A separation between the boundary shape and the boundary functions in the parametric integral equation system for the 3D Stokes equation

Eugeniusz Zieniuk¹ · Krzysztof Szerszeń¹

Received: 18 July 2017 / Accepted: 1 March 2018 / Published online: 3 May 2018
© The Author(s) 2018

Abstract The paper introduces the analytical modification of the classic boundary integral equation (BIE) for Stokes equation in 3D. The performed modification allows us to obtain separation of the approximation boundary shape from the approximation of the boundary functions. As a result, the equations, called the parametric integral equation system (PIES) with formal separation between the boundary geometry and the boundary functions, are obtained. It enables us to independently choose the most convenient methods of boundary geometry modeling depending on its complexity without any intrusion into the approximation of boundary functions and vice versa. Furthermore, we investigated the possibility of the modeling of the domains bounded by rectangular and triangular parametric Bézier patches. The boundary functions are approximated by generalized Chebyshev series. In addition, numerical techniques for solving the obtained PIES have been developed. The effectiveness of the presented strategy for boundary representation by surface patches in connection with PIES has been studied in numerical examples.

Keywords Parametric integral equation system (PIES) · Boundary integral equation (BIE) · Stokes equation · Bézier surface patches

✉ Eugeniusz Zieniuk
ezieniuk@ii.uwb.edu.pl

Krzysztof Szerszeń
kszerszen@ii.uwb.edu.pl

¹ Faculty of Mathematics and Computer Science, University of Białystok, Konstantego Ciołkowskiego 1M, 15-245, Białystok, Poland

1 Introduction

The Stokes equation governs the motion of fluids where advective inertial forces are small compared with viscous forces [1–5]. In order to numerically solve this equation, in addition to the finite element method (FEM) [6–8] and meshless methods [9–13], another popular computational method – the boundary element method (BEM) [14–18] is naturally predisposed, where only the discretization of the boundary in the contact between the solid body and the fluid is required. Unfortunately, taking into account the specificity of BEM, the solutions on the boundary are directly obtained in discrete form at declared nodes of boundary elements. Furthermore, the improvement of accuracy in BEM is achieved through increasing the number of used elements or introducing higher-order elements. Hence, it requires multiple building of the element mesh structure with an increasing number of nodes. This seems to be particularly ineffective in the case of 3D problems, where a large number of elements may be redundant from the viewpoint of the accuracy of the boundary shape description.

In order to eliminate the above-mentioned limitations of the elements methods, an alternative approach for solving boundary value problems, which is called the parametric integral equation system (PIES), has been proposed. The concept of PIES provides a formal separation between boundary shape representation and approximation of the boundary values. In other words, it enables us to independently choose the most convenient methods for the boundary representation depending on its shape complexity without any intrusion into the approximation of boundary functions and vice versa. Furthermore, the above separation allows to eliminate in PIES the necessity for both boundary and domain discretization and introduces a new and more effective way to represent the shape of the boundary by the tools used for computer visualization of 3D objects, such as parametric surface patches. At the same time, it is possible to directly incorporate such patches into the mathematical formula of PIES used to solve boundary value problems [19] - only the control points necessary to define the shape of the boundary in the proposed approach, and their number depends on the complexity of the considered shape. This approach is different from the recently popular isogeometric analysis [20–22], because in PIES, the patches are applied at the level of analytical modification of boundary integral equation (BIE).

Previous research about the practical application of boundary representation by parametric surfaces in PIES in the case of 3D problems are concerned mainly with the boundary problems described by Laplace [23, 24] and Helmholtz [25] equations. Only preliminary studies for Stokes flow have so far been carried out and have been limited to boundaries described by flat Coons patches [26].

The main purpose of this paper is to present the modification of the classical BIE for the Stokes equations in 3D in order to analytically incorporate into its mathematical formalism the boundary representation with parametric patches. The performed modification allows us to obtain formal separation of the approximation boundary shape from the approximation of the boundary functions for the Stokes equation. Furthermore, we investigated the possibility of the modeling of the domains bounded by rectangular and triangular parametric Bézier patches. The shape of the boundary is, in this case, described by a limited and relatively small set of control points of the

patches. In turn, the boundary functions on an individual Bézier will be approximated by the Chebyshev series. In this case, improving the accuracy of numerical solutions in PIES is reduced to the increase of the number of terms in the Chebyshev series. Additionally, we propose a method for the numerical solution of the obtained PIES and solve several testing examples in order to verify the reliability of the analytical modification of BIE and the accuracy of the obtained solutions.

2 The strategy of boundary representation for flow problems

The boundary problems or boundary-initial problems require the consideration of the domain where the problems are defined. In the case of the FEM, the domain of the problem is modeled by finite elements, while in the case of the BEM, only the boundary of the considered domain is described by boundary elements. These elements, due to their elementary shape, allow for modeling different domains and boundaries and their number has a direct impact on the accuracy of the solutions.

However, regardless of the popularity of FEM and BEM, it is advisable to search for more efficient ways for domain and boundary description in boundary value problems. It seems to be interesting to model the boundary of the problem with tools from computer graphics, for example by parametric surfaces. Figure 1a shows the sample boundary modeled by eight triangular Bézier patches. To define the boundary, only the control points of triangular Bézier patches are used. In this case, the process of modification of the boundary shape is very simple, which is reduced to changing the position of one or more selected control points, as is shown in Fig. 1b. The modified domain is still modeled by eight patches.

The advantages of surface patches were also noticed by the researchers dealing with MEB, but they used them mainly as the representation of boundary elements that divided the boundary. These elements, however, due to the concepts of BEM, would be significantly more in number than the patches shown in Fig. 1. Moreover,

Fig. 1 Sample modeling and modification of 3D boundary representation with surface patches

the modification of such a boundary requires their rediscretization. In addition, considering the patches as boundary elements significantly reduces their advantages such as automatically providing the appropriate class of continuity on the edges of their intersection. This continuity is also automatically provided during the modification of the domain, as shown in Fig. 1b.

The present contribution aims to fully exploit the advantages of the patches by using a different strategy than in BEM. The strategy is not based on dividing the boundary into elements as in BIE, but on analytical modification of BIE to directly apply the way of boundary representation by the surface patches shown in Fig. 1. In other words, we want to incorporate the equations describing the surface patches into BIE in order to define the boundary, instead of generally defining the boundary as a symbol of a boundary integral as is performed into classical BIE. Such a definition of the boundary by a symbol of a boundary integral requires their dividing into elements during numerical solving of BIE.

The basic equations describing the surface patches used for the presented modification of classical BIE in Section 3 are presented below. In this way, it is possible to declare both flat as well as curved surfaces.

2.1 Rectangular Bézier patches

Rectangular Bézier patch is defined by a set of $(n + 1)(m + 1)$ control points P_{ij} (where $0 \leq i \leq n$ and $0 \leq j \leq m$) and is written as follows [27]

$$P(v, w) = \sum_{i=0}^m \sum_{j=0}^n P_{ij} B_i^m(v) B_j^n(w), \text{ for } 0 \leq v, w \leq 1, \quad (1)$$

where $B_i^m(v)$, $B_j^n(w)$ are Bernstein basis functions written in the form analogous for Bézier curves

$$\begin{aligned} B_i^m(v) &= \binom{m}{i} v^i (1-v)^{m-i}, \\ B_j^n(w) &= \binom{n}{j} w^j (1-w)^{n-j} \text{ for } i = 0, \dots, m, j = 0, \dots, n. \end{aligned} \quad (2)$$

Figure 2 shows the patch of degree 3 created by 16 control points.

The matrix form of the Bézier patch of degree 3 takes the following form

$$\begin{aligned} P(v, w) &= \sum_{i=0}^3 \sum_{j=0}^3 P_{ij} B_i^3(v) B_j^3(w) \\ &= \begin{bmatrix} B_0^3(v) & B_1^3(v) & B_2^3(v) & B_3^3(v) \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} B_0^3(w) \\ B_1^3(w) \\ B_2^3(w) \\ B_3^3(w) \end{bmatrix}. \end{aligned} \quad (3)$$

The main advantage of Bézier patches is the simplicity of defining and modifying surface with help of only control points. Examples of such modifications are shown in Fig. 3b.

Fig. 2 Rectangular bicubic Bézier patch with control points and the parametric domain v, w

2.2 Triangular Bézier patches and their rational modification

A triangular Bézier patch of degree n is defined by a set of $0.5(n+1)(n+2)$ control points P_{ijk} wherein indexes i, j, k must satisfy the following relationship $i+j+k = n$. The mathematical formulation of such a patch is dependent upon the barycentric coordinates $u = (v, w, u)$ inside the triangle and presented as [27]

$$P(u) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} P_{ijk} B_{ijk}^n(u), \quad (4)$$

Fig. 3 Two sample shapes of bicubic Bézier patch: initial flat surface (a) and the modified shape after moving control point P_{00} (b)

wherein $B_{ijk}^n(\mathbf{u})$ are the basis functions stated as follows

$$B_{ijk}^n(\mathbf{u}) = \frac{n!}{i!j!k!} v^i w^j u^k, \text{ for } 0 \leq v, w, u \leq 1, w + v + u = 1, \quad (5)$$

Figure 4 shows triangular patches of degrees 3 and 4 defined by 10 and 15 control points, respectively. For exact representation of the sphere, we need to use the rational modification of triangular patches. The rational Bézier patch is expressed by the following formula [27]

$$\mathbf{P}(\mathbf{u}) = \frac{\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} w_{ijk} \mathbf{P}_{ijk} B_{ijk}^n(\mathbf{u})}{\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} w_{ijk} B_{ijk}^n(\mathbf{u})} \quad (6)$$

In relation to formula (4), it also contains the weights w_{ijk} associated with each control point $\mathbf{P}_{ijk}$. Bernstein basis functions $B_{ijk}^n(\mathbf{u})$ are described by formula (5).

