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ORIGINAL PAPER

State estimation-based parameter identification for a class of nonlinear fractional-order systems

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Abstract Parametric identification is an important part of system theory since knowledge of the parameters allows the analysis and control of the system. The aim of this paper is to propose a novel robust (against measurement noise) parameter identification method for a class of nonlinear fractional-order systems. In order to solve the parametric identification we carry out this problem to a state estimation problem, we introduce a Fractional Algebraic Identifiability (FAI) property which allows to represent the system parameters as a function of the inputs and outputs of the system, this parameter identification method provides an online identification process (while the system is operating), we also propose a fractional-order differentiator which allows to reduce the effect of measurement noise as well as to provide the estimation of a fractional-order derivative of the system output. Moreover, we use the Mittag–Leffler boundedness to demonstrate the convergence of this method, a different approach for this stability analysis method is given in this paper. Finally, we illustrate the accuracy and robustness of our proposed method by means of the parametric identification of two nonlinear fractional-order systems: a time-varying nonlinear fractional-order system and a nonlinear fractional-order mathematical model of a simple pendulum.

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List of Symbols

${}_{0}^{C}D_{t}^{\alpha}$	Caputo fractional-order derivative
α	Fractional order
$\Gamma(\cdot)$	Gamma function
$E_{\alpha,\beta}$	Two parameters Mittag–Leffler type function
E_{α}	One parameter Mittag–Leffler type function
$\|\cdot\|$	Euclidean norm
$\lvert\cdot\rvert$	Absolute value
$X^{\intercal}$	Transpose of a matrix X
$\lambda_{\max(\min)}(X)$	Maximum (minimum) eigenvalue of matrix X
$I_{n\times m}$	Identity matrix of dimensions $n \times m$
$0_{n\times m}$	Zero matrix of dimensions $n \times m$

1 Introduction

The behavior of a dynamic system can be described through a mathematical model which is a set of equations, commonly a set of differential equations or partial differential equations, which describe the evolution of the system. Currently, with fractional calculus it is possible to characterize a system through fractional-order differential equations, these systems are known as fractional-order systems, newly, fractional-order mathematical models have become of great research interest because these models describe the behavior of a physical system better than a integer-order model, i.e., the characterization is better fitted to the experimental data. For instance, viscoelastic materials [1], virus models [2], electro mechanical systems [3], Lithium–Ion battery model [4], biological tissues [5], among others.

The knowledge of the system parameters is very important because it allows the analysis and control of the system. Actually, parametric identification in fractional-order systems has been a research topic, many authors have studied this topic from different perspectives, each approach has different advantages. Nevertheless, most proposals have something in common, the estimation of the system parameters is performed off-line, i.e., excite the system with a specific signal and collect the response data presented by the system in order to apply some kind of recursive algorithm and minimizing a cost function. One of the most commonly used approaches for parametric identification in fractional-order systems consists of representing the fractional-order derivative operator through an operational matrix, converting the fractional differential equation into an algebraic equation. For instance we can mention the works [6–9], where the authors use block pulse functions operational matrix. Moreover, in [10,11] the identification is based on Legendre wavelet operational matrix and Haar wavelet operational matrix, respectively. However, the representation used in these methods is an approximation of the fractional-order derivative operator. On the other hand, there are other works where the operational matrix approach is not used but they are still recursive algorithms and the identification is off-line, in [12] two methods are proposed, recursive least square with state variable filters and the prediction-error. Meanwhile, intelligent algorithms have also been proposed for the parametric identification of chaotic fractional-order systems such as the differential evolution algorithm [13], the particle swarm optimization algorithm [14] and neural networks [15].

As mentioned above, the methods that have been proposed for parametric identification in fractional-order systems are performed off-line, and most of the methods are proposed for time-invariant systems, but many physical systems exhibit time-varying behavior. As far as we know these situations have been poorly studied. For example for the on-line identification we just can mention the work [16] where the authors propose a parametric identification method for a class of linear fractional-order systems based on the modulating function method, and also consider noise at the output of the system, meanwhile, for time-varying parameters we mention the studies [17,18] where the authors propose identification methods for time-varying systems but they assume that the function form of the time-varying parameter is known and only estimate the coefficients.

Taking into account the general overview of parametric identification in fractional-order systems. In this paper we propose a novel robust (against measurement noise) parameter identification method for a class of nonlinear fractional-order systems. The proposed identification method is based on defining an auxiliary variable that will be considered as an extra state of the original system, this variable can depend on the states of the system and only one parameter. With the inclusion of the auxiliary variable dynamics, the original system is transformed into an augmented system where the dynamics of the extra state is considered unknown, thus the parametric identification problem is carried out to a state estimation problem.

In order to design the fractional-order estimator we introduce a Fractional Algebraic Identifiability (FAI) property which allows us to represent the system parameters as a function of the know information of the systems, in particular, the inputs and outputs of the system. The most natural way to propose a fractional-order estimator consists in not requiring a system copy, so the estimator that we propose is only constructed with a proportional and fractional-order integral correction of the estimation error. In addition we propose a fractional-order differentiator, which allows us to reduce the effect of measurement noise and provides an estimate of a fractional-order derivative of the system output, moreover, we can assign the initial conditions of the fractional-order estimator and the fractional-order differentiator freely.

The parameter identification method proposed in this work provides an on-line identification process, which means that while the system is working a parameter estimate is obtained continuously.

On the other hand, the mathematical analysis for the convergence of the proposed method is treated with the Mittag–Leffler boundedness, this method is used to analyze the stability in perturbed fractional-order systems. Wan and Jian [19], the definitions of this stability analysis method is given, nonetheless, in this paper a different approach to the Mittag–Leffler boundedness is given.

Considering the aforementioned, we present the main contributions of this paper:

1. Provide an alternative approach to the Mittag–Leffler boundedness and demonstrate its convergence.
2. A parameter identification method for a class of nonlinear fractional-order systems based on state estimation is presented.
3. A new property about identifiability in fractional-order systems is introduced.
4. The proposed parameter identification method is robust against measurement noise and provides an on-line identification process.
5. The proposed parameter identification method allows the estimation of time-varying parameters.
6. Since the parametric identification problem is driven to a state estimation problem, it is not necessary to fulfill the well-known persistent excitation condition.
7. All initial conditions for the identification method that we propose in this paper can be freely assigned.

The rest of this paper is organized as follows: Sect. 2 provides the mathematical background which includes some basic definitions of fractional-order derivatives, Mittag–Leffler type function and stability analysis in fractional-order systems. The main result of this work is presented in Sect. 3 which is divided into three parts. In Sect. 4 the parametric identification of a two nonlinear fractional-order systems: a time-varying nonlinear fractional-order system and a nonlinear fractional-order mathematical model of a simple pendulum, is performed and the effectiveness and robustness of the proposed method is proven by numerical simulation. Finally, Sect. 5 is devoted to the conclusion.

2 Mathematical background

In this section we give some basic definitions and previous results about fractional-order derivatives as well as state the concepts of some special functions. Furthermore, explain a new approach to the Mittag–Leffler boundedness.

Fractional-order derivatives

In contrast to ordinary calculus, fractional calculus has different definitions for the derivative operator. The typical definitions of the fractional-order derivative are The Caputo, Riemann–Liouville and Grünwald–Letnikov definition. In general the different definitions of the fractional-order derivative do not coincide, however, under certain conditions some definitions may coincide.

For example, consider the derivative of fractional order $\alpha \in \mathbb{R}^+$ satisfying $0 \leq n-1 < \alpha \leq n$ with $n \in \mathbb{N}$ of a function $f(t)$, then, if we consider a class of functions which have n continuous derivatives then the Grünwald–Letnikov and Riemann–Liouville definitions coincide, on the other hand, the Caputo and Riemman–Louville definition agree if the $n-1$ initial conditions are null. Moreover, the definitions of Grünwald–Letnikov, Riemann–Liouville and Caputo interpolate to derivatives of integer order, i.e., if $\alpha \to n$ then the n-th derivative of $f(t)$ is obtained [20].

In this paper we use the Caputo fractional-order derivative to represent fractional-order systems because the initial conditions have an interpretation in the sense of a physical system, which is a consequence caused by the fact that the meaning of the initial conditions coincides with a integer-order system. Moreover, the Caputo fractional-order derivative of a constant is zero and when the derivative operator is applied consecutively there is additivity in the derivative order.

Definition 1 [20] The Caputo derivative of fractional order $\alpha \in \mathbb{R}^+$ of a function $f(t) : [0, \infty) \to \mathbb{R}$ is defined as follows

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \quad (1) $$

where $0 \leq n-1 < \alpha \leq n$ is the fractional order with $n \in \mathbb{N}$ and $\Gamma(\cdot)$ is the gamma function.

Remark 1 In this paper we consider $n = 1$ thus $0 < \alpha \leq 1$, in addition, for simplicity we omit the time dependence of the variables.

2.1 Mittag–Leffler type function

The Mittag–Leffler type functions play a very important role in the theory of fractional calculus, since they are used to find and represent the solutions of a fractional-order differential equation.

Definition 2 [20] The special function defined by the power series

$$E_{\alpha,\beta}(z) = \sum_{\ell=0}^{\infty} \frac{z^{\ell}}{\Gamma(\ell\,\alpha + \beta)} \tag{2}$$

is known as two-parameter Mittag–Leffler function, where $\alpha > 0$ and $\beta, z \in \mathbb{C}$.

Definition 3 When $\beta = 1$ the one-parameter Mittag–Leffler function is obtained

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{\ell=0}^{\infty} \frac{z^{\ell}}{\Gamma(\ell\,\alpha + 1)} \tag{3}$$

The Mittag–Leffler type functions have many interesting properties and applications which can be reviewed in [21,22]. In our particular case, one of the properties that will be very useful is the following [23]

Property 1 The function $E_{\alpha,\beta}(-z)$ is completely monotonic if and only if $0 < \alpha \leq 1$ and $\beta \geq \alpha$, for all $z \geq 0$ with $\alpha > 0$, $\beta > 0$.

2.2 Stability analysis for fractional-order systems

In what follows, we consider the following class of nonlinear fractional-order system given by

$${}_{0}^{C}D_t^{\alpha} x \; = \; f(t,x) + \chi(t,x) \quad x(0) = x_0 \tag{4}$$

where $0 < \alpha \leq 1$ is the fractional order, $f : [0,\infty) \times \Omega \to \mathbb{R}^n$ is a piecewise continuous function in t and locally Lipschitz in x on $[0,\infty) \times \Omega$, the vector of initial conditions is $x_0 \in \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a domain that contains the origin. $\chi(t,x) : [0,\infty) \times \Omega \to \mathbb{R}^n$ is the uncertain vector function, assumed to be bounded $\sup_t \|\chi(t,x)\| = \bar{\chi} \in \mathbb{R}^+$

In contrast to Mittag–Leffler stability [24], Mittag–Leffler boundedness provides sufficient stability conditions in perturbed fractional-order systems.

Definition 4 The solution trajectories of the fractional-order system (4) are said to be Mittag–Leffler bounded if

$$\|x\| \;\leq\; \left[g(x_0)\, E_{\alpha}\left(-\vartheta\, t^{\alpha}\right) + \varphi\right]^{c} \tag{5}$$

where $0 < \alpha \leq 1$, $c, \vartheta, \varphi \in \mathbb{R}^+$ and $g(x) \geq 0$ is a locally Lipschitz function on x.

Theorem 1 *Let $\mathbb{B} \subset \mathbb{R}^n$ be a domain that contains the origin and $V(t,x) : [0,\infty) \times \mathbb{B} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz on x such that*

$$\gamma_1 \|x\|^{a} \;\leq\; V(t,x) \;\leq\; \gamma_2 \|x\|^{ab} \tag{6}$$
$${}_{0}^{C}D_t^{\alpha} V(t,x) \;\leq\; -\gamma_3 \|x\|^{ab} + \varpi \tag{7}$$

where $0 < \alpha \leq 1$ is the fractional order, $\gamma_1, \gamma_2, \gamma_3, a, b \in \mathbb{R}^+$, $\varpi \in \mathbb{R}^+$ represents the disturbances and uncertainties of the fractional-order system (4). Then, the solution trajectories of (4) are Mittag–Leffler bounded. Moreover, if $\mathbb{B} = \mathbb{R}^n$ they are globally Mittag–Leffler bounded.

Proof In light of (6) we can write (7) as follows

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha} V(t,x) &\leq -\frac{\gamma_3}{\gamma_2}\, V(t,x) + \varpi \\ &= -\frac{\gamma_3}{\gamma_2}\left(V(t,x) - \frac{\gamma_2}{\gamma_3}\,\varpi\right) \end{aligned} \tag{8}$$

Let $\bar{V}(t,x) := V(t,x) - \frac{\gamma_2}{\gamma_3}\varpi$ be a change of variable, then

$${}_{0}^{C}D_t^{\alpha} \bar{V}(t,x) \;\leq\; -\frac{\gamma_3}{\gamma_2}\, \bar{V}(t,x) \tag{9}$$

In this way, there exists a non-negative function $R(t)$ such that

$${}_{0}^{C}D_t^{\alpha} \bar{V}(t,x) + \frac{\gamma_3}{\gamma_2}\, \bar{V}(t,x) + R(t) \;=\; 0 \tag{10}$$

Taking into account the existence and uniqueness theorem for fractional-order differential equations (see [20]), it follows that

$$\begin{aligned} \bar{V}(t,x) &= \bar{V}_0\, E_{\alpha}\left(-\frac{\gamma_3}{\gamma_2}\, t^{\alpha}\right) \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left[-\frac{\gamma_3}{\gamma_2}(t-\tau)^{\alpha}\right] R(\tau)d\tau \end{aligned} \tag{11}$$

where $\bar{V}_0 = \bar{V}(0, x_0)$ is the initial condition of (10).

