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ORIGINAL PAPER

Bivariate general Appell interpolation problem

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Abstract
In this paper, the solution to a bivariate Appell interpolation problem proposed in a previous work is given. Bounds of the truncation error are considered. Ten new interpolants for real, regular, bivariate functions are constructed. Numerical examples and comparisons with bivariate Bernstein polynomials are considered.

Keywords Interpolation · Polynomial sequences · Appell polynomials · Bivariate Appell sequence

Mathematics Subject Classification (2010) 65D05 · 11B83 · 11B68

1 Introduction

Interpolation theory for real functions is a classic problem both in mathematical and numerical analysis. In fact, on the one hand, it is connected to representability of an analytic function $f(x)$ as a series $\sum_{n=0}^{\infty} c_n \phi_n(x)$, where $\{\phi_n\}_{n\in\mathbb{N}}$ is a prescribed sequence of functions, called basis functions, and c_n are real constants related to the function f [3, 6]. On the other hand, interpolation is fundamental in numerical approximation of functions, numerical quadrature and cubature, boundary value methods, etc. [12, 16, 17, 28]. In an interpolation problem the choice of basic functions, that is the system $\{\phi_n\}_{n\in\mathbb{N}}$, is very important.

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In this paper we will consider the bivariate interpolation problem proposed in [11]. We will give the unique solution expressed in the basis of the so-called bivariate, general Appell polynomial sequences [21]. It can be so formulated: let X be the linear space of bivariate, real continuous functions having continuous partial derivatives of all necessary orders, defined in a domain $D \subset \mathbb{R}^2$. Usually, for simplicity, $D = [0, 1] \times [0, 1]$. We look for, if there exists, the bivariate polynomial $i_n[f]$, $n \in \mathbb{N}$, such that, for any $f \in X$

$$L\left(\frac{\partial^k f}{\partial x^k}\right) = L\left(\frac{\partial^k i_n[f]}{\partial x^k}\right), \qquad k = 0, \ldots, N, \quad N \in \mathbb{N},$$

where L is a linear functional on X such that $L(1) \neq 0$.

Bivariate and, in general, multivariate interpolation has widely employed in the literature (see for example [18, 19, 23, 24, 27, 29, 30] and the references therein).

This paper is organized as follows. In order to make the work as autonomous as possible, Section 2 is a preliminary section. In fact, it includes some known definitions and results that we need in the paper. In Section 3, we find the unique solution of the bivariate interpolation problem mentioned above and give, also, a "complementary" polynomial interpolant. The remainder is analyzed in Section 4 by using the well-known Sard's formula. Then, in Section 5, we give some particular examples of interpolants that doesn't appear in the literature. Section 6 contains numerical examples of bivariate real functions approximations. Comparisons of the new interpolants with the bivariate Bernstein approximation is also given. Finally, in Section 7, we provide some concluding remarks.

2 Preliminaries

Let $A(t)$ and $\phi(y, t)$ be two power series such that

$$A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}, \qquad \phi(y, t) = \sum_{k=0}^{\infty} \varphi_k(y) \frac{t^k}{k!}, \tag{1}$$

with $\alpha_0 = \varphi_0(y) = 1$, $\alpha_k \in \mathbb{R}$, $k \geq 1$ and $\varphi_k(y)$ are real polynomials of degree k in the variable y.

The sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\varphi_k)_{k\in\mathbb{N}}$ generate the elements of the bivariate polynomial sequence $\{r_n\}^b_{n\in\mathbb{N}}$ (the superscript b stands for *bivariate*) satisfying [11]

$$\frac{\partial}{\partial x} r_n(x, y) = n\, r_{n-1}(x, y), \qquad n \geq 1 \tag{2a}$$

$$r_n(0, y) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} \varphi_{n-k}(y), \tag{2b}$$

$$r_0(x, y) = 1. \tag{2c}$$

Remark 1 Differential relation (2a) is known in the literature (see, for example, [1, 11, 16] and references therein), but in different contexts and with different approaches.

It has been proved [11] that $\forall n \in \mathbb{N}$

$$r_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} p_k(x, y) = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{n}{k} \binom{k}{i} \alpha_{k-i} x^i \varphi_{n-k}(y), \qquad (3)$$

where $\{p_k\}_{k\in\mathbb{N}}^b$ is the so-called elementary bivariate Appell sequence whose elements are defined as

$$p_n\,(x, y) = \sum_{k=0}^{n} \binom{n}{k} x^k \varphi_{n-k}(y), \qquad \forall n \in \mathbb{N}. \qquad (4)$$

It is also known that the elements of the bivariate sequences $\{r_k\}_{k\in\mathbb{N}}^b$ and $\{p_k\}_{k\in\mathbb{N}}^b$ have as generating functions

$$F\,(x, y; t) = A(t)e^{xt}\phi(y, t) \quad \text{and} \quad G\,(x, y; t) = e^{xt}\phi(y, t), \qquad (5)$$

respectively, that is,

$$A(t)e^{xt}\phi(y, t) = \sum_{n=0}^{\infty} r_n(x, y)\frac{t^n}{n!}, \qquad e^{xt}\phi(y, t) = \sum_{n=0}^{\infty} p_n(x, y)\frac{t^n}{n!}. \qquad (6)$$

For any $k \in \mathbb{N}$, let $\hat{p}_k(x, y)$ be the following polynomial of degree k

$$\hat{p}_k(x, y) = \sum_{i=0}^{k} \binom{k}{i} (x-1)^i \varphi_{k-i}(y) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} p_i(x, y).$$

Furthermore, let $(\beta_k)_{k\in\mathbb{N}}$ be the numerical sequence defined by

$$\sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} \beta_k = \delta_{n0}, \qquad (7)$$

that is, if $A(t)$ is as in (1), then

$$\frac{1}{A(t)} = \sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!}. \qquad (8)$$

The determinant forms for $r_n(x, y)$ and $p_n(x, y)$ [11], respectively, are fundamental in the sequel:

$$r_n\,(x, y) = (-1)^n \begin{vmatrix} p_0(x, y) & p_1(x, y) & p_2(x, y) & \cdots & p_n(x, y) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n}{1}\beta_{n-1} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n > 0, \qquad (9)$$

$$p_n(x,y) = (-1)^n \begin{vmatrix} \hat{p}_0(x,y) & \hat{p}_1(x,y) & \hat{p}_2(x,y) & \cdots & \hat{p}_n(x,y) \\ 1 & -1 & 1 & \cdots & (-1)^n \\ 0 & 1 & -\binom{2}{1} & \cdots & \binom{n}{1}(-1)^{n-1} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\binom{n}{n-1} \end{vmatrix}, \quad n > 0. \quad (10)$$

Moreover, the following recurrence relation holds:

$$r_0(x,y) = 1, \qquad r_n(x,y) = p_n(x,y) - \sum_{j=0}^{n-1} \binom{n}{j} \beta_{n-j} r_j(x,y), \quad n \geq 1. \quad (11)$$

From (7) we get

$$\alpha_0 = \frac{1}{\beta_0},$$
$$\alpha_k = \frac{(-1)^k}{\beta_0^{k+1}} \begin{vmatrix} \beta_1 & \beta_0 & 0 & \cdots\cdots & 0 \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & 0 \ \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots \ \ddots & \vdots \\ \beta_{k-1} & \binom{k-1}{k-2}\beta_{k-2} & \binom{k-1}{k-3}\beta_{k-3} & \cdots \ \ddots & \beta_0 \\ \beta_k & \binom{k}{k-1}\beta_{k-1} & \binom{k}{k-2}\beta_{k-2} & \cdots\cdots & \binom{k}{1}\beta_1 \end{vmatrix}, \quad k = 1, \ldots, n, \quad (12)$$

and, by symmetry, we can obtain β_k, $k = 0, \ldots, n$.

We define

$$\mathscr{S}_n := \text{span}\{p_0, \ldots, p_n\}, \qquad n \in \mathbb{N},$$

with p_k as in (4) or in (10) for all $k \in \mathbb{N}$. Observe that $\forall n \in \mathbb{N}$, $\{p_0, \ldots, p_n\}$ is a set of $n+1$ linear independent bivariate polynomials.

In the sequel of the paper any fundamental notation and hypothesis introduced so far will be used without references, unless otherwise specified.