After substituting $u = 1 - v - w$ into (4) and (6) and introducing additional constraints $0 \leq v, w \leq 1$ and $v + w \leq 1$, we can obtain a more convenient mathematical representation of such patches dependent only on two parameters v, w , which allows for a unified mathematical description for both rectangular and triangular surfaces in PIES.

Fig. 4 Triangular Bézier patches of degree: 3 (a) and 4 (b) with control points and the parametric domain v, w

Fig. 5 The spherical boundary represented with rational Bézier patches in PIES

The rational patches are used for exact representation of the sphere in Examples 2 and 3 in Section 7. As shown in Fig. 5, the sphere can be modeled by eight symmetrical patches of degree 4.

The single triangular patch which approximates one eight of the sphere of radius 1 with the center at the origin is defined by 15 control points with the following coordinates (placed in the first quadrant of Cartesian coordinates) [27]

$$\begin{aligned} &P_{004}\{0, 1, 0\} \quad P_{013}\{\alpha, 1, 0\} \quad P_{022}\{\beta, \beta, 0\} \quad P_{031}\{1, \alpha, 0\} \quad P_{040}\{1, 0, 0\} \\ &P_{103}\{0, 1, \alpha\} \quad P_{112}\{\gamma, 1, \gamma\} \quad P_{121}\{1, \gamma, \gamma\} \quad P_{130}\{1, 0, \alpha\} \\ &P_{202}\{0, \beta, \beta\} \quad P_{211}\{\gamma, \gamma, 1\} \quad P_{220}\{\beta, 0, \beta\} \\ &P_{301}\{0, \alpha, 1\} \quad P_{310}\{\alpha, 0, 1\} \\ &P_{400}\{0, 0, 1\} \end{aligned} \quad , \quad (7)$$

where $\alpha = (\sqrt{3} - 1)/\sqrt{3}$, $\beta = (\sqrt{3} + 1)/2\sqrt{3}$, $\gamma = 1 - (5 - \sqrt{2})(7 - \sqrt{3})/46$, with corresponding weights

$$w_{040} = 4\sqrt{3}(\sqrt{3} - 1), \quad w_{031} = 3\sqrt{2}, \quad w_{202} = 4, \quad w_{121} = \frac{\sqrt{3}}{\sqrt{2}}(3 + 2\sqrt{2} - \sqrt{3}). \quad (8)$$

The declaration of the full sphere requires to define 66 control points.

3 Modification of BIE for the Stokes equation in 3D

We consider 3D Stokes problem defined on domain Ω bounded by boundary D . Boundary integral equation (BIE) for the Stokes equation can be represented by the integral identity [28]

$$\bar{u}_j(\mathbf{x}) = \frac{-1}{8\pi\mu} \int_S U_{ji}(\mathbf{x}, \mathbf{y}) f_i(\mathbf{y}) dS(\mathbf{y}) + \frac{1}{8\pi} \int_S u_i(\mathbf{y}) T_{ijk}(\mathbf{x}, \mathbf{y}) n_k(\mathbf{y}) dS(\mathbf{y}), \quad (9)$$

$$\text{where } \bar{\mathbf{u}}_j(\mathbf{x}) = \begin{cases} \bar{\mathbf{u}}_j(\mathbf{x}) & \mathbf{x} \in \Omega \\ 0.5\bar{\mathbf{u}}_j(\mathbf{x}) & \text{for } \mathbf{x} \in S, f(\mathbf{y}) \equiv \sigma \cdot \mathbf{n}. \\ 0 & \mathbf{x} \notin \bar{\Omega} \end{cases}$$

In identity (9), the integrand $U_{ij}^*(\mathbf{x}, \mathbf{y})$ $i, j = 1, 2, 3$ is known as the fundamental solution for the Stokes equation and is represented by the following formula [28].

$$U_{ij}(\mathbf{x}, \mathbf{y}) = \delta_{ij} \frac{1}{r} + \frac{(x_i - y_i)(x_j - y_j)}{r^3}, \quad (10)$$

where μ is the dynamic viscosity and δ_{ij} is the Kronecker symbol, whereas

$$r = \left[(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 \right]^{0.5}. \quad (11)$$

In physical meaning, $U_{ij}(\mathbf{x}, \mathbf{y})$ can be interpreted as i th velocity component at $\mathbf{x}$ due to unit force in the j direction applied at $\mathbf{y}$.

In turn, $T_{ijk}(\mathbf{x}, \mathbf{y})$ $i, j, k = 1, 2, 3$ in (9) is the classical singular solution represented by traction kernels written as follows

$$T_{ijk}(\mathbf{x}, \mathbf{y}) = -6 \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5}. \quad (12)$$

The value of $\bar{\mathbf{u}}_j(\mathbf{x})$ depends on the location of $\mathbf{x}$ and $\bar{\mathbf{u}}_j(\mathbf{x}) = \mathbf{u}_j(\mathbf{x})$ for $\mathbf{x} \in \Omega$, $\bar{\mathbf{u}}_j(\mathbf{x}) = 0.5\mathbf{u}_j(\mathbf{x})$ for $\mathbf{x} \in S$, $\bar{\mathbf{u}}_j(\mathbf{x}) = 0$ for $\mathbf{x} \notin \bar{\Omega}$. If $\mathbf{x} \in S$, then formula (9) is the classical BIE for smooth boundary. Boundary functions $u_i(\mathbf{y})$, $f_i(\mathbf{y})$ in (9) in physical interpretation represent the distribution of the velocity and the normal traction fields on the boundary D , respectively.

The accuracy of the solutions in the domain Ω depends upon two factors: (1) the accuracy of the approximation of the boundary functions on the boundary and (2) the accuracy of the approximation of the shape of the boundary geometry. Hence, it seems reasonable to modify the traditional BIE in such a way as to obtain an independent boundary shape approximation from the approximation of the unknown functions in the numerical solution. In other words, it enables us to independently choose the most convenient methods of boundary geometry modeling depending on its complexity and also most convenient methods for the approximation of the boundary functions.

Due to the above, we take up the analytical modification of BIE first so that the boundary geometry is not defined by the boundary integral, but is included in the mathematical formalism of the expression obtained after the modification of BIE.

3.1 Fourier transformation of the integral identity

The modification of generalized integral identity (9) for 3D is more complex than the one obtained for the potential problems modeled by Laplace's equation [23]. To do that, we need to apply the Fourier transform to identity (9). As a result, we obtain

$$\hat{u}_j(i\xi) = \frac{-1}{8\pi\mu} \int_S \hat{U}_{ji}(i\xi, y) f_i(y) dS(y) + \frac{1}{8\pi} \int_S u_i(y) \hat{T}_{ijk}(i\xi, y) n_k(y) dS(y), \quad (13)$$

$$\text{and } \hat{u}_j(i\xi) = \begin{cases} \hat{u}_j(i\xi) & \xi \in \Omega' \\ 0.5\hat{u}_j(i\xi) & \text{for } \xi \in S' \\ 0 & \xi \notin \bar{\Omega}' \end{cases},$$

where Ω' and S' are the domains of the Fourier transforms for Ω and S , respectively.

In turn, $\hat{U}_{ji}(i\xi, y)$ is the transform of the fundamental solution, whereas $\hat{T}_{ijk}(i\xi, y)$ is the transform of the singular solution in the domain of the Fourier transforms for the Stokes equation presented in the form

$$\hat{U}_{ji}(i\xi, y) = L_{ji}(i\xi) e^{-i(\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3)}, \quad (14)$$

$$\hat{T}_{ijk}(i\xi, y) = \bar{T}_{ijk}(i\xi) e^{-i(\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3)}, \quad (15)$$

where

$$L_{ji}(i\xi) = \frac{1}{\mu |\xi|^4} \left\{ \delta_{ji} |\xi|^2 - \xi_j \xi_i \right\}, \quad |\xi|^2 = \left| \xi_1^2 + \xi_2^2 + \xi_3^2 \right| \quad (16)$$

and

$$\bar{T}_{ijk}(i\xi) = -i \left[\delta_{ik} \frac{\xi_j}{|\xi|^2} + \xi_k L_{ij}(i\xi) + \xi_i L_{kj}(i\xi) \right]. \quad (17)$$

After substituting (14) and (15) into (13), we obtain the following form of the Fourier transforms in the domain

$$\hat{u}_j(i\xi) = \frac{-1}{8\pi\mu} L_{ji}(i\xi) \tilde{f}_i(\xi) + \frac{1}{8\pi} \bar{T}_{ijk}(i\xi) \tilde{u}_i \tilde{n}_k(\xi), \quad (18)$$

where boundary functions $\tilde{f}_i(\xi)$ and $\tilde{u}_i \tilde{n}_k(\xi)$, $k = 1, 2, 3$ are represented by means of the following boundary integrals

$$\tilde{f}_i(\xi) = \int_S f_i(y) e^{-i(\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3)} dS(y), \quad (19)$$

$$\tilde{u}_i \tilde{n}_k(\xi) = \int_S u_i(y) n_k(y) e^{-i(\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3)} dS(y). \quad (20)$$

We can see the separation of the boundary from the domain in (18). The boundary is defined by the formalism of boundary integrals (19) and (20).