Since $(t-\tau)^{\alpha-1}$, $E_{\alpha,\alpha}\left[-\frac{\gamma_3}{\gamma_2}(t-\tau)^{\alpha}\right]$ and $R(\tau)$ are non-negative function, then we can write

$$\bar{V}(t,x) \leq \bar{V}_0\, E_{\alpha}\left(-\frac{\gamma_3}{\gamma_2}t^{\alpha}\right) \tag{12}$$

Therefore

$$V(t,x)-\frac{\gamma_2}{\gamma_3}\varpi \leq \left(V(0,x_0)-\frac{\gamma_2}{\gamma_3}\varpi\right) E_{\alpha}\left(-\frac{\gamma_3}{\gamma_2}t^{\alpha}\right) \tag{13}$$

In view of (6) we have that

$$\gamma_1\|x\|^a \leq \left(V(0,x_0)-\frac{\gamma_2}{\gamma_3}\varpi\right) E_{\alpha}\left(-\frac{\gamma_3}{\gamma_2}t^{\alpha}\right)+\frac{\gamma_2}{\gamma_3}\varpi \tag{14}$$

Furthermore, it is also true that $\gamma_1\|x_0\|^a \leq V(0,x_0) \leq \gamma_2\|x_0\|^{ab}$, thus

$$\|x\| \leq \left[\frac{1}{\gamma_1}\left(\gamma_2\,\|x_0\|^{ab}-\frac{\gamma_2}{\gamma_3}\varpi\right) E_{\alpha}\left(-\frac{\gamma_3}{\gamma_2}t^{\alpha}\right) + \frac{\gamma_2}{\gamma_1\,\gamma_3}\varpi\right]^{1/a} \tag{15}$$

According to Definition 4 the theorem is proven. □

Remark 2 Mittag–Leffler boundedness agrees with Mittag–Leffler stability if $\varpi = 0$.

Remark 3 It is clear that, if the trajectories of system (4) are Mittag–Leffler bounded, then $\|x\|$ has an upper bound, i.e., $\limsup_{t\to\infty}\|x\| = \varphi^c$. In other words, there is a compact set $B := \{x \in \Omega \mid \|x\| \leq \varphi^c\}$ where the trajectories of (4) converge asymptotically and holds for all initial condition x_0.

Remark 4 In view of inequality (13), the compact set defined by

$$\Psi := \left\{x \in \mathbb{B} \mid V(t,x) \leq \frac{\gamma_2}{\gamma_3}\varpi\right\} \tag{16}$$

is called Mittag–Leffler attractive set. If $\mathbb{B} = \mathbb{R}^n$ then it is known as globally Mittag–Leffler attractive set.

Lemma 1 *Let $x \in \mathbb{R}^n$ be a vector of differential functions then for all $t \geq 0$ the following inequality*

$${}_0^C D_t^{\alpha}\left(x^{\top} P\, x\right) \leq x^{\top} P\, {}_0^C D_t^{\alpha} x + \left({}_0^C D_t^{\alpha} x\right)^{\top} P\, x \tag{17}$$

is met, where $0 < \alpha \leq 1$ and $P \in \mathbb{R}^{n\times n}$ is symmetric matrix.

Proof The demonstration of this lemma can be reviewed in [25] □

Lemma 2 *The matrix defined as*

$$\begin{pmatrix} -\kappa_1 & -\kappa_2 \\ \kappa_3 & -\kappa_4 \end{pmatrix} \tag{18}$$

is Hurwitz for all positive constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$.

Proof Note that the characteristic polynomial of the matrix (18) is given by

$$p(\lambda) = \lambda^2 + (\kappa_1+\kappa_4)\,\lambda + (\kappa_1\,\kappa_4 + \kappa_2\,\kappa_3) \tag{19}$$

It is easy to see that if $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{R}^+$ then $p(\lambda)$ is a Hurwitz polynomial since the coefficients have the same sign and therefore matrix (18) is Hurwitz. □

3 Main result

In this section we present the main contribution of this work, which is divided into three parts:

1. In the first part, we carry out the parametric identification problem for a class of nonlinear fractional-order system to a state estimation problem, in this way it is sufficient to design a fractional-order estimator in order to obtain an estimate of the system parameters.
2. The second part shows a methodology in which through a change of variable we can partially or totally avoid the use of signals that we consider unknown such as fractional-order derivatives of the output and input of the system.
3. In the third part we consider the existence of measurement noise at the system output, in particular we propose a technique where we can considerably attenuate the effect of this exogenous signal, in addition we obtain an estimate of a fractional-order derivative of the output where the measurement noise is being attenuated.

Hence, these three parts provides a robust identification method against measurement noise for a class of nonlinear fractional-order system.

3.1 First part

Firstly, consider the following class of nonlinear fractional-order system

$$\begin{aligned} {}_0^C D_t^{\alpha} x &= f(x,u,\xi), \quad x(0) = x_0 \\ y &= h(x) \end{aligned} \tag{20}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^q$ is the system input, $y \in \mathbb{R}^m$ is the system output, $\xi \in \mathbb{R}^l$ denotes the set of original parameters (the components of ξ can be time-varying), $f : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^l \to \mathbb{R}^n$ is a locally Lipschitz function on x, u and ξ, $h : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function. Meanwhile, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, is the set of fractional orders which satisfy $0 < \alpha_i \leq 1$ with $i \in \{1, \ldots, n\}$. $x_0 \in \mathbb{R}^n$ is the vector of initial conditions.

Remark 5 If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ the nonlinear fractional-order system (20) is known as commensurate nonlinear fractional-order system, otherwise system (20) is called as incommensurate nonlinear fractional-order system. In this paper we assume that the fractional orders are known.

The aim of the parametric identification is to obtain an estimate of the elements that compose ξ, in the sequel we propose an algebraic technique to solve the parametric identification problem carrying it to a state estimation problem. In this way, we introduce the following property:

Definition 5 (*FAI*) A parameter ξ_i satisfies the Fractional Algebraic Identifiability (FAI) property if it can be represented as a function of the input and output of the system along with their fractional-order derivatives, that is

$$\xi_i = \phi_i \left(y, {}_0^C D_t^{\alpha_1} y, \ldots, {}_0^C D_t^{\alpha_1 + \cdots + \alpha_n} y, u, {}_0^C D_t^{\alpha_1} u, \ldots, {}_0^C D_t^{\alpha_1 + \cdots + \alpha_n} u \right) \tag{21}$$

where $\sum_{j=1}^{n} \alpha_j \leq 1$ and $\phi_i(\cdot)$ is a continuous function with $i \in \{1, \ldots, l\}$

Example 1 Let us consider the following nonlinear fractional-order system

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x_1 &= \xi\, x_1^2 + u \\ {}_0^C D_t^{\alpha_2} x_2 &= -x_1 - x_2 \\ y &= x_1 \end{aligned} \tag{22}$$

where $0 < \alpha_1, \alpha_2 \leq 1$ are the fractional orders, ξ is a parameter which can be time-variant.

Note that ξ fulfills the FAI property (21) since it can be represented as

$$\xi = \frac{1}{y^2} \left({}_0^C D_t^{\alpha_1} y - u \right) = \phi(y, {}_0^C D_t^{\alpha_1} y, u),\ y \neq 0 \tag{23}$$

Now, we define an auxiliary variable which can depend on the states and the unknown parameter $\eta_i(x, \xi_i)$, in order to define this auxiliary variable, two situations may occur:

1. If the parameter appears isolated inside of the nonlinear fractional-order system (20), i.e., it does not affect any state or a relation of them, then the auxiliary variable can be defined as the system parameter. for example if we consider ${}_0^C D_t^{\alpha} x = \sin(x) + \xi$, where ξ is a parameter, then the auxiliary variable can be defined as $\eta = \xi$.
2. If the parameter affects the states of the system or a relation of them, then the auxiliary variable can be defined as the relation of the parameter and the states, e.g., considering (22) we can define the auxiliary variable as $\eta = \xi x_1^2$.

We consider that the dynamics of this auxiliary variable is unknown. It is clear that the number of auxiliary variables that we need to define depends on the number of parameters that we need to estimate.

Then, the dynamic equation of auxiliary variables along with the original system (20) can be written in the following augmented form

$$\begin{aligned} {}_0^C D_t^{\alpha} x &= f(x, u, \eta), \quad x(0) = x_0 \\ {}_0^C D_t^{\mathring{\alpha}} \eta &= \Phi(\cdot), \quad \eta(0) = \eta_0 \\ y &= h(x) \end{aligned} \tag{24}$$

where $\eta \in \mathbb{R}^l$ is the vector of auxiliary variables, $\mathring{\alpha} = (\mathring{\alpha}_1, \mathring{\alpha}_2, \ldots, \mathring{\alpha}_l)$ is the set of fractional orders for the auxiliary variables which satisfy $0 < \mathring{\alpha}_i \leq 1$ with $i \in \{1, \ldots, l\}$, $\eta_0 \in \mathbb{R}^l$ is the vector of initial conditions for the auxiliary variables and $\Phi(\cdot)$ is an unknown vector function.

Note that the auxiliary variables can be considered as extra states of the augmented system (24), thus it is sufficient to estimate these extra states in order to obtain an estimate of the system parameters, therefore the parametric identification problem becomes a state estimation problem.

Now, given that we have to solve a state estimation problem, we establish the following property which will allow us to design an estimator for the auxiliary variables.

Definition 6 (*IFAO*) [26] A state variable η_i satisfies the Incommensurate Fractional Algebraic Observability (IFAO) property if it can be written as a function of the input and output of the system along with their fractional-order derivatives, that is

$$\eta_i = \psi_i \left(y, {}_0^C D_t^{\alpha_1} y, \ldots, {}_0^C D_t^{\alpha_1+\cdots+\alpha_n} y, u, {}_0^C D_t^{\alpha_1} u, \ldots, {}_0^C D_t^{\alpha_1+\cdots+\alpha_n} u \right) \tag{25}$$

where $\sum_{j=1}^{n} \alpha_j \leq 1$ and $\psi_i(\cdot)$ is a continuous function with $i \in \{1, \ldots, l\}$.

Remark 6 It is well known that the composition of the fractional-order derivatives of a function produces the derivative of the function where the fractional order is the sum of the fractional orders of each derivative applied consecutively, plus terms that refer to the fractional-order derivatives of the function evaluated at the initial instant (see [20]).

However in [27] it is shown that if the sum of the fractional orders is less than or equal to 1 the terms evaluated at initial instant disappear in Caputo's definition, that is why the relationships (21) and (25) are well defined.

On the other hand, if we would like to remove the condition $\sum_{j=1}^{n} \alpha_j \leq 1$ and instead have $\sum_{j=1}^{n} \alpha_j \leq \bar{n}$ with $\bar{n}$ the system order, then in our particular case it is sufficient to ensure that $dy(0)/dt = du(0)/dt = 0$ in order to have (21) and (25) valid (see [20]).

Remark 7 The auxiliary variable η_i associated with the parameter ξ_i satisfies the IFAO property if and only if the parameter ξ_i satisfies the FAI property.

Now, let us consider an unknown dynamics of the augmented system (24)

$${}_0^C D_t^{\mathring{\alpha}_i} \eta_i = \Phi_i(\cdot) \tag{26}$$

where $\mathring{\alpha}_i$ is an element of $\mathring{\alpha}$, η_i is an element of the vector η so it is an extra state of (24) and $\Phi_i(\cdot)$ is an element of the vector $\Phi(\cdot)$ consequently is a unknown function with $i \in \{1, \ldots, l\}$.

In order to ensure the existence and uniqueness of solution of the fractional-order differential Eq. (26) it is necessary to make an assumption about the unknown function $\Phi_i(\cdot)$.

Assumption 1 The unknown function $\Phi_i(\cdot)$ has an upper bound $0 < M < \infty$, such that $\sup_t \|\Phi_i(\cdot)\| = M$.

Remark 8 Since the parametric identification problem and the state estimation problem have no sense for unstable systems, then in this work we consider stable nonlinear fractional-order systems and bounded inputs that is why assumption 1 is reasonable.

Let us consider the following fractional-order system

$$\begin{aligned} {}_0^C D_t^{\mathring{\alpha}_i} \hat{\eta}_i &= k_1(\eta_i - \hat{\eta}_i) + k_2 \zeta_i \\ {}_0^C D_t^{\mathring{\alpha}_i} \zeta_i &= k_3(\eta_i - \hat{\eta}_i) - k_4 \zeta_i \end{aligned} \tag{27}$$

where $k_1, k_2, k_3, k_4 \in \mathbb{R}^+$, $\hat{\eta}_i$ is an estimate of extra state η_i and ζ_i is the fractional-order integral part of $\hat{\eta}_i$ with the initial conditions $\hat{\eta}_i(0) = \hat{\eta}_{i_0}$, $\zeta_i(0) = \zeta_{i_0}$ and $\hat{\eta}_{i_0}, \zeta_{i_0} \in \mathbb{R}$.