3 Bivariate general Appell interpolation

Let X be the linear space of bivariate real functions defined in $D \subset \mathbb{R}^2$ and belonging to $C^N(D)$. Note that $\forall n \in \mathbb{N}$, $\mathscr{S}_n \subset X$.

Let L be a linear functional on X with $L(1) \neq 0$. $\forall p_k \in \{p_\nu\}_{\nu\in\mathbb{N}}^b$, we set

$$L(p_k) = \beta_k, \qquad k = 0, \ldots, n, \quad \beta_0 = 1, \ \beta_k \in \mathbb{R}, \ k \geq 1. \quad (13)$$

We consider the numerical sequence $(\alpha_k)_{k\in\mathbb{N}}$ defined in (7), $(\beta_k)_{k\in\mathbb{N}}$ being as in (13). In addition, we consider the general bivariate Appell polynomial sequence $\{r_n\}_{n\in\mathbb{N}}^b$ defined equivalently as in (2a) or (9).

In this case we say that the bivariate polynomial sequence $\{r_n\}_{n\in\mathbb{N}}^b$ is associated with the functional L and, when necessary, we denote it by $\{r_n^L\}_{n\in\mathbb{N}}^b$.

The generating function of the bivariate Appell polynomial sequence $\{r_n^L\}_{n\in\mathbb{N}}^b$ is connected to the linear functional L by means of the following result.

Proposition 1 *For the generating function $F(x, y; t)$ of the general bivariate Appell sequence $\{r_n^L\}_{n\in\mathbb{N}}^b$ the following identity holds:*

$$F(x, y; t) = \frac{e^{xt}\phi(y, t)}{L^{x,y}(e^{xt}\phi(y, t))}, \tag{14}$$

where $L^{x,y}$ means that the linear functional L acts on $e^{xt}\phi(y, t)$ with respect to the variables x and y.

Proof Relation (14) follows from (5), (8), (13), the second equality in (6) and the linearity of the functional:

$$F(x, y; t) = \frac{e^{xt}\phi(y, t)}{\frac{1}{A(t)}} = \frac{e^{xt}\phi(y, t)}{\sum_{k=0}^{\infty}\beta_k \frac{t^k}{k!}} = \frac{e^{xt}\phi(y, t)}{L^{x,y}\left(\sum_{k=0}^{\infty} p_k(x, y)\frac{t^k}{k!}\right)} = \frac{e^{xt}\phi(y, t)}{L^{x,y}\left(e^{xt}\phi(y, t)\right)}.$$

For every $f \in X$ we look for, if there exists, the bivariate polynomial $i_n[f]$ such that $\forall k \in \mathbb{N}$, $k \leq n$,

$$L\left(\frac{\partial^k}{\partial x^k} i_n[f]\right) = L\left(\frac{\partial^k f}{\partial x^k}\right).$$

We call this problem *the general bivariate Appell interpolation problem*. If, $\forall n \in \mathbb{N}$, $i_n[f]$ exists, we call it *the bivariate Appell interpolant of f of order n* associated with the functional L. □

We note that this problem is very closely related to the corresponding univariate problem in [10, p. 101]. Therefore, we say that it is its "natural" bivariate extension.

In the sequel we will adopt the following notation for the derivatives of a bivariate function f:

$$f^{(i,j)} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}, \qquad f^{(0,0)} = f, \qquad f^{(i,j)}(\alpha, \beta) = f^{(i,j)}(x, y)\Big|_{(x,y)=(\alpha,\beta)}.$$

Theorem 1 (**The main theorem**) [11] *For every $f \in X$ the bivariate polynomial of total degree n given by*

$$i_n[f](x, y) = \sum_{i=0}^{n} L\left(f^{(i,0)}\right)\frac{r_i^L(x, y)}{i!} \tag{15}$$

is the unique element of $\mathscr{S}_n$ such that

$$L\left(i_n[f]^{(j,0)}\right) = L\left(f^{(j,0)}\right), \quad j = 0, \ldots, n, \tag{16}$$

that is, $i_n[f]$ is the bivariate Appell interpolant of f associated with the functional L.

Proof Let's define the linear functionals

$$L_0(f) = L(f), \qquad L_k(f) = L\left(f^{(k,0)}\right), \quad k = 1, \ldots, n.$$

We get

$$L_j\left(p_k\right) := L\left(p_k^{(j,0)}\right) = j!\binom{k}{j}\beta_{k-j}, \qquad j \leq k.$$

Then for the sequence $\left\{r_n^L\right\}_{n\in\mathbb{N}}^b$, from (9) we have $L_j\left(r_k^L\right) = k!\delta_{kj}$, $j = 0, \ldots, k$, that is the systems $\{L_n\}_{n\in\mathbb{N}}$ and $\left\{r_n^L\right\}_{n\in\mathbb{N}}^b$ are biorthogonal [14]. Hence the polynomial (15) satisfies the interpolation conditions (16). The uniqueness follows from the linear independence of the linear functionals L_j, $j = 0, \ldots, n$. □

Corollary 1 *For any $f \in \mathscr{S}_n$, $i_n[f](x, y) = f(x, y)$, $\forall(x, y) \in D$.*

Remark 2 In order to remove the calculation of $L(f)$ from the bivariate Appell interpolant of f (15), we consider an arbitrary fixed point $(u, v) \in D$. Then we get the bivariate polynomial

$$\begin{aligned} i_n^*[f](x, y) &= f(u, v) + i_n[f](x, y) - i_n[f](u, v) \\ &= f(u, v) + \sum_{k=1}^{n} L\left(f^{(k,0)}\right) \frac{r_k^L(x, y) - r_k^L(u, v)}{k!}. \end{aligned} \tag{17}$$

The polynomial $i_n^*[f](x, y)$ satisfies the interpolation conditions:

$$i_n^*[f](u, v) = f(u, v), \qquad L\left(i_n^*[f]^{(k,0)}\right) = L\left(f^{(k,0)}\right), \qquad k = 1, \ldots, n.$$

We note that the interpolant $i_n^*[f]$ replaces the calculation of the functional $L(f)$ by the evaluation of the function in a suitable point. Therefore, we call it *complementary Appell interpolant of f related to functional L.*

Remark 3 We observe that the Appell interpolant and its complementary interpolant can be considered finite Appell polynomial expansions of bivariate functions. They are the natural extensions to the bivariate polynomial case of the following univariate formulas, respectively, [10, Th. 7.1, p. 101]

$$P_{L,n}[f](x) = \sum_{i=0}^{n} \frac{L(f^{(i)})}{i!} a_{L,i}(x)$$

and [10, p. 103]

$$\widetilde{P}_{L,n}[f](x) = f(z) + \sum_{i=1}^{n} \frac{L(f^{(i)})}{i!}\left[a_{L,i}(x) - a_{L,i}(z)\right],$$

where $\left\{a_{L,i}\right\}_{i\in\mathbb{N}}$ is the univariate Appell polynomial sequence associated with the functional L.

4 Remainder for Appell interpolation

For the Appell bivariate interpolants $i_n[f]$ defined as in (15) and $i_n^*[f]$ defined as in (17), we consider the error at any $(x, y) \in D$.

Definition 1 For any $f \in X$, $(x, y) \in D$, the remainder for the interpolants (15) and (17) are, respectively, the linear functionals

$$E_n[f](x, y) = f(x, y) - i_n[f](x, y), \tag{18}$$

and

$$E_n^*[f](x, y) = f(x, y) - i_n^*[f](x, y). \tag{19}$$

Remark 4 We observe that for any $q \in \mathscr{S}_n$, $E_n[q](x, y) = 0$ and $E_n^*[q](x, y) = 0$, $\forall (x, y) \in D$. In this case we say that $E_n[f]$ and $E_n^*[f]$ have order n with respect to φ.

In order to estimate errors (18) and (19), in the following theorem we remember Sard's formula [17, 29].

