Interval versions of Milne's multistep methods

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Abstract The paper presents explicit interval multistep methods of Milne type, which may be considered as alternative methods to other known explicit interval multistep methods (of Adams-Bashforth and Nyström). It is proved that enclosures of solutions (in the form of intervals) obtained by these methods contain the exact solutions of the initial value problem. Numerical examples show that the widths of intervals obtained by proposed methods are smaller than those obtained by explicit interval multistep methods known so far.

Keywords Initial value problem · Multistep methods · Milne's methods · Interval Milne's methods · Floating-point interval arithmetic

1 Introduction

Interval arithmetic (see, e.g., [1, 10, 12, 34, 35, 43]) is the basic mathematical tool in verified numerical computing. This arithmetic is also important for solving the

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initial value problem, because the obtained enclosures of solutions include errors of methods. Realizing such interval methods in floating-point interval arithmetic, we can obtain enclosures at discrete points¹ (in the form of intervals), which contain all possible numerical errors, also representation errors and rounding errors. These intervals may also include data uncertainties. One inconvenience of this arithmetic is the wrapping effect, but some efforts reduce its effects (see, e.g., [9, 22, 23, 38, 40, 44]).

There are a number of interval methods for approximating the initial value problem. The first one was described by R. E. Moore in 1965 [33–35]. There are also known interval methods based on high-order Taylor series (see, e.g., [2, 3, 5, 15, 21, 39, 41]), explicit and implicit Runge-Kutta methods [7, 8, 20, 24, 27, 31, 43], explicit and implicit multistep methods [17–19, 25–27, 30, 43]. The last ones concern interval methods based on conventional methods of Adams-Bashforth, Adams-Moulton, Nyström, and Milne-Simpson types. These methods have been considered and compared by us, especially in [30]. Although we usually apply such methods of low orders, it appears that for a suitable choice of step sizes the obtained enclosures are as good as enclosures obtained by methods based on high-order Taylor series. Moreover, explicit interval multistep method (of Adams-Bashforth and Nyström types) can be used as predictors for implicit interval methods (of Adams-Moulton and Milne-Simpson types)—see [30] for details. From our analysis, it appears that the explicit multistep methods of Nyström type are a little bit better than those of Adams-Bashforth type—the widths of intervals are smaller.

In this paper, we propose interval versions of other explicit multistep methods called the Milne multistep methods. In Section 2, we repeat the conventional forms of such methods. In Section 3, we present their interval form, and prove a theorem that the exact solution is included in the enclosures (intervals) obtained and a theorem regarding the widths of these intervals. Some numerical examples are presented in Section 4. From these examples, it follows that the proposed interval multistep methods of Milne type give enclosures of solutions with smaller widths in comparison with interval methods of Nyström type of the same order. Although the considered method is only of fourth order, for a sufficiently small step size, it gives comparable results even with interval methods based on high-order Taylor series (this has been checked up on using the VNODE-LP package [36, 37]). Therefore, it is worth to take into account such a method when one solves the initial value problem.

2 The conventional multistep methods of Milne

It is well-known that the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0, \quad (1)$$

¹In general, these discrete points may be represented in the form of intervals.

where $t \in [0, a]$, $y \in \mathbb{R}$, and $f : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$, can be written in the equivalent integral form²

$$y(t) = y(t_{k-l}) + \int_{t_{k-l}}^t f(\tau, y(\tau)) d\tau, \quad t > t_{k-l}, \quad l = 1, 2, \dots,$$

from which we have

$$y(t_k) = y(t_{k-l}) + \int_{t_{k-l}}^{t_k} f(\tau, y(\tau)) d\tau. \quad (2)$$

To obtain multistep methods, we approximate the function $f(\tau, y(\tau))$ by an adequate interpolation polynomial and then we integrate this polynomial. For instance, if for $l = 1$, we take a polynomial of the degree $n - 1$, then we obtain the n -step explicit Adams-Bashforth method, while for a polynomial of the degree n we get the n -step implicit Adams-Moulton method. Taking $l = 2$ for a polynomial of the degree $n - 1$ we have the n -step explicit method of Nyström, and for a polynomial of the degree n —the n -step implicit method of Milne-Simpson (see, e.g., [4, 11, 13, 16, 32]).

The multistep explicit methods of Milne we can obtain if in (2) we take $l = 4$ and use a polynomial of the degree $n - 1$ (see, e.g., [16]). Let us denote such a polynomial by $P(\tau)$, and let

$$P(t_{k-j}) = f(t_{k-j}, y(t_{k-j})), \quad j = 1, 2, \dots, n.$$

If we exchange the variable τ for t in such a way that $\tau = t_{k-1} + th$, where $h = t_i - t_{i-1} = a/m$ for each $i = k - n + 1, k - n + 2, \dots, n$ (m denotes any integer greater or equal to n), then we can write the polynomial $P(\tau)$ as follows:

$$\begin{aligned} P(t_{k-1} + th) &= f(t_{k-1}, y(t_{k-1})) + t \nabla f(t_{k-1}, y(t_{k-1})) \\ &+ \dots + \frac{t(t+1) \dots (t+n-2)}{(n-1)!} \nabla^{n-1} f(t_{k-1}, y(t_{k-1})), \end{aligned}$$

where ∇ denotes the backward difference operator.³

We approximate the integrand in (2) by the polynomial $P(\tau)$, i.e., we substitute

$$f(\tau, y(\tau)) = P(\tau) + r(\tau),$$

where $r(\tau)$ denotes the interpolation error given by

$$r(t_{k-1} + th) = f^{(n)}(\xi(t)) h^n \frac{t(t+1) \dots (t+n-1)}{n!},$$

²We assume that the solution of (1) exists and is unique.

