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Controllability of the time-varying fractional dynamical systems with a single delay in control

K. S. Vishnukumar · S. M. Sivalingam · Hijaz Ahmad · V. Govindaraj

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Abstract In this article, we explored the controllability of fractional dynamical systems with a single delay in the control function with the Caputo fractional derivative. It is the first work in which the author studies the controllability of a time-varying fractional dynamical system with a delay in the control function. We develop the necessary and sufficient criteria for the solution representation of controllability of time-varying fractional linear dynamical systems by utilizing the Grammian matrix. We use Schauder's fixed point theorem to establish sufficient conditions for the controllability of time-varying nonlinear fractional dynamical systems. With the help of successive approximation techniques, numerical examples validate the theoretical results.

K. S. Vishnukumar · S. M. Sivalingam · V. Govindaraj
Department of Mathematics, National Institute of Technology Puducherry, Karaikal 609609, India

H. Ahmad (✉)
Operational Research Center in Healthcare, Near East University, TRNC Mersin 10, 99138 Nicosia, Turkey
e-mail: hijaz.ahmad@neu.edu.tr

H. Ahmad
Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia

H. Ahmad
Center for Applied Mathematics and Bioinformatics, Gulf University for Science and Technology, Mishref, Kuwait

H. Ahmad
Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon

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1 Introduction

In the last thirty years, fractional calculus is an advanced area of mathematical analysis that deals with non-integer orders of differentiation and integration. This advanced field provides a powerful tool to model complex systems that exhibit non-linear dynamics, such as those in physics, engineering, and economics [1,2]. It is an operator that comprises both integrals and derivatives of integer order as specific cases. Matychyn et al. [3] investigated an analytical solution of linear fractional systems with variable coefficients involving Riemann-Liouville and Caputo derivatives. Applications of fractional calculus are pervasive, using many fractional derivatives such as Caputo derivative, Riemann–Liouville derivative, Atangana–Baleanu derivative, Caputo–Fabrizio derivative, Hadamard derivative, and Grünwald–Letnikov, etc., and a significant number of researchers conduct research in fractional calculus [4–7]. Idczak et al. [8] provided the existence and uniqueness for the solution of Riemann-Liouville fractional Cauchy problem in $\mathbb{R}^n$. When constructing mathematical models for a wide range of distinct physical processes, the category of fractional differential equations of various forms plays an essential part. Baleanu et al. [9] introduced a Caputo-type fractional model to investigate the dynamics of COVID-19 pandemic and explore its fundamental behaviours. In, Baleanu et al. [10] proposed a new mathematical model in a generalized fractional framework for the investigation of an HIV/AIDS transmission dynamics.

The notion of controllability has played a pivotal role in the development of control theory and engineering. This area of study has a strong connection to the concepts of structural deconstruction and quadratic optimum, in addition to other ideas that are conceptually related [11–13]. In dynamical system theory, controllability has been suggested to be a qualitative feature [14,15]. This indicates that it is possible to steer the system from any starting state to any ending state by employing some admissible control and some short period of time. The notions of controllability, observability, stability, and stabilizability are central to the study of dynamical systems in the field of control theory. Matingon et al. [16] defined the controllability and observability properties of linear fractional differential systems of finite dimension, given either in state space form or in polynomial representation, and derived the structural results from both analytic and algebraic points of view. In 2013, Balachandran et al. [17] derived a set of sufficient conditions for controllability of nonlinear fractional dynamical system of order $1 < \zeta < 2$ in finite dimensional spaces. In this paper [18], Govindaraj et al. discussed the trajectory controllability of linear and nonlinear fractional dynamical systems represented by the fractional differential equations in the sense of Caputo fractional derivative by using the Mittag-Leffler function and Gronwall-Bellman inequality. In 2017, Govindaraj et al. [19] provided a set of equivalent conditions for observability and controllability of linear fractional dynamical systems represented by the fractional differential equation in the sense of Caputo fractional derivative of order $\zeta \in (0, 1]$ are established by functional approach. In 2023, Selvam et al. [20] investigated the controllability of dynamical systems in terms of the Ψ-Caputo fractional derivative. The Grammian matrix is used to get the necessary and sufficient controllability requirements for linear systems, which are characterized by the Mittag-Leffler functions, while the fixed point approach is used to arrive at adequate controllability criteria for nonlinear systems. The last few decades have seen an increase in papers discussing various controllability ideas for dynamical systems [21–26].

Delay differential equations are a specific type of differential equation that differ from the usual ones in that they incorporate time delays into the modeling process. This means that the derivative of the unknown function at a certain time depends not only on the function's current value but also on its past values. These kinds of processes can be discovered, among other locations, in the systems of rolling mills, communication lines, manufacturing facilities, and even chemical operations. The representation of these systems is done through delay, integral, and integro-differential equations. To solve delay differential equations, one would often use methods of computing, asymptotic approaches, and visualization techniques. However, the theory that supports delay differential equations has progressed to a more advanced stage of development, which is in comparison with the current state of the scientific literature on fractional delay differential equations, which has just reached its infancy stage of growth in the modern world. Many researchers have tried to solve these problems with state and control function delays [27–29].

Research was done on the controllability of linear structures with delay, nonlinear structures with delay, and integro-differential systems with delay. In [30], the author Wei gave the solution expression for fractional control systems with control delay, and provided the necessary and sufficient conditions for the controllability of fractional control systems with control delay. Several authors have investigated controllability results in linear and nonlinear fractional systems with control delays. In, Zhang et al. [31] discussed the reachability and controllability of fractional singular dynamical systems with control delay and a set of sufficient and necessary conditions of controllability for such systems are established based on the algebraic approach. In, Muni et al. [32] studied the controllability of finite-dimensional dynamical control systems modelled by fractional order $\zeta \in (0, 1)$ semilinear autonomous differential equations with a constant time delay in control function. Panneer Selvam et al. [33] studied the reachability of linear and non-linear fractional dynamical systems with multiple delays in control in the sense of the Ψ-Hilfer pseudo-fractional derivative. Vellappandi et al. [34] investigated the fractional optimal control problem with a single delay in the state by using the operator theoretic approach. In this approach, the author first reduced the delay fractional dynamical system into an equivalent operator equation. Then, by providing sufficient conditions to the operators, an optimal pair is proved for the abstract system. In, Panneer Selvam et al. [35] investigated the controllability of linear and non-linear fractional dynamical systems with distributed delays in control using the Ψ-Caputo fractional derivative. The following papers [36–40,42,43] related to fractional dynamical systems with control delay are provided here for reference.

The fundamental of linear control theory were esblished in the 1960s in the work of Kalman [44]. Reissig et al. [45] studied novel results on the controllability and observability of the linear discrete-time and continuous-time control system. In [46], Lü et al. devoted to analyzing the controllability problem of fractional ordinary and partial differential equations (ODE/PDE). Davison et al. [47] derived some sufficient conditions for global and local controllability of nonlinear time-varying systems with control appearing linearly. The controllability of time-varying fractional linear dynamical systems has been the subject of much research in recent years. This type of system is characterized by a linear fractional differential equation with time-varying coefficients, and it has many applications in various fields, such as aerospace engineering, control systems, and signal processing. The controllability of such systems is important because it allows us to design control strategies that can drive the system to a desired state. In Jolić [48] et al. studied the controllability and observability of linear time-varying fractional systems. Bourdin [49] addressed the existence and uniqueness results for solutions of non-linear Cauchy problems with vector fractional multi-order. Finally, Bourdin introduced notions of fractional state-transition matrices and derived fractional versions of the classical Duhamel formula. In, Sivalingam et al. [50] discussed time-varying impulsive fractional differential equations using theory of functional connections and neural network. Also, Sivalingam et al. provided the existence and uniqueness of the solution of the impulsive fractional system are proved theoretically using the Bourdin state transition matrix-based solution representation and the Banach fixed point theorem. However, qualitative characteristics of this study have not yet been studied. This characteristic inspires this research, which investigates the controllability of time-varying fractional dynamical systems with a single delay in control by using the Caputo fractional derivative. In this article, we consider a time-varying fractional linear dynamical system with a single delay in control guided by a Caputo fractional dynamical system of the form

$$
\begin{aligned}
&{}^{C}_{t_0}D^{\zeta}_{t}\, w(t) = \mathcal{P}(t)w(t) + \mathcal{Q}(t)\eta(t-\gamma),\\
&\qquad\qquad t \in [t_0, t_1],\ 0 < \zeta \le 1,\\
&w(t)|_{t=t_0} = w_0,\\
&\eta(t) = \eta_0(t), \quad t \in [t_0-\gamma, t_0).
\end{aligned}
$$

