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REVIEW

On resonances and transverse and longitudinal oscillations in a hoisting system due to boundary excitations

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Abstract In this paper, we study transverse and longitudinal oscillations and resonances in a hoisting system induced by boundary disturbances. The dynamics can be described by an initial-boundary value problem for a coupled system of nonlinear wave equations on a slowly time-varying spatial domain. It will be shown how the boundary excitations and the nonlinear terms influence transverse and longitudinal vibrations of the system. Firstly, due to the slow variation of the cable length, a singular perturbation problem arises. By using an interior layer analysis, many resonance manifolds are detected. Secondly, it will be shown that resonances in the system are caused not only by boundary disturbances but also by nonlinear interactions. Based on these observations, a three-timescales perturbation method is used to approximate the solution of the initial-boundary value problem analytically. It turns out that for special frequencies in the boundary excitations and for certain parameter values of the longitudinal stiffness and the conveyance mass, many oscillation modes jump up from small to large amplitudes in the transverse and longitudinal directions. Finally, numerical simulations are presented to verify the obtained analytical results.

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1 Introduction

Within the last decades, hoisting systems are widely used for transportation of objects to a large height or depth. Such systems consist of a drum, a head sheave, a driving motor, a flexible hoisting cable with time-varying length, and a hoisting conveyance moving along two guiding ropes. When the flexible hoisting cable's bending stiffness is not considered, the mathematical model for this system can be described as an axially moving string with a time variable length [1]. Compared with rigid structures, the flexible cable has many advantages, such as low costs, high speeds, and high load carrying capacities, which are applied in various engineering fields, for instance, elevators [2], marine risers [3,4], suspension bridges [5,6], medical rescue systems [7], etc. In hoisting processes [8,9], vibration-induced structural failure for hoisting cables may occur due to external disturbances such as airflows or earthquakes, or due to other internal or external excitations. These failures are usually related to internal or external resonances. Resonance refers to the phenomenon that a small periodic excitation can produce large vibrations when the frequency of the external or internal excitation is close to one of the natural frequencies of the system. In most cases resonance is harmful, it will not only lead to significant deformations and dynamic stress, but also leads to accidents. Therefore, it is important to develop advanced analytical models to figure out the nature of these large vibrations in moving media.

There is an abundance of papers on the analysis of moving flexible string models. Some researchers focus on traveling strings defined on a fixed domain, and other researchers focus on traveling strings subject to moving boundaries. For the analysis on the fixed domain: Zhu et al. in [10] considered a constrained translating cable with a spring-mass-dashpot located at any position along the cable. They determined asymptotic approximations of the eigenvalues of the system. Nguyen and Hong in [11] investigated an active control scheme for an axially moving cable system by employing the Lyapunov method. The controller suppresses the vibrations and regulates the transport velocity of the cable to track a desired moving velocity profile. Gaiko and van Horssen in [12] considered transverse vibrations of a traveling string with a spring-mass-dashpot boundary. They constructed approximations of the solutions and eigenvalues on a long timescale. For the analysis on the moving boundaries: Zhu in [13] considered the transverse vibration stability of a class of translating media with an arbitrarily varying length from energy standpoint. Sandilo and van Horssen in [14] studied the vertically translating string with a time varying length and a space–time-varying tension by an interior layer analysis. Gaiko and van Horssen in [15] discussed resonances and vibrations in an elevator cable system due to boundary sway by using a multiple timescales perturbation method. Wang et al. in [16] studied resonances of transversally vibrating cables for a fixed domain, in which resonances are induced by an external force and a time-dependent coefficient in a Robin boundary condition. This problem may serve as a simplified model describing longitudinal vibrations (see also [17]) as well as resonances in axially moving strings for which the length changes in time.

In recent years, researchers found that, due to external excitation and loading conditions, the nonlinear interactions between transverse and longitudinal string motions may influence the vibration behaviour in two directions when the hoisting conveyance is moving up or down. Some research has been conducted on similar types of problems by using numerical simulations. Crespo et al. in [18] introduced a stationary high-rise elevator cable system model and presented its numerical simulations. Wang et al. in [19] investigated a coupled dynamic model for a flexible guiding hoisting system and presented the response of the system by numerical simulations. In this paper, we will construct analytical approximate solutions for the nonlinear coupled transverse and longitudinal vibration string problem with time-varying length.

The hoisting system considered in this paper is described by a vertically translating string with a time-varying length and a mass attached at one of the ends of the string. The time-varying length of the string is given by $l(t) = l_0 + vt$, where l_0 is the cable initial length, and v is the longitudinal velocity of the hoisting cable, and where l_0, v are constants. It is assumed that the axial velocity of the string is small compared to nominal wave velocity, and that the string mass is small compared to cage mass. The system is excited at the upper end by small displacements in the horizontal and vertical directions from its equilibrium position caused by, for instance, wind forces (see Fig. 1). By Hamilton's principle, the model can be written as a coupled system of nonlinear wave equations (in transverse and longitudinal directions) on a slowly time-varying spatial domain. The string is excited at a boundary by two harmonic functions in the horizontal and in the vertical directions. The main objective of this paper is to study how the boundary excitations and nonlinear interactions between the two motion directions influence the vibration behaviour in the transverse and in the longitudinal directions for the moving string. In contrast to previous research, where only the transverse or the longitudinal vibration behaviour was studied, the coupled model is more accurate. However, the appearance of nonlinear and coupled terms increases the complexity of the system analysis. For the problem with nonlinearly coupled terms, and with moving boundary conditions, the traditional (analytical) methods, such as the method of separation of variables (SOV), and the (equivalent) Laplace transform method, can usually not be applied. In order to deal with these difficulties, perturbation methods and an internal layer analysis are used in this paper to approximate the vibrations and the resonances, including determining the resonance amplitudes and the size of the resonance zones. Based on this analysis, solutions of the coupled initial-boundary value problem for the transverse and the longitudinal motions can be predicted analytically. To the best of our knowledge, the results about analytical approximations of the solutions have not been

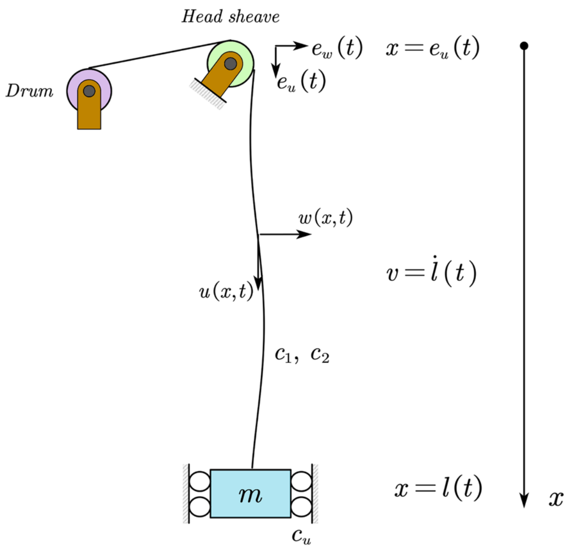

Fig. 1 Coupled transverse-longitudinal vibrating cable with time-varying cable length

Nomenclature	
$w(x,t)$	the transverse displacement
$u(x,t)$	the longitudinal displacement
$l(t)$	the length of the hoisting cable
$v = \dot{l}(t)$	the longitudinal velocity of the hoisting cable
$\hat{a} = \ddot{l}(t)$	the longitudinal acceleration of the hoisting cable
ρ	the linear density of the hoisting cable
m	the mass of the hoisting conveyance
EA	the longitudinal stiffness, E Young's elasticity modulus, A the cross-sectional area
$T(x,t)$	the spatiotemporally varying tension in hoisting cable
c_1, c_2	transverse and longitudinal viscous damping coefficients in hoisting cable
g	the standard gravity
E_{gs}	initial gravitational potential energy
c_u	longitudinal viscous damping coefficient in hoisting conveyance
$e_w(t), e_u(t)$	the transverse and longitudinal fundamental excitations at the top of the hoisting cable $e_w(t) = \beta_1 cos(\omega_1 t + \alpha)$, $e_u(t) = \beta_2 cos(\omega_2 t)$
β_1, β_2	the amplitudes of the transverse and longitudinal fundamental excitations
α	primary phase of the transverse fundamental excitation

proposed for the coupled transverse and longitudinal vibrations of the moving cable system until now.

The remaining part of this paper is organized as follows. In Sect. 2, the problem is formulated. In Sect. 3, the problem is reformulated from a partial differential equations formulation to an ordinary differential equations formulation by using the method of separation of variables. Many resonance manifolds for the transverse and longitudinal motions are detected by an inner layer analysis. In Sect. 4, approximate solutions are constructed analytically for the transverse and longitudinal motions by using a three-timescales perturbation method. In Sect. 5, some numerical approximations are presented by using a central finite difference scheme to validate the theoretical results from Sect. 4. Finally, in the last section we draw some conclusions.

2 Formulation of the physical system

2.1 Modelling of the problem

In this section, the mathematical model of the hoisting system is described and the equations for the transverse and the longitudinal motions of the system are derived and explained. By using Hamilton's principle [20], the mathematical problem for the vibrating cable (Fig. 1) can be written as an initial boundary value problem for the transverse vibration (see also Appendix A):

$$\begin{cases} \rho(w_{tt} + 2vw_{xt} + v^2 w_{xx} + \hat{a}w_x) - (Tw_x)_x \\ \qquad + c_1(w_t + vw_x) - EA(zw_x)_x = 0, \\ \qquad \beta_2\cos(\omega_2 t) < x < l(t),\ t > 0, \\ w(l(t),t) = 0, \quad w(\beta_2\cos(\omega_2 t), t) \\ \qquad = \beta_1\cos(\omega_1 t + \alpha), \quad t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \\ \qquad \beta_2 < x < l_0, \end{cases} \tag{1}$$

and as an initial boundary value problem for the longitudinal vibration:

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2 u_{xx} + \hat{a}u_x + \hat{a}) + c_2(u_t + vu_x) \\ \qquad -EAz_x = 0, \\ \qquad \beta_2\cos(\omega_2 t) < x < l(t),\ t > 0, \\ [m(u_{tt} + 2vu_{xt} + v^2 u_{xx} + \hat{a}u_x + \hat{a}) \\ \qquad + c_u(u_t + vu_x) + EAz]|_{x=l(t)} = 0, \\ u(\beta_2\cos(\omega_2 t), t) = \beta_2\cos(\omega_2 t), \quad t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \beta_2 < x < l_0, \end{cases} \tag{2}$$

where $z = u_x + \frac{1}{2}w_x^2$ and

$$T(x,t) = [m + \rho(l(t) - x)]g, \quad \beta_2\cos(\omega_2 t) \le x \le l(t). \tag{3}$$

In this paper, we use the following assumptions for the parameters and functions:

- The longitudinal velocity v is small compared to the wave velocities $\sqrt{\frac{EA}{\rho}}$ and $\sqrt{\frac{mg}{\rho}}$, that is, $v = \varepsilon v_0$;
- The nominal wave velocities $\sqrt{\frac{EA}{\rho}}$ and $\sqrt{\frac{mg}{\rho}}$ are of the same order of magnitude, that is, $\frac{EA}{mg} = O(1)$, $\sqrt{\frac{EA}{mg}} > 1$, and $\frac{EA}{mg}$ is not near 1, i.e. $\sqrt{\frac{EA}{mg}} - 1 >> O(\varepsilon)$;
- The cable mass ρL is small compared to the car mass m (L is the maximum length of the cable), that is, $\mu = \frac{\rho L}{m} = \varepsilon\mu_0$;
- The viscous damping parameters c_1, c_2, and c_u are small, that is, $c_1 = \varepsilon c_{1,0}$, $c_2 = \varepsilon c_{2,0}$, $c_u = \varepsilon c_{u,0}$;
- The fundamental excitations at the top of the hoisting rope are small, and the longitudinal excitation is smaller than the transverse excitation, that is, $\beta_1 = \varepsilon\beta_{1,0}$, $\beta_2 = \varepsilon^2\beta_{2,0}$;
- The initial conditions $w_0(x) = O(\varepsilon)$, $w_1(x) = O(\varepsilon)$, $u_0(x) = O(\varepsilon^2)$ and $u_1(x) = O(\varepsilon^2)$;
- For convenience, we only consider a non-accelerating cable, that is, the cable length $l(t) = l_0 + vt$ and $\hat{a} = 0$, where l_0 is the initial string length.

In the above assumptions, v_0, μ_0, $c_{1,0}$, $c_{2,0}$, $c_{u,0}$, $\beta_{1,0}$, $\beta_{2,0}$, α m, ρ, ω_1, ω_2, L, and l_0 are positive constants and are of order 1, and ε is a small parameter with $0 < \varepsilon << 1$.

