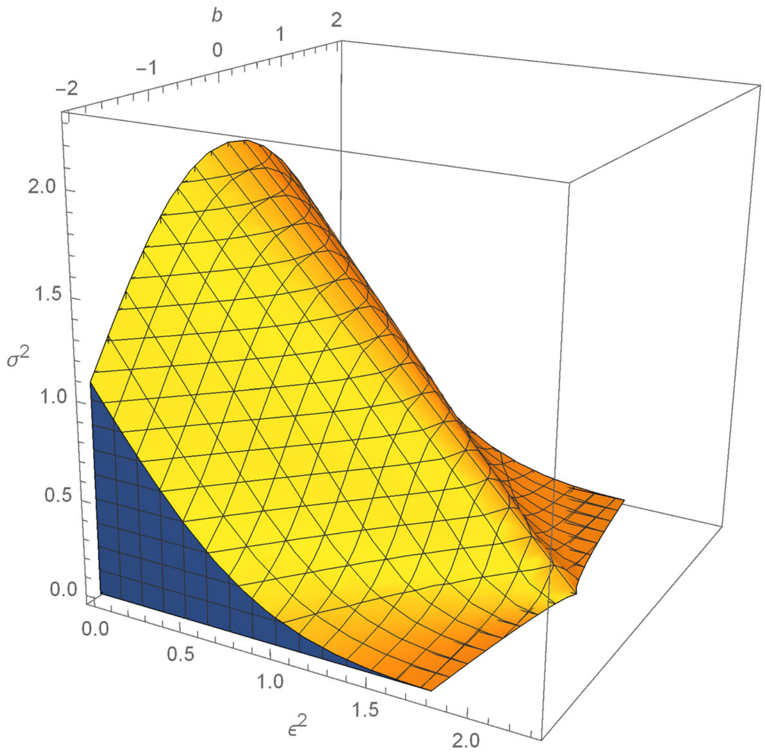

Fig. 2 MS-stability region of the system (15) with $\lambda = -1$

with $\lambda, \sigma, \varepsilon, b \in \mathbb{R}$. In this case, the matrix in (7) is

$$\mathcal{M}_{ll} = \begin{pmatrix} 2\lambda & \varepsilon^2 & 2b \\ \sigma^2 & 2\lambda & 0 \\ 0 & b & 2\lambda \end{pmatrix}$$

with characteristic polynomial (8) with

$$\begin{aligned} a_1 &= -6\lambda, \\ a_2 &= 12\lambda^2 - \varepsilon^2\sigma^2, \\ a_3 &= -4\lambda^3 - b^2\sigma^2 + \varepsilon^2\sigma^2\lambda, \end{aligned}$$

and the Routh-Hurwitz conditions (9) for its stability become

$$32\lambda^3 - (b^2 + 2\varepsilon^2\lambda)\sigma^2 < 0, \tag{20}$$

$$\lambda < 0, \tag{21}$$

$$4\lambda^3 + b^2\sigma^2 - \varepsilon^2\sigma^2\lambda < 0. \tag{22}$$

Since (21)–(22) imply (20), we conclude:

Theorem 5 *The two-dimensional test system* (19) *with* $\lambda < 0$ *is asymptotically MS-stable if and only if:*

$$b^2\sigma^2 + \lambda\left(4\lambda^2 - \varepsilon^2\sigma^2\right) < 0.$$

Remark 5 This result is new. The lateral noises have different effects on the MS-stability: If $\varepsilon = 0$, the MS-stability condition reduces to $b^2\sigma^2 + 4\lambda^3 < 0$ (see [18]), whereas if $\sigma = 0$, the MS-stability condition becomes $\lambda < 0$, i.e., the stability of the (non-normal) ordinary system does not vary if only a perturbation determined by a diffusion of the form

$$\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dW_t \tag{23}$$

acts. From the theorem, when $\sigma\varepsilon \neq 0$, both perturbations and their intensities have an effect on the MS-stability of the stochastic system, even when $b = 0$. This last case is remarkable, because the action of any single lateral noise has no effect on the stability of the original equation ($\lambda < 0$), whereas the simultaneous action has a "multiplicative" effect ($2\lambda + |\sigma\varepsilon| < 0$) .

Using the condition obtained in Theorem 5, a geometrical representation of the MS-stability region of the system (19) with $\lambda = -1$ in $\mathbb{R}^3_{(b,\sigma^2,\varepsilon^2)}$ is shown in Fig. 3.