Theorem 2 (**Sard's formula for bivariate functions**) *Let $f \in C^{n+1}\left(\overline{D}\right)$. Then for odd n, $n = 2k + 1$,*

$$\begin{aligned} f(x, y) = & \sum_{\nu=0}^{n} \sum_{\mu=0}^{\nu} \frac{1}{\nu!} \binom{\nu}{\mu} x^{\nu-\mu} y^{\mu} f^{(\nu-\mu,\mu)}(0, 0) \\ & + \frac{1}{n!} \sum_{\mu=0}^{k} \binom{n}{\mu} \left\{ y^{\mu} \int_0^x (x-u)^{n-\mu} f^{(n+1-\mu,\mu)}(u, 0)\, du \right. \\ & \left. + x^{\mu} \int_0^y (y-v)^{n-\mu} f^{(\mu,n+1-\mu)}(0, v)\, dv \right\} \\ & + \frac{1}{k!k!} \int_0^x \int_0^y (x-u)^k (y-v)^k f^{(k+1,k+1)}(u, v)\, du\, dv, \end{aligned}$$

and for even n, $n = 2k$,

$$\begin{aligned} f(x, y) = & \sum_{\nu=0}^{n} \sum_{\mu=0}^{\nu} \frac{1}{\nu!} \binom{\nu}{\mu} x^{\nu-\mu} f^{(\nu-\mu,\mu)}(0, 0) \\ & + \frac{1}{n!} \sum_{\mu=0}^{k}{}' \binom{n}{\mu} \left\{ y^{\mu} \int_0^x (x-u)^{n-\mu} f^{(n+1-\mu,\mu)}(u, 0)\, du \right. \\ & \left. + x^{\mu} \int_0^y (y-v)^{n-\mu} f^{(\mu,n+1-\mu)}(0, v)\, dv \right\} \\ & + \frac{1}{2(k-1)!k!} \int_0^x \int_0^y (x-u)^{k-1} (y-v)^{k-1} \left[(x-u) f^{(k+1,k)}(u, v) \right. \\ & \left. + (y-v) f^{(k,k+1)}(u, v) \right] du\, dv, \end{aligned}$$

where $\sum_{\mu=0}^{k}{}' a_\mu = \frac{1}{2}a_0 + \cdots + a_{k-1} + a_k$.

Now, let $T_n[f] = E_n[f]$ or $E_n^*[f]$.

Theorem 3 *Let* $f \in C^{n+1}\left(\overline{D}\right)$*. For the remainder* $T_n[f]$ *the following representation holds:*

$$T_n[f] = \frac{1}{n!}\sum_{\mu=0}^{\rho}{}^{\star}\binom{n}{\mu}\left[\int_0^\tau K_1^\mu(u)\, f^{(n+1-\mu,\mu)}(u,0)\,du + \int_0^\sigma K_2^\mu(v)\, f^{(\mu,n+1-\mu)}(0,v)\,dv\right] + \frac{1}{\rho!\rho!}\iint_D K_{\rho,\rho}(u,v) f^{(\rho+1,\rho+1)}(u,v)\,du\,dv$$

for $n = 2\rho+1$*;*

$$T_n[f] = \frac{1}{n!}\sum_{\mu=0}^{\rho}{}^{\star}\binom{n}{\mu}\left[\int_0^\tau K_1^\mu(u)\, f^{(n+1-\mu,\mu)}(u,0)\,du + \int_0^\sigma K_2^\mu(v)\, f^{(\mu,n+1-\mu)}(0,v)\,dv\right] + \frac{1}{2(\rho-1)!\rho!}\left\{\iint_D K_{\rho,\rho-1}(u,v) f^{(\rho+1,\rho)}(u,v)\,du\,dv + \iint_D K_{\rho-1,\rho}(u,v) f^{(\rho,\rho+1)}(u,v)\,du\,dv\right\}$$

for $n = 2\rho$*, where*

$$\sum{}^{\star} = \begin{cases} \sum, & n = 2\rho+1 \\ \sum', & n = 2\rho, \end{cases}$$

$\tau = \tau(y) > 0$, $\sigma = \sigma(x) > 0$ *are numbers such that* $(\tau, y), (x, \sigma) \in \partial D$. $K_1^\mu(u)$*,* $K_2^\mu(v)$*,* $K_{i,j}(u,v)$ *are the Sard kernel functions of* $T_n[f]$ [17] *defined as*

$$K_1^\mu(u) = T_n^{x,y}\left[y^\mu (x-u)_+^{n-\mu}\right], \qquad \mu = 0,\ldots,\rho,\ (x,y) \in \overline{D}$$

$$K_2^\mu(v) = T_n^{x,y}\left[x^\mu (y-v)_+^{n-\mu}\right], \qquad \mu = 0,\ldots,\rho,\ (x,y) \in \overline{D}$$

$$K_{\rho,\rho}(u,v) = T_n^{x,y}\left[(x-u)_+^\rho (y-v)_+^\rho\right], \qquad n = 2\rho+1, \quad (x,y) \in \overline{D}$$

$$K_{\rho,\rho-1}(u,v) = T_n^{x,y}\left[(x-u)_+^\rho (y-v)_+^{\rho-1}\right], \quad n = 2\rho, \quad (x,y) \in \overline{D},$$

$$K_{\rho-1,\rho}(u,v) = T_n^{x,y}\left[(x-u)_+^{\rho-1} (y-v)_+^{\rho}\right], \quad n = 2\rho, \quad (x,y) \in \overline{D},$$

where $T_n^{x,y}$ *is the functional* T_n *with respect to the variables* x *and* y.

Proof The error functional $T_n[f]$ vanishes at each element of $\mathscr{S}_n$, according to Remark 4. The result follows from Th. 3.2.2 in [17, p. 105]. □

If all the Sard kernels have constant sign, from the general mean value theorem, there exist points $(\xi_i, 0)$, $(0, \zeta_i)$ and (η, θ), (η_1, θ_1), (η_2, θ_2) in D such that, for $f \in C^{n+1}\left(\overline{D}\right)$,

$$T_n[f] = \frac{1}{n!}\sum_{\mu=0}^{\rho}{}^{\star}\binom{n}{\mu}\left[f^{(n+1-\mu,\mu)}(\xi_\mu, 0)\int_0^1 K_1^\mu(u)\,du + f^{(\mu,n+1-\mu)}\left(0, \zeta_\mu\right)\int_0^1 K_2^\mu(v)\,dv\right]$$
$$+\begin{cases} \dfrac{1}{\rho!\rho!} f^{(\rho+1,\rho+1)}(\eta, \theta)\displaystyle\iint_D K_{\rho,\rho}(u, v)\,du\,dv, & n = 2\rho+1,\\[2ex] \dfrac{1}{2(\rho-1)!\rho!}\left\{f^{(\rho+1,\rho)}(\eta_1, \theta_1)\displaystyle\iint_D K_{\rho,\rho-1}(u, v)\,du\,dv\right. & \\ \qquad\left. + f^{(\rho,\rho+1)}(\eta_2, \theta_2)\displaystyle\iint_D K_{\rho-1,\rho}(u, v)\,du\,dv\right\}, & n = 2\rho. \end{cases}$$

In order to get error bounds for $E_n[f]$ we can apply Holder's inequality, for example in the case of the sup norm $\|\cdot\|$. Let $f \in C^{n+1}\left(\overline{D}\right)$. We set

$$M_{i,j} = \sup_{(x,y)\in\overline{D}} \left| f^{(i,j)}(x, y)\right|. \tag{20}$$

Then we get

$$\left|T_n[f]\right| \le \frac{1}{n!}\sum_{\mu=0}^{\rho}{}^{\star}\binom{n}{\mu}\left[M_{n+1-\mu,\mu}\int_0^1 \left|K_1^\mu(u)\right| du + M_{\mu,n+1-\mu}\int_0^1 \left|K_2^\mu(v)\right| dv\right]$$
$$+\begin{cases} \dfrac{M_{\rho+1,\rho+1}}{\rho!\rho!}\displaystyle\iint_D \left|K_{\rho,\rho}(u, v)\right| du\,dv, & n = 2\rho+1,\\[2ex] \dfrac{1}{2(\rho-1)!\rho!}\left[M_{\rho+1,\rho}\displaystyle\iint_D \left|K_{\rho,\rho-1}(u, v)\right| du\,dv\right. & \\ \qquad\left. + M_{\rho,\rho+1}\displaystyle\iint_D \left|K_{\rho-1,\rho}(u, v)\right| du\,dv\right], & n = 2\rho. \end{cases}$$

Now we consider some important examples of functionals L. For each functional we determine the two interpolants $i_n[f]$ and $i_n^*[f]$, and an error bound in the case of $i_n[f]$ (for $i_n^*[f]$ an analogous bound can be obtained). In the expressions of the error bound we set $M = \max\limits_{\substack{i,j\ge 0\\ i+j\le n+1}} M_{i,j}$, with $M_{i,j}$ defined as in (20) and

$$R = \max_{\substack{1\le k\le n\\ (x,y)\in\overline{D}}} \left|r_k^L(x, y)\right|.$$