To put Eq. (1) and (2) into non-dimensional forms, the following dimensionless variables and parameters are used:

$$w^* = \frac{w}{L},\ u^* = \frac{u}{L},\ x^* = \frac{x}{L},$$
$$t^* = \frac{t}{L}\sqrt{\frac{mg}{\rho}},\ v^* = v\sqrt{\frac{\rho}{mg}},$$
$$\beta_1^* = \frac{\beta_1}{L},\ \beta_2^* = \frac{\beta_2}{L},$$
$$c_1^* = c_1\frac{L}{\sqrt{mg\rho}},\ c_u^* = c_u\frac{L}{m}\sqrt{\frac{\rho}{mg}},$$
$$\omega_1^* = L\omega_1\sqrt{\frac{\rho}{mg}},\ u_0^* = \frac{u_0}{L},\ u_1^* = \sqrt{\frac{\rho}{mg}}u_1,$$
$$l^* = \frac{l}{L},\ \mu = \frac{\rho L}{m},\ c_2^* = c_2\frac{L}{\sqrt{mg\rho}},$$
$$\omega_2^* = L\omega_2\sqrt{\frac{\rho}{mg}},\ w_0^* = \frac{w_0}{L},\ w_1^* = \sqrt{\frac{\rho}{mg}}w_1.$$

The initial boundary value problem for the transverse motion in non-dimensional form becomes:

$$\begin{cases} w_{tt} + 2vw_{xt} + v^2w_{xx} - w_{xx} - \mu(l(t) - x)w_{xx} \\ \quad +\mu w_x + c_1(w_t + vw_x) \\ \quad -\frac{EA}{mg}(zw_x)_x = 0, \quad \beta_2\cos(\omega_2 t) \\ \quad < x < l(t),\ t > 0, \\ w(l(t), t) = 0, \quad w(\beta_2\cos(\omega_2 t), t) \\ \quad = \beta_1\cos(\omega_1 t + \alpha), \quad t \ge 0, \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \\ \quad \beta_2 < x < l_0, \end{cases} \tag{4}$$

and the initial boundary value problem for the longitudinal motion in non-dimensional form becomes:

$$\begin{cases} u_{tt} + 2vu_{xt} + v^2u_{xx} + c_2(u_t + vu_x) - \frac{EA}{mg}u_{xx} \\ \quad -\frac{EA}{mg}(\frac{1}{2}w_x^2)_x = 0, \\ \quad \beta_2\cos(\omega_2 t) < x < l(t),\ t > 0, \\ [u_{tt} + 2vu_{xt} + v^2u_{xx} + c_u(u_t + vu_x) \\ \quad +\frac{\mu EA}{mg}z]|_{x=l(t)} = 0, \\ u(\beta_2\cos(\omega_2 t), t) = \beta_2\cos(\omega_2 t), \quad t \ge 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \\ \quad \beta_2 < x < l_0, \end{cases} \tag{5}$$

where the asterisks (indicating the dimensionless variables and parameters) are omitted in the problems (4) and (5) for convenience.

2.2 Transformations of the problem

In order to convert the time-varying spatial domain $[\beta_2 cos(\omega_2 t), l(t)]$ for x to a fixed domain [0,1] for ξ, a new independent spatial coordinate $\xi = \frac{x - \beta_2 cos(\omega_2 t)}{h(t)}$, where $h(t) = l(t) - \beta_2 cos(\omega_2 t)$, is introduced. After this spatial transformation, new dependent variables $\bar{w}(\xi, t) = w(x, t)$, $\bar{u}(\xi, t) = u(x, t)$, and all the partial derivatives have to be rewritten as follows:

$$\xi_t = -\frac{v\xi}{h(t)} + \beta_2\frac{\omega_2(1-\xi)sin(\omega_2 t)}{h(t)},$$
$$\xi_{tt} = \frac{v^2\xi}{h^2(t)} + \beta_2\left[\frac{\omega_2^2(1-\xi)cos(\omega_2 t)}{h(t)} - \frac{v\omega_2(1-2\xi)sin(\omega_2 t)}{h^2(t)}\right] - \beta_2^2\frac{\omega_2^2(1-\xi)sin^2(\omega_2 t)}{h^2(t)},$$

$$w_t = \bar{w}_\xi \xi_t + \bar{w}_t, \quad w_{tt} = \bar{w}_{\xi\xi}(\xi_t)^2 + 2\bar{w}_{\xi t}\xi_t + \bar{w}_\xi \xi_{tt} + \bar{w}_{tt},$$
$$w_x = \frac{1}{h(t)}\bar{w}_\xi, \quad w_{xx} = \frac{1}{h^2(t)}\bar{w}_{\xi\xi},$$
$$u_t = \bar{u}_\xi \xi_t + \bar{u}_t, \quad u_{tt} = \bar{u}_{\xi\xi}(\xi_t)^2 + 2\bar{u}_{\xi t}\xi_t + \bar{u}_\xi \xi_{tt} + \bar{u}_{tt},$$
$$u_x = \frac{1}{h(t)}\bar{u}_\xi, \quad u_{xx} = \frac{1}{h^2(t)}\bar{u}_{\xi\xi}.$$

Then, the equation for the transverse motion becomes:

$$\begin{cases} \bar{w}_{tt} + \frac{2v}{h(t)}\bar{w}_{\xi t} - \frac{1}{h^2(t)}\bar{w}_{\xi\xi} - \frac{\mu}{h(t)}(1-\xi)\bar{w}_{\xi\xi} \\ \quad + \frac{\mu}{h(t)}\bar{w}_\xi + c_1\bar{w}_t - \frac{EA}{mgh^3(t)}(\bar{u}_\xi \bar{w}_\xi)_\xi \\ \quad - \frac{EA}{mgh^4(t)}(\frac{1}{2}\bar{w}_\xi^3)_\xi - \frac{2v\xi}{h(t)}\bar{w}_{\xi t} = O(\varepsilon^2\bar{w}), \\ \quad 0 < \xi < 1, \ t > 0, \\ \bar{w}(1,t) = 0, \quad \bar{w}(0,t) = \beta_1\cos(\omega_1 t + \alpha), \quad t \geq 0, \\ \bar{w}(\xi,0) = \bar{w}_0(\xi), \quad \bar{w}_t(\xi,0) = \bar{w}_1(\xi), \quad 0 < \xi < 1, \end{cases} \tag{6}$$

where $\bar{w}_0(\xi) = w_0(\xi l_0 + \beta_2(1-\xi))$ and $\bar{w}_1(\xi) = w_1(\xi l_0 + \beta_2(1-\xi))$. It should be observed that the order of the term $-\frac{EA}{mgh^3(t)}(\bar{u}_\xi \bar{w}_\xi)_\xi$ in (6) is unknown a priori due to possibly occurring resonances. So, we keep this term explicitly in the equation, and analyse it later. The equation for the longitudinal motion then becomes:

$$\begin{cases} \bar{u}_{tt} + \frac{2v}{h(t)}\bar{u}_{\xi t} + c_2\bar{u}_t - \frac{EA}{mgh^2(t)}\bar{u}_{\xi\xi} - \frac{EA}{mgh^3(t)}\bar{w}_\xi \bar{w}_{\xi\xi} \\ \quad - \frac{2v\xi}{h(t)}\bar{u}_{\xi t} = O(\varepsilon^2\bar{u}), \quad 0 < \xi < 1, \ t > 0, \\ [\bar{u}_{tt} + c_u\bar{u}_t + \frac{\mu EA}{mgh(t)}\bar{u}_\xi + \frac{\mu EA}{2mgh^2(t)}\bar{w}_\xi^2]|_{\xi=1} \\ \quad = O(\varepsilon^2\bar{u}), \\ \bar{u}(0,t) = \beta_2\cos(\omega_2 t), \quad t \geq 0, \\ \bar{u}(\xi,0) = \bar{u}_0(\xi), \quad \bar{u}_t(\xi,0) = \bar{u}_1(\xi), \quad 0 < \xi < 1, \end{cases} \tag{7}$$

where $\bar{u}_0(\xi) = u_0(\xi l_0 + \beta_2(1-\xi))$ and $\bar{u}_1(\xi) = u_1(\xi l_0 + \beta_2(1-\xi))$. The orders of the term $-\frac{EA}{mgh^3(t)}\bar{w}_\xi\bar{w}_{\xi\xi}$ and the term $\frac{\mu EA}{2mgh^2(t)}\bar{w}_\xi^2(1,t)$ in (7) are unknown a priori due to possible resonances. So, we keep these terms in the equation, and analyse them later.

In order to eliminate the time-variable coefficients in $\frac{1}{h^2(t)}\bar{w}_{\xi\xi}$ and in $\frac{EA}{mgh^2(t)}\bar{u}_{\xi\xi}$ in the initial boundary problems (6) and (7), the Liouville–Green transformation (see also the WKBJ method [25] [26]) is introduced with $\frac{ds}{dt} = \frac{1}{l(t)}$. In accordance with a new time variable s, all the partial derivatives have to be rewritten as follows:

$$s = \frac{1}{\varepsilon v_0} ln\left(\frac{l(t)}{l_0}\right), \quad l(t) = \hat{l}(s) = l_0 e^{\varepsilon v_0 s},$$
$$\chi(s) = \frac{l_0(e^{\varepsilon v_0 s} - 1)}{\varepsilon v_0},$$
$$\bar{w}_t = \frac{1}{\hat{l}}\tilde{w}_s, \quad \bar{w}_{\xi t} = \frac{1}{\hat{l}}\tilde{w}_{\xi s}, \quad \bar{w}_{tt} = \frac{1}{\hat{l}^2}\tilde{w}_{ss} - \frac{v}{\hat{l}^2}\tilde{w}_s,$$
$$\bar{u}_t = \frac{1}{\hat{l}}\tilde{u}_s, \quad \bar{u}_{\xi t} = \frac{1}{\hat{l}}\tilde{u}_{\xi s}, \quad \bar{u}_{tt} = \frac{1}{\hat{l}^2}\tilde{u}_{ss} - \frac{v}{\hat{l}^2}\tilde{u}_s.$$

Substituting these derivatives into the problem (6), the initial boundary value problem for the transverse motion becomes:

$$\begin{cases} \tilde{w}_{ss} - \tilde{w}_{\xi\xi} = v\tilde{w}_s - 2v\tilde{w}_{\xi s} - c_1\hat{l}\tilde{w}_s - \mu\hat{l}\tilde{w}_\xi \\ \quad + \mu\hat{l}(1-\xi)\tilde{w}_{\xi\xi} + \frac{EA}{mg\hat{l}}(\tilde{u}_\xi\tilde{w}_\xi)_\xi \\ \quad + \frac{EA}{mg\hat{l}^2}(\frac{1}{2}\tilde{w}_\xi^3)_\xi + 2v\xi\tilde{w}_{\xi s} + O(\varepsilon^2\tilde{w}), \\ \quad 0 < \xi < 1, \ s > 0, \\ \tilde{w}(1,s) = 0, \quad s \geq 0, \\ \tilde{w}(0,s) = \beta_1 cos(\omega_1\chi(s) + \alpha), \quad s \geq 0, \\ \tilde{w}(\xi,0) = \bar{w}_0(\xi), \quad \tilde{w}_s(\xi,0) = l_0\bar{w}_1(\xi), \\ \quad 0 < \xi < 1. \end{cases} \tag{8}$$

The initial boundary value problem for the longitudinal motion becomes:

$$\begin{cases} \tilde{u}_{ss} - \frac{EA}{mg}\tilde{u}_{\xi\xi} = v\tilde{u}_s - 2v\tilde{u}_{\xi s} - c_2\hat{l}\tilde{u}_s + \frac{EA}{mg\hat{l}}\tilde{w}_\xi\tilde{w}_{\xi\xi} \\ \quad + 2v\xi\tilde{u}_{\xi s} + O(\varepsilon^2\tilde{u}), \\ \quad 0 < \xi < 1, \ s > 0, \\ \tilde{u}_{ss}(1,s) = [-\frac{\mu EA\hat{l}}{mg}\tilde{u}_\xi + v\tilde{u}_s - c_u\hat{l}\tilde{u}_s \\ \quad - \frac{\mu EA}{2mg}\tilde{w}_\xi^2]\,|_{\xi=1} + O(\varepsilon^2\tilde{u}), \\ \tilde{u}(0,s) = \beta_2 cos(\omega_2\chi(s)), \quad s \geq 0, \\ \tilde{u}(\xi,0) = \bar{u}_0(\xi), \quad \tilde{u}_s(\xi,0) = l_0\bar{u}_1(\xi), \quad 0 < \xi < 1. \end{cases} \tag{9}$$

The initial boundary value problem (9) can further be rewritten as

$$\begin{cases} \tilde{u}_{ss} - \frac{EA}{mg}\tilde{u}_{\xi\xi} = v\tilde{u}_s - 2v\tilde{u}_{\xi s} - c_2\hat{l}\tilde{u}_s + \frac{EA}{mg\hat{l}}\tilde{w}_\xi\tilde{w}_{\xi\xi} \\ \quad + 2v\xi\tilde{u}_{\xi s} + O(\varepsilon^2\tilde{u}), \\ \quad 0 < \xi < 1, \ s > 0, \\ \tilde{u}_{\xi\xi}(1,s) = [-\mu\hat{l}\tilde{u}_\xi + \frac{mg}{EA}(c_2 - c_u)\hat{l}\tilde{u}_s - \frac{\mu}{2}\tilde{w}_\xi^2 \\ \quad - \frac{1}{\hat{l}}\tilde{w}_\xi\tilde{w}_{\xi\xi}]\,|_{\xi=1} + O(\varepsilon^2\tilde{u}), \\ \tilde{u}(0,s) = \beta_2 cos(\omega_2\chi(s)), \quad s \geq 0, \\ \tilde{u}(\xi,0) = \bar{u}_0(\xi), \quad \tilde{u}_s(\xi,0) = l_0\bar{u}_1(\xi), \\ \quad 0 < \xi < 1. \end{cases} \tag{10}$$