4 MS-stability analysis of a non-normal three-dimensional test equation

Following the destabilizing perturbation structures presented in [2] (see also [3]), we consider the three-dimensional test system with non-normal drift

$$\begin{aligned} d\begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix} &= \begin{pmatrix} \lambda & b & 0 \\ 0 & \lambda & b \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix} dt + \begin{pmatrix} 0 & \sigma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix} dW_t^1 \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix} dW_t^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix} dW_t^3, \end{aligned} \tag{24}$$

with parameters $\lambda, \sigma, b \in \mathbb{R}$. In this case, the linear system in (6) can be written

$$\begin{pmatrix} \frac{dp_{11}(t)}{dt} \\ \frac{dp_{22}(t)}{dt} \\ \frac{dp_{33}(t)}{dt} \\ \frac{dp_{12}(t)}{dt} \\ \frac{dp_{13}(t)}{dt} \\ \frac{dp_{23}(t)}{dt} \end{pmatrix} = \begin{pmatrix} 2\lambda & \sigma^2 & 0 & 2b & 0 & 0 \\ 0 & 2\lambda & \sigma^2 & 0 & 0 & 2b \\ \sigma^2 & 0 & 2\lambda & 0 & 0 & 0 \\ 0 & b & 0 & 2\lambda & b & 0 \\ 0 & 0 & 0 & 0 & 2\lambda & b \\ 0 & 0 & b & 0 & 0 & 2\lambda \end{pmatrix} \begin{pmatrix} p_{11}(t) \\ p_{22}(t) \\ p_{33}(t) \\ p_{12}(t) \\ p_{13}(t) \\ p_{23}(t) \end{pmatrix} \tag{25}$$

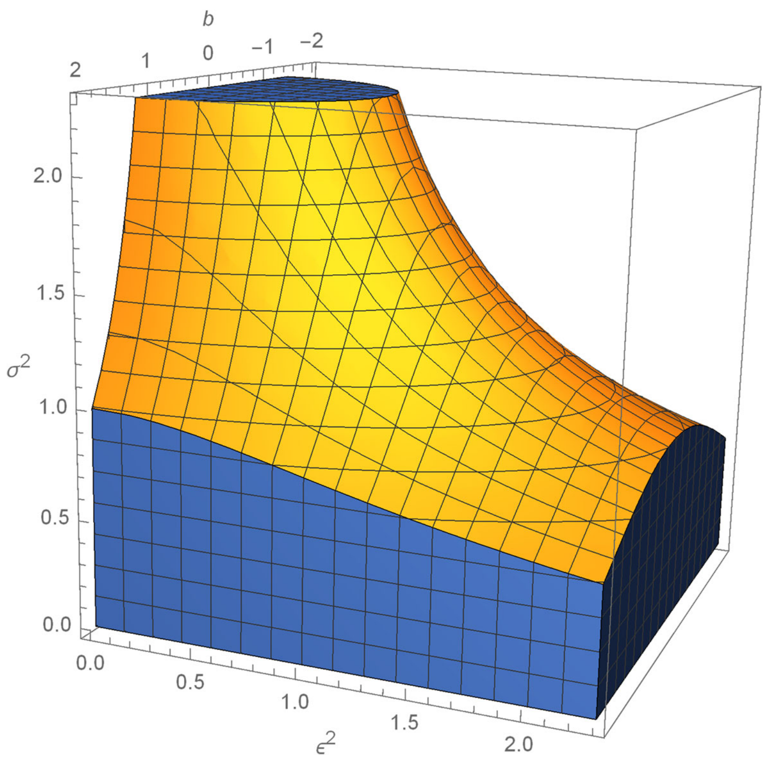

Fig. 3 MS-stability region of the system (19) with $\lambda = -1$

where $p_{ij} = \mathrm{E}\left[X^i(t)X^j(t)\right]$. The characteristic polynomial of the matrix in (25) is $P(x) = (x - 2\lambda)\, Q(x)$ where
$Q(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$ with

$$\begin{array}{l} a_1 = -10\lambda \\ a_2 = 40\lambda^2 \\ a_3 = -(\sigma^6 + 80\lambda^3) \\ a_4 = -4b^2\sigma^4 + 4\lambda(\sigma^6 + 20\lambda^3) \\ a_5 = -2\left(3b^4\sigma^2 - 4b^2\sigma^4\lambda + 2\lambda^2(\sigma^6 + 8\lambda^3)\right). \end{array} .$$