In our further considerations, the integral (20) is used to describe the transformation $\tilde{u}_i \tilde{n}_k(\xi)$ on the boundary S . The unknown identity function $u_i(\mathbf{y})$ in (20) can be defined by means of the following Fourier formula

$$u_i(\mathbf{y}) = \frac{1}{8\pi^3} \int_{R^3} e^{i(\omega_1 y_1 + \omega_2 y_2 + \omega_3 y_3)} \hat{u}_i(i\boldsymbol{\omega}) d\boldsymbol{\omega}, \quad \boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3), \quad (21)$$

where integrand $\hat{u}_i(\boldsymbol{\omega})$ is written as

$$\hat{u}_i(i\boldsymbol{\omega}) = \frac{-2}{8\pi\mu} L_{ij}(i\boldsymbol{\omega}) \tilde{f}_j(\boldsymbol{\omega}) + \frac{2}{8\pi} \bar{T}_{jik}(i\boldsymbol{\omega}) \tilde{u}_j \tilde{n}_k(\boldsymbol{\omega}). \quad (22)$$

Formula (22) is the particular case of the transform (18) for a smooth boundary. Therefore, formula (22) is related to variable $\boldsymbol{\omega}$.

3.2 The definition of the smooth contour integral by surface patches

After including (21) in (20), we obtain the convolution integral equation in the domain of the Fourier transform

$$\tilde{u}_i \tilde{n}_k(\xi) = \int_{R^3} K_k(\boldsymbol{\omega} - \xi) \hat{u}_i(i\boldsymbol{\omega}) d\boldsymbol{\omega}, \quad (23)$$

where the kernel is

$$K_k(\boldsymbol{\gamma}) = \frac{1}{8\pi^3} \int_S e^{i(\gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3)} n_k(\mathbf{y}) dS(\mathbf{y}), \quad \boldsymbol{\gamma} \equiv \{\gamma_1, \gamma_2, \gamma_3\}, \quad \gamma_i = \omega_i - \xi_i. \quad (24)$$

The contour integral in (24) takes into consideration the boundary geometry S . For a further transformation of the above analytical formulas, we need to specify the boundary S . We assume that the boundary is defined with a finite number of curved surfaces S_l ($l = 1, 2, \dots, n$). They can be described by parametric patches from computer graphics [19] as presented during analytical modification of BIE.

Therefore, having considered the boundary geometry defined by surface patches in kernel (24), we obtain

$$K_k(\boldsymbol{\gamma}) = \sum_{l=1}^n K_k^{(l)}(\boldsymbol{\gamma}), \quad (25)$$

where

$$K_k^{(l)}(\boldsymbol{\gamma}) = \frac{1}{8\pi^3} \int_{S_l} e^{i(\gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3)} n_k^{(l)}(\mathbf{y}) dS(\mathbf{y}). \quad (26)$$

The boundary transforms $\tilde{u}_i \tilde{n}_k(\xi)$ occurring on the left-hand side of (23) may be represented in the following form

$$\tilde{u}_i \tilde{n}_k(\xi) = \sum_{l=1}^n \tilde{u}_i \tilde{n}_k^{(l)}(\xi), \quad (27)$$

where

$$\tilde{u}_i \tilde{n}_k^{(l)}(\xi) = \int_{S_l} e^{i(\gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3)} \tilde{u}_i(\mathbf{y}) \tilde{n}_k^{(l)}(\mathbf{y}) dS(\mathbf{y}). \quad (28)$$

We mark the boundary transforms $\tilde{f}_j(\omega)$ and $\tilde{u}_j \tilde{n}_k(\omega)$ on individual patches on the right-hand side of (22)

$$\tilde{f}_j(\omega) = \sum_{r=1}^n \tilde{f}_j^{(r)}(\omega), \quad \tilde{u}_j \tilde{n}_k(\omega) = \sum_{r=1}^n \tilde{u}_j \tilde{n}_k^{(r)}(\omega). \quad (29)$$

After substituting (29), (28), and (25) into (23), we obtain the following system of the convolution integral equations

$$0.5 \tilde{u}_i \tilde{n}_k^{(l)}(\xi) = \int_{R^3} \mathbf{K}_k^{(l)}(\omega - \xi) \sum_{r=1}^n \hat{u}_i^{(r)}(i\omega) d\omega, \quad l = 1, 2, \dots, n, \quad (30)$$

where

$$\hat{u}_i^{(r)}(i\omega) = \frac{-1}{8\pi\mu} L_{ij}(i\omega) \tilde{f}_j^{(r)}(\omega) + \frac{1}{8\pi} \bar{T}_{jik}(i\omega) \tilde{u}_j \tilde{n}_k^{(r)}(\omega). \quad (31)$$

3.3 Parametric patches for definition of the contour integrals

Next, we assume that patches $\mathbf{P}_l(v_1, w_1)$ are mathematically defined by the parametric functions $\mathbf{P}_l(v_1, w_1) = [P_l^{(1)}(v_1, w_1), P_l^{(2)}(v_1, w_1), P_l^{(3)}(v_1, w_1)]^T$, with parameters v_1, w_1 . Hence, these functions must be taken into account in boundary transforms (26), (27), and (29), in which the boundary is still defined in a general way by means of segmental contour integrals over $\mathbf{P}_l$.

The kernel (26) for the boundary defined by parametric patches is described by the following formula $y_i = P_l^i(v_1, w_1)$, $i = 1, 2, 3$

$$K_k^{(l)}(\boldsymbol{\gamma}) = \frac{1}{8\pi^3} \int_{\bar{v}_{l-1}}^{\bar{v}_l} \int_{\bar{w}_{l-1}}^{\bar{w}_l} e^{i[\gamma_1 P_l^{(1)}(v_1, w_1) + \gamma_2 P_l^{(2)}(v_1, w_1) + \gamma_3 P_l^{(3)}(v_1, w_1)]} n_k^{(l)}(v_1, w_1) J_l(v_1, w_1) dv_1 dw_1, \quad (32)$$

while the integral (28) in the following form

$$\begin{aligned} \tilde{u}_i \tilde{n}_k^{(l)}(\xi) &= \int_{\bar{v}_{l-1}}^{\bar{v}_l} \int_{\bar{w}_{l-1}}^{\bar{w}_l} e^{-i\{\xi_1 P_l^{(1)}(v_1, w_1) + \xi_2 P_l^{(2)}(v_1, w_1) + \xi_3 P_l^{(3)}(v_1, w_1)\}} \\ &\quad u_i^{(l)}(v_1, w_1) n_k^{(l)}(v_1, w_1) J_l(v_1, w_1) dv_1 dw_1 \end{aligned} \quad (33)$$

where $\bar{v}_{l-1}, \bar{v}_l, \bar{w}_{l-1}, \bar{w}_l$ are, respectively, the beginning and ending of the interval of the domain (indexed by l) with parameters v_1, w_1 . Whereas integral functions

$$\begin{aligned} n_k^{(l)}(v_1, w_1) &= n_k^{(l)}[P_l^{(1)}(v_1, w_1), P_l^{(2)}(v_1, w_1), P_l^{(3)}(v_1, w_1)], \\ u_i^{(l)}(v_1, w_1) &= u_i^{(r)}[P_l^{(1)}(v_1, w_1), P_l^{(2)}(v_1, w_1), P_l^{(3)}(v_1, w_1)], \end{aligned} \quad (34)$$

are obtained after substituting for variables y_i in surface patches $P_l^i(v_1, w_1)$, while the $J_l(v_1, w_1)$ is the Jacobian.

4 Parametric Integral Equation System (PIES)

After applying the formulas for the inverse of the Fourier transform to (30) and the corresponding transformations, the following expression requiring further integration is obtained

$$\begin{aligned} & \frac{0.5}{8\pi^3} \int_{R^3} \int_{\bar{v}_{l-1}}^{\bar{v}_l} \int_{\bar{w}_{l-1}}^{\bar{w}_l} e^{-i\{\xi_1\lambda_1+\xi_2\lambda_2+\xi_3\lambda_3\}} u_i^{(l)}(v_1, w_1) n_k^{(l)}(v_1, w_1) J_l(v_1, w_1) dv_1 dw_1 d\xi \\ &= \frac{1}{8\pi^3} \int_{R^3} \int_{R^3} \frac{1}{8\pi^3} \int_{\bar{v}_{l-1}}^{\bar{v}_l} \int_{\bar{w}_{l-1}}^{\bar{w}_l} e^{i\{\xi_1\lambda_1+\xi_2\lambda_2+\xi_3\lambda_3\}} n_k^{(l)}(v_1, w_1) J_l(v_1, w_1) \\ & \quad e^{i\{\omega_1 P_l^{(1)}(v_1, w_1) + \omega_2 P_l^{(2)}(v_1, w_1) + \omega_3 P_l^{(3)}(v_1, w_1)\}} \sum_{r=1}^n \hat{u}_i^{(r)}(i\omega) dv_1 dw_1 d\omega d\xi \end{aligned} \quad (35)$$

where $\lambda_i = x_i - P_l^{(i)}(v_1, w_1)$, $i = 1, 2, 3$.