In order to eliminate the non-homogenous terms in the boundary conditions in (8) and in (10), the following transformations are used:

$$\tilde{w}(\xi,s)=\beta_1(1-\xi)\cos(\omega_1\chi(s)+\alpha)+\hat{w}(\xi,s), \tag{11}$$

$$\begin{aligned}\tilde{u}(\xi,s)=\hat{u}(\xi,s)&+\frac{\xi^2}{2}\left[-\mu\hat{l}\hat{u}_\xi+\frac{mg}{EA}(c_2-c_u)\hat{l}\hat{u}_s-\frac{\mu}{2}\hat{w}_\xi^2\right]|_{\xi=1}\\&+\frac{\xi^2}{2}\left[-\frac{1}{\hat{l}}\hat{w}_\xi\hat{w}_{\xi\xi}\right]|_{\xi=1}\\&+\frac{\xi^2}{4}\mu[\hat{w}_{\xi\xi}^2+\hat{w}_\xi\hat{w}_{\xi\xi\xi}]\,|_{\xi=1}\\&-\frac{\xi^2}{4}\frac{mg}{EA}(c_2-c_u)[\hat{w}_{\xi s}\hat{w}_{\xi\xi}+\hat{w}_\xi\hat{w}_{\xi\xi s}]\,|_{\xi=1}\\&+\frac{\xi^2\mu\beta_1}{2}\cos(\omega_1\chi(s)+\alpha)\hat{w}_\xi(1,s)\\&+\frac{\xi^2\beta_1}{2\hat{l}}\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}(1,s)\\&-\frac{\xi^2}{4}\frac{mg}{EA}(c_2-c_u)\beta_1\\&[\omega_1\hat{l}sin(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}(1,s)\\&-\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi s}(1,s)]\\&-\frac{\xi^2\mu\beta_1}{4}\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi\xi}(1,s)\\&+\beta_2\cos(\omega_2\chi(s)),\end{aligned} \tag{12}$$

Then, the initial boundary value problem in the transverse direction becomes:

$$\begin{cases}\hat{w}_{ss}-\hat{w}_{\xi\xi}=v\hat{w}_s-2v\hat{w}_{\xi s}-c_1\hat{l}\hat{w}_s-\mu\hat{l}\hat{w}_\xi\\\qquad+\mu\hat{l}(1-\xi)\hat{w}_{\xi\xi}+\frac{EA}{mg\hat{l}}(\hat{u}_\xi\hat{w}_\xi)_\xi\\\qquad+\frac{EA}{mg\hat{l}^2}(\frac{1}{2}\hat{w}_\xi^3)_\xi+2v\xi\hat{w}_{\xi s}\\\qquad+\beta_1(1-\xi)\omega_1^2\hat{l}^2\cos(\omega_2\chi(s)+\alpha)\\\qquad-\frac{EA}{mg\hat{l}^2}\hat{w}_\xi(1,s)\hat{w}_{\xi\xi}(1,s)\\\qquad\times[\hat{w}_\xi+\xi\hat{w}_{\xi\xi}]+h.o.t.,\\\qquad\quad 0<\xi<1,\ s>0,\\\hat{w}(1,s)=0,\quad\hat{w}(0,s)=0,\quad s\geq 0,\\\hat{w}(\xi,0)=\tilde{w}(\xi,0)-\beta_1(1-\xi)\cos(\alpha),\\\hat{w}_s(\xi,0)=\tilde{w}_s(\xi,0)+\beta_1\omega_1 l_0(1-\xi)\sin(\alpha),\\\qquad\quad 0<\xi<1,\end{cases} \tag{13}$$

where according to the initial condition assumptions, the terms in "h.o.t.", consisting of $O(\varepsilon\hat{u})$, $O(\varepsilon^2\hat{w})$, $O(\varepsilon\hat{w}^2)$, $O(\varepsilon\hat{u}\hat{w})$ and $O(\varepsilon\hat{w}^3)$, cannot influence the lowest order of the solution of problem (13) on timescales of $O(\frac{1}{\varepsilon})$.
The initial boundary value problem in the longitudinal direction then becomes:

$$\begin{cases}\hat{u}_{ss}-\frac{EA}{mg}\hat{u}_{\xi\xi}=v\hat{u}_s-2v\hat{u}_{\xi s}-c_2\hat{l}\hat{u}_s+2v\xi\hat{u}_{\xi s}\\\qquad+(c_2-c_u)\hat{l}\hat{u}_s(1,s)\\\qquad-\frac{EA}{mg}\mu\hat{l}\hat{u}_\xi(1,s)+\frac{\mu\hat{l}\xi^2}{2}\frac{EA}{mg}\hat{u}_{\xi\xi\xi}(1,s)\\\qquad-\frac{\xi^2}{2}(c_2-c_u)\hat{l}\hat{u}_{s\xi\xi}(1,s)+O(\varepsilon^2\hat{u})\\\qquad-\frac{\xi^2}{2\hat{l}}[\hat{w}_{\xi\xi\xi}(1,s)\hat{w}_{\xi\xi}(1,s)\\\qquad+\hat{w}_\xi\hat{w}_{\xi\xi\xi\xi}(1,s)]\\\qquad-\frac{\xi^2}{\hat{l}}\hat{w}_{\xi s}(1,s)\hat{w}_{\xi\xi s}(1,s)+\frac{EA}{mg\hat{l}}\hat{w}_\xi\hat{w}_{\xi\xi}\\\qquad-\frac{EA}{mg\hat{l}}\hat{w}_\xi(1,s)\hat{w}_{\xi\xi}(1,s)+O(\varepsilon\hat{w}^2)\\\qquad-\frac{EA}{mg\hat{l}}\beta_1\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}\\\qquad+\frac{EA}{mg\hat{l}}\beta_1\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}(1,s)\\\qquad+\frac{\xi^2}{2\hat{l}}\frac{EA}{mg}\beta_1\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi\xi\xi}(1,s)\\\qquad-\xi^2\beta_1\omega_1\sin(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi s}(1,s)\\\qquad+O(\varepsilon^2\hat{w})+\beta_2\omega_2^2\hat{l}^2\cos(\omega_2\chi(s))\\\qquad+O(\varepsilon^3),\quad 0<\xi<1,\ s>0,\\\hat{u}_{\xi\xi}(1,s)=O(\varepsilon^2\hat{u}),\quad\hat{u}(0,s)=0,\quad s\geq 0,\\\hat{u}(\xi,0)=\tilde{u}(\xi,0)-\frac{\xi^2}{2}[-\mu\hat{l}\tilde{u}_\xi(1,0)\\\qquad+\frac{mg}{EA}(c_2-c_u)\hat{l}\tilde{u}_s(1,0)\\\qquad-\frac{\mu}{2}\tilde{w}_\xi^2(1,0)-\frac{1}{\hat{l}}\tilde{w}_\xi(1,0)\tilde{w}_{\xi\xi}(1,0)]\\\qquad+O(\varepsilon^3),\\\hat{u}_s(\xi,0)=\tilde{u}_s(\xi,0)-\frac{\xi^2}{2}[-\mu\hat{l}\tilde{u}_{\xi s}(1,0)\\\qquad+\frac{mg}{EA}(c_2-c_u)\hat{l}\tilde{u}_{\xi\xi}(1,0)\\\qquad-\mu\tilde{w}_\xi(1,0)\tilde{w}_{\xi s}(1,0)-\frac{1}{\hat{l}}\tilde{w}_{\xi s}(1,0)\tilde{w}_{\xi\xi}(1,0)\\\qquad-\frac{1}{\hat{l}}\tilde{w}_\xi(1,0)\tilde{w}_{\xi\xi s}(1,0)]+O(\varepsilon^3),\quad 0<\xi<1,\end{cases} \tag{14}$$

where according to the initial condition assumptions, the terms in $O(\varepsilon\hat{w}^2)$, $O(\varepsilon^2\hat{u})$, $O(\varepsilon^2\hat{w})$, and $O(\varepsilon^3)$ cannot influence the lowest order of the solution $\hat{u}(\xi,s)$ in problem (14) on timescales of $O(\frac{1}{\varepsilon})$, so they can be neglected in the further analysis. In the following sections, the solutions of $\hat{w}(\xi,s)$, $\hat{u}(\xi,s)$ in problem (13) and (14) will be approximated by using an interior layer analysis and a three-timescales perturbation method.

3 Inner layer analysis

It will be shown that an interior layer analysis (including a rescaling and balancing procedure) leads to a description of an (un-)expected resonance manifold and leads to timescales which describe the solutions of the partial differential equations (13) and (14) sufficiently accurately. To derive the solutions $\hat{w}(\xi,s)$ and $\hat{u}(\xi,s)$ in problem (13) and (14), firstly the method of separation of variables is employed. In accordance with the method of separation of variables, the general solution of the transverse problem (13) can be expanded in the following form:

$$\hat{w}(\xi, s) = \sum_{n=1}^{\infty} T_n(s) \sin(n\pi\xi), \tag{15}$$

and the general solution of the longitudinal problem (14) can be expanded in the following form:

$$\hat{u}(\xi, s) = \sum_{n=1}^{\infty} Y_n(s) \sin(n\pi\xi). \tag{16}$$

Substituting (15) into the initial boundary value problem (13), and substituting (16) into problem (14), further by multiplying the obtained equations with $\sin(k\pi\xi)$, and by integrating with respect to ξ from $\xi = 0$ to $\xi = 1$, and by using the orthogonality properties of the sin-functions on $0 < \xi < 1$, we obtain the following ordinary differential equations for $T_k(s)$ (with $k = 1, 2, 3, ...$) in the transverse direction:

$$T_{k,ss} + k^2\pi^2 T_k = \hat{\chi}(s), \tag{17}$$

where

$$\begin{aligned}
\hat{\chi}(s) &= \varepsilon v_0 T_{k,s} - \varepsilon c_{1,0}\hat{l} T_{k,s} - \sum_{n=1}^{\infty} \varepsilon c^1_{n,k}\mu_0\hat{l}(n\pi)^2 T_n \\
&+ \sum_{n=1}^{\infty} c^2_{n,k}(-2v_0 n\pi T_{n,s} - \mu_0\hat{l} n\pi T_n) \\
&+ \varepsilon \sum_{n=1}^{\infty} c^3_{n,k} 2v_0 n\pi T_{n,s} \\
&+ \frac{EA\pi^3}{2mg\hat{l}}\left[\sum_{p=k+1}^{\infty} kp(k-p)Y_p T_{p-k} - \sum_{p=1}^{k-1} kp(k-p)Y_p T_{k-p}\right] \\
&- \frac{EA\pi^3}{2mg\hat{l}} \sum_{p=1}^{\infty} kp(k+p)Y_p T_{k+p} \\
&+ \frac{3EA}{8mg\hat{l}^2} \sum_{n+p=k+1}^{n+p=\infty} np\pi^3(k-n-p)T_n T_p T_{n+p-k} \\
&+ \frac{3EA}{8mg\hat{l}^2}\left[\sum_{n-p=-\infty}^{n-p=k-1} np\pi^3(k-n+p)T_n T_p T_{k-n+p} + \sum_{n,p=1}^{\infty} np\pi^3(k+n+p)T_n T_p T_{k+n+p}\right] \\
&+ \frac{3EA}{8mg\hat{l}^2}\left[\sum_{p-n=-\infty}^{p-n=k-1} np\pi^3(k+n-p)T_n T_p T_{k+n-p} - \sum_{p-n=k+1}^{p-n=\infty} np\pi^3(p-k-n)T_n T_p T_{p-k-n}\right] \\
&+ \frac{3EA}{8mg\hat{l}^2}\Big[\sum_{n+p=2}^{n+p=k-1} np\pi^3(k-n-p)T_n T_p T_{k-n-p} + \sum_{n-p=k+1}^{n-p=\infty} np\pi^3(k-n+p)T_n T_p T_{n-p-k}\Big] \\
&+ \varepsilon\beta_{2,0} d_k \omega_2^2 \hat{l}^2 \cos(\omega_2\chi(s) + \alpha) + h.o.t.,
\end{aligned}$$

$$\begin{aligned}
T_k(0) &= \frac{\int_0^1 \hat{w}_0(\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin(k\pi\xi)\sin(k\pi\xi)d\xi} = F_k, \\
T_{k,s}(0) &= \frac{\int_0^1 \hat{w}_1(\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin(k\pi\xi)\sin(k\pi\xi)d\xi} = G_k,
\end{aligned} \tag{18}$$

where $F_k = O(\varepsilon)$ and $G_k = O(\varepsilon)$. $c^1_{n,k}$, $c^2_{n,k}$, $c^3_{n,k}$ and d_k are given by:

$$\begin{aligned}
c^1_{n,k} &= \frac{\int_0^1 (1-\xi)\sin(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}, \\
c^2_{n,k} &= \frac{\int_0^1 \cos(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}, \\
c^3_{n,k} &= \frac{\int_0^1 \xi\cos(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}, \\
d_k &= \frac{\int_0^1 (1-\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}.
\end{aligned} \tag{19}$$