³Given the sequence $\{p_n\}$, $n = 0, 1, \dots$, the backward difference ∇p_n is defined by $\nabla p_n = p_n - p_{n-1}$ for $n \geq 1$. Higher powers are defined recursively by $\nabla^k p_n = \nabla^{k-1}(\nabla p_n)$.

and where $\xi(t)$ is an intermediate point in $[t_{k-n}, t_{k-1}]$. Here, we assume that the function f has a continuous derivative of order n . After integration, we have

$$y(t_k) = y(t_{k-4}) + h \sum_{j=0}^{n-1} \gamma_j \nabla^j f(t_{k-1}, y(t_{k-1})) + h^{n+1} \varphi_n, \quad (3)$$

where

$$\nabla^j f(t_{k-1}, y(t_{k-1})) = \sum_{m=0}^j (-1)^m \binom{j}{m} f(t_{k-1-m}, y(t_{k-1-m})), \quad (4)$$

$$\gamma_0 = 4, \quad \gamma_j = \frac{1}{j!} \int_{-3}^1 s(s+1) \dots (s+j-1) ds, \quad j = 1, 2, \dots, n-1,$$

and where $h^{n+1} \varphi_n$ is an error term. Let us note that using (4) we can write (3) in the form

$$y(t_k) = y(t_{k-4}) + \sum_{j=1}^n \beta_{nj} f(t_{k-j}, y(t_{k-j})) + h^{n+1} \varphi_n, \quad (5)$$

$$k = n, n+1, \dots, m,$$

where

$$\beta_{nj} = (-1)^{j-1} \sum_{p=j-1}^{n-1} \binom{p}{j-1} \gamma_p, \quad j = 1, 2, \dots, n.$$

In conventional Milne's methods (like in other conventional multistep methods) we omit the error term, and after replacing the unknown values $y(t_{k-n})$, $y(t_{k-n+1}), \dots, y(t_{k-1})$ with approximations $y_{k-n}, y_{k-n+1}, \dots, y_{k-1}$ we are given the formula for finding y_k . The most known formula of Milne is of the form ($n = 4$)

$$y_k = y_{k-4} + \frac{4h}{3} (2f_{k-1} - f_{k-2} + 2f_{k-3}), \quad (6)$$

where $f_{k-j} = f(t_{k-j}, y_{k-j})$, $j = 1, 2, 3$. This formula is of fourth order. Since for $n = 4$, we have $\gamma_3 = 0$, and hence $\beta_{44} = 0$, the term containing f_{k-4} does not occur what reduces the number of calculations.

In an interval version of Milne's methods the error term is very important, since it contains the error of the method. In (3) and (5), we have

$$h^{n+1} \varphi_n = h^{n+1} \frac{1}{n!} \int_{-3}^1 s(s+1) \dots (s+n-1) f^{(n)}(\xi(s)) ds, \quad (7)$$

which we write in the form

$$h^{n+1}\varphi_n = h^{n+1} [\gamma_n^* \psi(\eta^*, y(\eta^*)) + \gamma_n^{**} \psi(\eta^{**}, y(\eta^{**}))], \quad (8)$$

where $\psi(\eta, y(\eta)) \equiv f^{(n)}(\eta, y(\eta)) \equiv y^{(n+1)}(\eta)$, η^* and η^{**} are intermediate points (not necessary equal) in the interval considered, and

$$\gamma_n^* = \frac{1}{n!} \int_{-3}^0 s(s+1) \dots (s+n-1) ds, \quad \gamma_n^{**} = \frac{1}{n!} \int_0^1 s(s+1) \dots (s+n-1) ds.$$

In both integrals, the integrand does not change sign over the intervals $[-3, 0]$ and $[0, 1]$, respectively. Let us note that we cannot write the error term (7) in the form

$$h^{n+1}\varphi_n = h^{n+1}\gamma_n \psi(\eta, y(\eta)),$$

since the function $s(s+1) \dots (s+n-1)$ changes sign over the interval $[-3, 1]$ (for general integer n , not only for $n=4$).

For $n=4$ we have $\gamma_4^* = -27/720$, $\gamma_4^{**} = 251/720$ and the exact formula corresponding to (6) is of the form

$$\begin{aligned} y(t_k) = & y(t_{k-4}) + \frac{4h}{3} [2f(t_{k-1}, y(t_{k-1})) - f(t_{k-2}, y(t_{k-2})) + 2f(t_{k-3}, y(t_{k-3}))] \\ & - \frac{h^5}{720} [27\psi(\eta^*, y(\eta^*)) - 251\psi(\eta^{**}, y(\eta^{**}))], \quad k = n, n+1, \dots, m, \end{aligned} \quad (9)$$

where $\psi(\eta, y(\eta)) \equiv f^{(4)}(\eta, y(\eta)) \equiv y^{(5)}(\eta)$.