Also, we consider the time-varying fractional nonlinear dynamical system with single delay in control governed by Caputo fractional dynamical system of the form

$$
\begin{aligned}
&{}^{C}_{t_0}D^{\zeta}_{t}\, w(t) = \mathcal{P}(t)w(t) + \mathcal{Q}(t)\eta(t-\gamma)\\
&\qquad\qquad + f(t, w(t), \eta(t)), \quad t \in [t_0, t_1],\\
&w(t_0) = w_0,\\
&\eta(t) = \eta_0(t), \quad t \in [t_0-\gamma, t_0),
\end{aligned}
$$

where expression ${}^{C}_{t_0}D^{\zeta}_{t}\, w(t)$ defines the Caputo fractional derivative of order $0 < \zeta \le 1$. The vectors $w \in \mathbb{R}^k$ and $\eta \in \mathbb{R}^l$ are denotes the state and control respectively. The entries of matrices $\mathcal{P}(t)_{k\times k}$ and $\mathcal{Q}(t)_{k\times l}$ are matrices valued continuous function over $\mathbb{R}$ and the function $f : [t_0, t_1] \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ is continuous. The aim of this paper is to study control problems on a finite time interval $[t_0, t_1]$, in which the goal is to find a control function η that will steer the solution of the system from a given initial state $w(t_0) = w_0$ to a desired final state $w(t_1) = w_1$ during the interval $[t_0, t_1]$. The novelty of this study is that it generalizes the controllability of fractional dynamical systems. If we take $\mathcal{P}(t) = \mathcal{P}$, $\mathcal{Q}(t) = \mathcal{Q}$ and $\zeta \in (0, 1)$, the controllability of the time-varying fractional dynamical system of equations becomes controllability of the time-invariant the fractional dynamical system. Also, if $\mathcal{P}(t) = \mathcal{P}$, $\mathcal{Q}(t) = \mathcal{Q}$ and $\zeta = 1$, the controllability of the time-varying fractional system of equations becomes the controllability of the time-invariant ordinary dynamical system. The need for high control energy and computational resources limits the application of the proposed method to real-world systems. Currently, the authors are working to overcome this limitation.

The article's structure is as follows: In Sect. 2, we provide several definitions of fractional derivatives to facilitate the development of the article. Section 3 concentrated on the controllability of time-varying Caputo fractional linear dynamical systems with a single delay.

Following that, we used Schauder's fixed point theorem to expand our results to include the concept of nonlinear dynamical systems discussed in Sect. 4. The study's conclusions are presented in the form of a few examples that demonstrate the findings of the theory.

2 Preliminaries and basic results

Definition 1 [1] The left and right Riemann–Liouville fractional integrals of order $\zeta > 0$ for a function $q : [0, \infty) \to \mathbb{R}$ are defined as

$$ {}_{t_0}I_t^{\zeta} q(t) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t} (t-\varrho)^{\zeta-1} q(\varrho) d\varrho, \quad t > t_0 $$

and

$$ {}_{t}I_{t_1}^{\zeta} q(t) = \frac{1}{\Gamma(\zeta)} \int_{t}^{t_1} (\varrho - t)^{\zeta-1} q(\varrho) d\varrho, \quad t < t_1 $$

respectively.

Definition 2 [1] The left and right Caputo fractional derivative of order $\zeta > 0$ for a function $q \in C^{(\omega)}([0,\infty))$ and $\omega - 1 < \zeta \leq \omega$, $\omega \in \mathbb{N}$ are defined as

$$ {}^{C}_{t_0}D_t^{\zeta} q(t) = \frac{1}{\Gamma(\omega-\zeta)} \int_{t_0}^{t} (t-\varrho)^{\omega-\zeta-1} q^{(\omega)}(\varrho) d\varrho, \quad t > t_0 $$

and

$$ {}^{C}_{t}D_{t_1}^{\zeta} q(t) = \frac{1}{\Gamma(\omega-\zeta)} \int_{t}^{t_1} (\varrho - t)^{\omega-\zeta-1} q^{(\omega)}(\varrho) d\varrho, \quad t < t_1 $$

respectively.

Remark 1 Consider the time-varying fractional linear dynamical system

$$ {}^{C}_{t_0}D_t^{\zeta} w(t) = \mathcal{P}(t)w(t) + \mathcal{Q}(t)\eta(t), \; 0 < \zeta \leq 1, $$
$$ w(t)|_{t=t_0} = w_0, \tag{1} $$

where $w(t) \in \mathbb{R}^k$ is the state vector, $\eta(t) \in \mathbb{R}^l$ is the control vector and $\mathcal{P}(t) : [t_0, t_1] \to \mathbb{R}^{k\times k}$, $\mathcal{Q}(t) : [t_0, t_1] \to \mathbb{R}^{k\times l}$ are matrix valued continuous function on $[t_0, t_1]$. The solution to (1) expressed as

$$ w(t) = \Lambda(t, t_0)w_0 + \int_{t_0}^{t} \Delta(t, \phi)\mathcal{Q}(\phi)\eta(\phi)d\phi, $$

where $\Lambda(\lambda, .) : [\lambda, t_1] \to \mathbb{R}^{k\times k}$, $\lambda \in [t_0, t_1]$ and $\Delta(\lambda, .) : (\lambda, t_1) \to \mathbb{R}^{k\times k}$ are the left Riemann–Liouville and left Caputo state transition matrix and are given by the following initial value problem [49]

$$ {}^{RL}_{\lambda}D_t^{\zeta} \Delta(t,\lambda) = \mathcal{P}(t)\Delta(t,\lambda), $$
$$ {}_{\lambda}I_t^{1-\zeta} \Delta(t,\lambda)|_{t=\lambda} = \mathbb{I}, $$

and

$$ {}^{C}_{\lambda}D_t^{\zeta} \Lambda(t,\lambda) = \mathcal{P}(t)\Lambda(t,\lambda), $$
$$ \Lambda(t,t)| = \mathbb{I}, $$

where $\mathbb{I}$ represents the $n \times n$ identity matrix. The state transition matrix $\Delta(t,\lambda)$ satisfies the following inequality, which will be useful for developing the main theorem.

Theorem 1 [49] *If we assume that $t_1 > t_0$, then $\exists$ a $\mathcal{Z} \geq 0 \ni |\Delta_{ij}(t,\lambda)| \leq (t-\lambda)^{\zeta-1}\mathcal{Z}$ for almost everywhere $t_0 \leq \lambda < t \leq t_1$ and for every $i, j \in 1, 2, \ldots, n$.*

Remark 2 If $\mathcal{P}(t) = \mathcal{P}$ (constant matrix), the Riemann-Liouville state transition matrix and the Caputo state transition matrix are expressed in the following manner:

$$ \Delta(t,\lambda) = (t-\lambda)^{\zeta-1}\mathbb{E}_{\zeta,\zeta}(\mathcal{P}(t-\lambda)^{\zeta}), $$
$$ \Lambda(t,\lambda) = \mathbb{E}_{\zeta}\left(\mathcal{P}(t-\lambda)^{\zeta}\right). $$

Remark 3 Consider the time-varying fractional nonlinear dynamical system

$$ {}^{C}_{t_0}D_t^{\zeta} w(t) = P(t)w(t) + Q(t)\eta(t) + f(t, w(t), \eta(t)), \quad t \in [t_0, t_1], $$
$$ w(t_0) = w_0, \tag{2} $$

where $f : [t_0, t_1] \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ is a continuous function. For a given control $\eta(t)$, the solution of the dynamical system (2) is

$$ w(t) = \Lambda(t, t_0)w_0 + \int_{t_0}^{t} \Delta(t, \phi)\mathcal{Q}(\phi)\eta(\phi)d\phi + \int_{t_0}^{t} \Delta(t,\phi) f(\phi, w(\phi), \eta(\phi))d\phi. $$

3 Linear systems

In the following section of the study, we discuss the controllability of the time-varying Caputo fractional linear dynamical system with a single delay. Let us look at the time-varying fractional linear dynamical system

$$\begin{aligned} {}^{C}_{t_0}D^{\zeta}_{t} w(t) &= \mathcal{P}(t)w(t) + \mathcal{Q}(t)\eta(t-\gamma), \; 0 < \zeta \le 1, \\ w(t)|_{t=t_0} &= w_0, \\ \eta(t) &= \eta_0(t) \; \forall t \in [t_0-\gamma, t_0), \end{aligned} \tag{3}$$

where $w(t) \in \mathbb{R}^k$ is the state vector, $\eta(t) \in \mathbb{R}^l$ is the control vector and $\mathcal{P}(t) : [t_0, t_1] \to \mathbb{R}^{k\times k}$, $\mathcal{Q}(t) : [t_0, t_1] \to \mathbb{R}^{k\times l}$ are matrix valued continuous function on $[t_0, t_1]$, $\gamma > 0$ is time control delay and $\eta_0(t)$ the initial control function.