Example 1 (*Evaluating functional*) Assuming

$$L(f) = f(x_0, y_0), \qquad (x_0, y_0) \in \overline{D} \quad fixed,$$

the bivariate Appell interpolating polynomial becomes

$$i_n[f](x,y) = f(x_0,y_0) + \sum_{k=1}^{n} f^{(k,0)}(x_0,y_0)\frac{r_k^L(x,y)}{k!}. \tag{21}$$

Remembering that $r_k^L(x,y) = \sum_{i=0}^{k}\binom{k}{i}\alpha_{k-i}p_i(x,y)$, where α_k are related to β_k by the relation (7) and by substituting in (21), we get

$$i_n[f](x,y) = \sum_{i=0}^{n}\frac{f^{(i,0)}(x_0,y_0)}{i!}\alpha_i + \sum_{k=1}^{n} I_k^{n,\alpha}\frac{p_k(x,y)}{k!}, \quad I_k^{n,\alpha} = \sum_{i=k}^{n}\frac{f^{(i,0)}(x_0,y_0)}{(i-k)!}\alpha_{i-k}. \tag{22}$$

From Theorem 3 we get the following estimate:

$$\begin{aligned}\left|E_n[f]\right| \le & \frac{M}{(n+1)!}\left[3+R\left(2^{n+1}-1\right)\right] + \frac{M(2+R)}{(n+1)!}\sum_{\mu=1}^{\rho}\binom{n+1}{\mu} \\ & + \frac{M}{(\rho+1)!(\rho+1)!}\left[1+R\left(2^{\rho+1}-1\right)\right],\end{aligned} \qquad n = 2\rho+1$$

and

$$\begin{aligned}\left|E_n[f]\right| \le & \frac{M}{2(n+1)!}\left[3+R\left(2^{n+1}-1\right)\right] + \frac{M}{(n+1)!}\sum_{\mu=1}^{\rho}\binom{n+1}{\mu}(2+R) \\ & + \frac{M}{(\rho+1)!\rho!}\left[1+R\left(2^{\rho}+2^{\rho-1}-1\right)\right],\end{aligned} \qquad n = 2\rho.$$

We call polynomial (21), or equivalently (22), *general partial Taylor formula* for f at the initial point (x_0, y_0).

Example 2 (*Integral functional*) Assuming

$$L(f) = \int_0^1\int_0^1 f(x,y)\,dx\,dy, \tag{23}$$

the bivariate Appell interpolating polynomial related to the integral functional (23) becomes

$$i_n[f](x,y) = \int_0^1\int_0^1 f(x,y)dxdy + \sum_{k=1}^{n}\frac{r_k^L(x,y)}{k!}\int_0^1\left[f^{(k-1,0)}(1,y) - f^{(k-1,0)}(0,y)\right]dy. \tag{24}$$

After easy calculations we get

$$i_n[f](x,y)=\sum_{k=0}^{n}\frac{p_k(x,y)}{k!}I_k^{n,\alpha},\qquad I_k^{n,\alpha}=\sum_{i=k}^{n}\frac{\alpha_{i-k}}{(i-k)!}\int_0^1\Delta f^{(i-1,0)}(y)\,dy, \tag{25}$$

being $\Delta f(y)=f(1,y)-f(0,y)$.

From Theorem 3 we obtain the following estimate:

$$\begin{aligned}\left|E_n[f]\right| \le \frac{M\,2^{2\rho}}{(2\rho+1)!}\left[\frac{2}{\rho+2}+2^{2\rho+1}\,R+\frac{2^{\rho+1}\,R}{(\rho+2)(\rho+3)}\right]+ \\ +\frac{M}{(\rho+1)!\rho!}\left[\frac{1}{\rho+1}+\frac{2^{\rho+1}\,R}{\rho+2}\right],\end{aligned}\qquad n=2\rho+1$$

and

$$\begin{aligned}\left|E_n[f]\right| \le \frac{M\,2^{2\rho}}{(2\rho)!}\left[\frac{2}{\rho+1}+2^{2\rho}\,R+\frac{2^{\rho+1}\,R}{(\rho+1)(\rho+2)}\right]+ \\ +\frac{M}{2(\rho-1)!(\rho+1)!}\left[\frac{2}{\rho}+\frac{2^{\rho+1}\,R}{\rho}+\frac{2^{\rho}\,R}{(\rho+2)}\right],\end{aligned}\qquad n=2\rho.$$

We call the interpolant (24) (or (25)) *integral of first forward difference.*

The complementary interpolant associated with the functional (23) is

$$i_n^*[f](x,y)=f(u,v)+\sum_{k=1}^{n}\frac{r_k^L(x,y)-r_k^L(u,v)}{k!}\int_0^1\left[f^{(k-1)}(1,y)-f^{(k-1)}(0,y)\right]dy.$$

Example 3 (*Arithmetic mean functional*) Let (x_0,y_0) be an arbitrary fixed point of D. Assuming

$$L(f)=\frac{f(x_0+1,y_0+1)+f(x_0+1,y_0)+f(x_0,y_0+1)+f(x_0,y_0)}{4}=\mathscr{M}_0(f), \tag{26}$$

we get the bivariate Appell interpolating polynomial

$$i_n[f](x,y)=\mathscr{M}_0(f)+\sum_{k=1}^{n}\mathscr{M}_0\left(f^{(k,0)}\right)\frac{r_k^L(x,y)}{k!}.$$

After easy calculations we obtain

$$i_n[f](x,y)=\sum_{k=0}^{n}\frac{p_k(x,y)}{k!}K_k^{n,\alpha},$$

where

$$\begin{aligned}K_k^{n,\alpha}=\tfrac{1}{4}\textstyle\sum_{i=k}^{n}\frac{\alpha_{i-k}}{(i-k)!}\,\Big[f^{(i,0)}(x_0+1,y_0+1)+f^{(i,0)}(x_0+1,y_0) \\ +f^{(i,0)}(x_0,y_0+1)+f^{(i,0)}(x_0,y_0)\Big]\end{aligned}$$

and α_i are as in (1).

From Theorem 3 we get the following estimate:

$$\left| E_n[f] \right| \le \frac{M}{(n+1)!}\sum_{\mu=0}^{\rho}\binom{n+1}{\mu}\left[2+ R\left(2^{n-\mu+1}+2^{\mu-1}-\frac{1}{2}\right)\right] + \frac{M}{\left((\rho+1)!\right)^2}\left[1+R\left(2^{\rho+1}-1\right)\right], \qquad n=2\rho+1$$

and

$$\left| E_n[f] \right| \le \frac{M}{2(n+1)!}\left[3+R\left(2^{n+1}-1\right)\right] + \frac{M}{(n+1)!}\sum_{\mu=1}^{\rho}\binom{n+1}{\mu}\left[2+ R\left(2^{n-\mu+1}+2^{\mu-1}-\frac{1}{2}\right)\right] + \frac{M}{\rho!(\rho+1)!}\left[1+R\left(2^{\rho}+2^{\rho-1}-1\right)\right], \qquad n=2\rho.$$

The complementary interpolant is

$$i_n^*[f](x,y) = f(u,v) + \sum_{k=1}^{n}\frac{p_k(x,y)-p_k(u,v)}{k!}K_k^{n,\alpha}.$$

5 Some particular examples of bivariate Appell interpolation

Now we will give some particular examples of interpolants in the case $D=[0,1]\times[0,1]$.

A. Let $\phi(y,t)=e^{yt}$.

It is known [11] that the related bivariate elementary Appell sequence is $\{p_k\}^b_{k\in\mathbb{N}}$, with

$$p_k(x,y) = (x+y)^k, \qquad \forall k\in\mathbb{N}. \tag{27}$$

In the literature [4, 5, 7, 15] $p_k(x,y)$ is denoted also by $H_n^{(1)}(x,y)$ and called bivariate Hermite polynomial.

Now we consider different functionals.

A1. **Evaluating functional**.