Further, the differential Eq. (17) can be written as:

$$\begin{aligned}
T_{k,ss} + k^2\pi^2 T_k
&= \varepsilon\Big[v_0 T_{k,s} - c_{1,0}\hat{l} T_{k,s} - \sum_{n=1}^{\infty} c^1_{n,k}\mu_0\hat{l}(n\pi)^2 T_n \\
&+ \sum_{n=1}^{\infty} c^2_{n,k}(-2v_0 n\pi T_{n,s} - \mu_0\hat{l} n\pi T_n) \\
&+ \sum_{n=1}^{\infty} c^3_{n,k} 2v_0 n\pi T_{n,s} \\
&+ \frac{EA\pi^3}{2\varepsilon mg\hat{l}} \sum_{p=k+1}^{\infty} kp(k-p)Y_p T_{p-k} \\
&- \frac{EA\pi^3}{2\varepsilon mg\hat{l}} \sum_{p=1}^{k-1} kp(k-p)Y_p T_{k-p} \\
&- \frac{EA\pi^3}{2\varepsilon mg\hat{l}} \sum_{p=1}^{\infty} kp(k+p)Y_p T_{k+p} \\
&+ \beta_{1,0} d_k \omega_1^2 \hat{l}^2 \cos(\omega_1\chi(s) + \alpha)\Big] + h.o.t.,
\end{aligned} \tag{20}$$

where $T_k(0)$ and $T_{k,s}(0)$ are given by (18), $c^1_{n,k}$, $c^2_{n,k}$, $c^3_{n,k}$ and d_k are given by (19). Note that the term "h.o.t."(including $T_nT_pT_j$ in (17)) cannot influence the lowest order of the solution of the differential Eq. (20) on timescales of $O(\frac{1}{\varepsilon})$. This can be seen as follows. When the addition or subtraction of the three subscripts in $T_nT_pT_j$ equals to k or $-k$, then for the given initial conditions of $O(\varepsilon)$, $T_nT_pT_j$ leads to $O(\varepsilon^2)$ contributions in the solution of the differential Eq. (20) on timescales of $O(\frac{1}{\varepsilon})$; otherwise, $T_nT_pT_j$ will lead to contributions of $O(\varepsilon^3)$ in the solution of the differential Eq. (20) on timescales of $O(\frac{1}{\varepsilon})$.

Similarly, we obtain the following differential equations for Y_k (with $k = 1, 2, 3, ...$) in the longitudinal direction:

$$\begin{aligned}
Y_{k,ss} &+ \frac{EA}{mg}k^2\pi^2 Y_k \\
&= \varepsilon(v_0 - c_{2,0}\hat{l})Y_{k,s} + \sum_{n=1}^{\infty} 2\varepsilon v_0 n\pi d^1_{n,k} Y_{n,s} \\
&\quad - \sum_{n=1}^{\infty} \frac{EA}{mg}\mu\hat{l}n\pi d^3_{n,k} Y_n \\
&\quad + \sum_{j=1}^{k-1} d^4_{k-j,j} T_{k-j}T_j + \sum_{j=k+1}^{\infty} d^4_{j-k,j} T_{j-k}T_j \\
&\quad - \sum_{j=1}^{\infty} d^4_{j+k,j} T_{j+k}T_j \\
&\quad + \beta_1 \tilde{d}^4_{k,k} T_k \cos(\omega_1\chi(s) + \alpha) \\
&\quad + \beta_2\omega_2^2\hat{l}^2 d_{1,k}\cos(\omega_2\chi(s)) + h.o.t., \\
Y_k(0) &= \frac{\int_0^1 \hat{u}_0(\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin(k\pi\xi)sin(k\pi\xi)d\xi} = f_k, \\
Y_{k,s}(0) &= \frac{\int_0^1 \hat{u}_1(\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin(k\pi\xi)sin(k\pi\xi)d\xi} = g_k, \qquad (21)
\end{aligned}$$

where $f_k = O(\varepsilon^2)$ and $g_k = O(\varepsilon^2)$. $d^1_{n,k}$, $d^3_{n,k}$, $d^4_{n,j}$, $d_{1,k}$, $\tilde{d}^4_{k,k}$ are given by:

$$\begin{aligned}
d^1_{n,k} &= \frac{\int_0^1(\xi - 1)\cos(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}, \\
d^3_{n,k} &= \frac{\int_0^1(1 + \frac{(n\pi\xi)^2}{2})\cos(n\pi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}, \\
d^4_{n,j} &= \frac{EAnj^2\pi^3}{2mg\hat{l}}, \quad d_{1,k} = \frac{\int_0^1 \sin(k\pi\xi)d\xi}{\int_0^1 \sin^2(k\pi\xi)d\xi}, \\
\tilde{d}^4_{k,k} &= \frac{EA}{mg\hat{l}}k^2\pi^2. \qquad (22)
\end{aligned}$$

Before approximately solving the ordinary differential Eqs. (20) and (21), according to an inner layer analysis process (see also [16]), we can make the following remarks beforehand. For the given initial conditions for Y_k (which are of $O(\varepsilon^2)$), the terms in the right-hand side of Eq. (21) can lead to different contributions in the solution Y_k on timescales of $O(\frac{1}{\varepsilon})$. The first three terms in the right-hand side of Eq. (21) only lead to contributions of $O(\varepsilon^2)$. The coupled, nonlinear terms including T_pT_j can lead to contributions up to $O(\frac{1}{\varepsilon}T_pT_j)$, and the term with frequency ω_1 lead to contributions up to $O(\sqrt{\varepsilon}T_k)$. The term with frequency ω_2 can lead to contributions up to $O(\varepsilon\sqrt{\varepsilon})$. Since T_k may increase from the initial state order of $O(\varepsilon)$ to lower orders, the orders of the terms including T_pT_j determine that of the solution Y_k. Therefore, the solution of Eq. (21) can be approximated as $Y_k = O(\frac{1}{\varepsilon}T_pT_j)$. Similarly, we obtain that for the given initial conditions for T_k (which are of $O(\varepsilon)$), terms in the right-hand side of Eq. (20) can also have different contributions to the solution T_k on timescales of $O(\frac{1}{\varepsilon})$. The first five terms in the right-hand side of Eq. (20) only can lead to contributions of $O(\varepsilon)$. Based on the fact that $Y_k = O(\frac{1}{\varepsilon}T_pT_j)$, terms including Y_pT_j can lead to contributions up to $O(\varepsilon)$, and the last term with frequency ω_1 can lead to contributions up to $O(\sqrt{\varepsilon})$. This implies that in Eq. (20) only the external forcing with frequency ω_1 produces resonance, and leads to a jump in the solution T_k from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Further, it follows from Eq. (21) that the coupled terms including T_pT_j produce maximum amplitude responses, and the amplitude responses depend on the solution T_k of Eq. 20).

After the above made observations, to obtain the (un-)expected resonance manifolds which describe the solutions of ordinary differential Eqs. (20) and (21) sufficiently accurately, the following standard transformations are introduced:

$$\begin{aligned}
T_k(s) &= A_{1,k}(s)\sin(k\pi s) + B_{1,k}(s)\cos(k\pi s), \\
T_{k,s}(s) &= k\pi A_{1,k}(s)\cos(k\pi s) - k\pi B_{1,k}(s)\sin(k\pi s), \\
Y_k(s) &= C_{1,k}(s)\sin(\lambda_k s) + D_{1,k}(s)\cos(\lambda_k s), \\
Y_{k,s}(s) &= \lambda_k C_{1,k}(s)\cos(\lambda_k s) - \lambda_k D_{1,k}(s)\sin(\lambda_k s),
\end{aligned}$$

where $\lambda_k = \sqrt{\frac{EA}{mg}}k\pi$. The transverse problem (20) can now be rewritten in the following form (where the dot $\cdot$ represents differentiation with respect to s):

$$\begin{aligned}
\dot{A}_{1,k} &= \varepsilon(v_0 - c_{1,0}\hat{l})A_{1,k}\cos^2(k\pi s) \\
&\quad - \varepsilon(v_0 - c_{1,0}\hat{l})B_{1,k}\sin(k\pi s)\cos(k\pi s)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon \sum_{n=1}^{\infty} \Bigg[\left(-c_{n,k}^{1} \mu_0 \hat{l} \frac{n^2 \pi}{k} \right) A_{1,n} \\
&+ \left(2c_{n,k}^{2} v_0 \frac{n^2 \pi}{k} \right) B_{1,n} \\
&- \left(c_{n,k}^{2} \mu_0 \hat{l} \frac{n}{k} \right) A_{1,n} \\
&- \left(2c_{n,k}^{3} v_0 \frac{n^2 \pi}{k} \right) B_{1,n} \Bigg] \sin(n\pi s) \cos(k\pi s) \\
&+\varepsilon \sum_{n=1}^{\infty} \Bigg[\left(-c_{n,k}^{1} \mu_0 \hat{l} \frac{n^2 \pi}{k} \right) B_{1,n} \\
&+ \left(2c_{n,k}^{2} v_0 \frac{n^2 \pi}{k} \right) A_{1,n} \\
&- \left(c_{n,k}^{2} \mu_0 \hat{l} \frac{n}{k} \right) B_{1,n} \\
&+ \left(2c_{n,k}^{3} v_0 \frac{n^2 \pi}{k} \right) A_{1,n} \Bigg] \cos(n\pi s) \cos(k\pi s) \\
&+ \frac{EA\pi^2}{2mg\hat{l}} \sum_{p=k+1}^{\infty} p(k-p) \\
&[C_{1,p} A_{1,p-k} \sin(\lambda_p s) \sin((p-k)\pi s) \\
&+ C_{1,p} B_{1,p-k} \sin(\lambda_p s) \cos((p-k)\pi s) \\
&+ D_{1,p} A_{1,p-k} \cos(\lambda_p s) \sin((p-k)\pi s) \\
&+ D_{1,p} B_{1,p-k} \cos(\lambda_p s) \cos((p-k)\pi s)] \cos(k\pi s) \\
&- \frac{EA\pi^2}{2mg\hat{l}} \sum_{p=1}^{k-1} p(k-p) \\
&[C_{1,p} A_{1,k-p} \sin(\lambda_p s) \sin((k-p)\pi s) \\
&+ C_{1,p} B_{1,k-p} \sin(\lambda_p s) \cos((k-p)\pi s) \\
&+ D_{1,p} A_{1,k-p} \cos(\lambda_p s) \sin((k-p)\pi s) \\
&+ D_{1,p} B_{1,k-p} \cos(\lambda_p s) \cos((k-p)\pi s)] \cos(k\pi s) \\
&- \frac{EA\pi^2}{2mg\hat{l}} \sum_{p=1}^{\infty} p(k+p) \\
&[C_{1,p} A_{1,p+k} \sin(\lambda_p s) \sin((p+k)\pi s) \\
&+ C_{1,p} B_{1,p+k} \sin(\lambda_p s) \cos((p+k)\pi s) \\
&+ D_{1,p} A_{1,p+k} \cos(\lambda_p s) \sin((p+k)\pi s) \\
&+ D_{1,p} B_{1,p+k} \cos(\lambda_p s) \cos((p+k)\pi s)] \cos(k\pi s) \\
&+ \beta_1 \frac{d_k \omega_1^2 \hat{l}^2}{2} [\cos(k\pi s + \omega_1 \chi(s) + \alpha) \\
&+ \cos(k\pi s - \omega_1 \chi(s) - \alpha)] \\
&+ h.o.t,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\dot{B}_{1,k} = &-\varepsilon(v_0 - c_{1,0}\hat{l}) A_{1,k} \cos(k\pi s) \sin(k\pi s) \\
&+ \varepsilon(v_0 - c_{1,0}\hat{l}) B_{1,k} \sin^2(k\pi s) \\
&-\varepsilon \sum_{n=1}^{\infty} \Bigg[\left(-c_{n,k}^{1} \mu_0 \hat{l} \frac{n^2 \pi}{k} \right) A_{1,n} \\
&+ \left(2c_{n,k}^{2} v_0 \frac{n^2 \pi}{k} \right) B_{1,n} \\
&- \left(c_{n,k}^{2} \mu_0 \hat{l} \frac{n}{k} \right) A_{1,n} \\
&- \left(2c_{n,k}^{3} v_0 \frac{n^2 \pi}{k} \right) B_{1,n} \Bigg] \sin(n\pi s) \sin(k\pi s) \\
&-\varepsilon \sum_{n=1}^{\infty} \Bigg[\left(-c_{n,k}^{1} \mu_0 \hat{l} \frac{n^2 \pi}{k} \right) B_{1,n} \\
&+ \left(2c_{n,k}^{2} v_0 \frac{n^2 \pi}{k} \right) A_{1,n} \\
&- \left(c_{n,k}^{2} \mu_0 \hat{l} \frac{n}{k} \right) B_{1,n} \\
&+ \left(2c_{n,k}^{3} v_0 \frac{n^2 \pi}{k} \right) B_{1,n} \Bigg] \cos(n\pi s) \sin(k\pi s) \\
&- \frac{EA\pi^2}{2mg\hat{l}} \sum_{p=k+1}^{\infty} p(k-p) \\
&[C_{1,p} A_{1,p-k} \sin(\lambda_p s) \sin((p-k)\pi s) \\
&+ C_{1,p} B_{1,p-k} \sin(\lambda_p s) \cos((p-k)\pi s) \\
&+ D_{1,p} A_{1,p-k} \cos(\lambda_p s) \sin((p-k)\pi s) \\
&+ D_{1,p} B_{1,p-k} \cos(\lambda_p s) \cos((p-k)\pi s)] \sin(k\pi s) \\
&+ \frac{EA\pi^2}{2mg\hat{l}} \sum_{p=1}^{k-1} p(k-p) \\
&[C_{1,p} A_{1,k-p} \sin(\lambda_p s) \sin((k-p)\pi s) \\
&+ C_{1,p} B_{1,k-p} \sin(\lambda_p s) \cos((k-p)\pi s) \\
&+ D_{1,p} A_{1,k-p} \cos(\lambda_p s) \sin((k-p)\pi s) \\
&+ D_{1,p} B_{1,k-p} \cos(\lambda_p s) \cos((k-p)\pi s)] \sin(k\pi s) \\
&+ \frac{EA\pi^2}{2mg\hat{l}} \sum_{p=1}^{\infty} p(k+p) \\
&[C_{1,p} A_{1,p+k} \sin(\lambda_p s) \cos((p+k)\pi s) \\
&+ C_{1,p} B_{1,p+k} \sin(\lambda_p s) \cos((p+k)\pi s) \\
&+ D_{1,p} A_{1,p+k} \cos(\lambda_p s) \sin((p+k)\pi s) \\
&+ D_{1,p} B_{1,p+k} \cos(\lambda_p s) \cos((p+k)\pi s)] \sin(k\pi s) \\
&- \beta_1 \frac{d_k \omega_1^2 \hat{l}^2}{2} [\sin(k\pi s + \omega_1 \chi(s) + \alpha) \\
&+ \sin(k\pi s - \omega_1 \chi(s) - \alpha)] \\
&+ h.o.t.
\end{aligned}
\tag{24}
$$