Based on the aforementioned and defining the estimation error as

$$\tilde{\eta}_i = \eta_i - \hat{\eta}_i \tag{28}$$

Then we can establish the following theorem:

Theorem 2 *Let us consider the nonlinear fractional-order system (20) and suppose that the system parameters fulfill the FAI property such that the system (20) can be expanded to the augmented form (24). Furthermore, if assumption 1 is satisfied then the fractional-order system (27) is a fractional Proportional Integral (PI) estimator for the unknown dynamics (26), whose estimation error is globally Mittag–Leffler bounded and therefore the estimation error converges asymptotically and remains in the compact set*

$$\bar{B}_1 = \left\{ \varepsilon \in \mathbb{R}^2 \;\middle|\; \|\varepsilon\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)\, M}{\sqrt{\lambda_{\min}(Q)\,\varrho - \varrho^2}} \right\}$$

where $\varepsilon = (\, \tilde{\eta}_i \;\; \zeta_i \,)^{\mathsf{T}}$, $P, Q \in \mathbb{R}^{2\times 2}$ *are symmetric positive definite matrices and* $\varrho \in \mathbb{R}^+$, $0 < \varrho < \lambda_{\min}(Q)$.

Proof Compute the derivative of fractional order $\mathring{\alpha}_i$ of (28) we have

$${}_0^C D_t^{\mathring{\alpha}_i} \tilde{\eta}_i = {}_0^C D_t^{\mathring{\alpha}_i} \eta_i - {}_0^C D_t^{\mathring{\alpha}_i} \hat{\eta}_i \tag{29}$$

From (26) and (27) it follows that

$${}_0^C D_t^{\mathring{\alpha}_i} \tilde{\eta}_i = \Phi_i(\cdot) - k_1 \tilde{\eta}_i - k_2 \zeta_i \tag{30}$$

Let us define the vector $\varepsilon = (\, \tilde{\eta}_i \;\; \zeta_i \,)^{\mathsf{T}}$, then we can write the following matrix form

$${}_0^C D_t^{\mathring{\alpha}_i} \varepsilon = K \varepsilon + \aleph \tag{31}$$

where

$$K = \begin{pmatrix} -k_1 & -k_2 \\ k_3 & -k_4 \end{pmatrix}; \quad \aleph = \begin{pmatrix} \Phi_i(\cdot) \\ 0 \end{pmatrix} \tag{32}$$

Consider the quadratic function

$$V(\varepsilon) = \varepsilon^{\mathsf{T}} P \varepsilon \tag{33}$$

where $P \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix. It is clear that $V(0) = 0$ and $V(\varepsilon) \neq 0$ for all $\varepsilon \neq 0$.

Taking into account fractional-order system (31) and Lemma 1, we have that

$$\begin{aligned} {}_0^C D_t^{\mathring{\alpha}_i} V(\varepsilon) &\leq \varepsilon^\top P \, {}_0^C D_t^{\mathring{\alpha}_i} \varepsilon + \left({}_0^C D_t^{\mathring{\alpha}_i} \varepsilon\right)^\top P \varepsilon \\ &= \varepsilon^\top P \,(K \varepsilon + \aleph) + \left(\varepsilon^\top K^\top + \aleph^\top\right) P \varepsilon \\ &= \varepsilon^\top \left(P K + K^\top P\right) \varepsilon + 2 \varepsilon^\top P \aleph \end{aligned} \tag{34}$$

Since matrix K is Hurwitz (see Lemma 2), then there exists a symmetric positive definite matrix $Q \in \mathbb{R}^{2\times 2}$ such that

$$P K + K^\top P = -Q \tag{35}$$

Thus, we can express the fractional-order differential inequality (34) as follows

$${}_0^C D_t^{\mathring{\alpha}_i} V(\varepsilon) \leq -\varepsilon^\top Q \varepsilon + 2 \varepsilon^\top P \aleph \tag{36}$$

Now, by Rayleigh–Ritz inequality $\lambda_{\min}(Q)\|\varepsilon\|^2 \leq \varepsilon^\top Q\varepsilon \leq \lambda_{\max}(Q)\|\varepsilon\|^2$, we have that $-\varepsilon^\top Q\varepsilon \leq -\lambda_{\min}(Q)\|\varepsilon\|^2$.

On the other hand, using Cauchy–Schwarz inequality it follows that

$$|2 \varepsilon^\top P \aleph| \leq 2 \|\varepsilon^\top\| \, \|P\| \, \|\aleph\| \leq 2 \lambda_{\max}(P) \|\varepsilon\| \, \|\aleph\| \tag{37}$$

Since assumption 1 is satisfied, then $\|\aleph\| \leq M$, thus

$$|2 \varepsilon^\top P \aleph| \leq 2 \omega \|\varepsilon\| \tag{38}$$

where $\omega = \lambda_{\max}(P) M$.

In this way, we can write (36) as follows

$$\begin{aligned} {}_0^C D_t^{\mathring{\alpha}_i} V(\varepsilon) &\leq -\lambda_{\min}(Q) \|\varepsilon\|^2 + 2 \omega \|\varepsilon\| \\ &= -\lambda_{\min}(Q) \|\varepsilon\|^2 + 2 \omega \|\varepsilon\| \\ &\quad + \varrho \|\varepsilon\|^2 - \varrho \|\varepsilon\|^2 \\ &= -\left[\lambda_{\min}(Q) - \varrho\right] \|\varepsilon\|^2 - \varrho \|\varepsilon\|^2 \\ &\quad + 2 \omega \|\varepsilon\| \end{aligned} \tag{39}$$

with $0 < \varrho < \lambda_{\min}(Q)$.

Using the fact that for $d_1, d_2 \in \mathbb{R}$ we have $(d_1 - d_2)^2 \geq 0$ then $-d_1^2 + 2d_1 d_2 \leq d_2^2$, it follows that

$${}_0^C D_t^{\mathring{\alpha}_i} V(\varepsilon) \leq -\theta \|\varepsilon\|^2 + \mu \tag{40}$$

where $\theta = \lambda_{\min}(Q) - \varrho$ and $\mu = \omega^2/\varrho$.

Taking into account that the quadratic function (33) also satisfies the Rayleigh–Ritz inequality $\lambda_{\max}(P)\|\varepsilon\|^2 \leq V(\varepsilon) \leq \lambda_{\max}(P)\|\varepsilon\|^2$ and by theorem 1 we obtain

$$\begin{aligned} \|\varepsilon\| \leq \Bigg[& \frac{1}{\lambda_{\min}(P)} \left(\lambda_{\max}(P) \|\varepsilon_0\|^2 \right. \\ & \left. - \frac{\lambda_{\max}(P)}{\theta} \mu \right) E_{\mathring{\alpha}_i}\left(-\frac{\theta}{\lambda_{\max}(P)} t^{\mathring{\alpha}_i}\right) \\ & + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)\,\theta} \mu \Bigg]^{1/2} \end{aligned} \tag{41}$$

where $\varepsilon_0 = \varepsilon(0)$ is the initial condition of fractional-order system (31).

Therefore, the estimation error is globally Mittag–Leffler bounded, finally according to Remark 3 by inspection of (41) we have that $\varphi_1 = \lambda_{\max}(P)\mu/\lambda_{\min}(P)\theta$ and $c_1 = 1/2$, therefore the estimation error converges asymptotically and remains in the compact set

$$\bar{B}_1 = \left\{ \varepsilon \in \mathbb{R}^2 \;\middle|\; \|\varepsilon\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P) M}{\sqrt{\lambda_{\min}(Q)\,\varrho - \varrho^2}} \right\} \tag{42}$$

for all initial condition $\varepsilon_0 \in \mathbb{R}^2$. □

Remark 9 The globally Mittag–Leffler attractive set is defined by (see Remark 4)

$$\mathring{\Psi} = \left\{ \varepsilon \in \mathbb{R}^2 \mid V(\varepsilon) \leq \frac{\lambda_{\max}(P)^3 M^2}{\lambda_{\min}(Q)\,\varrho - \varrho^2} \right\} \tag{43}$$

Remark 10 It should be noted that the proposed fractional-order estimator (27) may not require the complete system information, like the well-known Luenberger estimator and Kalman filter.

Remark 11 In view of (42) a selection criterion can be obtained for the estimator gains (27). Fixing $P = I$, then from (35) it follows that

$$K + K^\top = -Q \Rightarrow Q = \begin{pmatrix} 2k_1 & k_2 - k_3 \\ k_2 - k_3 & 2k_4 \end{pmatrix} \tag{44}$$

if $k_2 = k_3$ and $k_1 > k_4$ then

$$\bar{B}_1 = \left\{ \varepsilon \in \mathbb{R}^2 \mid \|\varepsilon\| \leq \frac{M}{\sqrt{2k_4\,\varrho - \varrho^2}} \right\} \tag{45}$$

By setting $\varrho = k_4$ we obtain

$$\bar{B}_1 = \left\{ \varepsilon \in \mathbb{R}^2 \mid \|\varepsilon\| \leq \frac{M}{k_4} \right\} \tag{46}$$

In words, according to (46) the size of the compact set $\bar{B}_1$ is inversely proportional to the gain k_4, so the gain selection criterion would be to set the gain k_4 and fulfill the conditions $k_1 > k_4$ and $k_2 = k_3$.

Bear in mind that obtaining an estimate of the auxiliary variable η_i through the fractional PI estimator (27) also yields an estimate of the parameter ξ_i associated with η_i.

3.2 Second part

Note that the IFAO property suggests the use of signals that are considered unknown such as the fractional-order derivatives of the output and input of the system, in what follows we give a simple proposal to partially or totally avoid the use of these signals, through a change of variable which can be performed iteratively.

Suppose that the extra state η_i satisfies the IFAO property (25) so it can be written as follows

$$\eta_i = \sum_{\imath=1}^{m} \delta_{1,\imath}\, {}_0^C D_t^{\alpha_{1,\imath}} y_\imath + \sum_{\jmath=1}^{q} \delta_{2,\jmath}\, {}_0^C D_t^{\alpha_{2,\jmath}} u_\jmath + s_i(y,u) \tag{47}$$

where $\delta_{1,\imath}, \delta_{2,\jmath} \in \mathbb{R}$, $0 < \alpha_{1,\imath}, \alpha_{2,\jmath} \leq 1$ are the fractional orders and $s_i(u,y)$ is a continuous function assumed to be bounded.

Remark 12 In the special case when we can be sure that $dy_\imath(0)/dt = du_\jmath(0)/dt = 0$ with $\imath \in \{1,\ldots,m\}$ and $\jmath \in \{1,\ldots,q\}$ then we have that $0 < \alpha_{1,\imath}, \alpha_{2,\jmath} \leq \bar{n}$ where $\bar{n}$ is the system order.

Without loss of generality we consider the case when $m = q = 1$, in order to illustrate clearly the process of change of variable. Then

$$\eta_i = \delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}} y_1 + \delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}} u_1 + s_i(y,u) \tag{48}$$

Since η_i satisfies IFAO then we can find a common factor $\bar{\alpha}$ between $\alpha_{1,1}$ and $\alpha_{2,1}$ such that

$$\begin{aligned}\eta_i &= \delta_{1,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \right) \\ &\quad + \delta_{2,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right) + s_i(y,u)\end{aligned} \tag{49}$$

where $0 < \bar{\alpha} \leq 1$, now, we are interested in the form that the fractional PI estimator will have, so replacing (49) in (27) and fixing $\mathring{\alpha}_i = \bar{\alpha}$ we obtain

$$\begin{aligned}{}_0^C D_t^{\bar{\alpha}} \hat{\eta}_i &= k_1\delta_{1,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \right) \\ &\quad + k_1\delta_{2,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right) \\ &\quad + k_1 s_i(y,u) - k_1\hat{\eta}_i + k_2\zeta_i\end{aligned} \tag{50}$$

$$\begin{aligned}{}_0^C D_t^{\bar{\alpha}} \zeta_i &= k_3\delta_{1,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \right) \\ &\quad + k_3\delta_{2,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right) \\ &\quad + k_3 s_i(y,u) - k_3\hat{\eta}_i - k_4\zeta_i\end{aligned} \tag{51}$$

Let us consider the following change of variable

$$\hat{\eta}_i = \sigma_{1,i} + k_1\delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 + k_1\delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \tag{52}$$

$$\zeta_i = \rho_{1,i} + k_3\delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 + k_3\delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \tag{53}$$

where $\sigma_{1,i}$ and $\rho_{1,i}$ are auxiliary functions which are considered sufficiently smooth, so computing the derivative of (52) and (53) of fractional order $\bar{\alpha}$, we obtain

$$\begin{aligned}{}_0^C D_t^{\bar{\alpha}} \hat{\eta}_i &= {}_0^C D_t^{\bar{\alpha}} \sigma_{1,i} + k_1\delta_{1,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \right) \\ &\quad + k_1\delta_{2,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right)\end{aligned} \tag{54}$$

$$\begin{aligned}{}_0^C D_t^{\bar{\alpha}} \zeta_i &= {}_0^C D_t^{\bar{\alpha}} \rho_{1,i} + k_3\delta_{1,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \right) \\ &\quad + k_3\delta_{2,1}\, {}_0^C D_t^{\bar{\alpha}} \left({}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right)\end{aligned} \tag{55}$$

Replacing (52), (53), (54) in (50) and (52),(53), (55) in (51) we have that

$$\begin{aligned}{}_0^C D_t^{\bar{\alpha}} \sigma_{1,i} &= k_1 s_i(y,u) \\ &\quad - k_1 \left(\sigma_{1,i} + k_1\delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 + k_1\delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right) \\ &\quad + k_2 \left(\rho_{1,i} + k_3\delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 + k_3\delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right)\end{aligned} \tag{56}$$

$$\begin{aligned}{}_0^C D_t^{\bar{\alpha}} \rho_{1,i} &= k_3 s_i(y,u) \\ &\quad - k_3 \left(\sigma_{1,i} + k_1\delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 + k_1\delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right) \\ &\quad - k_4 \left(\rho_{1,i} + k_3\delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 + k_3\delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1 \right)\end{aligned} \tag{57}$$

which leads to

$$
\begin{aligned}
{}_0^C D_t^{\bar{\alpha}} \sigma_{1,i} \;&=\; -k_1 \sigma_{1,i} + k_2 \rho_{1,i} + k_1 s_i(y,u) \\
&+ \left(k_2 k_3 - k_1^2\right) \delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \\
&+ \left(k_2 k_3 - k_1^2\right) \delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
{}_0^C D_t^{\bar{\alpha}} \rho_{1,i} \;&=\; -k_3 \sigma_{1,i} - k_4 \rho_{1,i} + k_3 s_i(y,u) \\
&+ (-k_1 k_3 - k_3 k_4)\, \delta_{1,1}\, {}_0^C D_t^{\alpha_{1,1}-\bar{\alpha}} y_1 \\
&+ (-k_1 k_3 - k_3 k_4)\, \delta_{2,1}\, {}_0^C D_t^{\alpha_{2,1}-\bar{\alpha}} u_1
\end{aligned} \tag{59}
$$

Note that we can obtain $\hat{\eta}_i$ and ζ_i through (58) and (59) along with (52) and (53) with the initial conditions $\sigma_{1,i}(0) = \sigma_{1,i_0} \in \mathbb{R}$ and $\rho_{1,i}(0) = \rho_{1,i_0} \in \mathbb{R}$.