3 Explicit interval multistep methods of Milne type

Let us denote:

- Δ_t and Δ_y – bounded sets in which the function $f(t, y)$, occurring in (1), is defined, i.e.,

$$\Delta_t = \{t \in \mathbb{R} : 0 \leq t \leq a\}, \quad \Delta_y = \{y \in \mathbb{R} : \underline{b} \leq y \leq \bar{b}\},$$

- $F(T, Y)$ – an interval extension⁴ of $f(t, y)$,
- $\Psi(T, Y)$ – an interval extension of $\psi(t, y(t)) \equiv f^{(n)}(t, y(t)) \equiv y^{(n+1)}(t)$,

and let us assume:

- $F(T, Y)$ is defined and continuous over all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- $F(T, Y)$ is monotonic with respect to inclusion, i.e.,

$$T_1 \subset T_2 \wedge Y_1 \subset Y_2 \Rightarrow F(T_1, Y_1) \subset F(T_2, Y_2),$$

⁴An interval extension of the function $f: \mathbb{R} \times \mathbb{R} \supset \Delta_t \times \Delta_y \rightarrow \mathbb{R}$ we call a function $F: \mathbb{IR} \times \mathbb{IR} \supset \mathbb{I}\Delta_t \times \mathbb{I}\Delta_y \rightarrow \mathbb{IR}$ such that $(t, y) \in (T, Y) \Rightarrow f(t, y) \in F(T, Y)$, where $\mathbb{IR}$ denotes the space of real intervals.

- for each $T \subset \Delta_t$ and for each $Y \subset \Delta_y$ there exists a constant $\Lambda > 0$ such that

$$w(F(T, Y)) \leq \Lambda(w(T) + w(Y)),$$

where $w(A)$ denotes the width of interval A ,

- $\Psi(T, Y)$ is defined for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- $\Psi(T, Y)$ is monotonic with respect to inclusion.

Assuming that $y(0) \in Y_0$, and the intervals Y_k such that $y(t_k) \in Y_k$ for $k = 1, 2, \dots, n-1$ are known, for (5) we propose the following interval version

$$Y_k = Y_{k-4} + h \sum_{j=1}^n \beta_{nj} F_{k-j} + h^{n+1} (\gamma_n^* \Psi_n + \gamma_n^{**} \Psi_n), \quad (10)$$

$$k = n, n+1, \dots, m,$$

where

$$F_{k-j} = F(T_{k-j}, Y_{k-j}),$$

$$\Psi_n = \Psi(T_{k-1} + [-(n-1)h, h], Y_{k-1} + [-(n-1)h, h] F(\Delta_t, \Delta_y)),$$

and where $\Psi(T, Y)$ denotes an interval extension of $f^{(n)}(t, y(t))$. For the interval version of (6), in (10), we should take $n = 4$. The additional starting intervals Y_k ($k = 1, 2, \dots, n-1$) one can obtain by applying an interval one-step method, for example an interval method of Runge-Kutta type (see, e.g., [7, 8, 20, 27, 31, 43]) or interval methods based on Taylor series (see, e.g., [2, 3, 5, 15, 21, 39]).

Let us note that in (10), we cannot write $(\gamma_n^* + \gamma_n^{**}) \Psi_n$ instead of $\gamma_n^* \Psi_n + \gamma_n^{**} \Psi_n$, because in general $|\gamma_n^* + \gamma_n^{**}|$ may be different from $|\gamma_n^*| + |\gamma_n^{**}|$.

We can prove the following

Theorem 1 *If $y(0) \in Y_0$ and $y(t_i) \in Y_i$ for $i = 1, 2, \dots, n-1$, then for the exact solution $y(t)$ of the initial value problem (1)⁵ we have*

$$y(t_k) \in Y_k$$

for $k = n, n+1, \dots, m$, where $Y_k = Y(t_k)$ are obtained from the method (10).

Proof First, let us prove that if $(t_i, y(t_i)) \in (T_i, Y_i)$ for $i = k-n, k-n+1, \dots, k-1$, then for any $j = 0, 1, \dots, n-1$ we have

$$\nabla^j f(t_{k-1}, y(t_{k-1})) \in \nabla^j F(T_{k-1}, Y_{k-1}). \quad (11)$$

Since $F(T, Y)$ is an interval extension of $f(t, y)$, then

$$f(t_{k-1-p}, y(t_{k-1-p})) \in F(T_{k-1-p}, Y_{k-1-p}), \quad p = 0, 1, \dots, j.$$

⁵As previously, we assume that such a solution exists and is unique.

This implies that

$$\sum_{p=0}^j (-1)^p \binom{j}{p} f(t_{k-1-p}, y(t_{k-1-p})) \in \sum_{p=0}^j (-1)^p \binom{j}{p} F(T_{k-1-p}, Y_{k-1-p}). \quad (12)$$

But

$$\sum_{p=0}^j (-1)^p \binom{j}{p} F(T_{k-1-p}, Y_{k-1-p}) = \nabla^j F(T_{k-1}, Y_{k-1}). \quad (13)$$

From (4), (12), and (13), the relation (11) follows immediately.