Assumptions:

- Let γ be a time delay constant in the control function $\eta(\cdot)$ which satisfies $t_0 \le \gamma \le t_1$.
- Let $\eta_0(\cdot)$ be a $\mathbb{R}^l$ valued initial control function which is continuous and bounded on $[t_0-\gamma, t_0)$.
- Let $\mathcal{E} = \{k|k : [t_0, t_1] \to \mathbb{R}^k \times \mathbb{R}^l$ is continuous$\}$ be a complete normed space with respect to the norm $\|(k_1, k_2)\| = \|k_1\| + \|k_2\|$, where $\|k_1\| = sup\{|k_1(t)| : t \in [t_0, t_1]\}$ and $\|k_2\| = sup\{|k_2(t)| : t \in [t_0, t_1]\}$.

The solution of (3) is given by

$$\begin{aligned} w(t) = \Lambda(t, t_0)w_0 &+ \int_{t_0-\gamma}^{t_0} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi \\ &+ \int_{t_0}^{t-\gamma} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi. \end{aligned} \tag{4}$$

The controllability of time-varying fractional linear dynamical systems with delay in the control function has applications in various fields such as robotics [52,53], and aerospace. For instance, it can be used to ensure precise control over a robot arm's motion, ensuring it moves accurately and safely. In the field of aerospace, this method can be used to stabilize an aircraft [51], preventing it from losing control. By controlling such systems with delays in the control function, their performance and reliability can be improved. This improvement leads to more efficient and cost-effective solutions in these fields.

Definition 3 The system (3) is controllable on $[t_0, t_1]$, if for any initial state $w_0 \in R^k$ and final state $w_1 \in R^k$, there exists a control vector $\eta(t)$ such that $w(t_1) = w_1$.

Theorem 2 *The time-varying fractional linear dynamical system (3) is controllable over $t \in [t_0, t_1]$ if and only if the controllability Grammian*

$$\begin{aligned} \Re[t_0, t_1] = \int_{t_0}^{t_1-\gamma} &(t_1-(\phi+\gamma))^{2(1-\zeta)}\Delta(t_1, \phi+\gamma) \\ &\times \mathcal{Q}(\phi+\gamma)\mathcal{Q}^*(\phi+\gamma)\Delta^*(t_1, \phi+\gamma)d\phi \end{aligned} \tag{5}$$

is positive definite.

Proof Assume that the controllability Grammian $\Re[t_0, t_1]$ is positive definite. Then, we describe the control function

$$\eta(t) = \begin{cases} (t_1-(t+\gamma))^{2(1-\zeta)}\mathcal{Q}^*(t+\gamma)\Delta^*(t_1, t+\gamma) \\ \quad \times \Re^{-1}[t_0, t_1]\Big[w_1 - \Lambda(t_1, t_0)w_0 \\ \quad - \displaystyle\int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi\Big] & if \; t \in [t_0, t_1-\gamma], \\ 0 & if \; t \in (t_1-\gamma, t_1]. \end{cases} \tag{6}$$

By substituting Eqs. (5) and (6) in (4), we get

$$\begin{aligned} w(t_1) &= \Lambda(t_1, t_0)w_0 + \int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi \\ &\quad \times \int_{t_0}^{t_1-\gamma} \Delta(t_1, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi \\ &= \Lambda(t_1, t_0)w_0 + \int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi \\ &\quad + \int_{t_0}^{t_1-\gamma} \Delta(t_1, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\Big((t_1-(\phi+\gamma))^{2(1-\zeta)} \\ &\quad \times \mathcal{Q}^*(\phi+\gamma)\Delta^*(\phi+\gamma, t_1)\Re^{-1}[t_0, t_1] \\ &\quad \times \Big[w_1 - \Lambda(t_1, t_0)w_0 - \int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma) \\ &\quad \times B(\phi+\gamma)\eta_0(\phi)d\phi\Big]\Big)d\phi \\ &= \Lambda(t_1, t_0)w_0 + \int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi \end{aligned}$$

$$+\int_{t_0}^{t_1-\gamma}(t_1-(\phi+\gamma))^{2(1-\zeta)}\Delta(t_1,\phi+\gamma)$$
$$\times\mathcal{Q}(\phi+\gamma)\mathcal{Q}^*(\phi+\gamma)\Delta^*(t_1,\phi+\gamma)d\phi$$
$$\times\Re^{-1}[t_0,t_1]\Bigg[w_1-\Lambda(t_1,t_0)w_0-\times\int_{t_0-\gamma}^{t_0}\Delta(t_1,\phi+\gamma)B(\phi+\gamma)\eta_0(\phi)d\phi\Bigg]$$
$$=\Lambda(t_1,t_0)w_0+\int_{t_0-\gamma}^{t_0}\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\times\eta_0(\phi)d\phi\Bigg[w_1-\Lambda(t_1,t_0)w_0-\int_{t_0-\gamma}^{t_0}\Delta(t_1,\phi+\gamma)B(\phi+\gamma)\eta_0(\phi)d\phi\Bigg]=w_1.$$

Thus, at time $t = t_1$, the control function $\eta(t)$ steers the system from the initial state w_0 to the final state $w_1 \in \mathbb{R}^k$. As a result, the system (3) is controllable on $[t_0, t_1]$.

In contrast, if the controllability Grammian $\Re[t_0, t_1]$ is not positive definite, then there exists a non-zero vector β such that $\beta^*\Re[t_0, t_1]\beta = 0$. That is

$$\beta^*\int_{t_0}^{t_1-\gamma}(t_1-(\phi+\gamma))^{2(1-\zeta)}\Delta(t_1,\phi+\gamma)$$
$$\mathcal{Q}(\phi+\gamma)\mathcal{Q}^*(\phi+\gamma)\Delta^*(\phi+\gamma,t_1)d\phi$$
$$\beta=0.$$

Hence

$$\beta^*\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)=0,\ on\ [t_0,t_1].$$

Let $w_0 = [\Lambda(t_1, t_0)]^{-1}\beta$. The fact that the system (3) is controllable on $[t_0, t_1]$ allows us to identify a control function $\eta(t)$ that steers the system (3) from the initial state w_0 to the final state w_1 in the interval $[t_0, t_1]$. The result is

$$w(t_1)=\Lambda(t_1,t_0)w_0+\int_{t_0-\gamma}^{t_0}\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi$$
$$+\int_{t_0}^{t_1-\gamma}\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi$$

$$0=\beta+\int_{t_0-\gamma}^{t_0}\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi$$
$$+\int_{t_0}^{t_1-\gamma}\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi.$$

Consequently

$$0=\beta^*\beta+\int_{t_0-\gamma}^{t_0}\beta^*\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi$$
$$+\int_{t_0}^{t_1-h}\beta^*\Delta(t_1,\phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi.$$

Eventually, the second and third terms are zero, concluding that $\beta^*\beta = 0$ is a contradiction for $\beta \neq 0$. Hence, $\Re[t_0, t_1]$ is positive definite.