The main advantage of using these relations in order to obtain $\hat{\eta}_i$ and ζ_i instead of using (50) and (51) directly is that we avoid the use of the derivative of the output and input of the system of fractional order $\alpha_{1,1}$ and $\alpha_{2,1}$, respectively. Instead we need the derivative of the input and output of fractional order $\alpha_{1,1} - \bar{\alpha}$ and $\alpha_{2,1} - \bar{\alpha}$.

This process of changing the variable can be applied iteratively until the fractional-order derivatives of the output and input of the system satisfies $0 \leq \alpha_{1,1} - p\bar{\alpha} \leq 1$ and $0 \leq \alpha_{2,1} - p\bar{\alpha} \leq 1$, respectively with $p \in \mathbb{N}\backslash\{0\}$.

Note that it may still be necessary to use fractional-order derivatives of the input and the output of the system, however, we can design a fractional-order differentiator in order to obtain these signals that in principle are unknown, this fractional-order differentiator will be explained later.

In general, after the process of change of variable we can represent the fractional PI estimator as follows

$$
\begin{aligned}
{}_0^C D_t^{\bar{\alpha}} \sigma_{p,i} &= -k_1 \sigma_{p,i} + k_2 \rho_{p,i} + k_1 s_i(y,u) \\
&\quad + \sum_{\imath=1}^{m} \bar{\delta}_{1,\imath}\, {}_0^C D_t^{\alpha_{1,\imath}-p\bar{\alpha}} y_\imath \\
&\quad + \sum_{\jmath=1}^{q} \bar{\delta}_{2,\jmath}\, {}_0^C D_t^{\alpha_{2,\jmath}-p\bar{\alpha}} u_\jmath
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
{}_0^C D_t^{\bar{\alpha}} \rho_{p,i} &= -k_3 \sigma_{p,i} - k_4 \rho_{p,i} + k_3 s_i(y,u) \\
&\quad + \sum_{\imath=1}^{m} \bar{\delta}_{3,\imath}\, {}_0^C D_t^{\alpha_{1,\imath}-p\bar{\alpha}} y_\imath \\
&\quad + \sum_{\jmath=1}^{q} \bar{\delta}_{4,\jmath}\, {}_0^C D_t^{\alpha_{2,\jmath}-p\bar{\alpha}} u_\jmath
\end{aligned} \tag{61}
$$

where $\sigma_{p,i}$ and $\rho_{p,1}$ are auxiliary functions which are considered sufficiently smooth, $\bar{\delta}_{1,\imath}, \bar{\delta}_{2,\jmath}, \bar{\delta}_{3,\imath}, \bar{\delta}_{4,\jmath} \in \mathbb{R}$, with the initial conditions $\sigma_{p,i}(0) = \sigma_{p,i_0} \in \mathbb{R}$ and $\rho_{p,i}(0) = \rho_{p,i_0} \in \mathbb{R}$.

Naturally the question arises, after the change of variable the fractional PI estimator is stable?, in this sense we state the following theorem.

Theorem 3 *The trajectories of fractional-order system formed by Eqs. (60) and (61) are globally Mittag–Leffler bounded.*

Proof The Eqs. (60) and (61) can be written in matrix form

$$
{}_0^C D_t^{\bar{\alpha}} \epsilon \;=\; \bar{K} \epsilon + \Delta \tag{62}
$$

where

$$
\epsilon = \begin{pmatrix} \sigma_{p,i} \\ \rho_{p,i} \end{pmatrix}; \quad \bar{K} = \begin{pmatrix} -k_1 & k_2 \\ -k_3 & -k_4 \end{pmatrix} \tag{63}
$$

$$
\Delta = \begin{pmatrix} k_1 s_i(y,u) + \sum_{\imath=1}^{m} \bar{\delta}_{1,\imath}\, {}_0^C D_t^{\alpha_{1,\imath}-p\bar{\alpha}} y_\imath + \sum_{\jmath=1}^{q} \bar{\delta}_{2,\jmath}\, {}_0^C D_t^{\alpha_{2,\jmath}-p\bar{\alpha}} u_\jmath \\ k_3 s_i(y,u) + \sum_{\imath=1}^{m} \bar{\delta}_{3,\imath}\, {}_0^C D_t^{\alpha_{1,\imath}-p\bar{\alpha}} y_\imath + \sum_{\jmath=1}^{q} \bar{\delta}_{4,\jmath}\, {}_0^C D_t^{\alpha_{2,\jmath}-p\bar{\alpha}} u_\jmath \end{pmatrix} \tag{64}
$$

Let us consider the quadratic function

$$
W(\epsilon) \;=\; \epsilon^\intercal \epsilon \tag{65}
$$

Note that $W(0) = 0$ and $W(\epsilon) > 0$ for all $\epsilon \neq 0$.

Let us compute the derivative of (65) of fractional order $\bar{\alpha}$ along the fractional-order system (62) (see Lemma 1)

$$
\begin{aligned}
{}_0^C D_t^{\bar{\alpha}} W(\epsilon) &\leq \epsilon^\intercal \left(\bar{K}\epsilon + \Delta\right) + \left(\epsilon^\intercal \bar{K}^\intercal + \Delta^\intercal\right)\epsilon \\
&= \epsilon^\intercal (\bar{K} + \bar{K}^\intercal)\epsilon + \epsilon^\intercal \Delta + \Delta^\intercal \epsilon
\end{aligned} \tag{66}
$$

It is easy to verify that the matrix $\bar{K}$ is Hurwitz that is why the following relation is met

$$
\bar{K} + \bar{K}^\intercal \;=\; -\bar{Q} \tag{67}
$$

where $\bar{Q} \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix.

On the other hand, using the fact that for any two matrices in their appropriate dimensions the inequality

$$
X^\intercal Y + Y^\intercal X \;\leq\; w X^\intercal X + \frac{1}{w} Y^\intercal Y \tag{68}
$$

is satisfied for $w \in \mathbb{R}^+$. In view of (67) and (68) we can write the fractional-order differential inequality (66) as follows

$$
{}_0^C D_t^{\bar{\alpha}} W(\epsilon) \;\leq\; -\epsilon^\intercal \bar{Q} \epsilon + w \epsilon^\intercal \epsilon + \frac{1}{w} \Delta^\intercal \Delta \tag{69}
$$

Due the fact that matrix $\bar{Q}$ satisfies the Rayleigh–Ritz inequality then

$$
{}_0^C D_t^{\bar{\alpha}} W(\epsilon) \;\leq\; -\left[\lambda_{\min}(\bar{Q}) - w\right] \|\epsilon\|^2 + \frac{1}{w} \|\Delta\|^2 \tag{70}
$$

Since we work with stable fractional-order systems and bounded inputs (see Remark 8), then there exists an upper bound $\sup_t \|\Delta\| = \bar{\Delta} \in \mathbb{R}$ such that

$$ {}_0^C D_t^{\bar{\alpha}} W(\epsilon) \leq -\left[\lambda_{\min}(\bar{Q}) - w\right] \|\epsilon\|^2 + \frac{1}{w}\bar{\Delta}^2 \quad (71) $$

According to Theorem 1, if $0 < w < \lambda_{\min}(\bar{Q})$ we can conclude that the trajectories of fractional-order system formed by (60) and (61) are globally Mittag–Leffler bounded. □

So far we have proposed a parametric identification scheme in fractional-order systems, carrying it to a state estimation problem, on the other hand we also propose a change of variable which can be performed iteratively in order to partially or totally avoid the use of signals that we consider unknown such as fractional-order derivatives of the system inputs and outputs. However, we have left aside the effect that measurement noise has on the estimation of the parameters, so in the sequel we deal with this problem by means of a fractional-order differentiator.

3.3 Third part

First of all, let us consider an output of the nonlinear fractional-order system (20)

$$ y_r = \bar{y}_r + \upsilon \quad (72) $$

where $\bar{y}_r = h_r(x)$ is the system output without noise, υ is bounded measurement noise with $r \in \{1, \ldots, m\}$.

Now, we consider a special form of fractional PI estimator (27) which was proposed in [28], this form is obtained by setting $k_2 = 1$ and $k_4 = 0$

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} \hat{\eta} &= k_1(\eta - \hat{\eta}) + \zeta \\ {}_0^C D_t^{\breve{\alpha}} \zeta &= k_3(\eta - \hat{\eta}) \end{aligned} \quad (73) $$

where $0 < \breve{\alpha} \leq 1$ is the fractional order, with the initial conditions $\hat{\eta}(0) = \hat{\eta}_0$, $\zeta(0) = \zeta_0$ and $\hat{\eta}_0, \zeta_0 \in \mathbb{R}$. if we select $\eta = y_r$ then (73) has the following form

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} \hat{\eta} &= k_1(\bar{y}_r - \hat{\eta}) + \zeta + k_1\upsilon \\ {}_0^C D_t^{\breve{\alpha}} \zeta &= k_3(\bar{y}_r - \hat{\eta}) + k_3\upsilon \end{aligned} \quad (74) $$

We define the following error variables

$$ e_1 := \bar{y}_r - \hat{\eta}, \quad e_2 := {}_0^C D_t^{\breve{\alpha}} \bar{y}_r - \zeta \quad (75) $$

Computing the derivatives of (75) of fractional order $\breve{\alpha}$ we get

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} e_1 &:= {}_0^C D_t^{\breve{\alpha}} \bar{y}_r - {}_0^C D_t^{\breve{\alpha}} \hat{\eta}, \\ {}_0^C D_t^{\breve{\alpha}} e_2 &:= {}_0^C D_t^{2\breve{\alpha}} \bar{y}_r - {}_0^C D_t^{\breve{\alpha}} \zeta \end{aligned} \quad (76) $$

Taking into account (74), (75) and (76), we can obtain the following fractional-order system

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} e_1 &= -k_1 e_1 + e_2 - k_1\upsilon \\ {}_0^C D_t^{\breve{\alpha}} e_2 &= -k_3 e_1 - k_3\upsilon + {}_0^C D_t^{2\breve{\alpha}} \bar{y}_r \end{aligned} \quad (77) $$

Or in matrix form

$$ {}_0^C D_t^{\breve{\alpha}} e = K_e e + K_\upsilon \upsilon + \Xi \quad (78) $$

where

$$ e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad K_e = \begin{pmatrix} -k_1 & 1 \\ -k_3 & 0 \end{pmatrix}, $$
$$ K_\upsilon = \begin{pmatrix} -k_1 \\ -k_3 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0 \\ {}_0^C D_t^{2\breve{\alpha}} \bar{y}_r \end{pmatrix} \quad (79) $$

The following theorem gives sufficient conditions for the stability of the fractional-order system (78)

Theorem 4 *If there exists a symmetric positive definite matrix $\bar{P}$ and positive constants w_1, w_2, w_3, such that*

$$ \begin{pmatrix} \bar{P}K_e + K_e^\intercal \bar{P} + w_1\bar{P} & \bar{P}K_\upsilon & \bar{P} \\ K_\upsilon^\intercal \bar{P} & -w_2 & 0_{1\times 2} \\ \bar{P} & 0_{2\times 1} & -w_3 I_{2\times 2} \end{pmatrix} < 0 \quad (80) $$

then the trajectories of fractional-order system (78) are globally Mittag–Leffler bounded.

Proof Consider the following quadratic function

$$ \bar{W}(e) = e^\intercal \bar{P} e \quad (81) $$

where $\bar{P}$ is a symmetric positive definite matrix, it is clear that $\bar{W}(e)$ is radially unbounded and $\bar{W}(0) = 0$.