The MS-stability analysis of system (24) is reduced to study the sign of the spectral abscissa of the matrix in (25). Note that 2λ is an eigenvalue of $P(x)$. On the other hand, the Routh-Hurwitz conditions for $Q(x)$,

$$a_1 > 0, a_5 > 0, a_3 < a_1a_2, a_3^2 + a_1^2a_4 < a_1(a_2a_3 + a_5),$$
$$a_4(-a_1a_2a_3 + a_3^2 + a_1^2a_4) + a_1(a_2^2 - 2a_4)a_5 + a_5^2 < a_2a_3a_5,$$

become

$$\begin{aligned}\lambda &< 0, && (26)\\ 3b^4\sigma^2 - 4b^2\sigma^4\lambda + 2\lambda^2(\sigma^6 + 8\lambda^3) &< 0, && (27)\\ 320\lambda^3 - \sigma^6 &< 0, && (28)\\ \sigma^6(\sigma^6 + 120\lambda^3) - 20\lambda(3b^4\sigma^2 + 16b^2\sigma^4\lambda + 896\lambda^5) &< 0, && (29)\\ 9b^8\sigma^4 + 96b^6\sigma^6\lambda + \lambda(\sigma^6 - 64\lambda^3)(\sigma^6 + 64\lambda^3)^2 & && \\ +88b^4\sigma^2\lambda^2(\sigma^6 + 192\lambda^3) - b^2\sigma^4(\sigma^{12} + 256\sigma^6\lambda^3 + 12288\lambda^6) &< 0. && (30)\end{aligned}$$

Let us see that (26)–(27) imply the rest of conditions. Condition (28) is obvious from (26). On the other hand, from (26) to (27), we have

$$3b^4\sigma^2 + 2\lambda^2(\sigma^6 + 8\lambda^3) < 0, \tag{31}$$

and then

$$\sigma^6 + 8\lambda^3 < 0. \tag{32}$$

From (31)

$$9b^8\sigma^4 - 4\lambda^4(\sigma^6 + 8\lambda^3)^2 < 0, \tag{33}$$

and from (32)

$$\sigma^{12} - 64\lambda^6 < 0. \tag{34}$$

Writing the left part of (29) as

$$\sigma^6(\sigma^6 + 8\lambda^3) + 152\sigma^6\lambda^3 - 20\lambda(3b^4\sigma^2 + 2\lambda^2(\sigma^6 + 8\lambda^3)) - \lambda^2(320b^2\sigma^4 + 17600\lambda^4)$$

and the left part of (30) as

$$\begin{aligned} 9b^8\sigma^4 &- 4\lambda^4(\sigma^6 + 8\lambda^3)^2 + \lambda^2(\sigma^6 + 8\lambda^3)(-4032\lambda^5 + 88b^4\sigma^2 - 256b^2\sigma^4\lambda)\\ &- b^2\sigma^{16} + 68\lambda^4(\sigma^{12} - 64\lambda^6) + \lambda\sigma^6(\sigma^{12} + 96b^6) + 16192b^4\sigma^2\lambda^5\\ &- \lambda^6(225280\lambda^4 + 10240b^2\sigma^4), \end{aligned}$$

and using (26), (32)–(34), it is clear that all the addends on the left parts of (29) and (30) are negative. Then (26)–(27) imply (29) and (30).

Theorem 6 *The test system (24) with $\lambda < 0$ is asymptotically MS-stable if and only if*

$$3b^4\sigma^2 - 4b^2\sigma^4\lambda + 2\lambda^2(\sigma^6 + 8\lambda^3) < 0.$$

Up to our knowledge, the condition in Theorem 6 is new. Using it, a graphical representation of the domain of MS-stability of the SDE (24) can be given in $\mathbb{R}^3_{(\lambda,b,\sigma^2)}$ (see Fig. 4).

5 Conclusions

A survey of the stability effects on a linear differential system with the introduction of multiplicative noises with different structures has been presented. The chosen