Let $L(f)$ be defined as

$$L(f)=f(0,0), \qquad \forall f\in X.$$

In this case $\forall k\in\mathbb{N}$ we get

$$\beta_k = L(p_k) = L\left((x+y)^k\right) = \begin{cases} 1 & k=0 \\ 0 & k>0. \end{cases}$$

Consequently, from (7),

$$\alpha_k = \begin{cases} 1 & k = 0 \\ 0 & k > 0. \end{cases}$$

Then the general partial Taylor formula, that in this case we denote by $t_n[f]$, becomes

$$t_n[f](x, y) = f(0, 0) + \sum_{k=1}^{n} f^{(k,0)}(0, 0)\frac{(x + y)^k}{k!}.$$

We call this polynomial *partial-Taylor formula at starting point* $(0, 0)$.

We observe that this formula is quite different from the generalized Taylor formula in [16].

From Example 1 we get the following estimate:

$$\begin{aligned} \left| E_n[f] \right| \le & \frac{2M}{(n+1)!}\left(2^n+2+\sum_{\mu=1}^{\rho}\binom{n+1}{\mu}\left(2^{\mu-1}+1\right)\right) \\ & +\frac{2M}{(\rho+1)!(\rho+1)!}\left(2^{\rho}+1\right), \end{aligned} \qquad n = 2\rho+1,$$

and

$$\begin{aligned} \left| E_n[f] \right| \le & \frac{2M}{(n+1)!}\left(2^{n-1}+1+\sum_{\mu=1}^{\rho}\binom{n+1}{\mu}\left(2^{\mu-1}+1\right)\right) \\ & +\frac{2M}{(\rho+1)!\rho!}\left(2^{\rho-1}+2^{\rho-2}+1\right), \end{aligned} \qquad n = 2\rho,$$

where $M = \max\limits_{i,j\ge 0, i+j\le n+1} M_{i,j}$, with $M_{i,j}$ defined as in (20).

A2. **Integral functional.**

For any $f \in X$ let's consider the integral functional L as in (23). From (13) we have

$$\forall k \in \mathbb{N}, \qquad \beta_k = L(p_k) = \int_0^1\int_0^1 (x + y)^k\, dx\, dy = \frac{2\left(2^{k+1} - 1\right)}{(k + 1)(k + 2)}.$$

The bivariate Appell sequence associated with L is given by the recurrence formula (see (11))

$$r_0(x, y) = 1, \quad r_k(x, y) = (x+y)^k - 2\sum_{j=0}^{k-1}\frac{k!\left(2^{k-j+1}-1\right)}{j!(k-j+2)!}r_j(x, y), \quad k > 0,$$

or, equivalently, by the determinant form

$$r_0(x,y)=1, \quad r_k(x,y)=(-1)^n \begin{vmatrix} 1 & x+y & (x+y)^2 & \cdots & (x+y)^k \\ 1 & 1 & \frac{7}{6} & \cdots & \frac{2(2^{k+1}-1)}{(k+1)(k+2)} \\ 0 & 1 & 2 & \cdots & \frac{2(2^k-1)}{k+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & k \end{vmatrix}, \quad k>0.$$

From Proposition 1, the generating function of the sequence $\{r_n\}_{n\in\mathbb{N}}^b$ is

$$F(x,y;t)=\frac{e^{xt}e^{yt}}{\int_0^1\int_0^1 e^{(x+y)t}dx\,dy}=\frac{t^2}{\left(e^t-1\right)^2}e^{(x+y)t}. \tag{28}$$

In order to give an explicit expression of r_n, from (28) we can write

$$\left(\frac{t}{e^t-1}\right)^2 e^{(x+y)t}=\sum_{n=0}^{\infty} r_n(x,y)\frac{t^n}{n!}.$$

Hence

$$r_n(x,y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(2)}(x)y^{n-k}, \tag{29}$$

$B_n^{(2)}(x)$ being the Bernoulli polynomial of order 2 [8, 13, 22, 26] defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^2 e^{xt}=\sum_{n=0}^{\infty} B_n^{(2)}(x)\frac{t^n}{n!}.$$

Bernoulli polynomial of order 2 can be written also in terms of Bernoulli numbers of order 2, as follows

$$B_n^{(2)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(2)}x^{n-k}, \tag{30}$$

with $B_k^{(2)}$ given by

$$\left(\frac{t}{e^t-1}\right)^2=\sum_{k=0}^{\infty} B_k^{(2)}\frac{t^k}{k!}. \tag{31}$$

From (31) we get

$$B_k^{(2)}=\sum_{i=0}^{k}\binom{k}{i}B_i B_{k-i},$$

where B_j is the jth Bernoulli number. Finally, from (29), (30) and (31), after calculations, we get

$$r_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(2)}(x) y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y). \tag{32}$$

We note that $\left\{B_n^{(2)}\right\}_{n\in\mathbb{N}}$ is a univariate Appell sequence; therefore, from (29), according to Theorem 6.13 in [10, p. 90], we get

$$r_n(x, y) = B_n^{(2)}(x + y).$$

The bivariate Appell sequence $\{r_k\}_{k\in\mathbb{N}}^b$ defined in (29) or, equivalently in (32), does not appear in the literature. We call it *natural bivariate Bernoulli polynomial sequence of order 2* and we denote it by $\left\{\mathscr{B}_k^{(2)}\right\}_{k\in\mathbb{N}}^b$. Hence, we have

$$\mathscr{B}_n^{(2)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) = B_n^{(2)}(x + y). \tag{33}$$

The first natural bivariate Bernoulli polynomials of order 2 are

$$\mathscr{B}_0^{(2)}(x, y) = 1, \qquad \mathscr{B}_1^{(2)}(x, y) = x + y - 1,$$
$$\mathscr{B}_2^{(2)}(x, y) = (x + y)^2 - 2(x + y) + \frac{5}{6},$$
$$\mathscr{B}_3^{(2)}(x, y) = (x + y)^3 - 3(x + y)^2 + \frac{5}{2}(x + y) - \frac{1}{2},$$
$$\mathscr{B}_4^{(2)}(x, y) = (x + y)^4 - 4(x + y)^3 + 5(x + y)^2 - 2(x + y) + \frac{1}{10}.$$

Other properties of these polynomials will be studied in a future work.

The integral Appell interpolant is

$$i_n[f](x, y) = \int_0^1\int_0^1 f(x, y)dxdy + \sum_{k=1}^{n} \frac{\mathscr{B}_k^{(2)}(x,y)}{k!} \int_0^1 \left[f^{(k-1)}(1, y) - f^{(k-1)}(0, y)\right] dy. \tag{34}$$

Formula (34) can be called, also, *polynomial expansion of a bivariate real function in natural bivariate Bernoulli polynomials of order 2.*

The complementary integral Appell interpolant is

$$i_n^*[f](x, y) = f(0, 0) + \sum_{k=1}^{n} \frac{\mathscr{B}_k^{(2)}(x,y) - \mathscr{B}_k^{(2)}(0,0)}{k!} \int_0^1 \left[f^{(k-1)}(1, y) - f^{(k-1)}(0, y)\right] dy. \tag{35}$$

Remark 5 Formulas (34) and (35) are the bivariate extensions of univariate formulas (8.68) and (8.77) in [10] that are, respectively,

$$P_n(x) = \int_0^1 P_n(x) + \sum_{k=1}^{n} \frac{P_n^{(k-1)}(1) - P_n^{(k-1)}(0)}{k!} B_k(x),$$

and

$$f(x) = f(0) + \sum_{k=1}^{n} \frac{B_k(x) - B_k}{k!} \left[f^{(k-1)}(1) - f^{(k-1)}(0) \right] + R_n[f](x).$$

A3. **Arithmetic mean functional.**
For any $f \in X$ let's consider the functional $\mathscr{M}_0(f)$ defined in (26). In particular, for $(x_0, y_0) = (0, 0)$, we get

$$\mathscr{M}_0(f) = \frac{f(1,1) + f(1,0) + f(0,1) + f(0,0)}{4}. \quad (36)$$

Therefore, $\beta_0 = 1$ and $\forall k \in \mathbb{N}$, $k \geq 1$,

$$\beta_k = \mathscr{M}_0\left((x+y)^k\right) = \frac{1 + 2^{k-1}}{2}.$$

Then, the bivariate Appell sequence associated with the functional $\mathscr{M}_0$ is given by the recurrence formula

$$r_0(x, y) = 1, \quad r_n(x, y) = (x+y)^n - \sum_{j=0}^{n-1} \binom{n}{j} \frac{1+2^{n-j-1}}{2} r_j(x, y), \quad n \geq 1,$$

or, equivalently, by

$$r_n(x, y) = (-1)^n \begin{vmatrix} 1 & x+y & (x+y)^2 & (x+y)^3 & \cdots & (x+y)^n \\ 1 & 1 & \frac{3}{2} & \frac{5}{2} & \cdots & \frac{2^{n-1}+1}{2} \\ 0 & 1 & 2 & 3 & \cdots & n\frac{2^{n-2}+1}{2} \\ 0 & 0 & 1 & 1 & & \binom{n}{2}\frac{2^{n-3}+1}{2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \binom{n}{n-1} \end{vmatrix}, \quad n > 0.$$