The derivative of $\bar{W}(e)$ of fractional order $\breve{\alpha}$ along with (78) is given by (see Lemma 1)

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} \bar{W}(e) &\leq e^\intercal \bar{P}\left(K_e e + K_\upsilon \upsilon + \Xi\right) \\ &\quad + \left(e^\intercal K_e^\intercal + K_\upsilon^\intercal \upsilon + \Xi^\intercal\right)\bar{P}e \\ &= e^\intercal\left(\bar{P}K_e + K_e^\intercal \bar{P}\right)e \\ &\quad + e^\intercal \bar{P}K_\upsilon \upsilon + K_\upsilon^\intercal \upsilon \bar{P}e + e^\intercal \bar{P}\Xi + \Xi^\intercal \bar{P}e \end{aligned} \quad (82) $$

since υ is a scalar then $\upsilon = \upsilon^\intercal$ so

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} \bar{W}(e) &\leq e^\intercal\left(\bar{P}K_e + K_e^\intercal \bar{P}\right)e + e^\intercal \bar{P}K_\upsilon \upsilon \\ &\quad + \upsilon^\intercal K_\upsilon^\intercal \bar{P}e + e^\intercal \bar{P}\Xi + \Xi^\intercal \bar{P}e \end{aligned} \quad (83) $$

which leads to

$$
{}_0^C D_t^{\check{\alpha}} \bar{W}(e) \leq \begin{pmatrix} e^\intercal \\ \upsilon^\intercal \\ \Xi^\intercal \end{pmatrix}^\intercal \begin{pmatrix} \bar{P}K_e + K_e^\intercal \bar{P} & \bar{P}K_\upsilon & \bar{P} \\ K_\upsilon^\intercal \bar{P} & 0 & 0_{1\times 2} \\ \bar{P} & 0_{2\times 1} & 0_{2\times 2} \end{pmatrix} \begin{pmatrix} e \\ \upsilon \\ \Xi \end{pmatrix} \quad (84)
$$

Adding and subtracting the terms $w_1 e^\intercal \bar{P} e$, $w_2 \upsilon^\intercal \upsilon$ and $w_3 \Xi^\intercal \Xi$ to (84) we obtain

$$
{}_0^C D_t^{\check{\alpha}} \bar{W}(e) \leq \begin{pmatrix} e^\intercal \\ \upsilon^\intercal \\ \Xi^\intercal \end{pmatrix}^\intercal \begin{pmatrix} \bar{P}K_e + K_e^\intercal \bar{P} + w_1 \bar{P} & \bar{P}K_\upsilon & \bar{P} \\ K_\upsilon^\intercal \bar{P} & -w_2 & 0_{1\times 2} \\ \bar{P} & 0_{2\times 1} & -w_3 I_{2\times 2} \end{pmatrix} \begin{pmatrix} e \\ \upsilon \\ \Xi \end{pmatrix} - w_1 e^\intercal \bar{P} e + w_2 \upsilon^\intercal \upsilon + w_3 \Xi^\intercal \Xi \quad (85)
$$

if we ensure

$$
\begin{pmatrix} \bar{P}K_e + K_e^\intercal \bar{P} + w_1 \bar{P} & \bar{P}K_\upsilon & \bar{P} \\ K_\upsilon^\intercal \bar{P} & -w_2 & 0_{1\times 2} \\ \bar{P} & 0_{2\times 1} & -w_3 I_{2\times 2} \end{pmatrix} < 0 \quad (86)
$$

Then

$$
{}_0^C D_t^{\check{\alpha}} \bar{W}(e) \leq -w_1 e^\intercal \bar{P} e + w_2 \upsilon^\intercal \upsilon + w_3 \Xi^\intercal \Xi \quad (87)
$$

Since we deal with stable fractional-order systems (see Remark 8) and we consider bounded measurement noise, then we have that

$$
\sup_t \|\upsilon\| = \bar{\upsilon} \in \mathbb{R}, \quad \sup_t \|\Xi\| = \bar{\Xi} \in \mathbb{R} \quad (88)
$$

And therefore

$$
{}_0^C D_t^{\check{\alpha}} \bar{W}(e) \leq -w_1 \bar{W}(e) + w_2 \bar{\upsilon}^2 + w_3 \bar{\Xi}^2 \quad (89)
$$

Since the quadratic function (81) satisfies the Rayleigh–Ritz inequality, then by Theorem 1 the trajectories of fractional-order system (78) are globally Mittag–Leffler bounded. □

Remark 13 It is easy to verify that the matrix K_e is Hurwitz for all positive constants k_1 and k_3 that is why the LMI (80) is feasible.

We are now interested in how to select constants w_1, w_2 and w_3, taking into account Theorem 1 and Remark 3, by inspection of (89) we have that

$$
\varphi_2 = \frac{w_2 \bar{\upsilon}^2 + w_3 \bar{\Xi}^2}{\lambda_{\min}(\bar{P}) w_1}, \quad c_2 = \frac{1}{2} \quad (90)
$$

which leads to

$$
\limsup_{t\to\infty} \|e\| = \left(\frac{w_2 \bar{\upsilon}^2 + w_3 \bar{\Xi}^2}{\lambda_{\min}(\bar{P}) w_1} \right)^{1/2} \quad (91)
$$

This means that the solution trajectories of (78) converge asymptotically and remain in the compact set

$$
\bar{B}_2 = \left\{ e \in \mathbb{R}^2 \mid \|e\| \leq \left(\frac{w_2 \bar{\upsilon}^2 + w_3 \bar{\Xi}^2}{\lambda_{\min}(\bar{P}) w_1} \right)^{1/2} \right\} \quad (92)
$$

Remark 14 Note that the size of the compact set (92) is determined by the values of the constants w_1, w_2 and w_3, on the one hand w_1 acts inversely proportional while w_2 and w_3 act directly proportional. In this way, if we want to reduce the size of (92) it is enough to increase w_1 and decrease w_2 and w_3 in such a way that the LMI (80) is still feasible.

Remark 15 With all the above mentioned, it is clear that (73) is a fractional-order differentiator and it is justified that ζ is a fractional-order integral part of $\hat{\eta}$.

Remark 16 The two advantages of the fractional-order differentiator (73) are: Reduction of the effect of measurement noise on the output and it allows to obtain a fractional-order derivative of the output.

Finally, we can summarize in a series of steps the parametric identification method proposed in this paper.

1. Take a nonlinear fractional-order system.
2. Prove that the system parameters ξ_i satisfy the FAI property (21). If the parameters do not satisfy (21), then they cannot be identified under the method that we propose in this work.
3. Define the auxiliary variables $\eta_i(x, \xi_i)$ associated to the system parameters ξ_i and write the augmented form (24), since the parameters satisfy FAI then the associated auxiliary variables satisfy IFAO (see Remark 7).
4. Using the expression of the IFAO property of each auxiliary variable, a fractional PI estimator such as (27) is constructed for each of them.

5. If the fractional PI estimator for each auxiliary variable contain fractional-order derivatives of the inputs and outputs of the system, then we can use the change of variable explained in the second part to partially or totally avoid the use of these signals that are considered unknown.
6. If after the change of variable one can only partially avoid the use of fractional-order derivatives of the output and input of the system, then a fractional-order differentiator as presented in (73) is designed in order to estimate these signals.
7. Later, if there is measurement noise at the system output then a fractional-order differentiator such as (73) is designed in order to reduce the effect of noise on the system output.
8. Finally, when the estimate of the auxiliary variables is obtained, then we will obtain an estimate of the system parameters, thus the problem of parametric identification for a class of nonlinear fractional-order systems is solved.

4 Parametric identification

Here we apply the methodology shown in the previous section, performing the parameter identification of two nonlinear fractional-order systems, the first is a time-varying nonlinear fractional-order system, the second system is the nonlinear fractional-order mathematical model of a simple pendulum, in these examples we consider the presence of measurement noise at the output.

These examples show the main advantages of the estimation method proposed in this work, such as the on-line parameter identification which allows to identify time-varying parameters and the robustness of the estimator against measurement noise.

4.1 Example 1: time-varying nonlinear fractional-order system

Let us consider the following time-varying nonlinear fractional-order system

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x_1 &= -\xi_1 x_1 + u \\ {}_0^C D_t^{\alpha_2} x_2 &= \xi_2 x_1 - x_2 + \cos(x_1 + x_2) \\ y &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \tag{93}$$

where $\alpha_1 = \alpha_2 = 0.995$ are the fractional orders, ξ_1 and ξ_2 represent the system parameters, the input and output of the system are denoted by $u = 1$ and y, respectively, with zero initial conditions, i.e., $x_1(0) = x_2(0) = 0$.

It should be noted that both parameters ξ_1 and ξ_2 satisfy the FAI property, i.e., these can be represented as follows

$$\xi_1 = \frac{1}{y_1}\left(-{}_0^C D_t^{\alpha_1} y_1 + u\right) = \phi_1(y_1, {}_0^C D_t^{\alpha_1} y_1, u) \tag{94}$$

$$\begin{aligned} \xi_2 &= \frac{1}{y_1}\left({}_0^C D_t^{\alpha_2} y_2 + y_2 - \cos(y_1 + y_2)\right) \\ &= \phi_2\left(y_1, y_2, {}_0^C D_t^{\alpha_1} y_2\right) \end{aligned} \tag{95}$$

where $y_1 = x_1$ and $y_2 = x_2$, it is clear that if $y_1 = 0$ the parameters of the system lose the FAI property (94) and (95) since there will be a singularity problem, nonetheless, this problem can be treated through a characteristic function which will be explained later. However, as will be seen below, although there may be singularities in the FAI property, the auxiliary variables that are associated to the system parameters will not present this problem in the IFAO property, so they can be estimated.

Now, let us defined the auxiliary variables $\eta_1 = \xi_1 x_1$ and $\eta_2 = \xi_2 x_1$, since ξ_1 and ξ_2 satisfy the FAI property, then, η_1 and η_2 satisfy IFAO property (see Remark 7).

$$\eta_1 = -{}_0^C D_t^{\alpha_1} y_1 + u = \psi_1\left({}_0^C D_t^{\alpha_1} y_1, u\right) \tag{96}$$

$$\begin{aligned} \eta_2 &= {}_0^C D_t^{\alpha_2} y_2 + y_2 - \cos(y_1 + y_2) \\ &= \psi_2(y_1, y_2, {}_0^C D_t^{\alpha_2} y_2) \end{aligned} \tag{97}$$

Thus, the fractional-order system (93) can be written in the augmented form (24) as follows

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x_1 &= -\eta_1 + u \\ {}_0^C D_t^{\alpha_2} x_2 &= \eta_2 - x_2 + \cos(x_1 + x_2) \\ {}_0^C D_t^{\mathring{\alpha}_1} \eta_1 &= \Phi_1(\cdot) \\ {}_0^C D_t^{\mathring{\alpha}_2} \eta_2 &= \Phi_2(\cdot) \\ y &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \tag{98}$$

where $0 < \mathring{\alpha}_1, \mathring{\alpha}_2 \leq 1$ are the fractional orders for the dynamic equations of the auxiliary variables which will be assigned later. $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ are unknown functions.

Remark 17 Appendix A gives the stability analysis of the closed-loop system (93) with a constant input, where it is demonstrated that the trajectories are globally Mittag–Leffler bounded so that Assumption 1 is satisfied.

We can now design fractional-order estimators for the auxiliary variables η_1 and η_2.

Fractional PI estimator for η_1

Let us consider the following fractional PI estimator

$$ {}_0^C D_t^{\mathring{\alpha}_1} \hat{\eta}_1 = k_{11}(\eta_1 - \hat{\eta}_1) + k_{21}\zeta_1 \quad (99)$$

$$ {}_0^C D_t^{\mathring{\alpha}_1} \zeta_1 = k_{31}(\eta_1 - \hat{\eta}_1) - k_{41}\zeta_1 \quad (100)$$

where $k_{11}, k_{21}, k_{31}, k_{41} \in \mathbb{R}^+$ with the initial conditions $\hat{\eta}_1(0) = \hat{\eta}_{1_0}$, $\zeta_1(0) = \zeta_{1_0}$ and $\hat{\eta}_{1_0}, \zeta_{1_0} \in \mathbb{R}$.

Replacing (96) into (99) and (100) we obtain

$$ {}_0^C D_t^{\mathring{\alpha}_1} \hat{\eta}_1 = -k_{11}\hat{\eta}_1 + k_{21}\zeta_1 - k_{11}\, {}_0^C D_t^{\alpha_1} y_1 + k_{11}u $$
$$ {}_0^C D_t^{\mathring{\alpha}_1} \zeta_1 = -k_{31}\hat{\eta}_1 - k_{41}\zeta_1 - k_{31}\, {}_0^C D_t^{\alpha_1} y_1 + k_{31}u \quad (101)$$

Note that the value of $\mathring{\alpha}_1$ can be chosen freely only considering that $0 < \mathring{\alpha}_1 \leq 1$ must be satisfied, because ${}_0^C D_t^{\alpha_1} y_1$ can be obtained through a fractional-order differentiator such as the one presented in (73), for simplicity we take $\mathring{\alpha}_1 = \alpha_1$.

Fractional PI estimator for η_2

Consider the fractional PI estimator defined as

$$ {}_0^C D_t^{\mathring{\alpha}_2} \hat{\eta}_2 = k_{12}(\eta_2 - \hat{\eta}_2) + k_{22}\zeta_2 \quad (102)$$

$$ {}_0^C D_t^{\mathring{\alpha}_2} \zeta_2 = k_{32}(\eta_2 - \hat{\eta}_2) - k_{42}\zeta_2 \quad (103)$$

where $k_{12}, k_{22}, k_{32}, k_{42} \in \mathbb{R}^+$ with the initial conditions $\hat{\eta}_2(0) = \hat{\eta}_{2_0}$, $\zeta_2(0) = \zeta_{2_0}$ and $\hat{\eta}_{2_0}, \zeta_{2_0} \in \mathbb{R}$.