After integrating both sides of (35) with respect to ξ and then to v_1 and w_1 and introducing some transformations, (35), after simplifying the left and right sides, finally takes the form

$$0.5u_i^{(l)}(\bar{v}, \bar{w}) = \frac{1}{8\pi^3} \int_{R^3} e^{i\{\omega_1 P_l^{(1)}(\bar{v}, \bar{w}) + \omega_2 P_l^{(2)}(\bar{v}, \bar{w}) + \omega_3 P_l^{(3)}(\bar{v}, \bar{w})\}} \sum_{r=1}^n \hat{u}_i^{(r)}(i\omega) d\omega, \quad l=1, 2, \dots, n. \quad (36)$$

Further integration of the right side of (36) with respect to ω requires consideration of the integrand $\hat{u}^{(r)}(\omega)$ in explicit form. This function is represented by formula (31) in which $\tilde{f}_j^{(r)}(\omega)$ and $\tilde{u}_j \tilde{n}_k^{(r)}(\omega)$ on surface patches r are presented in a similar form as previously written for patches l in (32) and (33) as follows

$$\tilde{f}_j^{(r)}(\omega) = \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} e^{-i\{\omega_1 P_r^{(1)}(v, w) + \omega_2 P_r^{(2)}(v, w) + \omega_3 P_r^{(3)}(v, w)\}} \tilde{f}_j^{(r)}(v, w) J_r(v, w) dv dw, \quad (37)$$

$$\tilde{u}_j \tilde{n}_k^{(r)}(\omega) = \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} e^{-i\{\omega_1 P_r^{(1)}(v, w) + \omega_2 P_r^{(2)}(v, w) + \omega_3 P_r^{(3)}(v, w)\}} u_j^{(r)}(v, w) n_k^{(r)}(v, w) J_r(v, w) dv dw, \quad (38)$$

where

$$\begin{aligned} f_j^{(r)}(v, w) &= f_j^{(r)}[P_r^{(1)}(v, w), P_r^{(2)}(v, w), P_r^{(3)}(v, w)], n_k^{(r)}(v, w) \\ &= n_k^{(r)} P_r^{(1)}(v, w), P_r^{(2)}(v, w), P_r^{(3)}(v, w)], \\ u_j^{(r)}(v, w) &= u_j^{(r)}(v, w)[P_r^{(1)}(v, w), P_r^{(2)}(v, w), P_r^{(3)}(v, w)], \end{aligned}$$

and $J_r(v, w)$ is the Jacobian while $n_k^{(r)}(v, w)$ is the normal vector. The Jacobian is determined for all patches considered in the paper by the following formula

$$J_r(v, w) = [A_1^2(v, w) + A_2^2(v, w) + A_3^2(v, w)]^{0.5}. \quad (39)$$

Functions $A_1(v, w)$, $A_2(v, w)$, $A_3(v, w)$ in (39) represent the combination of the partial derivatives of functions $P_r(v, w) = [P_r^{(1)}(v, w), P_r^{(2)}(v, w), P_r^{(3)}(v, w)]^T$ written as

$$\begin{aligned} A_1(v, w) &= \frac{\partial P_r^{(2)}(v, w)}{\partial w} \frac{\partial P_r^{(3)}(v, w)}{\partial v} - \frac{\partial P_r^{(2)}(v, w)}{\partial v} \frac{\partial P_r^{(3)}(v, w)}{\partial w}, \\ A_2(v, w) &= \frac{\partial P_r^{(3)}(v, w)}{\partial w} \frac{\partial P_r^{(1)}(v, w)}{\partial v} - \frac{\partial P_r^{(3)}(v, w)}{\partial v} \frac{\partial P_r^{(1)}(v, w)}{\partial w}, \\ A_3(v, w) &= \frac{\partial P_r^{(1)}(v, w)}{\partial w} \frac{\partial P_r^{(2)}(v, w)}{\partial v} - \frac{\partial P_r^{(1)}(v, w)}{\partial v} \frac{\partial P_r^{(2)}(v, w)}{\partial w}. \end{aligned} \quad (40)$$

Formulas (39) and (40) are general and give the possibility of determining the Jacobian for any patches, not only for the ones presented in Section 2. The components of the normal vector are obtained using the following general formula

$$n_k^{(r)}(v, w) = \frac{A_m(v, w)}{J_r(v, w)}, m = 1, 2, 3. \quad (41)$$

The normal vectors of the surface patches should be directed outward the domain. Surface patches are defined by only control points. The way of defining these points has impact on the direction of the normal vector.

After substituting (37) and (38) into (31) and then the received expression in (36), we obtain the formula termed parametric integral equation system (PIES) in the following form

$$\begin{aligned} 0.5u_l(\bar{v}, \bar{w}) &= \sum_{r=1}^n \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} \left\{ \bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w) f_r(v, w) \right. \\ &\quad \left. - \bar{P}_{lr}^*(\bar{v}, \bar{w}, v, w) u_r(v, w) J_r \right\} (v, w) dv dw, \end{aligned} \quad (42)$$

$$v_{l-1} \leq \bar{v} \leq v_l, \quad w_{l-1} \leq \bar{w} \leq w_l, \quad v_{r-1} \leq v \leq v_r, \quad w_{r-1} \leq w \leq w_r,$$

where J_r is the Jacobian. The kernels in (42) are represented by functions $\bar{U}_{lr}^*$ and $\bar{P}_{lr}^*$.

PIES for 3D problems is formally defined on 2D parametric plane. Solution of (42) requires the calculation of integrals for functions $\bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w)$ and $\bar{P}_{lr}^*(\bar{v}, \bar{w}, v, w)$. These integrals are formally defined within the ranges $v_{r-1}, w_{r-1} \leq v, w \leq v_r, w_r$. This definition of integration intervals in the PIES is to show that the patches are connected to each other in the corner points by intervals of integration. The edges of these patches are directly connected by means of suitably selected control points

common to the adjacent patches. These points are directly included in the functions $\bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w)$ and $\bar{P}_{lr}^*(\bar{v}, \bar{w}, v, w)$.

The first integrand $\bar{U}_{lr}^*$ is represented by the following expression

$$\bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w) = \frac{-1}{64\pi^4} \int_{R^3} e^{i(\omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3)} \hat{L}_{ji}(i\omega) d\omega, \quad (43)$$

where

$$\begin{aligned} \eta_1 &= P_r^{(1)}(v, w) - P_l^{(1)}(\bar{v}, \bar{w}), & \eta_2 &= P_r^{(2)}(v, w) - P_l^{(2)}(\bar{v}, \bar{w}), \\ \eta_3 &= P_r^{(3)}(v, w) - P_l^{(3)}(\bar{v}, \bar{w}). \end{aligned}$$

Having calculated the integral (43), we obtain an expression in an explicit form, which due to its inclusion of the shape of boundary geometry in its formalism, is called a modified fundamental boundary solution for the Stokes equation

$$\bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w) = -\frac{1}{8\pi\mu\eta} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}, \quad (44)$$

where

$$\begin{aligned} U_{11} &= 1 + \frac{\eta_1^2}{\eta^2}, & U_{12} &= \frac{\eta_1\eta_2}{\eta^2}, & U_{13} &= \frac{\eta_1\eta_3}{\eta^2}, & U_{21} &= \frac{\eta_2\eta_1}{\eta^2}, & U_{22} &= 1 + \frac{\eta_2^2}{\eta^2}, \\ U_{23} &= \frac{\eta_2\eta_3}{\eta^2}, & U_{31} &= \frac{\eta_3\eta_1}{\eta^2}, & U_{32} &= \frac{\eta_3\eta_2}{\eta^2}, & U_{33} &= 1 + \frac{\eta_3^2}{\eta^2}. \end{aligned}$$

The second integrand $\bar{P}_{lr}^*(v_1, w_1, v, w)$ in (42) is represented by the

$$\bar{P}_{lr}^*(\bar{v}, \bar{w}, v, w) = \frac{1}{64\pi^4} \int_{R^3} e^{i(\omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3)} \bar{T}_{ijk}(i\omega) n_k(\omega) d\omega, \quad (45)$$

where

$$\begin{aligned} \eta_1 &= P_r^{(1)}(v, w) - P_l^{(1)}(\bar{v}, \bar{w}), & \eta_2 &= P_r^{(2)}(v, w) - P_l^{(2)}(\bar{v}, \bar{w}), \\ \eta_3 &= P_r^{(3)}(v, w) - P_l^{(3)}(\bar{v}, \bar{w}). \end{aligned}$$

After taking in (45) the integrand function (17) and integrating, we finally obtain

$$\bar{P}_{lj}^*(\bar{v}, \bar{w}, v, w) = \frac{6}{8\pi\eta^5} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}, \quad (46)$$

where

$$\begin{aligned} P_{11} &= \eta_1^3 n_1 + \eta_1^2 \eta_2 n_2 + \eta_1^2 \eta_3 n_3, & P_{12} &= \eta_1^2 \eta_2 n_1 + \eta_1 \eta_2^2 n_2 + \eta_1 \eta_2 \eta_3 n_3, \\ P_{13} &= \eta_1^2 \eta_3 n_1 + \eta_1 \eta_2 \eta_3 n_2 + \eta_1 \eta_3^2 n_3, & P_{21} &= \eta_1^2 \eta_2 n_1 + \eta_1 \eta_2^2 n_2 + \eta_1 \eta_2 \eta_3 n_3, \\ P_{22} &= \eta_1 \eta_2^2 n_1 + \eta_2^3 n_2 + \eta_2^2 \eta_3 n_3, & P_{23} &= \eta_1 \eta_2 \eta_3 n_1 + \eta_2^2 \eta_3 n_2 + \eta_2 \eta_3^2 n_3, \\ P_{31} &= \eta_1^2 \eta_3 n_1 + \eta_1 \eta_2 \eta_3 n_2 + \eta_1 \eta_3^2 n_3, & P_{32} &= \eta_1 \eta_2 \eta_3 n_1 + \eta_2^2 \eta_3 n_2 + \eta_2 \eta_3^2 n_3, \\ P_{33} &= \eta_1 \eta_3^2 n_1 + \eta_2 \eta_3^2 n_2 + \eta_3^3 n_3. \end{aligned}$$