To give the series expansion of $r_k^{\mathscr{M}}(x, y)$, from Proposition 1 we have

$$F(x, y; t) = \frac{e^{(x+y)t}}{\mathscr{M}_0\left(e^{(x+y)t}\right)} = \frac{4}{(e^t + 1)^2} e^{(x+y)t}.$$

Hence

$$\left(\frac{2}{e^t + 1}\right)^2 e^{(x+y)t} = \sum_{n=0}^{\infty} r_n(x, y) \frac{t^n}{n!}. \quad (37)$$

In order to give an explicit expression of r_n, from (37) we get

$$r_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(2)}(x) y^{n-k}, \tag{38}$$

$E_n^{(2)}(x)$ being the Euler polynomial of order 2 [2, 9] defined by the generating function

$$\left(\frac{2}{e^t + 1}\right)^2 e^{xt} = \sum_{n=0}^{\infty} E_n^{(2)}(x) \frac{t^n}{n!}$$

or by

$$E_n^{(2)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(2)}(0) x^{n-k}, \tag{39}$$

with $E_k^{(2)}(0)$ given by

$$\left(\frac{2}{e^t + 1}\right)^2 = \sum_{k=0}^{\infty} E_k^{(2)}(0) \frac{t^k}{k!}. \tag{40}$$

From Corollary 1.9 in [2] we get

$$E_k^{(2)}(0) = \sum_{i=0}^{k} \binom{k}{i} E_i(0) E_{k-i}(0),$$

where $E_j(0)$ is the value of the jth univariate Euler polynomial $E_j(x)$ at $x = 0$.

Finally, from (38), (39) and (40), after calculations, we get

$$r_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(2)}(x) y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} E_k(x) E_{n-k}(y). \tag{41}$$

$\left\{E_n^{(2)}\right\}_{n\in\mathbb{N}}$ is a univariate Appell sequence; therefore, from (38), according to Theorem 6.13 in [10, p. 90], we get

$$r_n(x, y) = E_n^{(2)}(x + y).$$

To the authors knowledge the bivariate sequence $\{r_k\}_{k\in\mathbb{N}}^b$ defined in (38) or, equivalently in (41), does not appear in the literature. We call it *bivariate natural Euler polynomial of order 2* and denote it by $\left\{\mathscr{E}_n^{(2)}\right\}_{k\in\mathbb{N}}^b$:

$$\mathscr{E}_n^{(2)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) E_{n-k}(y) = E_n^{(2)}(x + y). \tag{42}$$

The first bivariate natural Euler polynomials of order 2 are

$$\mathscr{E}_0^{(2)}(x,y)=1, \qquad \mathscr{E}_1^{(2)}(x,y)=x+y-1,$$
$$\mathscr{E}_2^{(2)}(x,y)=(x+y)^2-2(x+y)+\frac{1}{2},$$
$$\mathscr{E}_3^{(2)}(x,y)=(x+y)^3-3(x+y)^2+\frac{3}{2}(x+y)+\frac{1}{2},$$
$$\mathscr{E}_4^{(2)}(x,y)=(x+y)^4-4(x+y)^3+3(x+y)^2+2(x+y)-1.$$

The mean Appell interpolant is

$$\begin{aligned} i_n[f](x,y) &= \mathscr{M}_0(f)+\sum_{k=1}^{n}\mathscr{M}_0\left(f^{(k,0)}\right)\frac{\mathscr{E}_k^{(2)}(x,y)}{k!} \\ &= \frac{f(1,1)+f(1,0)+f(0,1)+f(0,0)}{4} \\ &\quad +\sum_{k=1}^{n}\frac{f^{(k,0)}(1,1)+f^{(k,0)}(1,0)+f^{(k,0)}(0,1)+f^{(k,0)}(0,0)}{4}\,\frac{\mathscr{E}_k^{(2)}(x,y)}{k!}. \end{aligned} \tag{43}$$

It is also the polynomial expansion of a bivariate real function in bivariate natural Euler polynomials of order 2. We note that interpolant (43) approximates a functions by only boundary values. In addition, it is the natural extension to the bivariate case of the univariate polynomial [10, p. 133]

$$P_n[f](x)=\frac{f(1)+f(0)}{2}+\sum_{i=1}^{n}\frac{f^{(i)}(1)+f^{(i)}(0)}{2}\,\frac{E_i(x)}{2}.$$

The complementary Appell interpolant is

$$i_n^*[f](x,y)=f(0,0)+\sum_{k=1}^{n}\mathscr{M}_0\left(f^{(k,0)}\right)\frac{\mathscr{E}_k^{(2)}(x,y)-\mathscr{E}_k^{(2)}(0,0)}{k!}.$$

B. Let $\phi(y,t)=e^{yt^2}$.

It is known [11] that in this case the elementary Appell sequence is $\{p_n\}_{n\in\mathbb{N}}^{b}$, with

$$p_n(x,y)=H_n^{(2)}(x,y)=n!\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{x^{n-2k}y^k}{k!(n-2k)!}. \tag{44}$$

The polynomials $H_n^{(2)}(x,y)$ are called Hermite-Kampé de Fériet (HKF) polynomials [4, 5, 7, 13, 20].

B1. **Evaluating functional**

Assuming $L(f)=f(0,0)$, $\forall k\in\mathbb{N}$ we have

$$\beta_k=L\left(H_k^{(2)}\right)=\begin{cases}1 & k=0\\ 0 & k\neq 0.\end{cases}$$

The Appell interpolant is

$$i_n[f](x, y) = f(0, 0) + \sum_{k=1}^{n} f^{(k,0)}(0, 0) \frac{H_k^{(2)}(x, y)}{k!}.$$

This means that the partial Taylor HKF-based polynomial provides also an expansion for a bivariate function in terms of HKF polynomials.

B2. **Integral functional.**

$\forall f \in X$ let's consider the integral functional as in (23). In order to determine the bivariate Appell sequence associated with functional (23), we get

$$\forall k \in \mathbb{N}, \qquad \beta_k = L\left(H_n^{(2)}\right) = k! \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{(j+1)!(k-2j+1)!}. \tag{45}$$

Thus we obtain the bivariate Appell sequence $\{r_n\}_{n\in\mathbb{N}}^b$ such that

$$r_0(x, y) = 1, \quad r_n(x, y) = H_n^{(2)}(x, y) - \sum_{j=0}^{n-1} \binom{n}{j} \beta_{n-j} r_j(x, y), \quad n \geq 1.$$

For the generating function of $\{r_n\}_{n\in\mathbb{N}}^b$, from Proposition 1 we get

$$F(x, y; t) = \frac{e^{xt+yt^2}}{\int_0^1 \int_0^1 e^{xt+yt^2}\, dx\, dy} = \frac{t^3 e^{xt+yt^2}}{(e^t - 1)\left(e^{t^2} - 1\right)}. \tag{46}$$

Hence

$$\frac{t^3 e^{xt+yt^2}}{(e^t - 1)\left(e^{t^2} - 1\right)} = \sum_{n=0}^{\infty} r_n(x, y) \frac{t^n}{n!}. \tag{47}$$

We call $\{r_n\}_{n\in\mathbb{N}}^b$ *bivariate Bernoulli HKF-based polynomial sequence* associated with functional (23) and denote it by $\{\mathscr{H}_k\}_{k\in\mathbb{N}}^b$. The first polynomials of this sequence are

$$\mathscr{H}_0(x, y) = 1, \quad \mathscr{H}_1(x, y) = x - \frac{1}{2}, \quad \mathscr{H}_2(x, y) = x^2 - x + 2y - \frac{5}{6},$$
$$\mathscr{H}_3(x, y) = x^3 - \frac{3}{2}x^2 - \frac{5}{2}x - 3y + 6xy + \frac{3}{2},$$
$$\mathscr{H}_4(x, y) = x^4 - 2x^3 - 5x^2 + 6x + 12y^2 + 12x^2y - 12xy - 10y + \frac{29}{30}.$$

From (47),

$$\mathscr{H}_n(x, y) = \sum_{j=0}^{n} \binom{n}{j} \alpha_{n-j} H_j^{(2)}(x, y), \qquad n \geq 1, \tag{48}$$

where

$$\frac{t^3}{(e^t - 1)\left(e^{t^2} - 1\right)} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}$$

and

$$\alpha_k = k! \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{B_{k-2j} B_j}{j!(k-2j)!},$$

with B_s the sth Bernoulli number.