Large transverse amplitude responses in (23) and (24), due to the external forcing with frequency ω_1, can be expected when $k\pi - \omega_1 \dot{\chi}(s) \approx 0$, or $k\pi + \omega_1 \dot{\chi}(s) \approx 0$. But since $k\pi > 0$ and $\omega_1 \dot{\chi}(s) > 0$, resonance only will occur when

$$\omega_1 l_0 \mathrm{e}^{\varepsilon v_0 s} \approx k\pi. \tag{25}$$

So, transverse resonances are expected for times s around $s^{(k)}$ with

$$s^{(k)} = \frac{1}{\varepsilon v_0} \ln\left(\frac{k\pi}{\omega_1 l_0}\right), \quad k\pi \geq \omega_1 l_0, \quad k = 1, 2, \ldots \tag{26}$$

To study the situation in the transverse resonance zone, we introduce time-like variables

$$\tau = \varepsilon s, \quad \phi_k(s) = k\pi s,$$
$$\varphi(s) = \omega_1 \chi(s) + \alpha,$$
$$\psi_k(s) = \phi_k(s) - \varphi(s),$$

and rescale $\tau - \tau^{(k)} = \delta(\varepsilon)\hat{\tau}$ with $\hat{\tau} = O(1)$ and $\tau^{(k)} = \varepsilon s^{(k)} = \frac{1}{v_0}\ln(\frac{k\pi}{\omega_1 l_0})$. Then

$$\begin{cases} \dot{\tau} = \varepsilon, \quad \dot{\hat{\tau}} = \frac{\varepsilon}{\delta(\varepsilon)}, \\ \dot{\phi_k} = k\pi, \\ \dot{\varphi} = \omega_1 l_0 \mathrm{e}^{v_0(\tau^{(k)}+\delta(\varepsilon)\hat{\tau})}, \\ \dot{\psi_k} = k\pi - \omega_1 l_0 \mathrm{e}^{v_0(\tau^{(k)}+\delta(\varepsilon)\hat{\tau})} \\ \quad\ = \delta(\varepsilon)\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}\hat{\tau}, \end{cases} \tag{27}$$

and $\dot{A}_{1,k}(s)$, $\dot{B}_{1,k}(s)$ are given by (23). It now follows that a balance in system (27) occurs when $\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon)$, and this implies that in the transverse resonance zone that $\delta(\varepsilon) = \sqrt{\varepsilon}$, i.e. the size of transverse resonance zone is $O(\frac{1}{\sqrt{\varepsilon}})$ for times s. So, together with $\tau - \tau^{(k)} = \delta(\varepsilon)\hat{\tau}$, it follows from (27) that

$$\hat{\tau} = \sqrt{\varepsilon}(s - s^{(k)}). \tag{28}$$

Further, from (27), we obtain $\psi_k(s) = \psi_k(s^{(k)}) + \frac{1}{2}\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}\varepsilon(s-s^{(k)})^2$. Hence, in the transverse resonance zone, we can write

$$\sin(\psi_k(s)) = \sin\left(\frac{1}{2}\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}\varepsilon(s - s^{(k)})^2 + \psi_k(s^{(k)})\right),$$
$$\cos(\psi_k(s)) = \cos\left(\frac{1}{2}\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}\varepsilon(s - s^{(k)})^2 + \psi_k(s^{(k)})\right), \tag{29}$$

where $\psi_k(s^{(k)}) = k\pi s^{(k)} - \frac{k\pi - \omega_1 l_0}{\varepsilon v_0} - \alpha$.

So far it can be concluded that the resonance responses for Y_k in (21) depend on the terms including $T_p T_j$, and the resonance responses for T_k in (20) depend on the terms with frequency ω_1. So, based on the inner layer analysis, the size of the resonance zones has been obtained, and this size will also be used as a new asymptotic scale to be introduced in the three-timescale perturbation method in the next section of this paper to study problems (20) and (21) in detail, and to construct asymptotic approximations of the solutions of the initial-boundary value problems (13) and (14).

4 Three-timescales perturbation method

In the previous section, it was shown that (under certain condition on the external frequency ω_1) resonances in the transverse direction can occur around time $s = \frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})$, and that resonances in the longitudinal direction depend on the solutions T_k of Eq. (20). For this reason, we rescale s by defining $s = \tilde{s} + \frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})$. Thus, problem (20) can be rewritten in $\tilde{s}$ as follows:

$$\begin{aligned} & T_{k,\tilde{s}\tilde{s}} + k^2\pi^2 T_k \\ & = \varepsilon\Bigg[v_0 T_{k,\tilde{s}} - c_{1,0}\hat{l}T_{k,\tilde{s}} - \sum_{n=1}^{\infty} c^1_{n,k}\mu_0\hat{l}(n\pi)^2 T_n \\ & + \sum_{n=1}^{\infty} c^2_{n,k}(-2v_0 n\pi T_{n,\tilde{s}} - \mu_0\hat{l}n\pi T_n) \\ & + \sum_{n=1}^{\infty} c^3_{n,k} 2v_0 n\pi T_{n,\tilde{s}} \\ & + \frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=k+1}^{\infty} kp(k-p)Y_p T_{p-k} \\ & - \frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=1}^{k-1} kp(k-p)Y_p T_{k-p} \\ & - \frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=1}^{\infty} kp(k+p)Y_p T_{k+p} \\ & + \beta_{1,0} d_k \omega_1^2 \hat{l}^2 \cos\left(\omega_1\chi\left(\tilde{s} + \frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right) + \alpha\right)\Bigg] + h.o.t., \\ & T_k\left(-\frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right) = F_k, \\ & T_{k,\tilde{s}}\left(-\frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right) = G_k, \end{aligned} \tag{30}$$

and problem (21) can be rewritten in $\tilde{s}$ as follows:

$$\begin{aligned} & Y_{k,\tilde{s}\tilde{s}} + \lambda_k^2 Y_k \\ & = \varepsilon(v_0 - c_{2,0}\hat{l})Y_{k,\tilde{s}} + \sum_{n=1}^{\infty} 2\varepsilon v_0 n\pi d^1_{n,k} Y_{n,\tilde{s}} \\ & - \sum_{n=1}^{\infty} \frac{EA}{mg}\mu\hat{l}n\pi d^3_{n,k} Y_n \\ & + \sum_{j=1}^{k-1} d^4_{k-j,j} T_{k-j} T_j + \sum_{j=k+1}^{\infty} d^4_{j-k,j} T_{j-k} T_j \\ & - \sum_{j=1}^{\infty} d^4_{j+k,j} T_{j+k} T_j \\ & + \beta_1 \tilde{d}^4_{k,k} T_k \cos\left(\omega_1\chi\left(\tilde{s} + \frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right) + \alpha\right) \\ & + \beta_2 \omega_2^2 \hat{l}^2 d_{1,k} \cos\left(\omega_2\chi\left(\tilde{s} + \frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right) + \alpha\right) \end{aligned}$$

$$
\begin{aligned}
&+h.o.t.,\\
&Y_k\left(-\frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right)=f_k,\\
&Y_{k,\tilde{s}}\left(-\frac{1}{\varepsilon v_0}\ln\left(\frac{k\pi}{\omega_1 l_0}\right)\right)=g_k.
\end{aligned}\tag{31}
$$

Next, we study problems (30) and (31) in detail under the assumption that ω_1 is such that a resonance zone exits for the k^{th} oscillation mode. The application of the straightforward expansion method to solve (30) and (31) will result in the occurrence of so-called secular terms which cause the approximations of the solutions to become unbounded on long timescales. For this reason, to remove secular terms, and to obtain approximations which are valid on long timescales, we introduce three timescales $s_0=\tilde{s}$, $s_1=\sqrt{\varepsilon}\tilde{s}$, $s_2=\varepsilon\tilde{s}$. The timescale $s_1=\sqrt{\varepsilon}\tilde{s}$ is introduced because of the size of the resonance zone which has been found in the previous section, and the other two timescales are the natural scalings for nonlinear equations such as (30) and (31). By using the three-timescales perturbation method, the functions $T_k(\tilde{s};\sqrt{\varepsilon})$ and $Y_k(\tilde{s};\sqrt{\varepsilon})$ are supposed to be functions of s_0, s_1 and s_2,

$$
\begin{aligned}
&T_k(\tilde{s};\sqrt{\varepsilon})=\widetilde{T}_k(s_0,s_1,s_2),\\
&Y_k(\tilde{s};\sqrt{\varepsilon})=\widetilde{Y}_k(s_0,s_1,s_2).
\end{aligned}
$$

By substituting $\widetilde{T}_k(s_0,s_1,s_2)$ and $\widetilde{Y}_k(s_0,s_1,s_2)$ into the differential Eq. (20), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$
\begin{aligned}
&\frac{\partial^2\tilde{T}_k}{\partial s_0^2}+k^2\pi^2\tilde{T}_k+2\sqrt{\varepsilon}\frac{\partial^2\tilde{T}_k}{\partial s_0\partial s_1}\\
&\quad+\varepsilon\left(2\frac{\partial^2\tilde{T}_k}{\partial s_0\partial s_2}+\frac{\partial^2\tilde{T}_k}{\partial s_1^2}\right)+2\varepsilon\sqrt{\varepsilon}\frac{\partial^2\tilde{T}_k}{\partial s_1\partial s_2}\\
&=\varepsilon\Bigg[(v_0-c_{1,0}\hat{l})\frac{\partial\tilde{T}_k}{\partial s_0}-\sum_{n=1}^{\infty}c_{n,k}^1\mu_0\hat{l}(n\pi)^2\tilde{T}_n\\
&\quad+\sum_{n=1}^{\infty}c_{n,k}^2\left(-2v_0n\pi\frac{\partial\tilde{T}_n}{\partial s_0}-\mu_0\hat{l}n\pi\tilde{T}_n\right)\\
&\quad+\sum_{n=1}^{\infty}c_{n,k}^3 2v_0n\pi\frac{\partial\tilde{T}_n}{\partial s_0}\Bigg]\\
&\quad+\varepsilon\sqrt{\varepsilon}\Bigg[(v_0-c_{1,0}\hat{l})\frac{\partial\tilde{T}_k}{\partial s_1}\\
&\quad+\sum_{n=1}^{\infty}c_{n,k}^2(-2v_0n\pi\frac{\partial\tilde{T}_n}{\partial s_1}+\sum_{n=1}^{\infty}c_{n,k}^3 2v_0n\pi\frac{\partial\tilde{T}_n}{\partial s_1}\Bigg]\\
&\quad+\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=k+1}^{\infty}kp(k-p)Y_pT_{p-k}\\
&\quad-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{k-1}kp(k-p)Y_pT_{k-p}\\
&\quad-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{\infty}kp(k+p)Y_pT_{k+p}\\
&\quad+\beta_1d_k\omega_1^2\hat{l}^2\cos(\omega_1\chi(s_0-a)+\alpha),\\
&\tilde{T}_k(a,b,c;\sqrt{\varepsilon})=F_k=\varepsilon\tilde{F}_k,\\
&\frac{\partial\tilde{T}_k}{\partial s_0}(a,b,c;\sqrt{\varepsilon})+\sqrt{\varepsilon}\frac{\partial\tilde{T}_k}{\partial s_1}(a,b,c;\sqrt{\varepsilon})\\
&\quad+\varepsilon\frac{\partial\tilde{T}_k}{\partial s_2}(a,b,c;\sqrt{\varepsilon})\\
&=G_k=\varepsilon\tilde{G}_k,
\end{aligned}\tag{32}
$$