Substituting (97) in (102) and (103), we obtain

$$ {}_0^C D_t^{\mathring{\alpha}_2} \hat{\eta}_2 = -k_{12}\hat{\eta}_2 + k_{22}\zeta_2 + k_{12}\, {}_0^C D_t^{\alpha_2} y_2 + k_{12} y_2 - k_{12}\cos(y_1 + y_2) $$
$$ {}_0^C D_t^{\mathring{\alpha}_2} \zeta_2 = -k_{32}\hat{\eta}_2 - k_{42}\zeta_2 + k_{32}\, {}_0^C D_t^{\alpha_2} y_2 + k_{32} y_2 - k_{32}\cos(y_1 + y_2) \quad (104)$$

Also in this case, $\mathring{\alpha}_2$ can be chosen free under the condition $0 < \mathring{\alpha}_2 \leq 1$, since ${}_0^C D_t^{\alpha_2} y_2$ can be obtained by means of a fractional-order differentiator, we choose $\mathring{\alpha}_2 = \alpha_2$.

Note that so far we have designed fractional PI estimators (101) and (104) which allow us to obtain an estimate of the auxiliary variables η_1 and η_2, respectively. However, we are interested in obtaining an estimate of the system parameters, these can be estimated from the definition of the auxiliary variables, namely

$$ \hat{\xi}_1 = \frac{\hat{\eta}_1}{y_1}, \quad \hat{\xi}_2 = \frac{\hat{\eta}_2}{y_1} \quad (105)$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ are estimates of the parameters ξ_1 and ξ_2, respectively.

It is clear that if $y_1 = 0$ then the relations of (105) present a singularity problem, as mentioned above, in order to avoid this problem we can use a characteristic function, which we define as follows

Definition 7 A characteristic function is defined as

$$ \pi_c(t) := \Theta + (1 - \Theta)\pi(t) \quad (106)$$

where $\pi(t)$ is any function and

$$ \Theta = \begin{cases} 1, & \text{if } |\pi(t)| \leq \varsigma \\ 0, & \text{if } |\pi(t)| > \varsigma \end{cases} \quad (107)$$

with $\varsigma > 0$ is a small constant.

Remark 18 Note that $\pi_c(t) \neq 0, \forall t$.

Thus, we can use $y_{1_c} = \Theta + (1-\Theta)y_1$ instead of y_1 in the relations (105) to avoid the singularity problem.

On the other hand, let us consider the existence of measurement noise at the outputs of the system, i.e.,

$$ y_1 = \bar{y}_1 + \upsilon_1 \quad (108)$$

$$ y_2 = \bar{y}_2 + \upsilon_2 \quad (109)$$

where $\bar{y}_1$ and $\bar{y}_2$ represent the noise-free outputs of the system and υ_1, υ_2 are the measurement noises.

Hence, we can design fractional-order differentiator such as the one presented in (73) to reduce the effect of noise.

For y_1;

$$ \begin{aligned} {}_0^C D_t^{\check{\alpha}_1} \hat{y}_1 &= l_{11}(y_1 - \hat{y}_1) + \mathring{y}_1 \\ {}_0^C D_t^{\check{\alpha}_1} \mathring{y}_1 &= l_{21}(y_1 - \hat{y}_1) \end{aligned} \quad (110)$$

where $0 < \check{\alpha}_1 \leq 1$ is the fractional order and $l_{11}, l_{21} \in \mathbb{R}^+$ with the initial conditions $\hat{y}_1(0) = \hat{y}_{1_0}$, $\mathring{y}_1(0) = \mathring{y}_{1_0}$ and $\hat{y}_{1_0}, \mathring{y}_{1_0} \in \mathbb{R}$.

For y_2:

$$\begin{aligned} {}_0^C D_t^{\breve{\alpha}_2} \hat{y}_2 &= l_{12}(y_2 - \hat{y}_2) + \mathring{y}_2 \\ {}_0^C D_t^{\breve{\alpha}_2} \mathring{y}_2 &= l_{22}(y_2 - \hat{y}_2) \end{aligned} \tag{111}$$

where $0 < \breve{\alpha}_2 \leq 1$ is the fractional order and $l_{12}, l_{22} \in \mathbb{R}^+$ with the initial conditions $\hat{y}_2(0) = \hat{y}_{2_0}, \mathring{y}_2(0) = \mathring{y}_{2_0}$ and $\hat{y}_{2_0}, \mathring{y}_{2_0} \in \mathbb{R}$.

Naturally the question arises, How to choose $\breve{\alpha}_1$ and $\breve{\alpha}_2$?, in the previous section we demonstrated that through (110) and (111) we can reduce the measurement noise at the outputs, as well as obtain the derivatives of y_1 and y_2 of fractional order $\breve{\alpha}_1$ and $\breve{\alpha}_2$, respectively. It is clear that we need to choose $\breve{\alpha}_1 = \alpha_1$ and $\breve{\alpha}_2 = \alpha_2$ in order to implement estimators (101) and (104).

Finally, we have the following estimation systems for the auxiliary variables η_1 and η_2

For the auxiliary variable η_1

$$\begin{cases} {}_0^C D_t^{\alpha_1} \hat{y}_1 = l_{11}(y_1 - \hat{y}_1) + \mathring{y}_1 \\ {}_0^C D_t^{\alpha_1} \mathring{y}_1 = l_{21}(y_1 - \hat{y}_1) \\ {}_0^C D_t^{\alpha_1} \hat{\eta}_1 = -k_{11}\hat{\eta}_1 + k_{21}\zeta_1 - k_{11}\mathring{y}_1 + k_{11}u \\ {}_0^C D_t^{\alpha_1} \zeta_1 = -k_{31}\hat{\eta}_1 - k_{41}\zeta_1 - k_{31}\mathring{y}_1 + k_{31}u \end{cases} \tag{112}$$

with the initial conditions $\hat{y}_{1_0}, \mathring{y}_{1_0}, \hat{\eta}_{1_0}, \zeta_{1_0}$.

For the auxiliary variable η_2

$$\begin{cases} {}_0^C D_t^{\alpha_2} \hat{y}_2 = l_{12}(y_2 - \hat{y}_2) + \mathring{y}_2 \\ {}_0^C D_t^{\alpha_2} \mathring{y}_2 = l_{22}(y_2 - \hat{y}_2) \\ {}_0^C D_t^{\alpha_2} \hat{\eta}_2 = -k_{12}\hat{\eta}_2 + k_{22}\zeta_2 + k_{12}\mathring{y}_2 \\ \qquad + k_{12}\hat{y}_2 - k_{12}\cos(\hat{y}_1 + \hat{y}_2) \\ {}_0^C D_t^{\alpha_2} \zeta_2 = -k_{32}\hat{\eta}_2 - k_{42}\zeta_2 + k_{32}\mathring{y}_2 \\ \qquad + k_{32}\hat{y}_2 - k_{32}\cos(\hat{y}_1 + \hat{y}_2) \end{cases} \tag{113}$$

with the initial conditions $\hat{y}_{2_0}, \mathring{y}_{2_0}, \hat{\eta}_{2_0}, \zeta_{2_0}$.

While the estimates of the system parameters are obtained by

$$\hat{\xi}_1 = \frac{\hat{\eta}_1}{\hat{y}_{1_c}}, \quad \hat{\xi}_2 = \frac{\hat{\eta}_2}{\hat{y}_{1_c}} \tag{114}$$

where $\hat{y}_{1_c} = \Theta_1 + (1 - \Theta_1)\hat{y}_1$ and

$$\Theta_1 = \begin{cases} 1, & \text{if } |\hat{y}_1| \leq \varsigma_1 \\ 0, & \text{if } |\hat{y}_1| > \varsigma_1 \end{cases} \tag{115}$$

with $\varsigma_1 > 0$ a small constant.

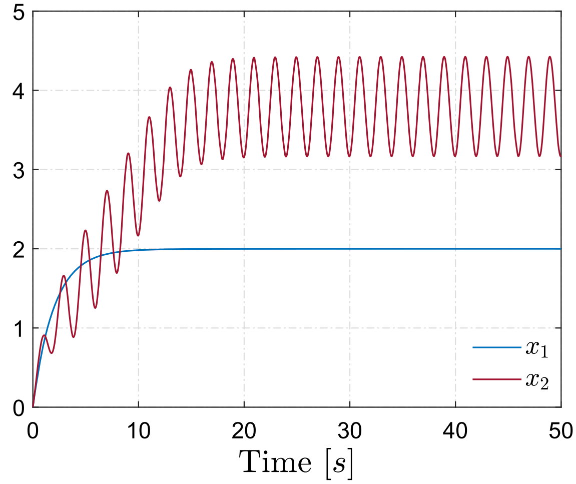


Fig. 1 Behavior of the time-varying fractional-order system (93)

Numerical simulation

Numerical simulation is carried out in Matlab-Simulink using the Dormand–Prince integration algorithm [29] with the integration step equal to 0.001 s.

In order to evaluate the accuracy of the identification method proposed in this paper, we consider the relative error as a performance criterion, which is defined as follows

$$e_r = \frac{|\xi - \hat{\xi}|}{|\xi|} \tag{116}$$

with $\xi \neq 0$, where $\hat{\xi}$ is an estimate of parameter ξ.

First, we show the behavior of the time-varying fractional-order system (93) considering the real value of the parameters $\xi_1 = 0.5$ and $\xi_2 = 1.5 + \sin(\pi t)$. Figure 1 depicts the behavior of the fractional-order system (93) verifying that the trajectories are bounded as shown mathematically in "Appendix A".

In the sequel we select the gains of systems (112) and (113), in order to identify the parameters ξ_1 and ξ_2.

For the system (112), in view of Theorem 4 and Remark 14 we take $l_{11} = 10$, $l_{21} = 25$, $w_{11} = 5$ and $w_{21} = w_{31} = 0.2$ which leads to

$$\bar{P}_1 = \begin{pmatrix} 0.0361 & -0.0069 \\ -0.0069 & 0.0024 \end{pmatrix} \tag{117}$$

On the other hand, taking into account Remark 11 we choose $k_{11} = 50$, $k_{21} = k_{31} = 5$ and $k_{41} = 45$.

For the system (113) in the same way we set $l_{12} = 14$, $l_{22} = 200$, $w_{12} = 5$, $w_{22} = 3$ and $w_{32} = 0.1$, then

$$\bar{P}_2 = \begin{pmatrix} 0.1907 & -0.0094 \\ -0.0094 & 0.0015 \end{pmatrix} \tag{118}$$

Also in this case, we select $k_{12} = 50$, $k_{22} = k_{32} = 5$ and $k_{42} = 45$.

The initial conditions for systems (112) and (113) are zero, i.e., $\hat{y}_{1_0} = \mathring{y}_{1_0} = \hat{\eta}_{1_0} = \zeta_{1_0} = 0$, $\hat{y}_{2_0} = \mathring{y}_{2_0} = \hat{\eta}_{2_0} = \zeta_{2_0} = 0$ and we fix $\varsigma_1 = 0.5$.

Now, in order to simulate the measurement noise at the outputs of the system we consider that

$$y_1 = x_1 + \upsilon_1 \tag{119}$$

$$y_2 = x_2 + \upsilon_2 \tag{120}$$

where υ_1 and υ_2 are a zero-mean white Gaussian noises, with Signal to Noise Ratio SNR = 20 dB.

Figure 2 shows the estimation of the system outputs and their fractional-order derivatives, as expected, a significant reduction of the measurement noise is achieved.

On the other hand, Fig. 3 shows the results of the parametric identification. Figure 3a and b depict the identification of the parameters ξ_1 and ξ_2, respectively, meanwhile Fig. 3c shows the relative errors.

According to the results obtained (see Figs. 2 and 3) we can say that through (112), (113), (114) and (115) we can obtain good estimates of the parameters ξ_1 and ξ_2 even when there is measurement noise. Thus, we can point out the main advantages of the parametric identification method proposed in this paper, i.e., it is a robust method against measurement noise, the identification is performed on-line so it allows the identification of time-varying parameters and the gain selection criteria are simple.

4.2 Example 2: nonlinear fractional-order mathematical model of a simple pendulum

A simple pendulum consists of a mass suspended from a pivot by a string or a rod of negligible mass (see Fig. 4).

We consider that there are an external force and the system is free of friction, it is well known that under these conditions the motion of the simple pendulum can be modeled through a differential equation of integer order, in particular of second order [30]. However this model can be generalized from the fractional-order derivatives which can be represented as

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x_1 &= x_2 \\ {}_0^C D_t^{\alpha_2} x_2 &= -\frac{g}{l}\sin(x_1) + u \end{aligned} \tag{121}$$

where $\alpha_1 = \alpha_2 = 0.95$ are the fractional orders, $x_1 = \theta$ is the angular displacement, $x_2 = \omega$ is the angular velocity, $u = 15$ is the system input, g represents the gravity acceleration and l denotes the length of the pendulum, with the initial conditions $x_1(0) = \pi/2$ and $x_2(0) = 0$.