Now, let us consider the formula (3) with (8) for $k = n$. We get

$$\begin{aligned} y(t_n) &= y(t_{n-4}) + h \sum_{j=0}^{n-1} \gamma_j \nabla^j f(t_{n-1}, y(t_{n-1})) \\ &\quad + h^{n+1} (\gamma_n^* \psi(\eta^*, y(\eta^*)) + \gamma_n^{**} \psi(\eta^{**}, y(\eta^{**}))), \end{aligned}$$

where $\eta^*, \eta^{**} \in [t_0, t_n]$. From the assumption, we have $y(t_{n-4}) \in Y_{n-4}$, and from (11) it follows that

$$h \sum_{j=0}^{n-1} \gamma_j \nabla^j f(t_{n-1}, y(t_{n-1})) \in h \sum_{j=0}^{n-1} \gamma_j \nabla^j F(T_{n-1}, Y_{n-1}).$$

Applying Taylor's formula, we have

$$y(\eta) = y(t_{n-1}) + (\eta - t_{n-1}) y'(t_{n-1} + \vartheta(\eta - t_{n-1})), \quad (14)$$

where $\vartheta \in [0, 1]$. Because $\eta \in [t_0, t_n]$ and $t_i = ih$ for $i = 0, 1, \dots, m$, we get

$$\eta - t_{n-1} \in [-(n-1)h, h]. \quad (15)$$

Moreover, $y'(t) = f(t, y(t))$. Since

$$f(t_{n-1} + \vartheta(\eta - t_{n-1}), y(t_{n-1} + \vartheta(\eta - t_{n-1}))) \in F(\Delta_t, \Delta_y),$$

then

$$y'(t_{n-1} + \vartheta(\eta - t_{n-1})) \in F(\Delta_t, \Delta_y). \quad (16)$$

Let us remind here that Δ_t and Δ_y are bounded sets in which the function $f(t, y)$ is defined, i.e. $t \in \Delta_t$ and $y \in \Delta_y$ for any t and y . Taking into account the above consideration, from the formula (14) we get

$$y(\eta) \in Y_{n-1} + [-(n-1)h, h] F(\Delta_t, \Delta_y).$$

But $\Psi(T, Y)$ is an interval extension of ψ . Since η can be either η^* or η^{**} , then applying (15) and (16), we have

$$\gamma_n^* \psi(\eta^*, y(\eta^*)) \in \gamma_n^* \Psi(T_{n-1} + [-(n-1)h, h], Y_{n-1} + [-(n-1)h, h] F(\Delta_t, \Delta_y))$$

and

$$\gamma_n^{**} \psi(\eta^{**}, y(\eta^{**})) \in \gamma_n^{**} \Psi(T_{n-1} + [-(n-1)h, h], Y_{n-1} + [-(n-1)h, h] F(\Delta_t, \Delta_y)).$$

Thus, we have shown that $y(t_n)$ belongs to the interval

$$\begin{aligned} Y_{n-4} + h \sum_{j=0}^{n-1} \gamma_j \nabla^j F(T_{n-1}, Y_{n-1}) + h^{n+1} (\gamma_n^* \Psi_n + \gamma_n^{**} \Psi_n) \\ = Y_{n-4} + h \sum_{j=1}^n \beta_{nj} F(T_{k-j}, Y_{k-j}) + h^{n+1} (\gamma_n^* \Psi_n + \gamma_n^{**} \Psi_n), \end{aligned}$$

but—according to (10)—this is the interval Y_n . This conclusion ends the proof for $k = n$. In a similar way, we can prove the theorem for $k = n + 1, n + 2, \dots, m$. $\square$

In [18, 25, 27], we have proved theorems regarding the widths of intervals Y_k ($k = n, n + 1, \dots, m$) for interval multistep methods of Adams-Bashforth and Nyström types. Using the same technique for the method (10), we can prove

Theorem 2 *If the intervals Y_k for $k = 0, 1, \dots, n - 1$ are known, $t_i = ih = T_i$, $i = 0, 1, \dots, m$, $h = a/m$, and Y_k for $k = n, n + 1, \dots, m$ are obtained from (10), then*

$$w(Y_k) \leq A \max_{q=0,1,\dots,n-1} w(Y_q) + B \max_{j=1,2,\dots,m-1} w(T_j) + Ch^n,$$

where $w(X)$ denotes the width of the interval X , and the nonnegative constants A , B and C are independent of h .

From this theorem, it follows that if $h \rightarrow 0$, then the width of Y_k ($k = n, n + 1, \dots, m$) depends theoretically only on the widths of known previous intervals Y_k ($k = 0, 1, \dots, n - 1$) and on the widths of T_j ($j = 1, 2, \dots, m - 1$; $w(T_0) = w([0, 0]) = 0$).⁶

It should be added that in classical theory of multistep methods, there are considered stability problems. Such concepts for interval multistep methods are not known at present. It seems to be interesting to have a definition of stability for interval multistep methods and consider stability problems for them.

⁶In practice, when all calculations are performed in floating-point interval arithmetic, the step size h cannot be arbitrary small since it may cause a growth of rounding errors and the intervals obtained may be wider.