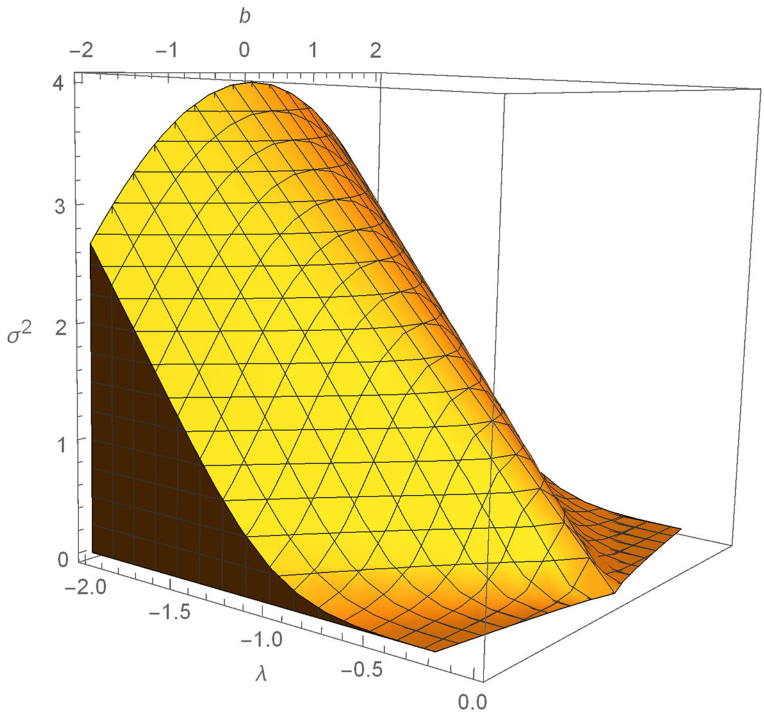

Fig. 4 MS-stability region of the system (24)

two-dimensional and three-dimensional linear test systems, found in the literature, are especially suitable for this study due to their simplicity (the number of involved parameters are manageable and let obtain closed-form equations) as well as due to their variety in the representation of different structures. Table 1 presents a summary of the obtained results. In the first column, the governing drift matrices A are shown; in all cases, the parameter b is a measure of non-normality and $b = 0$ correspond to a diagonal matrix with a single eigenvalue λ. The linear addends in the diffusion part, shown in the second column, are driven by matrices with different structures and contain only a single parameter representing the intensity of the noise.

Notice that, except in two cases, for systems with non-normal drift, the mean-square stability condition depends on b: for fixed $\lambda < 0$, as the value of b increases, the interval of intensities that give mean-square stable solutions shrinks. The exceptions are given (i) by perturbations with diagonal matrices, in which case the condition reduces to

$$2\lambda + \sigma^2 < 0; \tag{35}$$

and (ii) when a single perturbation of the form (23) acts. Note that in this case, the MS-stability condition is $\lambda < 0$, which means that the introduction of this kind of noise alone has no effect on the stability of the ordinary system. It is noticeable (see Theorem 5) that when this noise acts in conjunction with the opposite lateral noise, both perturbations and their intensities affect the MS-stability of the system.

Table 1 MS-stability conditions for the linear test equations (10), (11), (15), (19), and (24)

Drift A	Diffusion $B_1, \ldots, B_m$	MS-stability condition non-normal drift $b \neq 0$		MS-stability condition diagonal drift $b = 0$
$\begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad k = 1, \ldots, m \qquad \sigma^2 = \sum_{k=1}^{m} \sigma_k^2$	$2\lambda + \sigma^2 < 0$		$2\lambda + \sigma^2 < 0$
	$\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad k = 1, \ldots, m$	$2b^2\sigma^2 + (\sigma^2 - 2\lambda)^2(\sigma^2 + 2\lambda) < 0$		$2\lambda + \sigma^2 < 0$
	$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}$	$2b^2\varepsilon^2 + \left(\sigma^2 + \varepsilon^2 + 2\lambda\right)\left(\sigma^2 - \varepsilon^2 + 2\lambda\right)^2 < 0$		$2\lambda + \sigma^2 + \varepsilon^2 < 0$ (**)
	$\begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$	$b^2\sigma^2 + \lambda\left(4\lambda^2 - \varepsilon^2\sigma^2\right) < 0, \; \lambda < 0$		$2\lambda + \lvert\sigma\varepsilon\rvert < 0$ (*)
		$\varepsilon = 0$	$b^2\sigma^2 + 4\lambda^3 < 0$	$\lambda < 0$
		$\sigma = 0$	$\lambda < 0$	$\lambda < 0$
$\begin{pmatrix} \lambda & b & 0 \\ 0 & \lambda & b \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix}$	$3b^4\sigma^2 - 4b^2\sigma^4\lambda + 2\lambda^2(\sigma^6 + 8\lambda^3) < 0$		$2\lambda + \sigma^2 < 0$

For systems with diagonal drift ($b = 0$), a similar feature is observed for both lateral perturbations: when they act alone, the effect on the mean-square stability is null, whereas when they act simultaneously, both intensities affect the MS-stability (see (*) in Table 1). On the other hand, note that in (*) and (**), the condition can be normalized to (35): in (*), take for both intensities the same value $\varepsilon = \sigma$; for (**), take both intensities as $\sigma/\sqrt{2}$.

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Declarations

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