Kernels (46) and (44) allow including in their mathematical formalism the boundary shape generated by parametric patches $l, r = 1, 2, 3, \dots, n$ which are defined in Cartesian coordinates by the following relations

$$\begin{aligned}\eta_1 &= P_r^{(1)}(v, w) - P_l^{(1)}(\bar{v}, \bar{w}), & \eta_2 &= P_r^{(2)}(v, w) - P_l^{(2)}(\bar{v}, \bar{w}), \\ \eta_3 &= P_r^{(3)}(v, w) - P_l^{(3)}(\bar{v}, \bar{w}),\end{aligned}$$

and $\eta = [\eta_1^2 + \eta_2^2 + \eta_3^2]^{0.5}$, where $P_r^{(1)}(v, w)$, $P_r^{(2)}(v, w)$, $P_r^{(3)}(v, w)$ are scalar components of the surface patches $\mathbf{P}_r(v, w) = [P_r^{(1)}(v, w), P_r^{(2)}(v, w), P_r^{(3)}(v, w)]^T$ and depend upon parameters v and w .

In PIES through indexes $l, r = 1, 2, 3, \dots, n$, we take into account the appropriate relationships between the surfaces that define the boundary shape of the considered boundary problem. The relation between the patches modeling the closed boundary is written by a system of equations. In view of the fact that the unknowns in this system are functions over the patches, it is a system of integral equations.

We can note, that kernels (44) and (46) for the Stokes problem are identical as for the linear elastic problem for a perfectly incompressible material (Poisson's ratio $\nu = 0.5$) [29].

After solving PIES represented by the formula (42), we obtain the unknown functions $\mathbf{f}_r(v, w)$ or $\mathbf{u}_r(v, w)$ on the individual surface patches. PIES differs from classical BIE because in PIES the boundary is formally defined by surface patches. Therefore, PIES is physically defined not on the boundary as in BIE, but within parametric domain of these patches. This is due to the fact that modeling the boundary is done directly in PIES's kernels. They are called the fundamental and singular boundary solutions to distinguish them their classical fundamental and singular solutions, which are not include of the boundary. This advantage allows for the separation between approximation of the boundary shape and boundary functions. Thus, the approximation of solutions on the boundary is independent from the approximation of the boundary shape.

5 The integral identity for solutions in the domain

Having a solution on the boundary, we can obtain solutions in domain after appropriate adaptation of integral identity (18), where variable ω is replaced by ξ and boundary integrals $\tilde{f}_j(\xi)$ and $\tilde{u}_j \tilde{n}_k(\xi)$ are represented by formulas (29). Thus, we finally obtain

$$\hat{u}_j(i\xi) = \sum_{r=1}^n \left\{ \frac{-1}{8\pi\mu} L_{ji}(i\xi) \tilde{f}_j^{(r)}(\xi) + \frac{1}{8\pi} \bar{T}_{ijk}(i\xi) \tilde{u}_j \tilde{n}_k^{(r)}(\xi) \right\}, \quad (47)$$

where

$$\tilde{f}_j^{(r)}(\xi) = \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} e^{-i\{\xi_1 P_r^{(1)}(v, w) + \xi_2 P_r^{(2)}(v, w) + \xi_3 P_r^{(3)}(v, w)\}} f_j^{(r)}(v, w) J_r(v, w) dv dw, \quad (48)$$

$$\tilde{u}_j \tilde{n}_k^{(r)}(\xi) = \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} e^{-i\{\xi_1 P_r^{(1)}(v,w) + \xi_2 P_r^{(2)}(v,w) + \xi_3 P_r^{(3)}(v,w)\}} u_j^{(r)}(v,w) n_k^{(r)}(v,w) J_r(v,w) dv dw, \quad (49)$$

where v_{r-1} , w_{r-1} , v_r , w_r are, respectively, the beginning and ending of the domain with parameters v , w .

Having considered (48) in (49) and further in (47) and using the inverse Fourier transform, we obtain the following expression

$$u(x) = \sum_{r=1}^n \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} \left\{ \hat{U}_r^*(x, v, w) f_r(v, w) - \hat{P}_r^*(x, v, w) u_r(v, w) \right\} J_r(v, w) dv dw. \quad (50)$$

The first integrand $\hat{U}_r^*$ is calculated from the integral expression

$$\hat{U}_r^*(x, v, w) = \frac{-1}{64\pi^4} \int_{R^3} e^{i(\xi_1 \vec{r}_1 + \xi_2 \vec{r}_2 + \xi_3 \vec{r}_3)} L_{ji}(i\xi) d\xi, \quad (51)$$

where $\vec{r}_1 = P_r^{(1)}(v, w) - x_1$, $\vec{r}_2 = P_r^{(2)}(v, w) - x_2$ and $\vec{r}_3 = P_r^{(3)}(v, w) - x_3$.

After taking integral function $L_{ji}(i\xi)$ in (16) and integrating, we finally obtain

$$\hat{U}_r^*(x, v, w) = -\frac{1}{8\pi\mu\vec{r}} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} & \hat{U}_{13} \\ \hat{U}_{21} & \hat{U}_{22} & \hat{U}_{23} \\ \hat{U}_{31} & \hat{U}_{32} & \hat{U}_{33} \end{bmatrix}. \quad (52)$$

with

$$\hat{U}_{11} = 1 + \frac{\vec{r}_1^2}{\vec{r}^2}, \quad \hat{U}_{12} = \frac{\vec{r}_1 \vec{r}_2}{\vec{r}^2}, \quad \hat{U}_{13} = \frac{\vec{r}_1 \vec{r}_3}{\vec{r}^2}, \quad \hat{U}_{21} = \frac{\vec{r}_2 \vec{r}_1}{\vec{r}^2}, \quad \hat{U}_{22} = 1 + \frac{\vec{r}_2^2}{\vec{r}^2},$$

$$\hat{U}_{23} = \frac{\vec{r}_2 \vec{r}_3}{\vec{r}^2}, \quad \hat{U}_{31} = \frac{\vec{r}_3 \vec{r}_1}{\vec{r}^2}, \quad \hat{U}_{32} = \frac{\vec{r}_3 \vec{r}_2}{\vec{r}^2}, \quad \hat{U}_{33} = 1 + \frac{\vec{r}_3^2}{\vec{r}^2},$$

where

$$\vec{r}_1 = P_j^{(1)}(v, w) - x_1, \quad \vec{r}_2 = P_j^{(2)}(v, w) - x_2, \quad \vec{r}_3 = P_j^{(3)}(v, w) - x_3, \quad \vec{r} = [\vec{r}_1^2 + \vec{r}_2^2 + \vec{r}_3^2]^{0.5}.$$

The integrand $\hat{P}_r^*$ in (50) is represented as follows

$$\hat{P}_r^*(x, v, w) = \frac{1}{64\pi^4} \int_{R^3} e^{i\xi_1 \vec{r}_1 + \xi_2 \vec{r}_2 + \xi_3 \vec{r}_3} \bar{T}_{ijk}(i\xi) n_k(\xi) d\xi, \quad (53)$$

where individual elements are calculated by the following integral

$$\hat{\mathbf{P}}_r^*(\mathbf{x}, v, w) = \frac{6}{8\pi \vec{r}^5} \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13} \\ \hat{P}_{21} & \hat{P}_{22} & \hat{P}_{23} \\ \hat{P}_{31} & \hat{P}_{32} & \hat{P}_{33} \end{bmatrix}, \quad (54)$$

with

$$\begin{aligned} \hat{P}_{11} &= \vec{r}_1^3 n_1 + \vec{r}_1^2 \vec{r}_2 n_2 + \vec{r}_1^2 \vec{r}_3 n_3, & \hat{P}_{12} &= \vec{r}_1^2 \vec{r}_2 n_1 + \vec{r}_1 \vec{r}_2^2 n_2 + \vec{r}_1 \vec{r}_2 \vec{r}_3 n_3, \\ \hat{P}_{13} &= \vec{r}_1^2 \vec{r}_3 n_1 + \vec{r}_1 \vec{r}_2 \vec{r}_3 n_2 + \vec{r}_1 \vec{r}_3^2 n_3, & \hat{P}_{21} &= \vec{r}_1^2 \vec{r}_2 n_1 + \vec{r}_1 \vec{r}_2^2 n_2 + \vec{r}_1 \vec{r}_2 \vec{r}_3 n_3, \\ \hat{P}_{22} &= \vec{r}_1 \vec{r}_2^2 n_1 + \vec{r}_2^3 n_2 + \vec{r}_2^2 \vec{r}_3 n_3, & \hat{P}_{23} &= \vec{r}_1 \vec{r}_2 \vec{r}_3 n_1 + \vec{r}_2^2 \vec{r}_3 n_2 + \vec{r}_2 \vec{r}_3^2 n_3, \\ \hat{P}_{31} &= \vec{r}_1^2 \vec{r}_3 n_1 + \vec{r}_1 \vec{r}_2 \vec{r}_3 n_2 + \vec{r}_1 \vec{r}_3^2 n_3, & \hat{P}_{32} &= \vec{r}_1 \vec{r}_2 \vec{r}_3 n_1 + \vec{r}_2^2 \vec{r}_3 n_2 + \vec{r}_2 \vec{r}_3^2 n_3, \\ \hat{P}_{33} &= \vec{r}_1 \vec{r}_3^2 n_1 + \vec{r}_2 \vec{r}_3^2 n_2 + \vec{r}_3^3 n_3. \end{aligned}$$

Approximation of integral identity is described in detail in Appendix C.