Remark 4 If $\mathcal{P}(t) = \mathcal{P}$ and $\mathcal{Q}(t) = \mathcal{Q}$, then the time-varying fractional nonlinear dynamical system with Caputo fractional derivative transforms into time-invariant fractional nonlinear dynamical system. If we substitute $\mathcal{P}(t) = \mathcal{P}$ and $\mathcal{Q}(t) = \mathcal{Q}$, then the equation (11) becomes

$$\begin{aligned}{}^{C}_{t_0}D^{\zeta}_{t}w(t)&=\mathcal{P}w(t)+\mathcal{Q}\,\eta(t-\gamma),\\ &\quad t\in[t_0,t_1],\\ w(t_0)&=w_0.\end{aligned}\tag{7}$$

The solution of the dynamical system (7) under the specified control $\eta(t)$ is

$$w(t)=E_\zeta(\mathcal{P}(t-t_0)^\zeta)w_0+\int_{t_0-\gamma}^{t_0}(t-\phi-\gamma)^{\zeta-1}$$
$$\times E_{\zeta,\zeta}(\mathcal{P}(t-\phi-\gamma)^\zeta)\mathcal{Q}\,\eta_0(\phi)d\phi$$
$$+\int_{t_0}^{t-\gamma}(t-\phi-\gamma)^{\zeta-1}E_{\zeta,\zeta}(\mathcal{P}(t-\phi-\gamma)^\zeta)$$
$$\times\mathcal{Q}\,\eta(\phi)d\phi.$$

This solution coincides with the results obtained in Theorem 3.1 of an autonomous case in [32].

Remark 5 If $\mathcal{P}(t) = \mathcal{P}$, $\mathcal{Q}(t) = \mathcal{Q}$, and $\zeta = 1$, then the time-varying fractional nonlinear dynamical system with Caputo fractional derivative transforms into time-invariant ordinary nonlinear dynamical system. If we substitute $\mathcal{P}(v) = \mathcal{P}$, $\mathcal{Q}(v) = \mathcal{Q}$, and $\zeta = 1$, then the equation (11) becomes

$$\frac{dw}{dt} = \mathcal{P}w(t) + \mathcal{Q}\,\eta(t-\gamma), \quad t \in [t_0, t_1],$$
$$w(t_0) = w_0. \tag{8}$$

The solution of the dynamical system (8) under the specified control $\eta(t)$ is

$$w(t) = e^{\mathcal{P}(t-t_0)}w_0 + \int_{t_0-\gamma}^{t_0} e^{\mathcal{P}(t-\phi-\gamma)}\mathcal{Q}\,\eta_0(\phi)d\phi + \int_{t_0}^{t-\gamma} e^{\mathcal{P}(t-\phi-\gamma)}\mathcal{Q}\,\eta(\phi)d\phi.$$

This solution coincides with the results obtained in Theorem 3.1 of an autonomous case with N = 1 in Muni and George ([54], 2019).

4 Non-linear systems

In the following section of the study, we discuss the controllability of a time-varying Caputo fractional nonlinear dynamical system with a single delay. Consider the time-varying fractional nonlinear dynamical system

$$\begin{aligned} {}^{C}_{t_0}D^{\zeta}_{t}w(t) &= \mathcal{P}(t)w(t) + \mathcal{Q}(t)\eta(t-\gamma) \\ &\quad + f(t, w(t), \eta(t)),\ t \in [t_0, t_1], \\ w(t_0) &= w_0, \\ \eta(t) &= \eta_0(t), \quad t \in [t_0-\gamma, t_0). \end{aligned} \tag{9}$$

Given any $(\mu, \lambda) \in \mathcal{E}$

$$\begin{aligned} {}^{C}_{t_0}D^{\zeta}_{t}w(t) &= \mathcal{P}(t)w(t) + \mathcal{Q}(t)\eta(t-\gamma) \\ &\quad + f(t, \mu(t), \lambda(t)),\ t \in [t_0, t_1], \\ w(t_0) &= w_0. \end{aligned} \tag{10}$$

The solution of the dynamical system (10) under the specified control $\eta(t)$ is

$$w(t) = \Lambda(t, t_0)w_0 + \int_{t_0-\gamma}^{t_0} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi + \int_{t_0}^{t-\gamma} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi + \int_{t_0}^{t} \Delta(t, \phi) f(\phi, \mu(\phi), \lambda(\phi))d\phi.$$

Assumption:

$[\mathbb{L}]$: The continuous functions f satisfies $\lim_{|(w,\eta)|\longrightarrow\infty} \frac{|f(t, w, \eta)|}{|(w, \eta)|} = 0$ uniformly in $t \in [t_0, t_1]$.

Theorem 3 *If the nonlinear function $f(t, w(t), \eta(t))$ satisfy the assumption $[\mathbb{L}]$ and the equivalent linear system (3) is controllable on $[t_0, t_1]$, then the nonlinear system (9) is controllable on $[t_0, t_1]$.*

Proof Define $T : \mathcal{E} \to \mathcal{E}$ by $T(\mu, \lambda) = (w, \eta)$, where

$$\begin{aligned} \eta(t) &= (t_1 - (t+\gamma))^{2(1-\zeta)}\mathcal{Q}^*(t+\gamma)\Delta^*(t_1, t+\gamma) \\ &\quad \times \Re^{-1}[t_0, t_1]\Big[w_1 - \Lambda(t_1, t_0)w_0 \\ &\quad - \int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma)B(\phi+\gamma)\eta_0(\phi)d\phi \\ &\quad - \int_{t_0}^{t_1} \Delta(t_1, \phi) f(\phi, \mu(\phi), \lambda(\phi))d\phi\Big], \end{aligned}$$

and

$$w(t) = \Lambda(t, t_0)w_0 + \int_{t_0-\gamma}^{t_0} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi + \int_{t_0}^{t-\gamma} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta(\phi)d\phi + \int_{t_0}^{t} \Delta(t, \phi) f(\phi, \mu(\phi), \lambda(\phi))d\phi.$$

For simplicity, let's choose the following constants.

$$sup\,|f| = sup\{|f(\phi, \mu(\phi), \lambda(\phi))|\}, \qquad \tilde{\mathfrak{a}}_1 = (t_1 - (t+\gamma))^{(1-\zeta)}|\mathcal{Z}|,$$

$$\tilde{\mathfrak{a}}_2 = sup\{\|\Lambda(t, t_0)\,w_0\|\}, \qquad \tilde{\mathfrak{a}}_3 = \left\|\int_{t_0-\gamma}^{t_0} \Delta(t_1, \phi+\gamma)B(\phi+\gamma)\eta_0(\phi)d\phi\right\|,$$

$$\tilde{\mathfrak{a}} = max\{1, |\mathcal{Z}|(t-(t_0+\gamma))^{\zeta}\zeta^{-1}\|\mathcal{Q}\|\}, \qquad \tilde{\mathfrak{d}}_1 = 4\tilde{\mathfrak{a}}\tilde{\mathfrak{a}}_1\|\mathcal{Q}^*\|\,[\|w_1\| + \tilde{\mathfrak{a}}_2 + \tilde{\mathfrak{a}}_3]\,\Re^{-1}, \qquad \tilde{\mathfrak{d}}_2 = 4[\tilde{\mathfrak{a}}_2 + \tilde{\mathfrak{a}}_3],$$

$$\tilde{\mathfrak{c}}_1 = 4\|Q^*\|\|\tilde{\mathfrak{a}}\tilde{\mathfrak{a}}_1\|\Re^{-1}\||\mathcal{Z}|(t_1 - t_0))^{\zeta}\zeta^{-1}, \qquad \tilde{\mathfrak{c}}_2 = 4|\mathcal{Z}|(t_1-t_0)^{\zeta}\zeta^{-1},$$

$$\tilde{\mathfrak{c}} = max\{\tilde{\mathfrak{c}}_1, \tilde{\mathfrak{c}}_2\}, \qquad \tilde{\mathfrak{d}} = max\{\tilde{\mathfrak{d}}_1, \tilde{\mathfrak{d}}_2\}.$$