Let us consider that we know the angular displacement θ by measuring, i.e., $y = x_1$ in this way, we can write

$$\frac{g}{l} = \frac{1}{\sin(y)}\left(-{}_0^C D_t^{\alpha_2+\alpha_1} y + u\right) = \phi\left(y, {}_0^C D_t^{\alpha_2+\alpha_1} y, u\right) \tag{122}$$

it is clear that g/l satisfies the FAI property if $y \neq i\pi$ with $i \in \mathbb{N}$.

We define the auxiliary variable $\eta := (g/l)\sin(x_1)$, in view of (122) η satisfies IFAO property (see Remark 7) and can be represented as

$$\eta = -{}_0^C D_t^{\alpha_2+\alpha_1} y + u \tag{123}$$

Which leads to the following augmented system

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x_1 &= x_2 \\ {}_0^C D_t^{\alpha_2} x_2 &= -\eta + u \\ {}_0^C D_t^{\mathring{\alpha}} \eta &= \Phi(\cdot) \\ y &= x_1 \end{aligned} \tag{124}$$

where $0 < \mathring{\alpha} \leq 1$ is the fractional order for the dynamic equation of the auxiliary variable that will be fixed later, $\Phi(\cdot)$ is a unknown function.

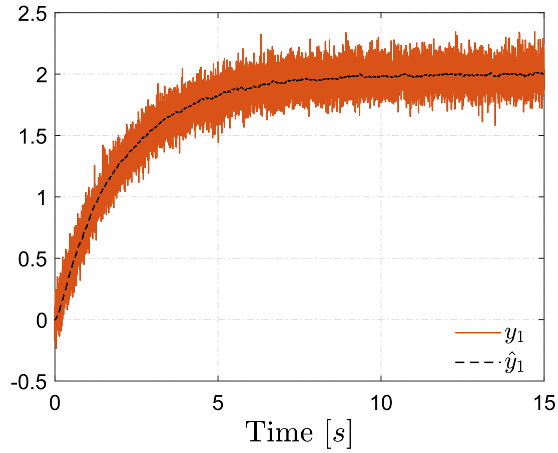
Remark 19 It is well known that the trajectories of the fractional-order system (121) have bounded oscillations that converge to the origin when $u = 0$, however when $u \neq 0$ but is a constant, the stability of the system is still preserved. therefore, Assumption 1 is fulfilled.

Now, we can design a fractional PI estimator for the auxiliary variable η. Let us consider the following fractional-order estimator

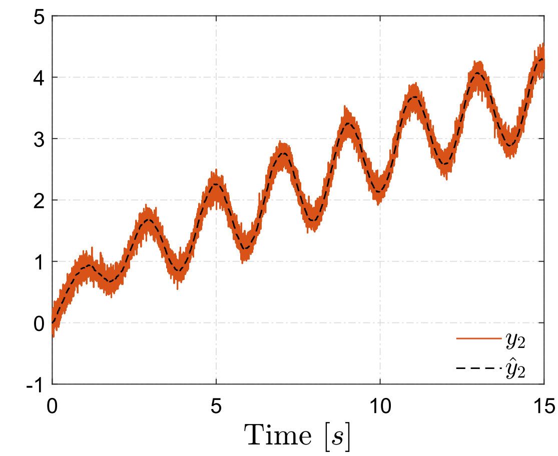
$${}_0^C D_t^{\mathring{\alpha}} \hat{\eta} = k_1(\eta - \hat{\eta}) + k_2\zeta \tag{125}$$

$${}_0^C D_t^{\mathring{\alpha}} \zeta = k_3(\eta - \hat{\eta}) - k_4\zeta \tag{126}$$

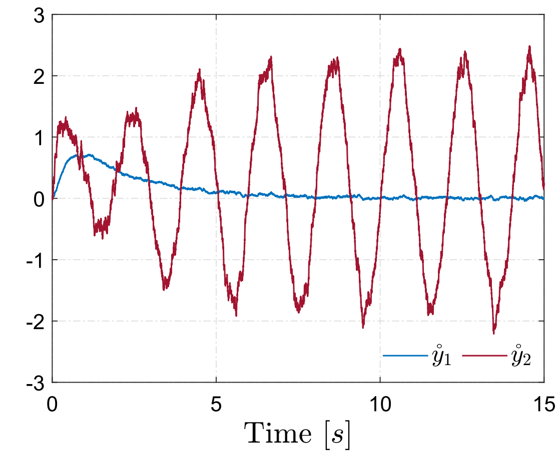
Fig. 2 Reduction of noise and estimation of fractional-order derivatives



(a) Reduction of the effect of noise on the output y_1

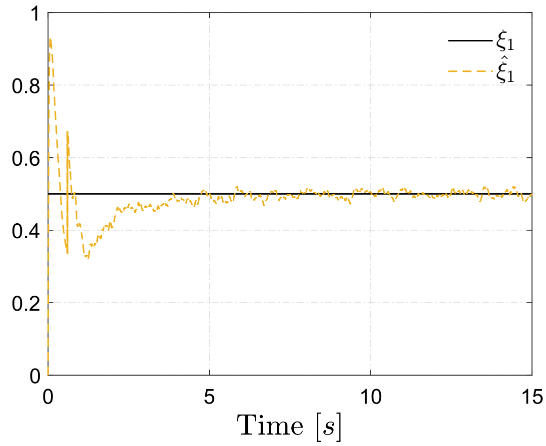


(b) Reduction of the effect of noise on the output y_2

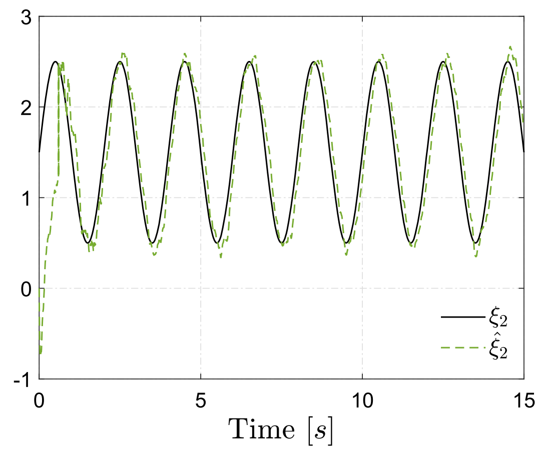


(c) Estimation of fractional-order derivatives of the outputs

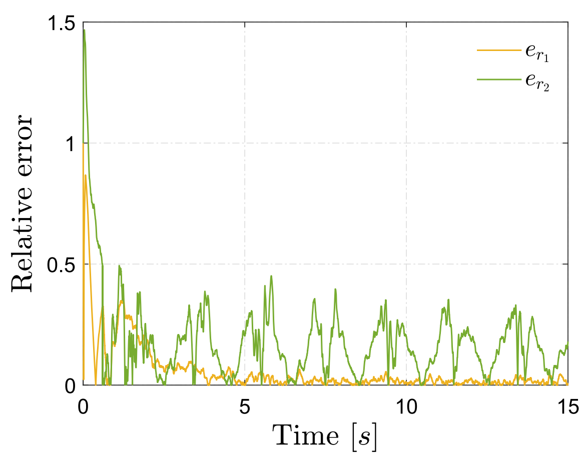
Fig. 3 Parametric identification of fractional-order system (93)



(a) Parameter ξ_1 and its estimate $\hat{\xi}_1$



(b) Parameter ξ_2 and its estimate $\hat{\xi}_2$



(c) Relative errors e_{r_1} and e_{r_2} for the estimation of the parameters ξ_1 and ξ_2, respectively

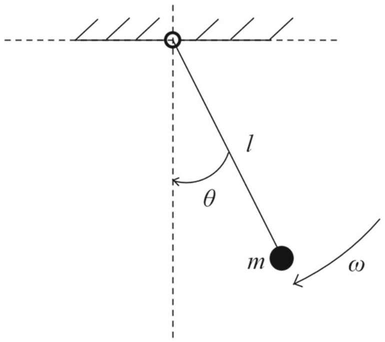


Fig. 4 Simple pendulum

where $k_1, k_2, k_3, k_4 \in \mathbb{R}^+$ with the initial conditions $\hat{\eta}(0) = \hat{\eta}_0$, $\zeta(0) = \zeta_0$ and $\hat{\eta}_0, \zeta_0 \in \mathbb{R}$.

Substituting (123) in (125) and (126) we have that

$$ {}_0^C D_t^{\mathring{\alpha}} \hat{\eta} = -k_1 \, {}_0^C D_t^{\alpha_2+\alpha_1} y + k_1 u - k_1 \hat{\eta} + k_2 \zeta \tag{127} $$

$$ {}_0^C D_t^{\mathring{\alpha}} \zeta = -k_3 \, {}_0^C D_t^{\alpha_2+\alpha_1} y + k_3 u - k_3 \hat{\eta} - k_4 \zeta \tag{128} $$

Note that $\alpha_2 + \alpha_1 > 1$ then we cannot design a fractional-order differentiator to estimate this unknown signal, however we can perform the change of variable that was explained in the previous section by fixing $\mathring{\alpha} = \alpha_2$, we define the following change of variable

$$ \hat{\eta} = \sigma - k_1 \, {}_0^C D_t^{\alpha_1} y \tag{129} $$

$$ \zeta = \rho - k_3 \, {}_0^C D_t^{\alpha_1} y \tag{130} $$

where σ and ρ are sufficiently smooth functions, which leads to

$$ {}_0^C D_t^{\alpha_2} \hat{\eta} = {}_0^C D_t^{\alpha_2} \sigma - k_1 \, {}_0^C D_t^{\alpha_2+\alpha_1} y \tag{131} $$

$$ {}_0^C D_t^{\alpha_2} \zeta = {}_0^C D_t^{\alpha_2} \rho - k_3 \, {}_0^C D_t^{\alpha_2+\alpha_1} y \tag{132} $$

Replacing (129), (130) and (131) into (127) we have that

$$ {}_0^C D_t^{\alpha_2} \sigma = -k_1 \sigma + k_2 \rho + (k_1^2 - k_2 k_3) \, {}_0^C D_t^{\alpha_1} y \tag{133} $$

Likewise, substituting (129), (130) and (132) in (128) we obtain

$$ {}_0^C D_t^{\alpha_2} \rho = -k_3 \sigma - k_4 \rho + (k_1 k_3 + k_3 k_4) \, {}_0^C D_t^{\alpha_1} y \tag{134} $$

Thus, after the change of variable the fractional PI estimator has the form

$$ \begin{cases} {}_0^C D_t^{\alpha_2} \sigma = -k_1 \sigma + k_2 \rho + (k_1^2 - k_2 k_3) \, {}_0^C D_t^{\alpha_1} y \\ {}_0^C D_t^{\alpha_2} \rho = -k_3 \sigma - k_4 \rho + (k_1 k_3 + k_3 k_4) \, {}_0^C D_t^{\alpha_1} y \\ \hat{\eta} = \sigma - k_1 \, {}_0^C D_t^{\alpha_1} y \end{cases} \tag{135} $$

with the initial conditions $\sigma(0) = \sigma_0$, $\rho(0) = \rho_0$ and $\sigma_0, \rho_0 \in \mathbb{R}$.

It should be noted that (135) does not depend on ${}_0^C D_t^{\alpha_2+\alpha_1} y$ anymore, instead ${}_0^C D_t^{\alpha_1} y$ needs to be known, but it can be estimated by means of a fractional-order differentiator. On the other hand, by definition of the auxiliary variable η we have that

$$ \frac{\hat{g}}{\hat{l}} = \frac{\hat{\eta}}{\sin(y)} \tag{136} $$

In order to avoid the singularity problems in (136), a characteristic function can be used (see Definition 7).

We are now interested in the effect that measurement noise has on the estimation of the auxiliary variable, so let us consider that

$$ y = \bar{y} + \upsilon \tag{137} $$

where $\bar{y}$ is the noise-free output and υ is the measurement noise.

Thus, let us consider the following fractional-order differentiator

$$ \begin{aligned} {}_0^C D_t^{\breve{\alpha}} \hat{y} &= l_1 (y - \hat{y}) + \mathring{y} \\ {}_0^C D_t^{\breve{\alpha}} \mathring{y} &= l_2 (y - \hat{y}) \end{aligned} \tag{138} $$

where $0 < \breve{\alpha} \leq 1$ is the fractional order, $l_1, l_2 \in \mathbb{R}^+$ with the initial conditions, $\hat{y}(0) = \hat{y}_0$, $\mathring{y}(0) = \mathring{y}_0$ and $\hat{y}_0, \mathring{y}_0 \in \mathbb{R}$.

As explained in the previous example, the choice of $\breve{\alpha}$ depends on the fractional-order derivative that we need to estimate. In this particular case, we need to estimate ${}_0^C D_t^{\alpha_1} y$ so we set $\breve{\alpha} = \alpha_1$, therefore, we obtain the following estimation system for the auxiliary variable η

$$ \begin{cases} {}_0^C D_t^{\alpha_1} \hat{y} = l_1 (y - \hat{y}) + \mathring{y} \\ {}_0^C D_t^{\alpha_1} \mathring{y} = l_2 (y - \hat{y}) \\ {}_0^C D_t^{\alpha_2} \sigma = -k_1 \sigma + k_2 \rho + (k_1^2 - k_2 k_3) \mathring{y} \\ {}_0^C D_t^{\alpha_2} \rho = -k_3 \sigma - k_4 \rho + (k_1 k_3 + k_3 k_4) \mathring{y} \\ \hat{\eta} = \sigma - k_1 \mathring{y} \end{cases} \tag{139} $$

with the initial conditions $\hat{y}_0, \mathring{y}_0, \sigma_0, \rho_0$.