6 Approximation of the boundary functions by Lagrange polynomials

The shape of the boundary in the proposed approach is theoretically described by parametric surfaces, while mathematical formalism of PIES is defined in the parametric domain of these patches. The separation of the simultaneous approximation of the boundary shape from the approximation of the boundary functions creates the possibility to consider these problems independently. In order to solve PIES, we can use one of the methods dedicated for solving the classical singular integral equations. We should remember that firstly, we must define the boundary in PIES.

For 3D problems, as well as for problems 2D, solution of PIES is reduced to approximation of the boundary functions $\mathbf{f}_r(v, w)$ and $\mathbf{u}_r(v, w)$ defined on individual surface patches with index r . The given boundary conditions as well as searched boundary functions on individual surface patches are approximated in the paper by the following series

$$\mathbf{f}_r(v, w) = \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \mathbf{f}_r^{(pt)} T_r^{(p)}(v) T_r^{(t)}(w), \quad \mathbf{u}_r(v, w) = \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \mathbf{u}_r^{(pt)} T_r^{(p)}(v) T_r^{(t)}(w), \quad (55)$$

where

$\mathbf{f}_r^{(pt)}$ and $\mathbf{u}_r^{(pt)}$ are the unknown coefficients,

N, M are the numbers of terms in the series,

$T_r^{(p)}(v)$ and $T_r^{(t)}(w)$ are given basis functions represented by the Chebyshev polynomials,

$\bar{n}_r = N \times M$ is the number of unknown coefficients (terms) in the series.

After substituting (55) into formula (42), we obtain the approximate expression of PIES

$$\begin{aligned}
 & 0.5 \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \mathbf{u}_l^{(pt)} T_l^{(p)}(\bar{v}) T_l^{(t)}(\bar{w}) \\
 &= \sum_{r=1}^n \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \left\{ f_r^{(pt)} \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} \bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w) - u_r^{(pt)} \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} \bar{P}_{lr}^*(\bar{v}, \bar{w}, v, w) \right\} \\
 & T_r^{(p)}(v) T_r^{(t)}(w) J_r(v, w) dv dw, l = 1, 2, 3, \dots, n,
 \end{aligned} \quad (56)$$

where integrands $\bar{U}_{lr}^*(\bar{v}, \bar{w}, v, w)$ and $\bar{P}_{lr}^*(\bar{v}, \bar{w}, v, w)$ are represented by formulas (44) and (46). In order to numerically solve formula (56), the pseudospectral method is used [30]. The numerical procedure for solving PIES is described in detail in Appendix A.

7 Practical application of the method

The application of the proposed approach is illustrated by three numerical examples. We show how to represent the boundary by the surfaces described in Section 2 and compare the accuracy of the results with known analytical solutions. The L_2 error norm is used to study the accuracy of solutions

$$\|e\|_{L_2} = \frac{1}{|\bar{\psi}^{(k)}|_{\max}} \sqrt{\frac{1}{K} \sum_{k=1}^K (\psi^{(k)} - \bar{\psi}^{(k)})^2} \times 100\%, \quad (57)$$

where $\psi^{(k)}$ represents a set of PIES solutions obtained at K sample points, while $\bar{\psi}^{(k)}$ are exact solutions. Singular and regular integrals are evaluated by Gaussian quadrature of order 40. Numerical evaluation of singular and regular integrals is described in detail in Appendix B.

7.1 Example 1

In the first example, we consider the Stokes equation inside a cubic domain with Dirichlet the boundary conditions given by the following functions [35]

$$\begin{aligned}
 u_1(x_1, x_2, x_3) &= x_1 + x_1^2 + x_1 x_2 + x_1^3 x_2, \\
 u_2(x_1, x_2, x_3) &= x_2 + x_1 x_2 + x_2^2 + x_1^2 x_2^2, \\
 u_3(x_1, x_2, x_3) &= -2x_3 - 3x_1 x_3 - 3x_2 x_3 - 5x_1^2 x_2 x_3.
 \end{aligned} \quad (58)$$

The above functions satisfy both the Stokes equation and represent the exact solution inside the domain. As shown in Fig. 6, the boundary of the cube is described by eight corner points P_i ($i = 1 \dots 8$) that define six rectangular Bézier patches of degree 1.

In Table 1, the impact of the arrangement and number of collocation points per surface patch in pseudospectral method on the accuracy of solutions is examined.

Fig. 6 Description of the boundary from Example 1 by 6 corner points

Two variants of arrangement of the collocation points were considered: uniform distribution and in accordance to the roots of the Chebyshev polynomials of the second kind.

The calculations have been repeated for four variants of number of collocation points $\bar{n}_r = N \times M$ varying from 9 to 36, which are related to the number of the approximation series in (55). We can observe low level of numerical errors, based on the L_2 norm, for all three components of the solutions, even for nine collocation points. It can be seen that the error norm is below 0.1% for all considered cases. Moreover, we notice that there is only a slight improvement in accuracy when the number of collocation points is increased from 16 to 25 and 36. The plots indicate smaller error for the Chebyshev distribution of collocation points when compared to uniform one. An exception here is the plot for u_3 component for 9 and 16 collocation points.

It should be noted that the relationship between the number of collocation points and the convergence of solutions is influenced by a number of factors such as evaluation of singular integrals. Moreover, when we increase the number of collocation points, the extreme points are located very close to the adjacent patch on which the integration is carried out, which may introduce the possibility of “quasi-singularity.”

Another factor is the choice of method for solving PIES. Here, one of the simplest method called the pseudospectral method was used. Therefore, it is advisable to examine other methods for this purpose. The separation of approximations significantly simplifies the use of other methods to solve it.

Furthermore, to improve the rate of convergence, we also considered Richardson’s extrapolation applied for sequence of results obtained previously for the uniform distribution of collocation points listed in column 4. The distance between collocation

Table 1 The impact of the number and arrangement of collocation points per surface patch on the accuracy of solutions obtained for 20 evenly spaced points along the cross-section ($x_1 = x_2 = x_3$) in comparison with the exact ones given by (58)

Number collocation points per surface patch	Total number of algebraic equations	Error norm L_2 of solutions in PIES		
		Chebyshev distribution	Uniform distribution	Richardson extrapolation
9	162	0.0136818	0.0146326	–
		0.0575261	0.0601321	
		0.0224659	0.019343	
16	188	0.0128337	0.013005	0.0122171 ^A
		0.0537826	0.0545982	0.0534259
		0.0209722	0.0196557	0.0208902
25	450	0.0127039	0.012755	0.0114525 ^B
		0.0533448	0.0533628	0.0526444
		0.0208317	0.0208434	0.0219482
36	648	0.0124129	0.0127259	0.0128431 ^C
		0.0529728	0.0532163	0.0533397
		0.0206531	0.0207822	0.0215263

^Athe results extrapolated from the PIES solutions computed for uniform distribution of 9 and 16 collocation points

^Bfor uniform distribution of 9, 16, and 25 collocation points

^Cfor uniform distribution of 9, 16, 25, and 36 collocation points

points is assumed as the extrapolation parameter. The results are shown in last column of Table 1. A slight improvement in the accuracy of the solutions was obtained in relation to the solutions from PIES for components u_1, u_2 and deterioration for component u_3 .

7.2 Example 2

In the next example, we examine the classical problem of the steady Stokes flow around a sphere. The exact analytical solution for velocity components u_1, u_2, u_3 of that flow generated by uniform parallel stream with velocity U_0 past the sphere of radius R is given by [1]

$$\begin{aligned}
 u_1(\mathbf{x}) &= U_0 R \left[\frac{3}{4r} + \frac{R}{4r^3} + \frac{3(r^2 - R^2)x_1^2}{4r^5} \right], \\
 u_2(x) &= 3U_0 R \frac{(r^2 - R^2)x_1 x_2}{4r^5}, \\
 u_3(x) &= 3U_0 R \frac{(r^2 - R^2)x_1 x_3}{4r^5}, \quad r^2 = \sqrt{x_1^2 + x_2^2 + x_3^2}.
 \end{aligned} \tag{59}$$

In addition to the number of collocation points and associated terms in series (55), the impact on the accuracy of the solutions is also the arbitrary selection of degrees in the Chebyshev series. The arbitrary selection of components is due to the fact already is described in Appendix A, wherein the total number of collocation points

	$T_r^{(0)}(w)$			$T_r^{(1)}(w)$			$T_r^{(2)}(w)$			$T_r^{(3)}(w)$			$T_r^{(4)}(w)$		
$T_r^{(0)}(v)$	A	B		A	B	C	A		C						
	D	E	F	D	E	F	D	E	F	D	E				
	G	H		G	H	I	G				H				
$T_r^{(1)}(v)$	A	B	C	A		C		B	C						
	D	E	F	D	E	F	D	E	F	D		F			
	G	H	I	G	H	I	G	H	I	G	H	I	G		I
$T_r^{(2)}(v)$	A		C		B			B	C						
	D	E	F	D	E	F					E	F			
	G			G	H	I	G	H	I	G	H	I	G	H	I
$T_r^{(3)}(v)$															
	D	E	F		F			E	F						
		H		G	H	I	G	H	I			I			
$T_r^{(4)}(v)$															
				G		I	G	H	I						

 Springer

in triangular patches is no longer a product of the number of rows and columns, as for rectangular patches. For this reason, it required adjustment to the total number of collocation points to the number of terms in series (55). This has been arbitrarily chosen with the symmetry of degrees of the Chebyshev polynomial used. It was noted that the stability of the results of the solutions also affects also the optimal distribution of collocation points in the triangular area.