Then

$$
\begin{aligned}
|\eta(t)| &\leq |\Re^{-1}|\|\mathcal{Q}^*\|\|\mathfrak{a}_1\|\left[|w_1| + \widetilde{\mathfrak{a}_2} + \widetilde{\mathfrak{a}_3} \right.\\
&\quad \left. + |\mathcal{Z}|(t_1 - t_0)^{\zeta}\zeta^{-1} sup|f|\right]\\
&\leq \frac{\widetilde{\mathfrak{c}_1}}{4\widetilde{\mathfrak{a}}} sup\,|f| + \frac{\widetilde{\mathfrak{d}_1}}{4\widetilde{\mathfrak{a}}}\\
&\leq \frac{1}{4\widetilde{\mathfrak{a}}}[\widetilde{\mathfrak{c}}\, sup\,|f| + \mathfrak{d}],
\end{aligned}
$$

and

$$
\begin{aligned}
|w(t)| &\leq [\widetilde{\mathfrak{a}_2} + \widetilde{\mathfrak{a}_3}] + |\mathcal{Z}|\|(t - (t_0 + \gamma))^{\zeta}\zeta^{-1}\|\\
&\quad \times \|\mathcal{Q}\|\left(\frac{\widetilde{\mathfrak{d}}}{4\widetilde{\mathfrak{a}}}\right.\\
&\quad \left. + \frac{\widetilde{\mathfrak{c}}}{4\widetilde{\mathfrak{a}}} sup\,|f|\right) + |\mathcal{Z}|(t_1 - t_0)^{\zeta}\zeta^{-1} sup\,|f|\\
&\leq \frac{\widetilde{\mathfrak{d}}}{4} + \frac{\widetilde{\mathfrak{d}}}{4} + \frac{\widetilde{\mathfrak{c}}}{4} sup\,|f| + \frac{\widetilde{\mathfrak{c}}}{4} sup\,|f|\\
&\leq \frac{\widetilde{\mathfrak{c}}}{2} sup\,|f| + \frac{\widetilde{\mathfrak{d}}}{2}\\
&\leq \frac{1}{2}[\widetilde{\mathfrak{c}}\, sup\,|f| + \mathfrak{d}].
\end{aligned}
$$

By hypothesis, the function f satisfies the following conditions [41]. For each set of two positive constants c and d, there exists a positive constant r such that, if $|\widetilde{s}| \leq r$, then

$$
\widetilde{\mathfrak{c}}\,|f(t, \widetilde{s})| + \widetilde{\mathfrak{d}} \leq \widetilde{r}, \forall t \in [t_0, t_1]. \tag{11}
$$

If $\widetilde{r}$ is a constant such that the inequality (11) is satisfied for certain values of c and d, then any r_1 such that $\widetilde{r} < r_1$ also satisfies the inequality (11). When $\|\mu\| \leq \frac{\widetilde{r}}{2}$ and $\|\lambda\| \leq \frac{\widetilde{r}}{2}$, we get $|\mu(t) + \lambda(t)| \leq \widetilde{r}\ \forall t \in [t_0, t_1]$. It implies that $\widetilde{c}\,|f(t, \widetilde{s})| + \widetilde{\mathfrak{d}} \leq r$. Hence, $|\eta(t)| \leq \frac{\widetilde{r}}{4\widetilde{a}}$, and therefore $\|\eta\| \leq \frac{\widetilde{r}}{4\widetilde{a}}$. Then $\|x\| \leq \frac{\widetilde{r}}{2}$ for every $t \in [t_0, t_1]$. Thus, we have proved that, if $\mathcal{E}(\widetilde{r}) = \left\{(w, \eta); \|w\| \leq \frac{\widetilde{r}}{4\widetilde{a}}, \|\eta\| \leq \frac{\widetilde{r}}{4\widetilde{a}}\right\}$, then T maps $\mathcal{E}(\widetilde{r})$ into itself. Since f is continuous, it implies that the operator is continuous, and by the application of Arzela-Ascoli's theorem, the operator is completely continuous. Since $\mathcal{E}(\widetilde{r})$ is compact and convex, the Schauder fixed point theorem guarantees that T has a fixed point $(\mu, \lambda) \in \mathcal{E}(\widetilde{r})$ such that $T(\mu, \lambda) = (\mu, \lambda) \equiv (w, \eta)$. Therefore

$$
\begin{aligned}
w(t) &= \Lambda(t, t_0)w_0 + \int_{t_0-\gamma}^{t_0} \Delta(t, \phi+\gamma)\mathcal{Q}(\phi+\gamma)\eta_0(\phi)d\phi\\
&\quad + \int_{t_0}^{t-\gamma} \Delta(t, \phi + \gamma)\mathcal{Q}(\phi + \gamma)\eta(\phi)d\phi\\
&\quad + \int_{t_0}^{t} \Delta(t, \phi) f(\phi, \mu(\phi), \lambda(\phi))d\phi.
\end{aligned}
$$

Thus, $w(t)$ is the solution of the system (10), and it is easy to show that $w(t_1) = w_1$. Hence, the control function $\eta(t)$ steers the system (10) from the initial state w_0 to the final state w_1 on $[t_0, t_1]$. This implies that, the system (10) is controllable on $[t_0, t_1]$.

5 Numerical examples

This section aims to demonstrate the development and use of the numerical technique. In this part, we give some computational results for the Caputo fractional derivative, a specific example of the time-varying linear and nonlinear dynamical system with a single delay in control. As a first example, we will investigate a time-varying linear fractional dynamical system with Caputo fractional derivative. The second and third examples considers a time-varying nonlinear fractional dynamical system of the Caputo type and uses a system of equations to model it.

Let us start the system with the initial state w_0. Our primary goal is to guide the system's state, $w(t)$, from its starting point, w_0, to the desired final state, w_1. To visualize this process, we have created two graphs: one without control and the other with control.

Firstly, we examine the graph without any control input. Based on this graph, it is clear that no trajectory exists between the starting state w_0 and the final state w_1 within the given interval $[t_0, t_1]$. This lack of a clear path highlights the need for a controlled approach.

Secondly, we present the graph with control. In this scenario, our primary objective is to determine the control function, denoted as $\eta(t)$, that will guide the state $w(t)$ from its initial state, w_0, to the desired final state, w_1, specifically within the predefined interval $[t_0, t_1]$. To achieve this, we employ the numerical technique of successive approximation.

Utilizing this technique, our control function $\eta(t)$ systematically adjusts the system's behavior over the specified interval $[t_0, t_1]$. This iterative approach allows us to regulate the control input at various points along the trajectory, ultimately steering the system toward the desired final state at w_1 from its starting point at w_0. In essence, this method empowers us to achieve the desired outcome of transitioning the system from one state to another within the specified interval. This task was unattainable without the application of control.

Example 1 Consider the time-varying fractional linear dynamical system without control

$$ {}_0^C D_t^{0.75} w(t) = \begin{bmatrix} t & -t \\ 0 & -t \end{bmatrix} w(t),\ t \in [0, 0.5], $$
$$ w(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}. \tag{12} $$

Comparing (12) to (3) yields $\zeta = 0.75$, $\mathcal{P}(t) = \begin{bmatrix} t & -t \\ 0 & -t \end{bmatrix}$, $\mathcal{Q}(t) = 0$, $w(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $t_0 = 0$, and $t_1 = 0.5$. In this system, we consider the final point is $w(0.5) = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$.

The linear Caputo and Riemann–Liouville state transition matrix for the system is

$$ \Lambda(t, \lambda) = \begin{bmatrix} 0.810649(t-\lambda) \times (0.761905t + 0.571429\lambda) + 1 & 0.816049(t-\lambda) \times (-0.761905t - 0.571429\lambda) \\ 0 & 0.810649(t-\lambda) \times (-0.761905t - 0.571429\lambda) + 1 \end{bmatrix}, $$

and

$$ \Delta(t, \lambda) = \begin{bmatrix} \dfrac{0.816049}{(t-\lambda)^{0.25} + (t-\lambda)^{0.75}} \times (0.691367t + 0.691367\lambda) & 0.816049(t-\lambda)^{0.75} \times (-0.691367t - 0.691367\lambda) \\ 0 & \dfrac{0.816049}{(t-\lambda)^{0.25} + (t-\lambda)^{0.75}} \times (-0.691367t - 0.691367\lambda) \end{bmatrix}. $$

The objective is to determine $\eta(t)$ that can control the state $w(t)$ from $w(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ to $w(0.5) = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$.