And the relationship of the system parameters can be obtained by

$$\frac{\hat{g}}{\hat{l}} = \frac{\hat{\eta}}{(\sin(\hat{y}))_c} \tag{140}$$

where $(\sin(\hat{y}))_c = \Theta_2 + (1 - \Theta_2)\sin(\hat{y})$ and

$$\Theta_2 = \begin{cases} 1, & \text{if } |\hat{y}_1| \leq \varsigma_2 \\ 0, & \text{if } |\hat{y}_1| > \varsigma_2 \end{cases} \tag{141}$$

with a small constant ς_2.

Numerical simulation

We consider that the real value of the parameters is $g = 9.81$ and $l = 0.5$, first of all, we have to set the gains of the system (139) in view of Theorem 4 and Remark 14, we select $l_1 = 15$, $l_2 = 200$, $w_1 = 6$, $w_2 = 3$, $w_3 = 0.1$ and we obtain

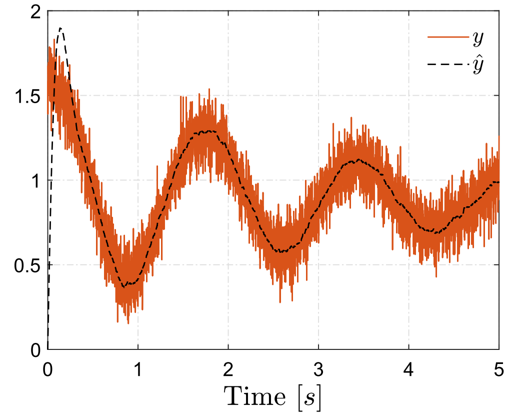
$$\bar{P} = \begin{pmatrix} 0.1887 & -0.0097 \\ -0.0097 & 0.0015 \end{pmatrix} \tag{142}$$

Taking into account, Remark 11 we select $k_1 = 15$, $k_2 = k_3 = 5$ and $k_4 = 10$. Additionally, we consider null initial conditions for the fractional-order system (139), i.e., $\hat{y}_0 = \mathring{y}_0 = \sigma_0 = \rho_0 = 0$.

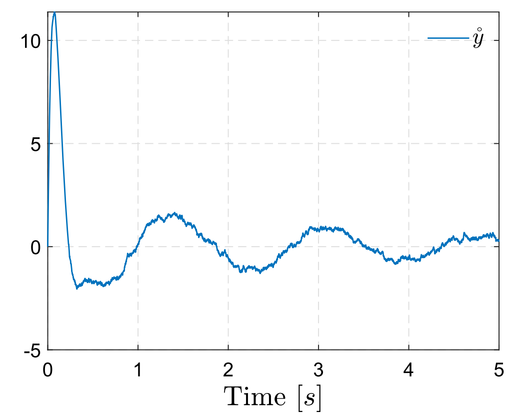
Also, in order to simulate the measurement noise at the output (137), we consider that υ is a zero mean white Gaussian noise, with SNR = 20 dB.

Figure 5 depicts the measurement noise attenuation as well as the estimation of the fractional-order derivative ${}_0^C D_t^{\alpha_1} y$, as in the previous example we obtain positive results.

On the other hand, Fig. 6 shows the estimation of g/l, we note that a suitable estimation is achieved in a short time even when there is measurement noise through (139) and (140), in addition we identified a mathematical model that describes the behavior of a physical system, therefore the identification method that we propose in this work can be easily implemented.



(a) Reduction of the effect of noise on the output y



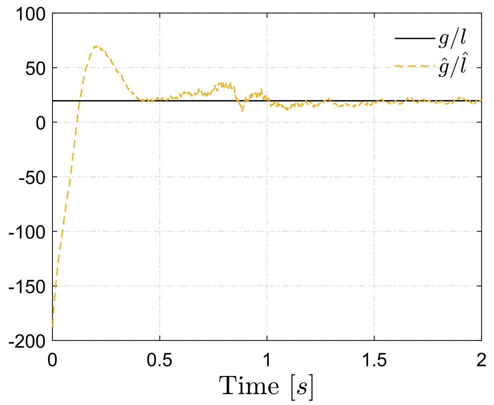
(b) Estimation of fractional-order derivative of the output

Fig. 5 Reduction of noise and estimation of fractional-order derivative

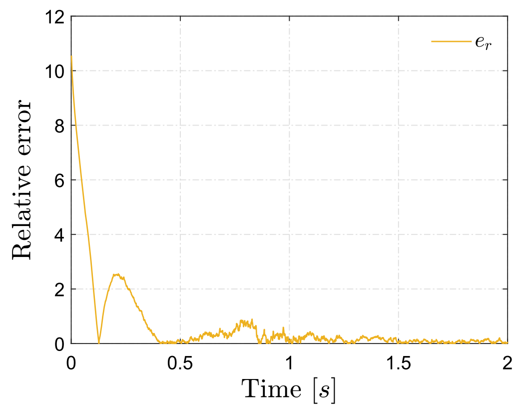
5 Conclusion

As a first point in this paper an alternative approach to the Mittag–Leffler boundedness is given and its convergence is proved, this method allows us to analyze the stability in perturbed fractional-order systems. Later we introduce a new property related to fractional algebraic identifiability which allowed us to translate the parameter identification problem into a state estimation problem. The parameter identification method that we propose presents certain advantages with respect to the methods that have been proposed by different authors, such as:

- The parametric identification method is applied to nonlinear fractional-order systems.
- The parameter identification process is performed on-line, i.e., while the system is working.
- It allows the identification of time-varying parameters.
- This method does not require the well-known persistent excitation condition.
- The parameter identification method is robust against measurement noise.
- The initial conditions considered in this method can be freely assigned.



(a) Parameter g/l and its estimate $\hat{g}/\hat{l}$



(b) Relative error e_r for the estimation of the parameter g/l

Fig. 6 Parametric identification of fractional-order mathematical model of a simple pendulum

On the other hand, since we translate the parameter identification to state estimation, we propose a fractional PI estimator in order to obtain an estimate of the extra states. Finally, the numerical examples show the accuracy and robustness of the method proposed in this work, where the identification of a time-varying nonlinear fractional-order system proves that our method can identify time-varying parameters, while the nonlinear fractional-order mathematical model of a simple pendulum shows that this method can be implemented for physical systems.

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Data availability All data generated or analyzed during this study are included in the article.

Declarations

Conflict of interest None.

Appendix A: Stability analysis of the time-varying nonlinear fractional-order system

In this section we will test the stability of the time-varying nonlinear fractional-order system (93).

Let us consider the time-varying nonlinear fractional-order system

$$\begin{aligned} {}_0^C D_t^{\alpha_1} x_1 &= -\xi_1 x_1 + u \\ {}_0^C D_t^{\alpha_2} x_2 &= \xi_2 x_1 - x_2 + \cos(x_1 + x_2) \end{aligned} \tag{A1}$$

where $\alpha_1 = \alpha_2 = 0.995$, $u = 1$, $\xi_1 = 0.5$ and $\xi_2 = 1.5 + \sin(\pi\, t)$ with the initial conditions $x_1(0) = x_2(0) = 0$.

Note that the system (A1) can be written in the following form

$${}_0^C D_t^{\alpha} x = A\, x + \Upsilon \tag{A2}$$

where $\alpha = \alpha_1 = \alpha_2$ and

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad A = \begin{pmatrix} -\xi_1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\Upsilon = \begin{pmatrix} u \\ \xi_2\, x_1 + \cos(x_1 + x_2) \end{pmatrix} \tag{A3}$$

We consider the quadratic function

$$V(x) = x^{\mathsf{T}} S\, x \tag{A4}$$

with $0 < S^\intercal = S \in \mathbb{R}^{2\times 2}$. Taking into account (A2) and from lemma 1 we have that

$$\begin{aligned} {}_0^C D_t^\alpha V(x) &\le x^\intercal S\, {}_0^C D_t^\alpha x + ({}_0^C D_t^\alpha x)^\intercal S x \\ &= x^\intercal S (A x + \Upsilon) + (x^\intercal A^\intercal + \Upsilon^\intercal) S x \\ &= x^\intercal (S A + A^\intercal S) x + x^\intercal S \Upsilon + \Upsilon^\intercal S x \end{aligned} \tag{A5}$$

Since A is Hurwitz then the equation

$$S A + A^\intercal S = -\varkappa I_{2\times 2} \tag{A6}$$

is met, where $\varkappa \in \mathbb{R}^+$.

On the other hand, in view of (68) we have that

$$x^\intercal S \Upsilon + \Upsilon^\intercal S x \le \bar{w} x^\intercal S S x + \frac{1}{\bar{w}} \Upsilon^\intercal \Upsilon \tag{A7}$$

where $\bar{w} \in \mathbb{R}^+$, which leads to

$${}_0^C D_t^\alpha V(x) \le -\varkappa x^\intercal x + \bar{w} x^\intercal S S x + \frac{1}{\bar{w}} \Upsilon^\intercal \Upsilon \tag{A8}$$

Since S is a symmetric positive definite matrix we can write

$${}_0^C D_t^\alpha V(x) \le -\left[\varkappa - \bar{w} \lambda_{\max}^2(S)\right] \|x\|^2 + \frac{1}{\bar{w}} \|\Upsilon\|^2 \tag{A9}$$

Bear in mind that the quadratic function (A4) satisfies the Rayleigh–Ritz inequality, then if we select $\varkappa > \bar{w}\lambda_{\max}^2(S)$ and $\|\Upsilon\|$ has an upper bound $\bar{\Upsilon} = \sup_t(\|\Upsilon\|)$, therefore according to Theorem 1, the trajectories of system (A1) are globally Mittag–Leffler bounded.

We now try to find an upper bound for $\|\Upsilon\|$, firstly note that

$$\Upsilon = \begin{pmatrix} u \\ \xi_2 x_1 + \cos(x_1 + x_2) \end{pmatrix} \le \begin{pmatrix} 1 \\ \xi_2 x_1 + 1 \end{pmatrix} \tag{A10}$$

So

$$\|\Upsilon\| \le \sqrt{1^2 + (\xi_2 x_1 + 1)^2} \le 1 + |\xi_2 x_1 + 1| \tag{A11}$$

Taking into account that parameter $|\xi_2| \le 2.5$ and using Triangle and Cauchy–Schwarz inequalities it follows that

$$\|\Upsilon\| \le 2 + 2.5\,|x_1| \tag{A12}$$

Nevertheless, we have that the behavior of x_1 is independent of x_2 so we can find the solution for x_1. We take the dynamic equation of x_1, namely

$${}_0^C D_t^{\alpha_1} x_1 = -\xi_1 x_1 + u \tag{A13}$$

The solution of (A13) is given by

$$x_1 = x_1(0) E_{\alpha_1}\left(-\xi_1 t^{\alpha_1}\right) + \int_0^t (t-\tau)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}\left[-\xi_1 (t-\tau)^{\alpha_1}\right] u(\tau) d\tau \tag{A14}$$

Since $x_1(0) = 0$ and using Triangle and Cauchy–Schwarz inequalities it follows that

$$|x_1| \le \int_0^t |(t-\tau)^{\alpha_1 - 1}| |E_{\alpha_1,\alpha_1}\left[-\xi_1 (t-\tau)^{\alpha_1}\right]| d\tau \tag{A15}$$

Bear in mind that, $(t-\tau)^{\alpha_1 - 1}$ and $E_{\alpha_1,\alpha_1}[-\xi_1 (t-\tau)^{\alpha_1}]$ are non-negative functions, then

$$|x_1| \le \int_0^t (t-\tau)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}\left[-\xi_1 (t-\tau)^{\alpha_1}\right] d\tau \tag{A16}$$

Now we consider the following properties of the Mittag–Leffler functions:

Property 2 [20] The following equality holds for $\beta > 0$ and $\upsilon > 0$

$$\int_0^t \tau^{\beta - 1} E_{\alpha,\beta}(-\upsilon \tau^\alpha) d\tau = t^\beta E_{\alpha,\beta+1}(-\upsilon t^\alpha) \tag{A17}$$

Theorem 5 [20] *If* $0 < \alpha < 2$, $\beta \in \mathbb{C}$ *and* $\nu \in \mathbb{R}$ *such that*

$$\frac{\pi \alpha}{2} < \nu < \min(\pi, \pi\alpha) \tag{A18}$$

then for $\iota \in \mathbb{Z}$ *with* $\iota \ge 1$ *the following expansion holds:*

$$E_{\alpha,\beta}(z) = -\sum_{j=1}^{\iota} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O\left(|z|^{-(\iota+1)}\right) \tag{A19}$$

with $|z| \to \infty$, $\nu \le |arg(z)| \le \pi$. □

Using (A17) we can write (A16) as follows

$$|x_1| \le t^{\alpha_1} E_{\alpha_1,\alpha_1+1}(-\xi_1 t^{\alpha_1}) \tag{A20}$$

when $t \to \infty$ we use (A19) with $\nu = 3\pi\alpha_1/4$ then

$$\lim_{t\to\infty} |x_1| \le \lim_{t\to\infty} t^{\alpha_1} E_{\alpha_1,\alpha_1+1}(-\xi_1 t^{\alpha_1}) \tag{A21}$$

Hence $|x_1| \le 1/\xi_1$ this implies that the upper bound for $\|\Upsilon\|$ is given by

$$\|\Upsilon\| \le 2 + \frac{2.5}{\xi_1} = \bar{\Upsilon} \tag{A22}$$

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