Taking $n = 4$, from (10), we obtain the following interval version of the popular method (6) which includes the method error (compare (9)):

$$Y_k = Y_{k-4} + \frac{4h}{3} (2F_{k-1} - F_{k-2} + 2F_{k-3}) - \frac{h^5}{720} (27\Psi_4 - 251\Psi_4), \quad (17)$$

where

$$F_{k-j} = F(T_{k-j}, Y_{k-j}), \quad j = 1, 2, 3, \\ \Psi_4 = \Psi(T_{k-1} + [-3h, h], Y_{k-1} + [-3h, h] F(\Delta_t, \Delta_y)). \quad (18)$$

Because in (17) the function $\Psi(T, Y)$ is an interval extension of $f^{(4)}(t, y(t)) \equiv y^{(5)}(t)$, it is useful to have $f^{(4)} = f^{(4)}(t, y(t))$ expressed by its partial derivatives. We have

$$f^{(4)} = \frac{\partial^4 f}{\partial t^4} + 4 \frac{\partial^4 f}{\partial t^3 \partial y} f + 6 \frac{\partial^4 f}{\partial t^2 \partial y^2} f^2 + 4 \frac{\partial^4 f}{\partial t \partial y^3} f^3 + \frac{\partial^4 f}{\partial y^4} f^4 \\ + 6 \left(\frac{\partial^3 f}{\partial t^2 \partial y} + 2 \frac{\partial^3 f}{\partial t \partial y^2} f + \frac{\partial^3 f}{\partial y^3} f^2 \right) f' + 3 \frac{\partial^2 f}{\partial y^2} (f')^2 \quad (19) \\ + 4 \left(\frac{\partial^2 f}{\partial t \partial y} + \frac{\partial^2 f}{\partial y^2} f \right) f'' + \frac{\partial f}{\partial y} f''',$$

where

$$f' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f, \quad f'' = \frac{\partial^2 f}{\partial t^2} + 2 \frac{\partial^2 f}{\partial t \partial y} f + \frac{\partial^2 f}{\partial y^2} f^2 + \frac{\partial f}{\partial y} f', \\ f''' = \frac{\partial^3 f}{\partial t^3} + 3 \frac{\partial^3 f}{\partial t^2 \partial y} f + 3 \frac{\partial^3 f}{\partial t \partial y^2} f^2 + \frac{\partial^3 f}{\partial y^3} f^3 + 3 \left(\frac{\partial^2 f}{\partial t \partial y} + \frac{\partial^2 f}{\partial y^2} f \right) f' + \frac{\partial f}{\partial y} f''.$$

Mathematical software (e.g., Derive, Matlab, Mathematica) can be very helpful to find analytical forms of the above derivatives, and then their interval extensions can be easily determined. Although analytical form of $\Psi(T, Y)$ causes to be the interval method very fast, on the other hand one can use automatic differentiation (see, e.g., [10, 42]) to calculate a bound for the error term. The important advantage of the second approach is that only a formula or algorithm for a given function is required and an explicit formula for its derivative is not needed.

In this and previous sections, we have considered the one-dimensional initial value problem, but all the theory can be easily extended to the system of differential equations (see Examples 4 and 5 in the next section).

4 Numerical examples

In this section, we present some examples for the interval method (17), which we call M4, and compare the results obtained with other interval methods of the same order (known so far), namely:

Table 1 The approximate (with 16 digits after decimal point) exact solution to (20) with $\lambda = 0.5$

t	$y(t)$
0.2	1.1051709180756476
0.6	1.3488588075760031
1.0	1.6487212707001282

- the fourth-step interval method of Adams-Bashforth type (called AB4)

$$Y_k = Y_{k-1} + \frac{h}{24} (55F_{k-1} - 59F_{k-2} + 37F_{k-3} - 9F_{k-4}) + \frac{251h^5}{720} \Psi_4,$$

- the fourth-step interval method of Nyström type (called N4)

$$Y_k = Y_{k-2} + \frac{h}{3} (8F_{k-1} - 5F_{k-2} + 4F_{k-3} - F_{k-4}) + \frac{h^5}{720} (251\Psi_4 - 19\Psi_4),$$

where F_{k-j} ($j = 1, 2, 3, 4$) and Ψ_4 are given by (18).

In all examples, we have used an implementation of floating-point interval arithmetic in Delphi Pascal. This implementation has been written in the form of a unit called *IntervalArithmetic-32and64* (the current version of this unit is presented in [29]). This unit takes advantage of the Delphi Pascal floating-point *Extended* type and makes it possible to:

- represent any input numerical data in the form of a machine interval (for a real number the ends of this interval are equal or are two subsequent machine numbers),
- perform all calculations in floating-point interval arithmetic using overloading operators,
- use some standard interval functions,
- give results in the form of proper intervals (if the ends of an interval are not the same machine numbers, one can see the difference in the output).

All programs written in Delphi Pascal for the examples presented can be found in [28]. In [28], it is also included a Delphi Pascal Program for solving any initial value problem by the M4 method. This program requires the user to write a dynamical link library with definitions of appropriate interval functions and to determine starting intervals.

Example 1 Firstly, let us consider the commonly used test problem

$$y' = \lambda y, \quad y(0) = 1, \quad (20)$$

Table 2 Starting intervals for the problem (20) and the M4 method with $h = 0.001$

k	$t = kh \in T_k$	Y_k
1	0.001	[<u>1.0005001250208359</u> , <u>1.0005001250208360</u>]
2	0.002	[<u>1.0010005001667083</u> , <u>1.0010005001667084</u>]
3	0.003	[<u>1.0015011255627110</u> , <u>1.0015011255627111</u>]

Table 3 The solution of (20) obtained by the M4 method

$t = kh \in T_k$	Y_k	Width
0.2	[1.1051709180756476E+0000, 1.1051709180756477E+0000]	$\approx 3.16 \cdot 10^{-17}$
0.6	[1.3498588075760030E+0000, 1.3498588075760032E+0000]	$\approx 1.10 \cdot 10^{-16}$
1.0	[1.6487212707001280E+0000, 1.6487212707001283E+0000]	$\approx 2.15 \cdot 10^{-16}$

with the exact solution $y = \exp(\lambda t)$. For $\lambda = 0.5$ and $t = 0.2, 0.6, 1.0$, the numerical values of the solution are given in Table 1.