where $\tilde{F}_k=O(1)$ and $\tilde{G}_k=O(1)$. Similarly, by substituting $\widetilde{T}_k(s_0,s_1,s_2)$, $\widetilde{Y}_k(s_0,s_1,s_2)$ into the differential Eq. (21), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$
\begin{aligned}
&\frac{\partial^2\tilde{Y}_k}{\partial s_0^2}+\lambda_k^2\tilde{Y}_k+2\sqrt{\varepsilon}\frac{\partial^2\tilde{Y}_k}{\partial s_0\partial s_1}\\
&\quad+\varepsilon\left(2\frac{\partial^2\tilde{Y}_k}{\partial s_0\partial s_2}+\frac{\partial^2\tilde{Y}_k}{\partial s_1^2}\right)+2\varepsilon\sqrt{\varepsilon}\frac{\partial^2\tilde{Y}_k}{\partial s_1\partial s_2}\\
&=\varepsilon\Bigg[(v_0-c_{2,0}\hat{l})\frac{\partial\tilde{Y}_k}{\partial s_0}+\sum_{n=1}^{\infty}2v_0n\pi d_{n,k}^1\frac{\partial\tilde{Y}_k}{\partial s_0}\\
&\quad-\sum_{n=1}^{\infty}\frac{EA}{mg}\mu_0\hat{l}n\pi d_{n,k}^3\tilde{Y}_n\\
&\quad+\beta_{1,0}\tilde{d}_{k,k}^4\tilde{T}_k\cos(\omega_1\chi(s_0-a)+\alpha)\Bigg]\\
&\varepsilon\sqrt{\varepsilon}\left[(v_0-c_{2,0}\hat{l})\frac{\partial\tilde{Y}_k}{\partial s_1}+\sum_{n=1}^{\infty}2v_0n\pi d_{n,k}^1\frac{\partial\tilde{Y}_k}{\partial s_1}\right]\\
&\quad+\sum_{j=1}^{k-1}d_{k-j,j}^4\tilde{T}_{k-j}\tilde{T}_j+\sum_{j=k+1}^{\infty}d_{j-k,j}^4\tilde{T}_{j-k}\tilde{T}_j\\
&\quad-\sum_{j=1}^{\infty}d_{j+k,j}^4\tilde{T}_{j+k}\tilde{T}_j\\
&\tilde{Y}_k(a,b,c;\sqrt{\varepsilon})=f_k=\varepsilon^2\tilde{f}_k,\\
&\frac{\partial\tilde{Y}_k}{\partial s_0}(a,b,c;\sqrt{\varepsilon})+\sqrt{\varepsilon}\frac{\partial\tilde{Y}_k}{\partial s_1}(a,b,c;\sqrt{\varepsilon})\\
&\quad+\varepsilon\frac{\partial\tilde{Y}_k}{\partial s_2}(a,b,c;\sqrt{\varepsilon})
\end{aligned}
$$

$$= g_k = \varepsilon^2 \tilde{g}_k, \quad (33)$$

where $\lambda_k = \sqrt{\frac{EA}{mg}} k\pi$, $\tilde{f}_k = O(1)$, $\tilde{g}_k = O(1)$, and

$$a = -\frac{1}{\varepsilon v_0} \ln\left(\frac{k\pi}{\omega_1 l_0}\right),$$
$$b = -\frac{\sqrt{\varepsilon}}{\varepsilon v_0} \ln\left(\frac{k\pi}{\omega_1 l_0}\right),$$
$$c = -\frac{1}{v_0} \ln\left(\frac{k\pi}{\omega_1 l_0}\right). \quad (34)$$

Since the functions $\tilde{T}_k$ and $\tilde{Y}_k$ can increase in s from the initial state orders to $O(\sqrt{\varepsilon})$ as has been shown in the previous section, a three-timescales perturbation method will be used, and $\tilde{T}_k(s_0, s_1, s_2)$ and $\tilde{Y}_k(s_0, s_1, s_2)$ will be approximated by the following formal asymptotic expansions:

$$\tilde{T}_k(s_0, s_1, s_2) = \sqrt{\varepsilon}\tilde{T}_{k,0}(s_0, s_1, s_2) + \varepsilon \tilde{T}_{k,1}(s_0, s_1, s_2) + \varepsilon\sqrt{\varepsilon}\tilde{T}_{k,2}(s_0, s_1, s_2) + O(\varepsilon^2), \quad (35)$$
$$\tilde{Y}_k(s_0, s_1, s_2) = \sqrt{\varepsilon}\tilde{Y}_{k,0}(s_0, s_1, s_2) + \varepsilon \tilde{Y}_{k,1}(s_0, s_1, s_2) + \varepsilon\sqrt{\varepsilon}\tilde{Y}_{k,2}(s_0, s_1, s_2) + O(\varepsilon^2), \quad (36)$$

where $\tilde{T}_{k,0}, \tilde{T}_{k,1}, \tilde{T}_{k,2}, \tilde{Y}_{k,0}, \tilde{Y}_{k,1}, \tilde{Y}_{k,2}$ are all functions of $O(1)$. In the transverse direction, by substituting (35) and (36) into problem (32), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain: the $O(\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 \tilde{T}_{k,0}}{\partial s_0^2} + k^2\pi^2 \tilde{T}_{k,0} = 0,$$
$$\tilde{T}_{k,0}(a, b, c) = 0, \qquad \frac{\partial \tilde{T}_{k,0}}{\partial s_0}(a, b, c) = 0, \quad (37)$$

the $O(\varepsilon)$-problem:

$$\frac{\partial^2 \tilde{T}_{k,1}}{\partial s_0^2} + k^2\pi^2 \tilde{T}_{k,1} + 2\frac{\partial^2 \tilde{T}_{k,0}}{\partial s_0 \partial s_1}$$
$$= \frac{EA\pi^3}{2mg\hat{l}} \sum_{p=k+1}^{\infty} kp(k-p)\tilde{Y}_{p,0}\tilde{T}_{p-k,0}$$
$$-\frac{EA\pi^3}{2mg\hat{l}} \sum_{p=1}^{k-1} kp(k-p)\tilde{Y}_{p,0}\tilde{T}_{k-p,0}$$
$$-\frac{EA\pi^3}{2mg\hat{l}} \sum_{p=1}^{\infty} kp(k+p)\tilde{Y}_{p,0}\tilde{T}_{k+p,0}$$
$$+\beta_{1,0} d_k \omega_1^2 \hat{l}^2 \cos(\omega_1 \chi (s_0 - a) + \alpha),$$
$$\tilde{T}_{k,1}(a, b, c) = \tilde{F}_k, \qquad \frac{\partial \tilde{T}_{k,1}}{\partial s_0}(a, b, c) + \frac{\partial \tilde{T}_{k,0}}{\partial s_1}(a, b, c) = \tilde{G}_k, \quad (38)$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 \tilde{T}_{k,2}}{\partial s_0^2} + k^2\pi^2 \tilde{T}_{k,2} + 2\frac{\partial^2 \tilde{T}_{k,1}}{\partial s_0 \partial s_1} + 2\frac{\partial^2 \tilde{T}_{k,0}}{\partial s_0 \partial s_2} + \frac{\partial^2 \tilde{T}_{k,0}}{\partial s_1^2}$$
$$= (v_0 - c_{1,0}\hat{l})\frac{\partial \tilde{T}_{k,0}}{\partial s_0} - \sum_{n=1}^{\infty} c_{n,k}^1 \mu_0 \hat{l}(n\pi)^2 \tilde{T}_{n,0}$$
$$+\sum_{n=1}^{\infty} c_{n,k}^2 \left(-2v_0 n\pi \frac{\partial \tilde{T}_{n,0}}{\partial s_0} - \mu_0 \hat{l} n\pi \tilde{T}_{n,0}\right)$$
$$+\sum_{n=1}^{\infty} c_{n,k}^3 2v_0 n\pi \frac{\partial \tilde{T}_{n,0}}{\partial s_0},$$
$$\tilde{T}_{k,2}(a, b, c) = 0,$$
$$\frac{\partial \tilde{T}_{k,2}}{\partial s_0}(a, b, c) + \frac{\partial \tilde{T}_{k,1}}{\partial s_1}(a, b, c) + \frac{\partial \tilde{T}_{k,0}}{\partial s_2}(a, b, c) = 0. \quad (39)$$

The solution of the $O(\sqrt{\varepsilon})$-problem (37) can be written as:

$$\tilde{T}_{k,0}(s_0, s_1, s_2) = A_k(s_1, s_2)\cos(k\pi s_0) + B_k(s_1, s_2)\sin(k\pi s_0), \quad (40)$$

where $A_k(b, c) = 0$, $B_k(b, c) = 0$, and where $A_k(s_1, s_2)$, $B_k(s_1, s_2)$ can be obtained explicitly by solving the $O(\varepsilon)$-problem (38) and the $O(\varepsilon\sqrt{\varepsilon})$-problem (39). We will study these problems later in this section.

In the longitudinal direction, by substituting (35) and (36) into problem (33), and after equating the coefficients of like powers in ε, we obtain: the $O(\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_0^2} + \lambda_k^2 \tilde{Y}_{k,0} = 0,$$
$$\tilde{Y}_{k,0}(a, b, c) = 0, \qquad \frac{\partial \tilde{Y}_{k,0}}{\partial s_0}(a, b, c) = 0, \quad (41)$$

the $O(\varepsilon)$-problem:

$$\frac{\partial^2 \tilde{Y}_{k,1}}{\partial s_0^2} + \lambda_k^2 \tilde{Y}_{k,1} + 2\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_0 \partial s_1}$$
$$= \left[\sum_{j=1}^{k-1} d_{k-j,j}^4 T_{k-j,0} T_{j,0} + \sum_{j=k+1}^{\infty} d_{j-k,j}^4 T_{j-k,0} T_{j,0}\right.$$

$$-\sum_{j=1}^{\infty} d^4_{j+k,j} T_{j+k,0} T_{j,0} \Bigg],$$

$$\tilde{Y}_{k,1}(a,b,c) = 0, \quad \frac{\partial \tilde{Y}_{k,1}}{\partial s_0}(a,b,c) + \frac{\partial \tilde{Y}_{k,0}}{\partial s_1}(a,b,c) = 0, \tag{42}$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 \tilde{Y}_{k,2}}{\partial s_0^2} + \lambda_k^2 \tilde{Y}_{k,2} + 2\frac{\partial^2 \tilde{Y}_{k,1}}{\partial s_0 \partial s_1} + 2\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_0 \partial s_2} + \frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_1^2}$$
$$= (v_0 - c_{2,0}\hat{l})\frac{\partial \tilde{Y}_{k,0}}{\partial s_0} + \sum_{n=1}^{\infty} 2v_0 n\pi d^1_{n,k} \frac{\partial \tilde{Y}_{n,0}}{\partial s_0}$$
$$-\sum_{n=1}^{\infty} \frac{EA}{mg}\mu_0 \hat{l} n\pi d^3_{n,k} \tilde{Y}_{n,0}$$
$$+\beta_{1,0}\tilde{d}^4_{k,k}\tilde{T}_{k,0}\cos(\omega_1\chi(s_0 - a) + \alpha),$$

$$\tilde{Y}_{k,2}(a,b,c) = 0, \quad \frac{\partial \tilde{Y}_{k,2}}{\partial s_0}(a,b,c) + \frac{\partial \tilde{Y}_{k,1}}{\partial s_1}(a,b,c) + \frac{\partial \tilde{Y}_{k,0}}{\partial s_2}(a,b,c) = 0, \tag{43}$$

where $\lambda_k = \sqrt{\frac{EA}{mg}}k\pi$.