Remark 6 From (7) and (12), for $k = 1, \ldots, n$, we get the following identity:

$$k! \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{B_{k-2j} B_j}{j!(k-2j)!} = (-1)^k \begin{vmatrix} \beta_1 & 1 & 0 & \cdots\cdots & 0 \\ \beta_2 & \binom{2}{1}\beta_1 & 1 & 0 \;\cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots\;\ddots & \vdots \\ \beta_{k-1} & \binom{k-1}{k-2}\beta_{k-2} & \binom{k-1}{k-3}\beta_{k-3} & \cdots\;\ddots & 1 \\ \beta_k & \binom{k}{k-1}\beta_{k-1} & \binom{k}{k-2}\beta_{k-2} & \cdots\cdots & \binom{k}{1}\beta_1 \end{vmatrix},$$

where β_j, $j = 1, \ldots, k$, are defined as in (45).

We note that the numbers α_j, $j = 0, \ldots, n$, and the sequence $\{\mathscr{H}_n\}_{n\in\mathbb{N}}^b$ appear in [13], but in a different context.

The bivariate Appell interpolant related to the sequence $\{\mathscr{H}_k\}_{k\in\mathbb{N}}^b$ is

$$i_n[f](x,y) = \int_0^1\int_0^1 f(x,y)\,dx\,dy + \sum_{k=1}^{n} \frac{\mathscr{H}_k(x,y)}{k!} \int_0^1\int_0^1 f^{(k,0)}(x,y)dx\,dy,$$

which provides an expansion of a bivariate function in terms of HKF polynomials.

The complementary Appell interpolant is

$$i_n^*[f](x,y) = f(x_0,y_0) + \sum_{k=1}^{n} \frac{\mathscr{H}_k(x,y) - \mathscr{H}_k(x_0,y_0)}{k!} \int_0^1\int_0^1 f^{(k,0)}(x,y)dx\,dy.$$

B3. **Arithmetic mean functional**

Let's consider the functional as in (36). In this case

$$\beta_0 = 1, \quad \beta_k = \mathscr{M}_0\left(H_k^{(2)}\right) = \frac{1}{4}\left(1 + \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{k!}{j!(k-2j)!} + \begin{cases} \frac{n!}{(\frac{n}{2})!} & n \; even \\ 0 & n \; odd \end{cases}\right), \quad k \geq 1.$$

Then the bivariate Appell sequence associated with the functional $\mathscr{M}_0$ is given by the recurrence formula

$$r_n(x, y) = H_n^{(2)}(x, y) - \sum_{k=0}^{n-1} \binom{n}{k} \beta_{n-k} r_k(x, y).$$

From Proposition 1 the generating function is

$$F(x, y; t) = \frac{e^{xt+yt^2}}{\mathscr{M}_0\left(e^{xt+yt^2}\right)}.$$

Since $\mathscr{M}_0\left(e^{xt+yt^2}\right) = \dfrac{\left(e^t+1\right)\left(e^{t^2}+1\right)}{4}$, then

$$\frac{4\, e^{xt+yt^2}}{\left(e^t+1\right)\left(e^{t^2}+1\right)} = \sum_{n=0}^{\infty} r_n(x, y) \frac{t^n}{n!}.$$

Setting

$$\frac{4}{\left(e^t+1\right)\left(e^{t^2}+1\right)} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!},$$

we have

$$\alpha_k = k! \sum_{s=0}^{\left[\frac{k}{2}\right]} \frac{E_s(0) E_{k-2s}(0)}{s!(k-2s)!},$$

where $E_n(x)$ is the classic Euler polynomial of degree n. Then, from (3) we obtain

$$r_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} H_k^{(2)}(x, y). \tag{49}$$

Remark 7 The bivariate Appell sequence $\{r_n\}_{n\in\mathbb{N}}^b$ appears in [13], but in a different context. We call it bivariate Euler HKF-based polynomial sequence of order 2 and denote it by $\left\{\mathscr{K}_n^{(2)}\right\}_{n\in\mathbb{N}}^b$.

The first bivariate Euler HKF-based polynomials of order 2 are:

$$\mathscr{K}_0^{(2)}(x, y) = 1, \quad \mathscr{K}_1^{(2)}(x, y) = x - \frac{1}{2}, \quad \mathscr{K}_2^{(2)}(x, y) = x^2 - x + 2y - 1,$$
$$\mathscr{K}_3^{(2)}(x, y) = x^3 - \frac{3}{2}x^2 - 3x - 3y + 6xy + \frac{7}{4},$$
$$\mathscr{K}_4^{(2)}(x, y) = x^4 - 2x^3 - 6x^2 + 7x + 12y^2 + 12x^2y - 12xy - 12y.$$

Finally, the bivariate Appell interpolant is

$$i_n[f](x,y) = \mathscr{M}_0(f) + \sum_{k=1}^{n} \mathscr{M}_0\left(f^{(k,0)}\right) \frac{\mathscr{K}_k^{(2)}(x,y)}{k!}.$$

The complementary interpolant is

$$i_n^*[f](x,y) = f(0,0) + \sum_{k=1}^{n} \mathscr{M}_0\left(f^{(k,0)}\right) \frac{\mathscr{K}_k^{(2)}(x,y) - \mathscr{K}_k^{(2)}(0,0)}{k!}.$$

Remark 8 All the bivariate HKF-based Appell interpolants satisfy the known heat equation.

Remark 9 All the bivariate Appell interpolants connected to the arithmetic mean linear functional use only boundary values.

Table 1 contains the list of the considered polynomial sequences and the related Appell interpolants.

6 Numerical results

In order to verify the previous theoretical results we consider the comparison between some functions and the related bivariate Appell interpolant. Particularly, we consider the following functions

- $f_1(x,y) = \sin(x+y)$
- $f_2(x,y) = \ln(x+y+5)$
- $f_3(x,y) = e^{-\frac{x+2}{4y+9}}$

and their interpolants. For every function we calculate the maximum error

$$E_{\mathrm{Q}} = \max_{(x,y)\in[0,1]\times[0,1]} \left| f_k(x,y) - i_n[f_k](x,y) \right|,$$

with $n = 1, \ldots, 10$, $k = 1, 2, 3$ and Q$\in \left\{\text{A1, A2, A3, B1, B2, B3}\right\}$.

In order to compare the numerical results of our interpolant with other approximants, we consider the well-known bivariate Bernstein polynomial [25].

Tables 2, 3, and 4 show the results for f_1, f_2 and f_3 respectively. The last column of each table contains the maximum error in the case of approximation by means of bivariate Bernstein polynomials.

From the previous tables we can observe that the results obtained by the interpolants based on $H_n^{(1)}$ (cases A1, A2, A3) are satisfactory and comparable favorably with those obtained by Bernstein approximations. The interpolants based on $H_n^{(2)}$ (cases B1, B2, B3) need a more in depth study from a computational point of view, particularly taking into account stability and accuracy.