For $h = 0.001$, $\Delta_t = [0, 1]$, $\Delta_y = [1, 1.65]$ and starting points given in Table 2, where $\underline{x}$ denotes the largest machine number less or equal to x and $\bar{x}$ denotes the smallest machine number greater or equal to x , the intervals obtained by the M4 method are presented in Table 3. We can observe, according to the theorem in Section 3, that in each case the exact solution belongs to the interval solution obtained (compare Table 1).

Let us note that the step size h should be selected suitable (for each problem considered). To demonstrate the step size influence on the widths of interval solutions, let us consider our method also with step sizes $h = 0.0001, 0.0005, 0.005, 0.01$. The widths of interval solutions obtained for these step sizes are presented in Fig. 1. From this figure, it follows that the step size $h = 0.0001$ gives the smallest widths. For step sizes less than 0.001, we have more calculations and this fact causes a growth of rounding errors.

Example 2 Let us compare the results obtained by our method with other known methods of the same order. For the initial value problem (20), $h = 0.001$ and the methods AB4, N4, and M4, the intervals obtained at $t = 1$ are presented in Table 4. The widths of interval solutions are shown in Fig. 2. From the results obtained, it

Fig. 1 The widths of interval solutions of (20) for different step sizes

Table 4 A comparison of interval solutions at $t = 1$ for the problem (20) and different interval methods of order 4

Method	Y	Width
AB4	[1.6487212707001276E+0000, 1.6487212707001317E+0000]	$\approx 3.94 \cdot 10^{-15}$
N4	[1.6487212707001278E+0000, 1.6487212707001285E+0000]	$\approx 6.17 \cdot 10^{-16}$
M4	[1.6487212707001280E+0000, 1.6487212707001283E+0000]	$\approx 2.15 \cdot 10^{-16}$

follows that the fourth-step interval Milne's method is better than other four-order method considered.

On the other hand, the VNODE-LP package with an interval method based on high-order Taylor series [36, 37] at $t = 1$ produces the output

$$1.64872127070012[75, 87],$$

what in scientific notation can be written as

$$[1.6487212707001275E + 0000, 1.6487212707001287E + 0000].$$

We see that our methods give comparable results.

In the above examples, we can observe that the AB4 method gives the worst results. The same we have observed in a number of other examples considered by us. Thus, we omit this method in comparisons in further examples.

Example 3 The problem considered in Example 1 and 2 is very simple. Except of $\partial f / \partial y$, all other partial derivatives occurring in (19) are equal to zero. Thus, let us

Fig. 2 A comparison of interval widths for the fourth order interval methods

Table 5 Starting intervals for the problem (21) and the methods N4 and M4

k	$t = kh \in T_k$	Y_k
1	0.001	[<u>4.0009997500832942</u> , <u>4.0009997500832943</u>]
2	0.002	[<u>4.0019990006660423</u> , <u>4.0019990006660424</u>]
3	0.003	[<u>4.0029977522468409</u> , <u>4.0029977522468410</u>]]

Table 6 A comparison of interval solutions at $t = 1$ for the problem (21) and different interval methods of order 4

Method	Y	Width
N3	[4.8075923778847027E+0000, 4.8075923778847100E+0000]	$\approx 7.22 \cdot 10^{-15}$
M3	[4.8075923778847050E+0000, 4.8075923778847076E+0000]	$\approx 2.59 \cdot 10^{-15}$

Table 7 The approximate (with 16 digits after decimal point) exact solution of (24) and (25)

t	$y_1(t)$	$y_2(t)$
0.5	−1.6396588322319527	0.0026272853504447
1.0	−0.0164547811431673	−0.5235724095003080
1.5	1.6394937004270022	−0.0078815914545999
2.0	0.0329079051076236	0.5234933138616935

Table 8 Starting intervals for the problem (24) and (25) and the M4 method

k	$t = th \in T_k$	$Y_s = Y_{sk}$
1	0.001	$Y_1 = [-0.00513474154027594, -0.00513474154027593]$ $Y_2 = [\underline{0.52359620822543062}, \overline{0.52359620822543063}]$
2	0.002	$Y_1 = [-0.01026943272597990, -0.01026943272597989]$ $Y_2 = [\underline{0.52358850613200318}, \overline{0.52358850613200319}]$
3	0.003	$Y_1 = [-0.01540402320303372, -0.01540402320303371]$ $Y_2 = [\underline{0.52357566939354822}, \overline{0.52357566939354823}]$