However, when we apply the control function with a delay of $\gamma = 0.1$ given by $\eta(t - 0.1)$ with $\mathcal{Q}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ in (12), we get the time-varying fractional linear dynamical system is of the form

$$ {}_0^C D_t^{0.75} w(t) = \begin{bmatrix} t & -t \\ 0 & -t \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ t \end{bmatrix} \eta(t-0.1),\ t \in [0, 0.5], $$
$$ \eta(t) = 0,\ t \in [-0.1, 0). \tag{13} $$

The controllability Grammian is

$$ \Re[0, 0.5] = \int_0^{0.4} (0.5 - (\phi + 0.1))^{2(1-0.75)} \Delta(0.5, 0.1) \mathcal{Q}(\phi + 0.1)\mathcal{Q}^*(\phi + 0.1)\Delta^*(0.5, \phi + 0.1) d\phi $$
$$ = \begin{bmatrix} 0.39022 & 0.0774812 \\ 0.0774812 & 0.239467 \end{bmatrix} > 0 $$

which says that the controllability Grammian $\Re[0, 0.5]$ is positive definite and, by Theorem 2, the given system (13) is controllable on [0,0.5].
Thus, the control function

$$ \eta(t) = -162.484(0.4 - t)^{0.25} (0.190042 + 0.214259(0.4 - t)) + (-1.81828 + 0.957099(0.4 - t))t + (0.4 - t)t^2) $$

steer the system

$$ w(t) = \Lambda(t, 0)w_0 + \int_{-0.1}^{0} \Delta(t, \phi + 0.1)\mathcal{Q}(\phi + 0.1)\eta_0(\phi) d\phi + \int_0^{t-0.1} \Delta(t, \phi + 0.1)\mathcal{Q}(\phi + 0.1)\eta(\phi) d\phi $$

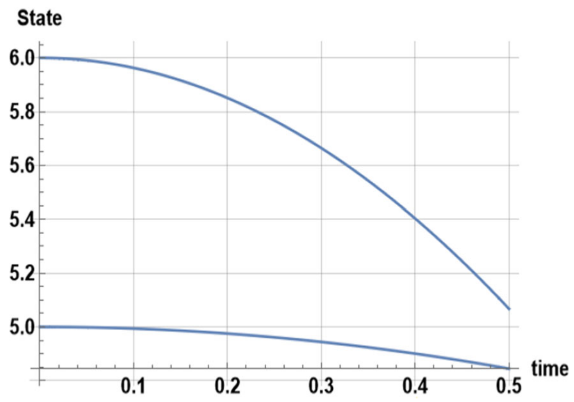

Fig. 1 The trajectory of the system (12) starts from the initial state $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and does not reach the final state $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$ in [0, 0.5]

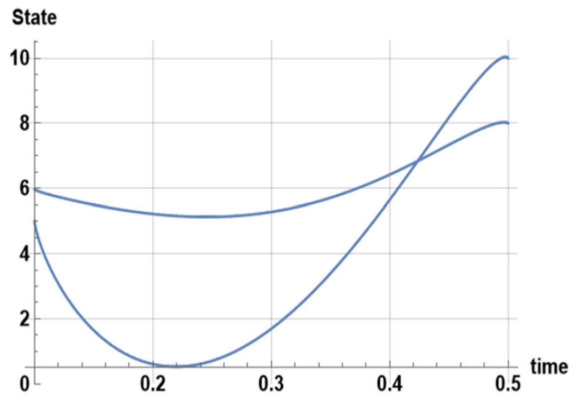

Fig. 2 The states of (13) follow a trajectory within the interval [0, 0.5] from initial point $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$

from the initial point $w(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ to the final point $w(0.5) = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$.

In Fig. 1, we examine the graph without any control input. It is clear that there is no trajectory between the initial point $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$. The simulated state trajectories $w(t)$ and steering control function $\eta(t)$ are shown in Figs. 2 and 3, respectively. From Fig. 2, we notice that the state of the linear system (13) starts initial point $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$ by exploiting the appropriate control function $\eta(t)$ during the interval

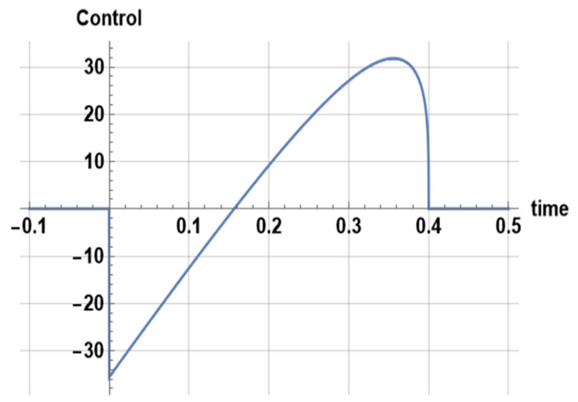

Fig. 3 The trajectory of the control function $\eta(t)$ is shown during the interval [0, 0.5]

[0, 0.5]. Observing that in Fig. 3, the control function $\eta(t)$ is essential for steering the state of the linear system (13) from its initial state to its final state.

Example 2 Consider the time-varying fractional nonlinear dynamical system without control

$$
\begin{aligned}
{}_0^C D_t^{0.6} w(t) &= \begin{bmatrix} -3t & 0 \\ 0 & -2t \end{bmatrix} w(t) \\
&\quad + \begin{bmatrix} \dfrac{w_1}{1+w_1^2+w_2^2} \\ \dfrac{w_1^2}{1+w_1^2+w_2^2} \end{bmatrix}, \; t \in [0, 0.5], \\
w(0) &= \begin{bmatrix} 15 \\ 16 \end{bmatrix}.
\end{aligned} \tag{14}
$$

Comparing (14) to (9) yields $\zeta = 0.6$, $\mathcal{P}(t) = \begin{bmatrix} -3t & 0 \\ 0 & -2t \end{bmatrix}$, $\mathcal{Q}(t) = 0$, $t_0 = 0$, $t_1 = 0.5$, $f(t, w(t), \eta(t)) = \begin{bmatrix} \dfrac{w_1}{1+w_1^2+w_2^2} \\ \dfrac{w_1^2}{1+w_1^2+w_2^2} \end{bmatrix}$, and $w(0) = \begin{bmatrix} 15 \\ 16 \end{bmatrix}$. In this system, we consider the final point is $w(0.5) = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$.

The nonlinear Caputo and Riemann–Liouville state transition matrix for the system is

$$\Lambda(t,\lambda)=\begin{bmatrix} 1+0.671505(-3.125t-1.875\lambda)(t-\lambda) & 0 \\ 0 & 1+0.671505(-2.08333t-1.25\lambda)(t-\lambda) \end{bmatrix},$$

and

$$\Delta(t,\lambda)=\begin{bmatrix} \frac{0.671505}{(t-\lambda)^{0.4}+(-2.43287t-2.43287\lambda)\times(t-\lambda)^{0.6}} & 0 \\ 0 & \frac{0.671505}{(t-\lambda)^{0.4}+(-1.62192t-1.62192\lambda)\times(t-\lambda)^{0.6}} \end{bmatrix}.$$

The objective is to determine $\eta(t)$ that can steer the state $w(t)$ from $w(0)=\begin{bmatrix}15\\16\end{bmatrix}$ to $w(0.5)=\begin{bmatrix}10\\8\end{bmatrix}$.

However, when we apply the control function with a delay of $\gamma=0.1$ given by $\eta(t-0.1)$ with $\mathcal{Q}(t)=\begin{bmatrix}1\\t\end{bmatrix}$ in (14), we get the time-varying fractional linear dynamical system is of the form

$$\begin{aligned} {}_0^C D_t^{0.6} w(t) &= \begin{bmatrix}-3t & 0\\ 0 & -2t\end{bmatrix} w(t) + \begin{bmatrix}1\\t\end{bmatrix}\eta(t-0.1) \\ &\quad + \begin{bmatrix}\frac{w_1}{1+w_1^2+w_2^2}\\ \frac{w_1^2}{1+w_1^2+w_2^2}\end{bmatrix}, t\in[0,0.5], \\ \eta(t) &= 0, t\in[-0.1,0). \end{aligned} \tag{15}$$

The controllability Grammian is

$$\begin{aligned} \Re[0,0.5] &= \int_0^{0.4} (0.5-(\phi+0.1))^{2(1-0.6)}\Delta(0.5,0.1) \\ &\quad \times \mathcal{Q}(\phi+0.1)\mathcal{Q}^*(\phi+0.1) \\ &\quad \times \Delta^*(0.5,\phi+0.1)d\phi \\ &= \begin{bmatrix}0.0798904 & 0.0318811\\ 0.0318811 & 0.0133449\end{bmatrix} > 0. \end{aligned}$$

Now, the controllability Grammian $\Re[0,0.5]$ is positive definite and, by Theorem 2, the corresponding linear system (15) is controllable on [0,0.5]. The nonlinear function $f(t,w(t),\eta(t))=\begin{bmatrix}\frac{w_1}{1+w_1^2+w_2^2}\\ \frac{w_1^2}{1+w_1^2+w_2^2}\end{bmatrix}$ is a bounded continuous function and it satisfy the assumption [$\mathbb{L}$]. According to Theorem 3, the nonlinear system (15) is controllable on [0, 0.5].