The solution of the $O(\sqrt{\varepsilon})$-problem (41) can be written as:

$$\tilde{Y}_{k,0}(s_0,s_1,s_2) = C_k(s_1,s_2)\cos(\lambda_k s_0) + D_k(s_1,s_2)\sin(\lambda_k s_0), \tag{44}$$

where $C_k(s_1,s_2)$, and $D_k(s_1,s_2)$ are still unknown functions in the slow variables s_1 and s_2, and these functions can be determined by avoiding secular terms in the $O(\varepsilon)-$ problem (42) and in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (43). By using the initial conditions in (41), it follows that $C_k(b,c) = D_k(b,c) = 0$. Now, we shall solve the $O(\varepsilon)-$ problem (42). By using (40) for $\tilde{T}_{k,0}$, and by using $d^1_{k,k} = -\frac{1}{2k\pi}$, which is given in (22), problem (42) can be written as:

$$\frac{\partial^2 \tilde{Y}_{k,1}}{\partial s_0^2} + \lambda_k^2 \tilde{Y}_{k,1}$$
$$= 2\lambda_k \frac{\partial C_k}{\partial s_1}\sin(\lambda_k s_0) - 2\lambda_k \frac{\partial D_k}{\partial s_1}\cos(\lambda_k s_0)$$
$$+\frac{1}{2}\sum_{j=1}^{k-1} d^4_{k-j,j}[(A_{k-j}A_j - B_{k-j}B_j)\cos(k\pi s_0)$$
$$+(A_{k-j}B_j + B_{k-j}A_j)\sin(k\pi s_0)$$
$$+(A_{k-j}A_j + B_{k-j}B_j)\cos((k-2j)\pi s_0)$$
$$+(A_{k-j}B_j - B_{k-j}A_j)\sin((k-2j)\pi s_0)]$$
$$+\frac{1}{2}\sum_{j=k+1}^{\infty} d^4_{j-k,j}[(A_{j-k}A_j + B_{j-k}B_j)$$
$$\times\cos(k\pi s_0) + (A_{j-k}B_j - B_{j-k}A_j)\sin(k\pi s_0)$$
$$+(A_{j-k}A_j - B_{j-k}B_j)\cos((2j-k)\pi s_0)$$
$$+(A_{j-k}B_j + B_{j-k}A_j)\sin((2j-k)\pi s_0)]$$
$$-\frac{1}{2}\sum_{j=1}^{\infty} d^4_{k+j,j}[(A_{k+j}A_j + B_{k+j}B_j)\cos(k\pi s_0)$$
$$+(A_jB_{j+k} - B_jA_{j+k})\sin(k\pi s_0)$$
$$+(A_{k+j}A_j - B_{k+j}B_j)\cos((k+2j)\pi s_0)$$
$$+(A_jB_{j+k} + B_jA_{j+k})\sin((k+2j)\pi s_0)],$$

$$\tilde{Y}_{k,1}(0,0,0) = 0, \quad \frac{\partial \tilde{Y}_{k,2}}{\partial s_0}(0,0,0) + \frac{\partial \tilde{Y}_{k,1}}{\partial s_1}(0,0,0) + \frac{\partial \tilde{Y}_{k,0}}{\partial s_2}(0,0,0) = 0. \tag{45}$$

It is obvious that the right-hand side of (45) contains resonant terms, such as $\sin(\lambda_k s_0)$ and $\cos(\lambda_k s_0)$. But the term in the right-hand side of (45) involving $\sin((2j-k)\pi s_0)$, $\cos((2j-k)\pi s_0)$, $\sin((k+2j)\pi s_0)$ or $\cos((k+2j)\pi s_0)$ is also a resonant term when there exist k, j_1, j_2, s.t., $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$. Actually, for any fixed parameter value of $\sqrt{\frac{EA}{mg}}$ with assumptions $\sqrt{\frac{EA}{mg}} = O(1)$ and $\sqrt{\frac{EA}{mg}} - 1 > O(\varepsilon)$, there always exist k s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$ with $j_1 = \frac{(1+\sqrt{\frac{EA}{mg}})k}{2}$ and $j_2 = \frac{(\sqrt{\frac{EA}{mg}}-1)k}{2}$. Therefore, to avoid secular terms in (45) the functions $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ have to satisfy the following:

- When k does not satisfy the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\frac{\partial C_k}{\partial s_1} = 0, \quad \frac{\partial D_k}{\partial s_1} = 0, \tag{46}$$

and $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ are given by:

$$C_k(s_1,s_2) = \overline{C}_k(s_2), \quad D_k(s_1,s_2) = \overline{D}_k(s_2). \tag{47}$$

- When k satisfies the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$
\begin{aligned}
\frac{\partial C_k}{\partial s_1} &= -\frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}B_{j_1}+B_{j_1-k}A_{j_1})\\
&\quad+\frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{j_2}B_{j_2+k}+B_{j_2}A_{j_2+k})\\
&= \tilde{P}_2(s_1,s_2),\\
\frac{\partial D_k}{\partial s_1} &= \frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}A_{j_1}-B_{j_1-k}B_{j_1})\\
&\quad-\frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{k+j_2}A_{j_2}-B_{k+j_2}B_{j_2})\\
&= \tilde{Q}_2(s_1,s_2),
\end{aligned}
\tag{48}
$$

and $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ can be obtained as:

$$
\begin{aligned}
C_k(s_1,s_2) &= \int_b^{s_1}\tilde{P}_2(\bar{\tau},s_2)d\bar{\tau}+\overline{C}_k(s_2),\\
D_k(s_1,s_2) &= \int_b^{s_1}\tilde{Q}_2(\bar{\tau},s_2)d\bar{\tau}+\overline{D}_k(s_2),
\end{aligned}
\tag{49}
$$

where the functions $\tilde{P}_2$ and $\tilde{Q}_2$ are given by (48), and where $\overline{C}_k(s_2)$ and $\overline{D}_k(s_2)$ in (47) and (49) are still unknown functions in the slow variable s_2. By $C_k(b,c) = 0$ and $D_k(b,c) = 0$, we obtain that $\overline{C}_k(c) = 0$ and $\overline{D}_k(c) = 0$. The undetermined behaviour with respect to s_2 can be used to avoid secular terms in the solution of the $O(\varepsilon\sqrt{\varepsilon})-$ problem (43).

According to (45), taking into account the secularity conditions, the general solution of the $O(\varepsilon)-$ problem (42) is given by

$$
\begin{aligned}
Y_{k,1}(s_0,s_1,s_2;\sqrt{\varepsilon}) &= E_k(s_0,s_1,s_2)\cos(\lambda_k s_0)\\
&\quad+H_k(s_0,s_1,s_2)\sin(\lambda_k s_0),
\end{aligned}
\tag{50}
$$

where

$$
E_k(a,b,c)=0,\quad H_k(a,b,c)=-\frac{\partial Y_{k,0}}{\partial s_1}(a,b,c).
\tag{51}
$$

Then, the $O(\varepsilon\sqrt{\varepsilon})-$ problem (43) can be written as:

$$
\begin{aligned}
&\frac{\partial^2\tilde{Y}_{k,2}}{\partial s_0^2}+\lambda_k^2\tilde{Y}_{k,2}\\
&=\Big[-2\frac{\partial^2 E_k}{\partial s_0\partial s_1}-2\lambda_k\frac{\partial H_k}{\partial s_1}-2\lambda_k\frac{\partial D_k}{\partial s_2}-\frac{\partial^2 C_k}{\partial s_1^2}\\
&\quad+(v_0-c_{2,0}\hat{l})\lambda_k D_k+2v_0k\pi d^1_{k,k}\lambda_k D_k\\
&\quad-\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}C_k\Big]\cos(\lambda_k s_0)\\
&\quad+\Big[-2\frac{\partial^2 H_k}{\partial s_0\partial s_1}+2\lambda_k\frac{\partial E_k}{\partial s_1}+2\lambda_k\frac{\partial C_k}{\partial s_2}-\frac{\partial^2 D_k}{\partial s_1^2}\\
&\quad-(v_0-c_{2,0}\hat{l})\lambda_k C_k-2v_0n\pi d^1_{k,k}\lambda_k C_k\\
&\quad-\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}D_k\Big]\sin(\lambda_k s_0),\\
&\quad+\beta_{1,0}\tilde{d}^4_{k,k}[A_k\cos(k\pi s_0)\\
&\quad+B_k\sin(k\pi s_0)]\cos(\omega_1\chi(s_0-a)+\alpha)\\
&\tilde{Y}_{k,2}(a,b,c)=0,\\
&\frac{\partial\tilde{Y}_{k,2}}{\partial s_0}(a,b,c)=-\frac{\partial\tilde{Y}_{k,1}}{\partial s_1}(a,b,c)\\
&\quad-\frac{\partial\tilde{Y}_{k,0}}{\partial s_2}(a,b,c).
\end{aligned}
\tag{52}
$$

Note that in the analysis of Sect. 3, the last term including $\cos(\omega_1\chi(s_0-a)+\alpha)$ in (52) can not affect the function $\tilde{Y}_{k,0}$. So, to avoid secular terms in the solution $\tilde{Y}_{k,2}$ in Eq. (52), the following different cases have to be considered:

- When k does not satisfy the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2j_2}{k}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$, then:

$$
\begin{aligned}
&-2\frac{\partial^2 E_k}{\partial s_0\partial s_1}-2\lambda_k\frac{\partial H_k}{\partial s_1}\\
&=2\lambda_k\frac{\partial\overline{D}_k}{\partial s_2}-(v_0-c_{2,0}\hat{l})\lambda_k\overline{D}_k\\
&\quad-2v_0k\pi d^1_{k,k}\lambda_k\overline{D}_k+\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\overline{C}_k,\\
&-2\frac{\partial^2 H_k}{\partial s_0\partial s_1}+2\lambda_k\frac{\partial E_k}{\partial s_1}\\
&=-2\lambda_k\frac{\partial\overline{C}_k}{\partial s_2}+(v_0-c_{2,0}\hat{l})\lambda_k\overline{C}_k\\
&\quad+2v_0k\pi d^1_{k,k}\lambda_k\overline{C}_k+\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\overline{D}_k.
\end{aligned}
\tag{53}
$$

- When k satisfies the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2j_2}{k}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$, then:

$$
\begin{aligned}
&-2\frac{\partial^2 E_k}{\partial s_0\partial s_1}-2\lambda_k\frac{\partial H_k}{\partial s_1}\\
&\quad-2\lambda_k\frac{\partial\int_0^{s_1}\tilde{Q}_2(\bar{\tau},s_2)d\bar{\tau}}{\partial s_2}-\frac{\partial\tilde{P}_2}{\partial s_1}\\
&\quad+(v_0-c_{2,0}\hat{l})\lambda_k\int_0^{s_1}\tilde{Q}_2(\bar{\tau},s_2)d\bar{\tau}\\
&\quad+2v_0k\pi d^1_{k,k}\lambda_k\int_0^{s_1}\tilde{Q}_2(\bar{\tau},s_2)d\bar{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\int_0^{s_1}\tilde{P}_2(\bar{\tau},s_2)d\bar{\tau}\\
&=2\lambda_k\frac{\partial\overline{D}_k}{\partial s_2}-(v_0-c_{2,0}\hat{l})\lambda_k\overline{D}_k\\
&-2v_0k\pi d^1_{k,k}\lambda_k\overline{D}_k+\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\overline{C}_k,\\
&-2\frac{\partial^2H_k}{\partial s_0\partial s_1}\\
&+2\lambda_k\frac{\partial E_k}{\partial s_1}+2\lambda_k\frac{\partial\int_0^{s_1}\tilde{P}_2(\bar{\tau},s_2)d\bar{\tau}}{\partial s_2}-\frac{\partial\tilde{Q}_2}{\partial s_1}\\
&-(v_0-c_{2,0}\hat{l})\lambda_k\int_0^{s_1}\tilde{P}_2(\bar{\tau},s_2)d\bar{\tau}\\
&-2v_0k\pi d^1_{k,k}\lambda_k\int_0^{s_1}\tilde{P}_2(\bar{\tau},s_2)d\bar{\tau}\\
&-\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\int_0^{s_1}\tilde{Q}_2(\bar{\tau},s_2)d\bar{\tau}\\
&=-2\lambda_k\frac{\partial\overline{C}_k}{\partial s_2}+(v_0-c_{2,0}\hat{l})\lambda_k\overline{C}_k\\
&+2v_0k\pi d^1_{k,k}\lambda_k\overline{C}_k+\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\overline{D}_k. \qquad (54)
\end{aligned}
$$

Solving (53) and (54) for E_k and H_k, we observe that the solution will be unbounded in s_0 and s_1, due to terms which are only depending on s_2. Therefore, to have secular-free solutions for E_k and H_k, the following conditions have to be imposed independently:

$$
\begin{aligned}
&2\lambda_k\frac{d\overline{D}_k}{ds_2}-(v_0-c_{2,0}\hat{l})\lambda_k\overline{D}_k-2v_0k\pi d^1_{k,k}\lambda_k\overline{D}_k\\
&+\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\overline{C}_k=0,\\
&-2\lambda_k\frac{d\overline{C}_k}{ds_2}+(v_0-c_{2,0}\hat{l})\lambda_k\overline{C}_k+2v_0k\pi d^1_{k,k}\lambda_k\overline{C}_k\\
&+\frac{EA}{mg}\mu_0\hat{l}k\pi d^3_{k,k}\overline{D}_k=0. \qquad (55)
\end{aligned}
$$

Due to $d^2_{k,k}=-\frac{1}{2k\pi}$, we then obtain from (55):

$$
\begin{aligned}
\overline{C}_k(s_2)&=e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}\\
&\left[\overline{C}_k(c)\cos\left(\frac{EA\mu_0\hat{l}k\pi d^3_{k,k}}{2mg\lambda_k}(s_2-c)\right)\right.\\
&\left.-\overline{D}_k(c)\sin\left(\frac{EA\mu_0\hat{l}k\pi d^3_{k,k}}{2mg\lambda_k}(s_2-c)\right)\right],\\
\overline{D}_k(s_2)&=e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}\\
&\left[\overline{C}_k(c)\sin\left(\frac{EA\mu_0\hat{l}k\pi d^3_{k,k}}{2mg\lambda_k}(s_2-c)\right)\right.\\
&\left.+\overline{D}_k(c)\cos\left(\frac{EA\mu_0\hat{l}k\pi d^3_{k,k}}{2mg\lambda_k}(s_2-c)\right)\right].
\end{aligned}
$$

Since $\overline{C}_k(c)=0$ and $\overline{D}_k(c)=0$, this implies that

$$
\overline{C}_k(s_2)=0,\quad \overline{D}_k(s_2)=0. \qquad (56)
$$

Now, all unknown functions in (44) can be determined, and the solution of the $O(\sqrt{\varepsilon})$-problem (41) can be written as:

$$
\begin{aligned}
\tilde{Y}_{k,0}(s_0,s_1,s_2)&=C_k(s_1,s_2)\cos(\lambda_ks_0)\\
&+D_k(s_1,s_2)\sin(\lambda_ks_0), \qquad (57)
\end{aligned}
$$

where $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ are given by (47), (49) and (56).