Table 9 The interval solution of (24) and (25) obtained by the M4 method

$t_k \in T_k$	$Y_s = Y_{sk}$	Width of Y_s
0.5	$Y_1 = [-1.6396588322332133E+0000, -1.6396588322308122E+0000]$	$\approx 2.40 \cdot 10^{-12}$
	$Y_2 = [2.6272853500601767E-0003, 2.6272853508293990E-0003]$	$\approx 7.69 \cdot 10^{-13}$
1.0	$Y_1 = [-1.6454781160720380E-0002, -1.6454781125617295E-0002]$	$\approx 3.51 \cdot 10^{-11}$
	$Y_2 = [-5.2357240950595249E-0001, -5.2357240949474017E-0001]$	$\approx 1.12 \cdot 10^{-11}$
1.5	$Y_1 = [1.6394937001873533E+0000, 1.6394937006670114E+0000]$	$\approx 4.80 \cdot 10^{-10}$
	$Y_2 = [-7.8815915311871515E-0003, -7.8815913780144385E-0003]$	$\approx 1.53 \cdot 10^{-10}$
2.0	$Y_1 = [3.2907901846203859E-0002, 3.2907908369053163E-0002]$	$\approx 6.52 \cdot 10^{-9}$
	$Y_2 = [5.2349331282029622E-0001, 5.2349331490324421E-0001]$	$\approx 2.08 \cdot 10^{-9}$

take into account a more complicated problem (the problem A5 from [6, p. 23])

$$y' = \frac{y-t}{y+t}, \quad y(0) = 4. \quad (21)$$

Here, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{2y}{(y+t)^2}, & \frac{\partial f}{\partial y} &= \frac{2t}{(y+t)^2}, \\ \frac{\partial^2 f}{\partial t^2} &= \frac{4y}{(y+t)^3}, & \frac{\partial^2 f}{\partial t \partial y} &= \frac{2(y-t)}{(y+t)^3}, & \frac{\partial^2 f}{\partial y^2} &= -\frac{4t}{(y+t)^3}, \\ \frac{\partial^3 f}{\partial t^3} &= -\frac{12y}{(y+t)^4}, & \frac{\partial^3 f}{\partial t^2 \partial y} &= \frac{4(t-2y)}{(y+t)^4}, \\ \frac{\partial^3 f}{\partial t \partial y^2} &= \frac{4(2t-y)}{(y+t)^4}, & \frac{\partial^3 f}{\partial y^3} &= \frac{12t}{(y+t)^4}, \\ \frac{\partial^4 f}{\partial t^4} &= \frac{48y}{(y+t)^5}, & \frac{\partial^4 f}{\partial t^3 \partial y} &= \frac{12(3y-t)}{(y+t)^5}, & \frac{\partial^4 f}{\partial t^2 \partial y^2} &= \frac{24(y-t)}{(y+t)^5}, \\ \frac{\partial^4 f}{\partial t \partial y^3} &= \frac{12(y-3t)}{(y+t)^5}, & \frac{\partial^4 f}{\partial y^4} &= -\frac{48t}{(y+t)^5}. \end{aligned}$$

Let $h = 0.001$, $\Delta_t = [0, 1]$, $\Delta_y = [4, \overline{4.9}]$ and let us take additional starting intervals presented in Table 5 (these intervals have been obtained by an interval version of conventional fourth order Runge-Kutta method [7, 20, 27, 43] with $h = 0.0001$). In Table 6, we present the results obtained at $t = 1$ by the methods N4 and M4. As previously, we see that the M4 method is better than the N4 method.

Our method can be also used for solving more than one-dimensional problems.

Example 4 Let us consider the motion of a simple pendulum described by

$$\varphi'' + u^2 \sin \varphi = 0, \quad (22)$$

where $\varphi = \varphi(t)$, $u = \sqrt{g/L}$, g is the gravitational acceleration at the Earth's surface, and L denotes the pendulum length. If we assume that the angle φ is small, i.e., $\sin \varphi \approx \varphi^7$, then the equation (22) can be reduced to the equation of simple harmonic motion

$$\varphi'' + u^2 \varphi = 0, \quad (23)$$

with the solution $\varphi(t) = \varphi_0 \cos(ut)$, where φ_0 is an initial angle. Denoting $y_1 = \varphi'$, $y_2 = \varphi$, and assuming that $\varphi'(0) = 0$, $\varphi(0) = \varphi_0$, we can transform (23) into the following systems of differential equations of the first order:

$$y_1' = -u^2 y_2, \quad y_2' = y_1, \quad (24)$$

with the initial conditions

$$y_1(0) = 0, \quad y_2(0) = \varphi_0. \quad (25)$$

For $g = 9.80665$, $L = 1$ and $\varphi_0 = \pi/6$, the exact solution at selected points t is presented in Table 7.

Assuming additional starting intervals as shown in Table 8, and applying the M4 method we have obtained interval approximations presented in Table 9. In the method we assumed that

$$g = [9.80665, \overline{9.80665}], \quad \varphi_0 = [\underline{0.52359877559829887}, \overline{0.52359877559829888}],$$

and took $h = 0.001$,

$$\Delta_t = [0, 2], \quad \Delta_{y_1} = [\underline{-1.8}, \overline{1.8}], \quad \Delta_{y_2} = [\underline{-0.6}, \overline{0.6}].$$

We can observe that for each t the exact solution belongs to the interval solutions obtained.