7.3 Example 3

Finally, we consider the flow induced by rotating with respect to axis $0x_3$ unit sphere with the following boundary conditions [13]

$$u(x)|_{\Gamma} = (-0.1x_2, 0.1x_1, 0.0) \quad (60)$$

The analytical solutions for velocity components in polar coordinates are expressed by the following expressions

$$u(x) = (-0.1x_2/r^3, 0.1x_1/r^3, 0.0). \quad (61)$$

The sphere is modeled in the same way as in Example 2. Table 4 shows solutions inside the cross-section $x_1 = x_3$, $x_2 = 1.0$ for the non-zero components of the velocity field u_1, u_2 . Table 3 shows the convergence of the solution for the same topologies of collocation points and selected terms in the series (55) as chosen in Example 2.

We can observe that the increase of the number of collocation points improves the results. However, the accuracy is also affected by the choice of terms in the series marked by the variants A–I. Nevertheless, with the significant increase in the number

Table 3 The impact of the number of collocation points and terms selected from series (55) on the convergence of solutions for Example 2

Number collocation points per surface patch	Total number of algebraic equations	Error norm L_2 of solutions in PIES for variants of arbitrarily selected terms of the series specified in Table 2		
6	144	Variant A	Variant B	Variant C
		3.43862e-009	3.43538e-009	0.0235342
		4.35759e-009	4.21145e-009	0.126248
10	240	6.85635e-009	6.4305e-009	0.182061
		Variant D	Variant E	Variant F
		3.93007e-009	1.65533e-008	8.01282e-009
		1.79126e-008	8.07833e-008	5.35142e-008
15	360	3.3176e-008	1.13912e-007	6.12621e-008
		Variant G	Variant H	Variant I
		1.48333e-006	1.47474e-006	0.003412
		2.53571e-006	2.8713e-006	0.00738831
		6.01477e-006	3.87735e-006	0.00840099

Table 4 The impact of the number of collocation points and terms selected from series (55) on the convergence of solutions for Example 3

Number collocation points per surface patch	Total number of algebraic equations	Error norm L_2 of solutions in PIES for variants of arbitrarily selected terms of the series specified in Table 2		
6	144	Variant A	Variant B	Variant C
		0.225931	0.205748	0.194358
		0.0378749	0.0506424	0.0746397
10	240	Variant D	Variant E	Variant F
		0.00547743	0.0383184	0.0258637
		0.00414645	0.0496445	0.0177999
15	360	Variant G	Variant H	Variant I
		0.000539158	0.000539158	0.00603757
		0.000381553	0.000381553	0.00362107

of collocation points, it was noted that the stability of the results was also influenced by the optimal location of the collocation points. However, additional extensive research is required to improve the stability of the solutions for a greater number of collocation points.

8 Conclusions

The paper presents the modification of the classical BIE for the Stokes equation in 3D to analytically include in its mathematical formalism the boundary representation with parametric patches and demonstrates the application of Bézier surfaces for the boundary description in 3D. This seems to be a promising alternative to traditional mesh generation procedures known from element methods. Furthermore, the parametric description of the geometry by Bézier patches in PIES is essential to include the demonstrated procedure into a real computer-aided design environment, where the design structures are usually represented in parametric form. The presented results show that the proposed boundary representation with PIES is successful in solving the boundary value problems on the example of the Stokes equation.

For the representation of the solution on the boundary in PIES, we used a generalized Chebyshev series of one variable. In the case of approximation of solutions for rectangular patches, such a generalization of a Chebyshev series is very simple, as you can deploy the same number of collocation points in every direction. In the case of the triangular patches it is more problematic, as discussed in the paper. This requires the elimination of some terms from the Chebyshev series, adjusting the number of unknown coefficients for the collocation points over triangular parametric space. We also note that the stability of the results is related to the arrangement of the collocation points in the triangular parametric domain. However, it requires further extensive research.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Appendix A: Numerical solution of PIES for 3D Stokes problem

In order to numerically solve formula (56), the pseudospectral method is used [30]. After applying (56) at the collocation points on each of the surface patches of the boundary, we obtain the system of algebraic equations from which we find values of the unknown coefficients $u_r^{(pt)}$ and $f_r^{(pt)}$ from (55). The collocation points are posed in the parametric reference (triangular or rectangular) domain v, w belongs to individual patches of the boundary. They can be placed at the roots of the Chebyshev polynomials in order to obtain the most accurate solutions, as shown by earlier studies [19]. The collocation points are defined in the parametric space domain v, w of every patch. The arrangement of collocation points in parametric space for rectangular and triangular patches are illustrated in Fig. 7.

The number of used collocation points is related directly to the number of terms in series (55), which approximate boundary functions. For rectangular patches, we assume that these points are formed in a regular structure as a Cartesian product of one dimensional grids of points defined by N rows and M columns. Their arrangement for $N = M = 2$ and $N = M = 3$ is shown in Fig. 7a, b. It is also possible to use more collocation points and also for a variant with $N \neq M$ as well as introduce their different number for each surface of the boundary. For the adopted values of N and M at each patch, the series (55) are generated with terms represented by the Cartesian product of one-dimensional Chebyshev polynomials up to degree $N - 1$ and $M - 1$.

This procedure for forming collocation points may also be used in the case of triangular patches, but requires some modifications. This is due to the fact that triangular patches are related with triangular parametric space where the collocation points must be arranged as shown in Fig. 7c, d. Additionally, the number of these points and terms in series (55) is no longer defined by multiplying the number of columns and rows as for rectangular patches. This is reflected in the formula (55), through the arbitrary choice of series terms that their number corresponds to the declared number of collocation points, with ensuring symmetry-degree polynomials.

For both rectangular and triangular patches, the collocation points are arranged corresponding to the roots of the Chebyshev polynomials of the second degree. This arrangement is, as shown in Fig. 7, characterized by their denser distribution on the edges. In the case of the plane of symmetry of the triangle by virtue of Chebyshev distribution further they moved away from the diagonal points of a triangle.

After writing down (56) at the collocation points $\bar{v}^{(c)}, \bar{w}^{(c)}$ located in the parametric domain $v_{l-1} < \bar{v}^{(c)} < v_l, w_{l-1} < \bar{w}^{(c)} < w_l$ of individual patches $P_i (i = 1, 2, \dots, n)$, we obtain the system of algebraic equations in the following form

$$Hu = Gp. \quad (\text{A.1})$$

Fig. 7 The distribution of 9 (a) and 16 (b) collocation points in a parametric domain for rectangular patches and for 6 (c) and 10 (d), for triangular patches

After solving (A.1), we can obtain values of the coefficients of terms (55), represented by vectors $[h_{lr}]$ or $[g_{lj}]$ (depending on the type of the posed boundary conditions).

Appendix B: Numerical integration

The matrices $\mathbf{H}$ and $\mathbf{G}$ are calculated on the basis of the following expressions requiring integration

$$\begin{aligned}
 [h_{lr}] = & 0.5 \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} T_l^{(p)}(\bar{v}^{(c)}) T_l^{(t)}(\bar{w}^{(c)}) + \\
 & \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \int_{v_{j-1}}^{v_j} \int_{w_{j-1}}^{w_j} \bar{\mathbf{P}}_{lr}^*(\bar{v}^{(c)}, \bar{w}^{(c)}, v, w) T_r^{(p)}(v) T_r^{(t)}(w) J_r(v, w) dv dw
 \end{aligned}
 \tag{B.1}$$

and

$$[g_{lj}] = \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \int_{\tilde{v}_{j-1}}^{\tilde{v}_j} \int_{\tilde{w}_{j-1}}^{\tilde{w}_j} \bar{U}_{lr}^*(\tilde{v}^{(c)}, \tilde{w}^{(c)}, v, w) T_r^{(p)}(v) T_r^{(t)}(w) J_r(v, w) dv dw. \quad (\text{B.2})$$

An important aspect, which has to be considered during numerical solving of PIES is efficient and accurate calculation of surface integrals appearing in (B.1) and (B.2). These integrals are defined over each surface patch.

The matrix elements in (B.1) and (B.2) in the case of $l \neq j$ requires the calculation of strongly and weak singular integrals. Integration intervals in (B.1) and (B.2) depend upon the parameterization of the patches used to model the boundary. In the case of the patches discussed in the paper, these intervals for $F(v, w) \equiv \{\bar{U}_{lr}^*(v, w), \bar{P}_{lr}^*(v, w)\}$, are normalized as written

$$\int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} F(v, w) dv dw = \begin{cases} \int_0^1 \int_0^1 F(v, w) dv dw & \text{for rectangular patches} \\ \int_0^1 \left\{ \int_0^{(1-w)} F(v, w, u) dw \right\} dv & \text{for triangular patches} \end{cases} \quad (\text{B.3})$$

where $u = 1 - v - w$.