Suppose that $w(0.5)=\begin{bmatrix}10\\8\end{bmatrix}$ is the final state and the associated control function $\eta(t)$ steers the state $w(t)$ of (15) from $w(0)=\begin{bmatrix}15\\16\end{bmatrix}$ to $w(0.5)=\begin{bmatrix}10\\8\end{bmatrix}$ on [0, 0.5]. The state vector $w(t)$ and control function $\eta(t)$ are approximated by the following iterative scheme

$$\begin{aligned} &\eta_{n+1}(t) \\ &= (0.5-(t+0.1))^{2(1-0.6)}\mathcal{Q}^*(t+0.1) \\ &\quad \times\Delta^*(0.5,t+0.1)\Re^{-1}[0,0.5] \\ &\quad \times\Bigg[w_1-\Lambda(0,0.5)w_0 \\ &\quad -\int_{-0.1}^{0}\Delta(0.5,\phi+0.1)\mathcal{Q}(\phi+0.1)\eta_0(\phi)d\phi \\ &\quad +\int_0^t \Delta(t,\phi)f(\phi,w_n(\phi),\eta_n(\phi))d\phi\Bigg], \end{aligned}$$

and

$$\begin{aligned} w_{n+1}(t) &= \Lambda(t,0)w_0 + \int_{-0.1}^{0}\Delta(t,\phi+0.1) \\ &\quad \times\mathcal{Q}(\phi+0.1)\eta_0(\phi)d\phi \\ &\quad +\int_0^{t-0.1}\Delta(t,\phi+0.1)\mathcal{Q}(\phi+0.1)\eta(\phi)d\phi \\ &\quad +\int_0^t \Delta(t,\phi)f(\phi,w_n(\phi),\eta_n(\phi))d\phi, \end{aligned}$$

n = 0, 1, 2,…respectively, where $w_0=\begin{bmatrix}15\\16\end{bmatrix}$.

In Fig. 4, we examine the graph without control input. It is clear that there is no trajectory between the initial point $\begin{bmatrix}15\\16\end{bmatrix}$ to final point $\begin{bmatrix}10\\8\end{bmatrix}$. The simulated state trajectories $w(t)$ and steering control function $\eta(t)$ are shown in Figs. 5 and 6, respectively. From Fig. 5, we notice that the state of the nonlinear system (15) starts initial point $\begin{bmatrix}15\\16\end{bmatrix}$ to final point $\begin{bmatrix}10\\8\end{bmatrix}$ by exploiting the appropriate control function $\eta(t)$ during the period

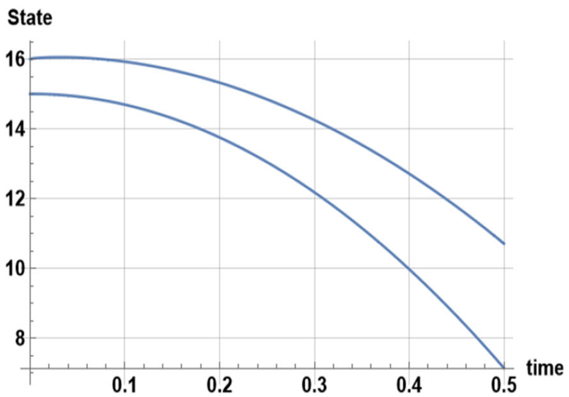

Fig. 4 The trajectory of the system (14) starts from the initial state $\begin{bmatrix} 15 \\ 16 \end{bmatrix}$ and does not reach the final state $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$ in [0, 0.5]

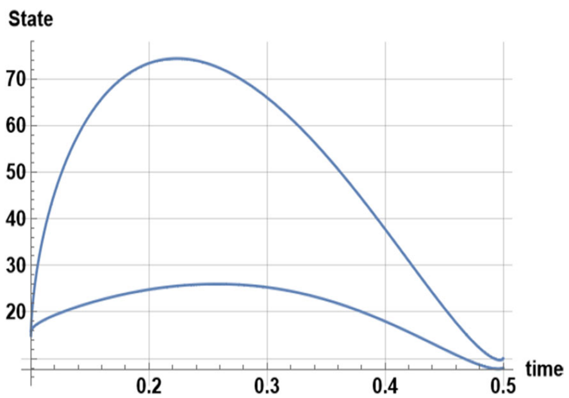

Fig. 5 The states of (15) follow a trajectory within the interval [0, 0.5] from initial point $\begin{bmatrix} 15 \\ 16 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 8 \end{bmatrix}$

[0, 0.5]. Observing that in Fig. 6, the control function $\eta(t)$ is essential for steering the state of the nonlinear system (15) from its initial state to its final state.

Example 3 Consider the time-varying fractional nonlinear dynamical system without control

$$
{}_0^C D_t^{0.95} w(t) = \begin{bmatrix} t+3 & t^2 & t+3 \\ t^2 & t+1 & t \\ t & t & t+8 \end{bmatrix} w(t) + \begin{bmatrix} \dfrac{w_1}{1+w_1^2+w_2^2+w_3^2} \\ \dfrac{w_2^2}{1+w_1^2+w_2^2+w_3^2} \\ \dfrac{w_3^2}{1+w_1^2+w_2^2+w_3^2} \end{bmatrix},
$$

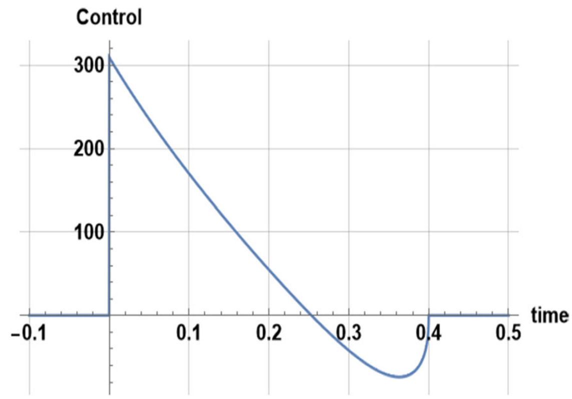

Fig. 6 The trajectory of the control function $\eta(t)$ is shown during the interval [0, 0.5]

$$
t \in [0, 1], \quad w(0) = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}. \tag{16}
$$

Comparing (16) to (9) yields $\zeta = 0.95$, $\mathcal{P}(t) = \begin{bmatrix} t+3 & t^2 & t+3 \\ t^2 & t+1 & t \\ t & t & t+8 \end{bmatrix}$, $\mathcal{Q}(t) = 0$, $t_0 = 0$, $t_1 = 1$, $f(t, w(t), \eta(t)) = \begin{bmatrix} \dfrac{w_1}{1+w_1^2+w_2^2+w_3^2} \\ \dfrac{w_2^2}{1+w_1^2+w_2^2+w_3^2} \\ \dfrac{w_3^2}{1+w_1^2+w_2^2+w_3^2} \end{bmatrix}$, and $w(0) = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$. In this system, we consider the final point is $w(1) = \begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$.

The nonlinear Caputo and Riemann–Liouville state transition matrix for the system is

$$\Lambda(t,\lambda)=\begin{bmatrix} 1+0.969506(t-\lambda)\times(3.15789+0.539811t+0.512821\lambda) & 0 & 0\\ 0 & 1+0.969506(t-\lambda)\times(1.05263+0.539811t+0.512821\lambda) & 0\\ 0 & 0 & 1+0.969506(t-\lambda)\times(8.42105+0.539811t+0.512821\lambda)\end{bmatrix},$$

and

$$\Delta(t,\lambda)=\begin{bmatrix} \frac{0.969506}{(t-\lambda)^{0.05}}+(t-\lambda)^{0.95}\times(3.21737+0.536229t+0.536229\lambda) & 0 & 0\\ 0 & \frac{0.969506}{(t-\lambda)^{0.05}}+(t-\lambda)^{0.95}\times(1.07246+0.536229t+0.536229\lambda) & 0\\ 0 & 0 & \frac{0.969506}{(t-\lambda)^{0.05}}+(t-\lambda)^{0.95}\times(8.57966+0.536229t+0.536229\lambda)\end{bmatrix}.$$

The objective is to determine $\eta(t)$ that can steer the state $w(t)$ from $w(0)=\begin{bmatrix}5\\7\\3\end{bmatrix}$ to $w(1)=\begin{bmatrix}10\\5\\16\end{bmatrix}$.