Example 5 Finally, let us consider a stiff problem (for a definition of stiffness see, e.g., [14]). Let us take the initial value problem E2 given in [6, p. 21]), i.e.,

$$\begin{aligned} y_1' &= y_2, & y_2' &= 5(1 - y_1^2)y_2 - y_1, \\ y_1(0) &= 2, & y_2(0) &= 0. \end{aligned} \quad (26)$$

Assuming

$$\begin{aligned} \Delta_t &= \{t \in \mathbb{R} : 0 \leq t \leq 1\}, \\ \Delta_{y_1} &= \{y_1 \in \mathbb{R} : 1.8 \leq y_1 \leq 2.1\}, & \Delta_{y_2} &= \{y_2 \in \mathbb{R} : -0.2 \leq y_2 \leq 0.1\}, \end{aligned} \quad (27)$$

$h = 0.0001$, and taking additional starting intervals given in Table 10 (these intervals have been obtained by an interval version of conventional fourth order Runge-Kutta method [7, 20, 27, 43] with $h = 0.00001$), we have obtained by the M4 method the results presented in Table 11.

If the integration interval is sufficiently small, the results obtained by the low (fourth) order N4 and M4 methods can be compared with the results obtained by methods based on high-order Tylor series. For example, at $t = 0.05$ the VNODE-LP package [37] gives

$$Y_1 = 1.99802342677384[48, 55], \quad Y_2 = -0.070355564016027[1, 3],$$

⁷For instance, for $|\varphi| < 5^\circ$ we have $|\sin \varphi - \varphi| < 1.1 \cdot 10^{-4}$.

Table 10 Starting intervals for the problem (26)

k	$t = th \in T_k$	$Y_s = Y_{sk}$
1	0.0001	$Y_1 = [\underline{1.999999900049981E+0000}, \overline{1.999999900049982E+0000}]$ $Y_2 = [\underline{-1.9985007463979870E-0004}, \overline{-1.9985007463979869E-0004}]$
2	0.0002	$Y_1 = [\underline{1.9999999600399701E+0000}, \overline{1.9999999600399702E+0000}]$ $Y_2 = [\underline{-3.9940059690355807E-0004}, \overline{-3.9940059690355806E-0004}]$
3	0.0003	$Y_1 = [\underline{1.9999999101348489E+0000}, \overline{1.9999999101348490E+0000}]$ $Y_2 = [\underline{-5.9865201382483130E-0004}, \overline{-5.9865201382483129E-0004}]$

Table 11 The interval solution of (26) obtained by the M4 method

$t_k \in T_k$	$Y_s = Y_{sk}$	Width of Y_s
0.2	$Y_1 = [\underline{1.9817243912807345E+0000}, \overline{1.9817243912807411E+0000}]$ $Y_2 = [\underline{-1.2774153125701151E-0001}, \overline{-1.2774153125691430E-0001}]$	$\approx 6.35 \cdot 10^{-15}$ $\approx 9.72 \cdot 10^{-14}$
0.6	$Y_1 = [\underline{1.9271919567926597E+0000}, \overline{1.9271919568934465E+0000}]$ $Y_2 = [\underline{-1.4072022792184367E-0001}, \overline{-1.4072022651593288E-0002}]$	$\approx 1.01 \cdot 10^{-10}$ $\approx 1.41 \cdot 10^{-9}$
1.0	$Y_1 = [\underline{1.8694384659159655E+0000}, \overline{1.8694392409135573E+0000}]$ $Y_2 = [\underline{-1.4824087078964731E-0001}, \overline{-1.4823088051413765E-0001}]$	$\approx 7.75 \cdot 10^{-7}$ $\approx 9.99 \cdot 10^{-6}$

Table 12 The Intervals obtained at $t = 0.05$ by the methods N4 and M4

Method	$Y_s = Y_{sk}$	Width of Y_s
N4	$Y_1 = [\underline{1.9980234267738451E+0000}, \overline{1.9980234267738457E+0000}]$ $Y_2 = [\underline{-7.0355564016030382E-0002}, \overline{-7.0355564016024156E-0002}]$	$\approx 5.20 \cdot 10^{-16}$ $\approx 6.22 \cdot 10^{-15}$
M4	$Y_1 = [\underline{1.9980234267738453E+0000}, \overline{1.9980234267738455E+0000}]$ $Y_2 = [\underline{-7.0355564016028083E-0002}, \overline{-7.0355564016026386E-0002}]$	$\approx 1.15 \cdot 10^{-16}$ $\approx 1.70 \cdot 10^{-15}$

while the methods N4 and M4 yield intervals presented in Table 12.

5 Conclusions

Interval methods executed in floating-point interval arithmetic yield approximations of solutions in the form of intervals, which contain all possible numerical errors, i.e., representation errors, rounding errors, truncation errors, and also data uncertainties. Interval multistep methods are of this kind of methods. They are simple, what causes that their execution time is short (for example, in comparison to interval methods of Runge-Kutta type).

So far, four classes of interval multistep methods have been known: explicit of Adams-Bashforth and Nyström, and implicit of Adams-Moulton and Milne-Simpson. In this paper, we presented a new explicit interval class based on conventional Milne's methods. We have paid our attention especially to the fourth-order method, since such a method requires only three interval function evaluations. Comparisons of this method to other known fourth order methods clearly show that our new method gives comparable results, and moreover is better than other explicit interval multistep methods. Although the method is of low order it gives comparable results even with interval methods based on high-order Taylor series.

As we have shown in Example 4 and 5, our method can be also used for solving more than one-dimensional problems. But in this case, one can obtain the effect of overestimation, which is called the wrapping effect (see, e.g., [10, p. 40]). We intend to consider and handle this problem in our future research.

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