Table 1 List of polynomial sequences and Appell interpolants

Functional	$\phi(y,t)$	$A(t)$	Polynomial sequence	Appell interpolant
$L(f)=f(0,0)$	e^{yt}	1	$H_k^{(1)}(x,y)$ (see (27))	$t_n[f](x,y)=\sum_{k=0}^n f^{(k,0)}(0,0)\frac{H_k^{(1)}(x,y)}{k!}$
$L(f)=f(0,0)$	e^{yt^2}	1	$H_k^{(2)}(x,y)$ (see (44))	$i_n[f](x,y)=\sum_{k=0}^n f^{(k,0)}(x_0,y_0)\frac{H_k^{(2)}(x,y)}{k!}$
$L(f)=\int_0^1\int_0^1 f(x,y)dx\,dy$	e^{yt}	$\left(\frac{t}{e^t-1}\right)^2$	$\mathscr{B}_k^{(2)}(x,y)$ (see (33))	$i_n[f](x,y)=\sum_{k=0}^n\frac{\mathscr{B}_k^{(2)}(x,y)}{k!}\int_0^1\int_0^1 f^{(k)}(x,y)\,dx\,dy$
				$i_n^*[f](x,y)=f(0,0)+\sum_{k=1}^n\frac{\mathscr{B}_k^{(2)}(x,y)-\mathscr{B}_k^{(2)}(0,0)}{k!}\int_0^1\int_0^1 f^{(k)}(x,y)\,dx\,dy$
$L(f)=\int_0^1\int_0^1 f(x,y)dx\,dy$	e^{yt^2}	$\frac{t^3}{(e^t-1)\left(e^{t^2}-1\right)}$	$\mathscr{H}_k(x,y)$ (see (48))	$i_n[f](x,y)=\sum_{k=0}^n\frac{\mathscr{H}_k(x,y)}{k!}\int_0^1\int_0^1 f^{(k)}(x,y)\,dx\,dy$
				$i_n^*[f](x,y)=f(x_0,y_0)+\sum_{k=1}^n\frac{\mathscr{H}_k(x,y)-\mathscr{H}_k(x_0,y_0)}{k!}\int_0^1\int_0^1 f^{(k,0)}(x,y)dx\,dy$
$L(f)=\mathscr{M}_{0,0}(f)$	e^{yt}	$\left(\frac{2}{e^t+1}\right)^2$	$\mathscr{E}_k^{(2)}(x,y)$ (see (42))	$i_n[f](x,y)=\sum_{k=0}^n\mathscr{M}_{0,0}\left(f^{(k,0)}\right)\frac{\mathscr{E}_k^{(2)}(x,y)}{k!}$
				$i_n^*[f](x,y)=f(0,0)+\sum_{k=1}^n\mathscr{M}_{0,0}\left(f^{(k,0)}\right)\frac{\mathscr{E}_k^{(2)}(x,y)-\mathscr{E}_k^{(2)}(0,0)}{k!}$
$L(f)=\mathscr{M}_{0,0}(f)$	e^{yt^2}	$\frac{4}{(e^t+1)\left(e^{t^2}+1\right)}$	$\mathscr{K}_k^{(2)}(x,y)$ (see (49))	$i_n[f](x,y)=\sum_{k=0}^n\mathscr{M}_{0,0}\left(f^{(k,0)}\right)\frac{\mathscr{K}_k^{(2)}(x,y)}{k!}$
				$i_n^*[f](x,y)=f(0,0)+\sum_{k=1}^n\mathscr{M}_{0,0}\left(f^{(k,0)}\right)\frac{\mathscr{K}_k^{(2)}(x,y)-\mathscr{K}_k^{(2)}(0,0)}{k!}$

Table 2 E_Q for $f_1(x, y) = \sin(x + y)$

n	E_{A1}	E_{A2}	E_{A3}	E_{B1}	E_{B2}	E_{B3}	E_{Bern}
1	1.09e+00	3.61e-01	2.31e-01	8.41e-01	5.25e-01	4.40e-01	2.01e-01
2	1.09e+00	4.54e-02	7.32e-02	8.41e-01	8.47e-01	7.64e-01	1.05e-01
3	2.42e-01	5.26e-03	3.52e-02	1.07e+00	7.31e-01	6.42e-01	7.13e-02
4	2.42e-01	8.51e-04	9.63e-03	1.07e+00	8.03e-01	6.55e-01	5.41e-02
5	2.40e-02	1.65e-04	4.78e-03	8.42e-01	7.97e-01	6.47e-01	4.35e-02
6	2.40e-02	2.28e-05	1.22e-03	8.42e-01	8.00e-01	6.23e-01	3.64e-02
7	1.36e-03	5.81e-06	6.07e-04	8.66e-01	8.00e-01	6.27e-01	3.13e-02
8	1.36e-03	1.21e-06	1.51e-04	8.66e-01	7.99e-01	6.25e-01	2.75e-02
9	5.00e-05	8.40e-07	7.40e-05	8.57e-01	7.99e-01	6.25e-01	2.45e-02
10	5.00e-05	6.85e-07	1.80e-05	8.57e-01	7.99e-01	6.28e-01	2.21e-02

Table 3 E_Q for $f_2(x, y) = \ln(x + y + 5)$

n	E_{A1}	E_{A2}	E_{A3}	E_{B1}	E_{B2}	E_{B3}	E_{Bern}
1	6.35e-02	1.25e-02	7.85e-03	1.82e-01	9.62e-02	9.15e-02	7.13e-03
2	1.64e-02	8.07e-04	1.20e-03	2.22e-01	1.08e-01	1.06e-01	3.57e-03
3	4.86e-03	3.33e-05	2.34e-04	2.22e-01	1.10e-01	1.08e-01	2.38e-03
4	1.53e-03	6.82e-06	6.16e-05	2.27e-01	1.10e-01	1.08e-01	1.79e-03
5	5.09e-04	2.19e-06	2.01e-05	2.27e-01	1.10e-01	1.08e-01	1.43e-03
6	1.73e-04	1.76e-06	7.70e-06	2.28e-01	1.10e-01	1.07e-01	1.19e-03
7	6.04e-05	1.70e-06	3.46e-06	2.28e-01	1.10e-01	1.07e-01	1.02e-03
8	2.14e-05	1.70e-06	1.77e-06	2.28e-01	1.10e-01	1.07e-01	8.93e-04
9	7.69e-06	1.70e-06	1.01e-06	2.28e-01	1.10e-01	1.07e-01	7.94e-04
10	2.79e-06	1.69e-06	6.54e-07	2.29e-01	1.10e-01	1.07e-01	7.14e-04

Table 4 E_Q for $f_3(x, y) = e^{-\frac{x+2}{4y+9}}$

n	E_{A1}	E_{A2}	E_{A3}	E_{B1}	E_{B2}	E_{B3}	E_{Bern}
1	1.71e-02	7.27e-03	7.57e-03	8.21e-03	3.87e-03	3.87e-03	6.11e-04
2	1.51e-02	7.22e-03	7.39e-03	6.73e-03	3.61e-03	3.51e-03	3.11e-04
3	1.53e-02	7.22e-03	7.38e-03	6.86e-03	3.63e-03	3.53e-03	2.08e-04
4	1.53e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	1.56e-04
5	1.52e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	1.25e-04
6	1.52e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	1.04e-04
7	1.52e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	8.94e-05
8	1.52e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	7.82e-05
9	1.52e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	6.95e-05
10	1.52e-02	7.22e-03	7.38e-03	6.85e-03	3.62e-03	3.53e-03	6.26e-05

7 Conclusions

In this paper we proposed a new type of linear interpolation for bivariate functions, called bivariate Appell interpolation. The interpolant conditions are not usual, but they are expressed in terms of a linear functional L, with $L(1) \neq 0$, on the space $C^N(X)$, $N > 1$, where X is a linear space of bivariate real functions defined in $D \subset \mathbb{R}^2$. We proved that for every $f \in C^N(X)$ there exists a unique bivariate polynomial $i_n[f](x, y)$ such that $L\left(i_n[f]^{(j,0)}\right) = L\left(f^{(j,0)}\right)$, $j = 0, \ldots, n$. To the bivariate Appell interpolant $i_n[f]$, which depends on the functional L, is associated the complementary interpolant $i_n^*[f]$, in which $L(f)$ is substituted by $f(u, v)$, being (u, v) an arbitrary fixed point. The truncation error for the bivariate interpolants are defined and bounds are given by Sard's Theorem. As examples we considered the bivariate Appell polynomials based on $H_n^{(i)}(x, y)$, $i = 1, 2$, and, for every family, three different linear functionals. So we obtained ten new bivariate interpolants for real, regular bivariate functions. We gave also numerical examples and comparisons with the bivariate Berstein polynomial. The comparison is advantageous except in the case of $H_n^{(2)}(x, y)$, for which further investigations are needed.

Further developments are possible. Beside the aforementioned computational aspects, the study of interpolant series for analytic functions with particular properties seems to be of interest. Other developments can be applications of interpolants, such as numerical cubature and numerical solution of boundary value problems for partial differential equations. Furthermore, theoretical attention can be given to the role of two variables in the definition of bivariate Appell extension.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Declarations

Conflict of interest The authors declare no competing interests.

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