Regular integrands

In the case of $l \neq j$ in (B.1) and (B.2), the surface integrals are regular. To evaluate regular integrands defined over rectangular patches, standard Gauss-Legendre quadrature is used [31]. We need to transform the physical domain of integration over rectangular surface patch to the parametric domain and then to the standard interval of the Gaussian coordinates.

In the case of integrals defined over the triangular patches, their values are also computed applying Gauss-Legendre quadratures with the transformation presented in [32] and written as

$$I = \sum_{i=1}^s \sum_{j=1}^s \left(\frac{1 + \xi_i}{8} \right) w_i w_j f \left(\frac{(1 + \xi_i)(1 + \eta_j)}{4}, \frac{(1 + \xi_i)(1 - \eta_j)}{4} \right), \quad (\text{B.4})$$

where ξ_i, η_j are Gaussian points in the ξ, η directions and w_i, w_j are the corresponding weights and s is the number of quadrature points.

Singular integrals

In the case of $l = j$ in (B.1) and (B.2), both the collocation points as well as the numerical quadrature points lie within the same parametric domain of the same patch and kernels are strongly for matrix $[h_{lr}]$ and weakly singular for matrix $[g_{lj}]$, respectively. In the case of weak singularity in kernel (B.2), we isolate the singularity by introducing an additional polar transformation consisting of subdividing the local

coordinate system in the parametric domain and using standard Gauss quadratures for rectangular and triangular patches. In the case of the kernel (B.1) with strong singularity, we use the method of eliminating this singularity described in [33], which are the improved version of Guiggiani's method [34].

Appendix C: Approximation of integral identity

After solving (A.1), we obtain the solutions on the boundary. The values of the solutions in the domain Ω at point $\mathbf{x} = (x_1, x_2, x_3)$ are computed by integral identity (50). After substituting approximating series (55) into (50), we give this identity in the following form

$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^n \sum_{p=0}^{N-1} \sum_{t=0}^{M-1} \left\{ \mathbf{f}_r^{(pt)} \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} \hat{\mathbf{U}}_r^*(\mathbf{x}, v, w) - \mathbf{u}_r^{(pt)} \int_{v_{r-1}}^{v_r} \int_{w_{r-1}}^{w_r} \hat{\mathbf{P}}_r^*(\mathbf{x}, v, w) \right\} T_r^{(p)}(v) T_r^{(t)}(w) J_r(v, w) dv dw. \quad (\text{C.1})$$

The integrand functions $\hat{\mathbf{U}}_r^*(\mathbf{x}, v, w)$ and $\hat{\mathbf{P}}_r^*(\mathbf{x}, v, w)$ are represented by (52) and (54). To determine the solution in the domain, only coefficients $\mathbf{u}_r^{(pt)}$ and $\mathbf{f}_r^{(pt)}$ for every surface patch which model the boundary have to be taken into account.

References

- Schlichting, H.: Boundary Layer Theory. McGraw-Hill, New York (1979)
- Youngren, G.K., Acrivos, A.: Stokes flow past a particle of arbitrary shape: a numerical method of solution. *J. Fluid Mech.* **69**(2), 377–403 (1975)
- Bringley, T.T., Peskin, C.S.: Validation of a simple method for representing spheres and slender bodies in an immersed boundary method for Stokes flow on an unbounded domain. *J. Comput. Phys.* **227**(11), 5397–5425 (2008)
- Cortez, R., Hoffmann, F.: A fast numerical method for computing doubly-periodic regularized Stokes flow in 3D. *J. Comput. Phys.* **258**, 1–14 (2014)
- Pironneau, O.: On optimum profiles in Stokes flow. *J. Fluid Mech.* **59**(1), 117–128 (1973)
- Zienkiewicz, O.C., Taylor, R.L.: The Finite Element Method. Butterworth-Heinemann, Oxford (2000)
- Backer, A.T., Brenner, S.C.: A mixed finite element method for the Stokes equations based a weakly over-penalized symmetric interior penalty approach. *J. Sci. Comput.* **58**(2), 290–307 (2014)
- Hou, L.S.: Error estimates for semidiscrete finite element approximations of the Stokes equations under minimal regularity assumptions. *J. Sci. Comput.* **16**(3), 287–317 (2001)
- Fang, J., Parriaux, A., Rentschler, M., Ancy, C.: Improved SPH methods for simulating free surface flows of viscous fluids. *Appl. Numer. Math.* **59**(2), 251–271 (2009)
- Zhang, L., Ouyang, J., Zhang, X.H.: On a two-level element-free Galerkin method for incompressible fluid flow. *Appl. Numer. Math.* **59**(8), 1894–1904 (2009)
- Oñate, E., Sacco, C., Idelsohn, S.: A finite point method for incompressible flow problems. *Comput. Vis. Sci.* **3**(1), 67–75 (2000)
- Wu, X.H., Tao, W.Q., Shen, S.P., Zhu, X.W.: A stabilized MLPG method for steady state incompressible fluid flow simulation. *J. Comput. Phys.* **229**(22), 8564–8577 (2010)
- Tan, F., Zhang, Y., Li, Y.: Development of a meshless hybrid boundary node method for Stokes flows. *Eng. Anal. Bound. Elem.* **37**(6), 899–908 (2013)
- Brebbia, C.A., Telles, J.C., Wrobel, L.C.: Boundary Element Techniques, Theory and Applications in Engineering. Springer, New York (1984)

15. Becker, A.A.: The Boundary Element Method in Engineering: a Complete Course. McGraw-Hill Book Companies, Cambridge (1992)
16. Beskos, D.E.: Boundary Element Methods in Mechanics. North-Holland, Amsterdam (1987)
17. Power, H., Wrobel, L.C.: Boundary integral methods in fluid mechanics. Computational Mechanics Publications (1995)
18. Muldowney, G.P., Higdon, J.J.L.: A spectral boundary element approach to three-dimensional Stokes flow. *J. Fluid Mech.* **298**, 167–192 (1995)
19. Zieniuk, E.: Computational method PIES for solving boundary value problems. PWN Warsaw. (in Polish) (2013)
20. Hughes, T.J.R., Cottrell, J.A., Bazilevs, Y.: Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Eng.* **194**(39), 4135–4195 (2005)
21. Cottrell, J.A., Hughes, T.J.R., Bazilevs, Y.: Isogeometric Analysis: Toward Integration of CAD and FEA. Wiley, New York (2009)
22. Scott, M.A., Simpson, R.N., Evans, J.A., Lipton, S., Bordas, S., Hughes, T.J.R., Sederberg, T.W.: Isogeometric boundary element analysis using unstructured T-splines. *Comput. Methods Appl. Mech. Eng.* **254**, 197–221 (2013)
23. Zieniuk, E., Szerszeń, K.: The PIES for solving 3D potential problems with domains bounded by rectangular bézier patches. *Eng. Comput.* **31**(4), 791–809 (2014)
24. Zieniuk, E., Szerszeń, K.: Triangular bézier surface patches in modeling shape of boundary geometry for potential problems in 3D. *Eng. Comput.* **29**(3), 517–527 (2013)
25. Zieniuk, E., Szerszeń, K.: Triangular bézier patches in modeling smooth boundary in exterior Helmholtz problems solved by PIES. *Archives of Acoustics* **34**(1), 51–61 (2009)
26. Zieniuk, E., Szerszen, K., Kapturczak, M.: A Numerical Approach to the Determination of 3D Stokes Flow in Polygonal Domains Using PIES. *Lecture Notes in Computer Sciences* 7203, part I, pp. 112–121. Springer, Berlin (2012)
27. Farin, G.: Curves and Surfaces for CAGD: A Practical Guide. Morgan Kaufmann Publishers, Burlington (2001)
28. Pozrikidis, C.: A Practical Guide to Boundary Element Methods with the Software Library BEMLIB. CRC Press, Boca Raton (2002)
29. Kokkinos, F.T., Reddy, J.N.: BEM And penalty FEM models for viscous incompressible fluids. *Comput. Struct.* **56**(5), 849–859 (1995)
30. Gottlieb, D., Orszag, S.A.: Numerical Analysis of Spectral Methods: Theory and Applications. SIAM, Philadelphia (1977)
31. Stroud, A.H.: Gaussian Quadrature Formulas. Prentice-Hall, New Jersey (1966)
32. Rathod, H.T., Nagaraja, K.V., Venkatesudu, B.: Symmetric Gauss Legendre quadrature formulas for composite numerical integration over a triangular surface. *Appl. Math. Comput.* **188**(1), 865–876 (2007)
33. Rong, J., Wen, L., Xiao, J.: Efficiency improvement of the polar coordinate transformation for evaluating BEM singular integrals on curved elements. *Eng. Anal. Bound. Elem.* **38**, 83–93 (2014)
34. Guiggiani, M., Krishnaswamy, G., Rudolphi, T.J., Rizzo, F.J.: A general algorithm for the numerical solution of hypersingular boundary integral equations. *J. Appl. Mech.* **59**(3), 604–614 (1992)
35. Young, D.L., Jane, S.C., Lin, C.Y., Chiu, C.L., Chen, K.C.: Solution of 2D and 3D Stokes laws using multiquadrics method. *Eng. Anal. Bound. Elem.* **28**(10), 1233–1243 (2004)