However, when we apply the control function with a delay of $\gamma=0.5$ given by $\eta(t-0.5)$ with $\mathcal{Q}(t)=\begin{bmatrix}-(t+1)\\-(t+2)\\-(t+5)\end{bmatrix}$ in (16), we get the time-varying fractional nonlinear dynamical system is of the form

$$\begin{aligned}{}_0^C D_t^{0.95}w(t)&=\begin{bmatrix}t+3 & t^2 & t+3\\ t^2 & t+1 & t\\ t & t & t+8\end{bmatrix}w(t)+\begin{bmatrix}-(t+1)\\-(t+2)\\-(t+5)\end{bmatrix}\eta(t-0.5)\\&\quad+\begin{bmatrix}\frac{w_1}{1+w_1^2+w_2^2+w_3^2}\\ \frac{w_2^2}{1+w_1^2+w_2^2+w_3^2}\\ \frac{w_3^2}{1+w_1^2+w_2^2+w_3^2}\end{bmatrix},\ t\in[0,1]\\ \eta(t)&=0,\ t\in[-0.5,0).\end{aligned}\tag{17}$$

The controllability Grammian is

$$\begin{aligned}\Re[0,1]&=\int_0^{0.5}(1-(\phi+0.5))^{2(1-0.95)}\Delta(1,0.5)\\&\qquad\mathcal{Q}(\phi+0.5)\mathcal{Q}^*(\phi+0.5)\Delta^*(1,\phi+0.5)d\phi\\&=\begin{bmatrix}5.89265 & 6.79895 & 33.7164\\ 6.79895 & 7.89123 & 38.4357\\ 33.7164 & 38.4357 & 197.846\end{bmatrix}>0\end{aligned}$$

which says that the controllability Grammian $\Re[0,1]$ is positive definite and, by Theorem 2, the corresponding linear system (17) is controllable on [0,1]. The nonlinear function $f(t,w(t),\eta(t))=\begin{bmatrix}\frac{w_1}{1+w_1^2+w_2^2+w_3^2}\\ \frac{w_2^2}{1+w_1^2+w_2^2+w_3^2}\\ \frac{w_3^2}{1+w_1^2+w_2^2+w_3^2}\end{bmatrix}$ is a bounded continuous function and it satisfy the assumption [$\mathbb{L}$]. According to Theorem 3, the nonlinear system (17) is controllable on [0, 1]. Suppose that

$w(1) = \begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$ is the final state and the associated control function $\eta(t)$ steers the state $w(t)$ of (17) from $w(0) = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$ to $w(1) = \begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$ on [0, 1]. The state vector $w(t)$ and control function $\eta(t)$ are approximated by the following iterative scheme

$$\begin{aligned}\eta_{n+1}(t) = {} & (1-(t+0.5))^{2(1-0.95)} \mathcal{Q}^*(t+0.5) \\ & \times \Delta^*(1, t+0.5)\Re^{-1}[0,1]\Big[w_1 - \Lambda(0,1)w_0 \\ & - \int_{-0.5}^{0} \Delta(1, \phi+0.5)\mathcal{Q}(\phi+0.5)\eta_0(\phi)d\phi \\ & + \int_0^t \Delta(t,\phi) f(\phi, w_n(\phi), \eta_n(\phi))d\phi\Big],\end{aligned}$$

and

$$\begin{aligned}w_{n+1}(t) = {} & \Lambda(t,0)w_0 + \int_{-0.5}^{0} \Delta(t, \phi+0.5)\mathcal{Q}(\phi+0.5)\eta_0(\phi)d\phi \\ & + \int_0^{t-0.5} \Delta(t, \phi+0.5)\mathcal{Q}(\phi+0.5)\eta(\phi)d\phi \\ & + \int_0^t \Delta(t,\phi) f(\phi, w_n(\phi), \eta_n(\phi))d\phi,\end{aligned}$$

n = 0, 1, 2,…respectively, where $w_0 = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$.

In Fig. 7, we examine the graph without control input. It is clear that there is no trajectory between the initial point $\begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$. The simulated state trajectories $w(t)$ and steering control function $\eta(t)$ are shown in Figs. 8 and 9, respectively. From Fig. 8, we notice that the state of the nonlinear system (17) starts from the initial point $\begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$ by exploiting the appropriate control function $\eta(t)$ during the period [0, 1]. Observing that in Fig. 9, the control function $\eta(t)$ is essential for steering the state of the nonlinear system (17) from its initial state to its final state.

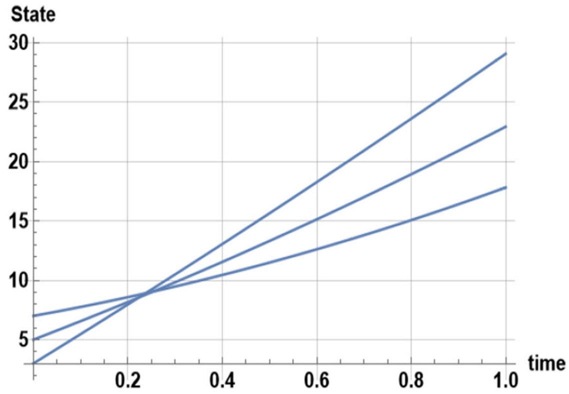

Fig. 7 The trajectory of the system (16) starts from the initial state $\begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$ and does not reach the final state $\begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$ in [0, 1]

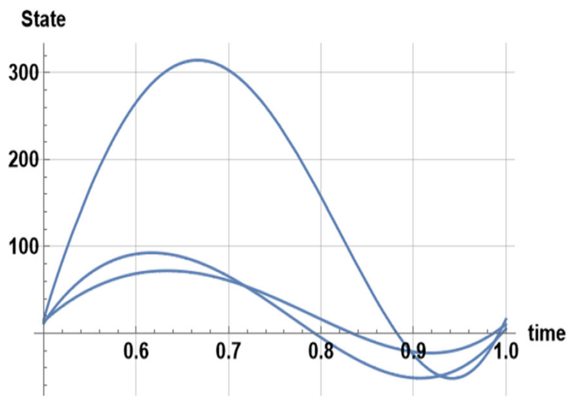

Fig. 8 The states of (17) follow a trajectory within the interval [0, 1] from initial point $\begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$ to final point $\begin{bmatrix} 10 \\ 5 \\ 16 \end{bmatrix}$

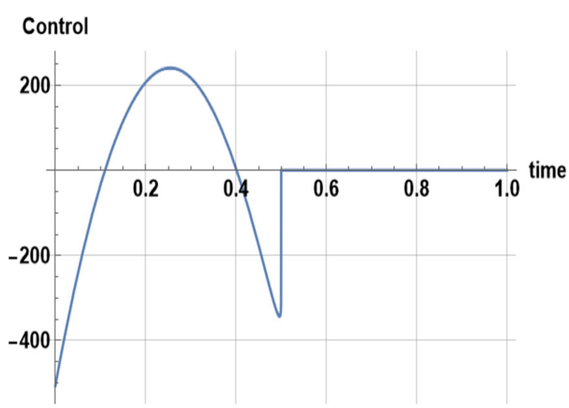

Fig. 9 The trajectory of the control function $\eta(t)$ is shown during the interval [0, 1]

6 Conclusion

In this article, we investigated the controllability of time-varying linear and non-linear fractional dynamical systems with single control delay in the sense of the Caputo fractional derivative. Initially, we obtained a necessary and sufficient condition for the controllability of the time-varying fractional linear system in terms of a controllability Grammian matrix. We have used the Schauder's fixed point theorem to establish sufficient conditions for the controllability of time-varying non-linear fractional dynamical systems. In order to point out the importance of the outcomes of our research, a relevant example has been included in this article. The authors also focus on studying the controllability of systems with impulses and stochastic disturbances in the presence of multiple and distributed delays.

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Data availibility The data used in this study is available/ mentioned in the manuscript.

Declarations

Competing interests The authors declare that they have no conflicts of interest.

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