Now, substituting (40) and (57) into the $O(\varepsilon)$-problem (38) for $\tilde{T}_{k,1}$, together with $c^1_{k,k}=\frac{1}{2}$, $c^2_{k,k}=0$ and $c^3_{k,k}=-\frac{1}{2k\pi}$ in (19), problem (38) becomes a nonlinear ordinary differential equation without coupling term:

$$
\begin{aligned}
&\frac{\partial^2\tilde{T}_{k,1}}{\partial s_0^2}+k^2\pi^2\tilde{T}_{k,1}\\
&=2k\pi\frac{\partial A_k}{\partial s_1}\cos(k\pi s_0)-2k\pi\frac{\partial B_k}{\partial s_1}\cos(k\pi s_0)\\
&\overbrace{+\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=k+1}^{\infty}kp(k-p)}\\
&\cdot\left[\frac{A_{p-k}D_p+B_{p-k}C_p}{2}\sin((\lambda_p+(p-k)\pi)s_0)\right.\\
&+\frac{A_{p-k}C_p-B_{p-k}D_p}{2}\cos((\lambda_p+(p-k)\pi)s_0)\\
&+\frac{A_{p-k}D_p-B_{p-k}C_p}{2}\sin((\lambda_p-(p-k)\pi)s_0)\\
&\left.+\frac{A_{p-k}C_p+B_{p-k}D_p}{2}\cos((\lambda_p-(p-k)\pi)s_0)\right]\\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{k-1}kp(k-p)\\
&\cdot\left[\frac{A_{k-p}D_p+B_{k-p}C_p}{2}\sin((\lambda_p+(k-p)\pi)s_0)\right.\\
&+\frac{A_{k-p}C_p-B_{k-p}D_p}{2}\cos((\lambda_p+(k-p)\pi)s_0)\\
&+\frac{A_{k-p}D_p-B_{k-p}C_p}{2}\sin((\lambda_p-(k-p)\pi)s_0)\\
&\left.+\frac{A_{k-p}C_p+B_{k-p}D_p}{2}\cos((\lambda_p-(k-p)\pi)s_0)\right]\\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{\infty}kp(k+p)\\
&\cdot\left[\frac{A_{k+p}D_p+B_{k+p}C_p}{2}\sin((\lambda_p+(k+p)\pi)s_0)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{+\frac{A_{k+p}C_p - B_{k+p}D_p}{2}\cos((\lambda_p + (k+p)\pi)s_0) + \frac{A_{k+p}D_p - B_{k+p}C_p}{2}\sin((\lambda_p - (k+p)\pi)s_0) + \frac{A_{k+p}C_p + B_{k+p}D_p}{2}\cos((\lambda_p - (k+p)\pi)s_0)\Big]}_{I}\\
&\underbrace{+\beta_{1,0}d_k\omega_1^2\hat{l}^2 cos(\omega_1\chi(s_0 - a) + \alpha)}_{II}, \qquad (58)
\end{aligned}
$$

where C_p and D_p are given by (47) and (49). The right-hand side of Eq. (58) contains resonant terms: for instance, at least one of the I terms is a resonant term when there exist k, p_1, p_2 s.t. $\frac{2k}{p_1} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2k}{p_2} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$. The II term with ω_1 can be resonant when $k\pi - \omega_1\dot{\chi}(s) \approx 0$ or $k\pi + \omega_1\dot{\chi}(s) \approx 0$. Obviously, the terms in (58) involving $\sin(k\pi s_0)$ or $\cos(k\pi s_0)$ are resonant.

Outside the resonance zone (or equivalently the resonance manifold), the corresponding timescales are $s_0 = \tilde{s}$ and $s_2 = \varepsilon\tilde{s}$ (without $s_1 = \sqrt{\varepsilon}\tilde{s}$). So to avoid secular terms in (58), A_k and B_k have to satisfy the following equations depending on the parameter values:

$$
\frac{\partial A_k}{\partial s_1} = 0, \quad \frac{\partial B_k}{\partial s_1} = 0, \qquad (59)
$$

which implies that:

$$
A_k(s_1, s_2) = \overline{A}_k(s_2), \quad B_k(s_1, s_2) = \overline{B}_k(s_2), \qquad (60)
$$

where $\overline{A}_k(s_2)$ and $\overline{B}_k(s_2)$ are still unknown functions in the slow variable s_2. Since $A_k(b, c) = 0$ and $B_k(b, c) = 0$, we obtain that $\overline{A}_k(c) = 0$ and $\overline{B}_k(c) = 0$. The undetermined behaviour with respect to s_2 can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (39). According to (58), taking into account the secularity conditions, the general solution of the $O(\varepsilon)-$ problem (38) can be written as

$$
\begin{aligned}
T_{k,1}(s_0, s_1, s_2; \sqrt{\varepsilon}) &= L_k(s_0, s_1, s_2)\cos(k\pi s_0)\\
&\quad + M_k(s_0, s_1, s_2)\sin(k\pi s_0),
\end{aligned} \qquad (61)
$$

where

$$
\begin{aligned}
L_k(a, b, c) = F_k, \quad M_k(a, b, c) &= -\frac{\partial T_{k,0}}{\partial s_1}(a, b, c) + G_k.
\end{aligned} \qquad (62)
$$

Then, together with $c^1_{k,k} = \frac{1}{2}$, $c^2_{k,k} = 0$ and $c^3_{k,k} = -\frac{1}{2k\pi}$ in (19), the $O(\varepsilon\sqrt{\varepsilon})-$ problem (39) can be written as

$$
\begin{aligned}
&\frac{\partial^2 \tilde{T}_{k,2}}{\partial s^2} + (k\pi)^2\tilde{T}_{k,2}\\
&= \Big[-2\frac{\partial^2 L_k}{\partial s_0\partial s_1} - 2k\pi\frac{\partial M_k}{\partial s_1} - 2k\pi\frac{\partial B_k}{\partial s_2} - \frac{\partial^2 A_k}{\partial s_1^2}\\
&\quad - c_{1,0}\hat{l}k\pi B_k - \frac{\mu_0\hat{l}(k\pi)^2}{2}A_k\Big]\cos(k\pi s_0)\\
&\quad + \Big[-2\frac{\partial^2 M_k}{\partial s_0\partial s_1} + 2k\pi\frac{\partial L_k}{\partial s_1} + 2k\pi\frac{\partial A_k}{\partial s_2} - \frac{\partial^2 B_k}{\partial s_1^2}\\
&\quad + c_{1,0}\hat{l})k\pi A_k - \frac{\mu_0\hat{l}(k\pi)^2}{2}B_k\Big]\sin(k\pi s_0),\\
&\tilde{Y}_{k,2}(0,0,0) = 0, \quad \frac{\partial \tilde{Y}_{k,2}}{\partial s}(0,0,0) = 0.
\end{aligned} \qquad (63)
$$

To avoid secular terms in $\tilde{T}_{k,2}$ in Eq. (63), the following conditions have to be imposed

$$
\begin{aligned}
&-2\frac{\partial^2 L_k}{\partial s_0\partial s_1} - 2k\pi\frac{\partial M_k}{\partial s_1}\\
&= 2k\pi\frac{\partial \overline{B}_k(s_2)}{\partial s_2} + c_{1,0}\hat{l}k\pi\overline{B}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{A}_k(s_2),\\
&-2\frac{\partial^2 M_k}{\partial s_0\partial s_1} + 2k\pi\frac{\partial L_k}{\partial s_1}\\
&= -2k\pi\frac{\partial \overline{A}_k(s_2)}{\partial s_2} - c_{1,0}\hat{l}k\pi\overline{A}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{B}_k(s_2).
\end{aligned} \qquad (64)
$$

By solving (64) for L_k and M_k, we observe that the solution will be unbounded in s_0 and s_1, due to terms which are only depending on s_2. Therefore, to have secular-free solutions for L_k and M_k, the following conditions have to be imposed independently

$$
\begin{aligned}
&2k\pi\frac{d\overline{B}_k(s_2)}{ds_2} + c_{1,0}\hat{l}\lambda_k\overline{B}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{A}_k(s_2) = 0,\\
&-2k\pi\frac{d\overline{A}_k(s_2)}{ds_2} - c_{1,0}\hat{l}\lambda_k\overline{A}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{B}_k(s_2) = 0,
\end{aligned} \qquad (65)
$$

we then obtain

$$\begin{aligned}&\overline{A}_k(s_2)\\&=e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}\left[\overline{A}_k(c)\cos\left(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c)\right)\right.\\&\quad\left.-\overline{B}_k(c)\sin\left(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c)\right)\right],\\&\overline{B}_k(s_2)\\&=e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}\left[\overline{A}_k(c)\sin\left(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c)\right)\right.\\&\quad\left.+\overline{B}_k(c)\cos\left(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c)\right)\right].\end{aligned}\tag{66}$$

Since $\overline{A}_k(c)=0$ and $\overline{B}_k(c)=0$, together with (66), this implies that

$$\overline{A}_k(s_2)=0,\quad \overline{B}_k(s_2)=0.\tag{67}$$

Now, outside the resonance zone, all these unknown functions in (40) have been determined in (60). So the solution of the $O(\sqrt{\varepsilon})$-problem (37) is $\tilde{T}_{k,0}(s_0,s_1,s_2)\equiv 0$.

Inside the resonance zone around $s=s^{(k)}$ (or equivalently, in the resonance manifold), according to the inner analysis as presented in Sect. 3, and to avoid secular terms in (58), A_k, B_k have to satisfy the following equations:

- When k does not satisfy the conditions that there always exist p_1, p_2 s.t. $\frac{2k}{p_1}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2k}{p_2}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$, then:

$$\begin{aligned}\frac{\partial A_k}{\partial s_1}&=\frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\sin\left(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}s_1^2+\psi_k(s^{(k)})\right),\\\frac{\partial B_k}{\partial s_1}&=-\frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\cos\left(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}s_1^2+\psi_k(s^{(k)})\right),\end{aligned}\tag{68}$$

which implies that

$$\begin{aligned}A_k(s_1,s_2)&=\frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\sin(\psi_k(s^{(k)}))\bar{C}_{Fr}(s_1)\\&\quad+\frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\cos(\psi_k(s^{(k)}))\bar{S}_{Fr}(s_1)\\&\quad+\overline{A}_k(s_2),\\B_k(s_1,s_2)&=-\frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\cos(\psi_k(s^{(k)}))\bar{C}_{Fr}(s_1)\\&\quad+\frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\sin(\psi_k(s^{(k)}))\bar{S}_{Fr}(s_1)\\&\quad+\overline{B}_k(s_2),\end{aligned}\tag{69}$$

where $\tilde{\alpha}=\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}$, and

$$\begin{aligned}\bar{C}_{Fr}(s)&=\int_{\sqrt{\frac{\tilde{\alpha}}{2}}b}^{\sqrt{\frac{\tilde{\alpha}}{2}}s}\cos(x^2)dx,\quad\text{and}\\\bar{S}_{Fr}(s)&=\int_{\sqrt{\frac{\tilde{\alpha}}{2}}b}^{\sqrt{\frac{\tilde{\alpha}}{2}}s}\sin(x^2)dx,\end{aligned}\tag{70}$$

which are the well-known Fresnel integrals. The presence of Fresnel functions $C_{Fr}(s_1)$ and $S_{Fr}(s_1)$ causes resonance jumps in the system. In (69), $\overline{A}_k(s_2)$ and $\overline{B}_k(s_2)$ are still unknown functions in the slow variable s_2. Since $A_k(b,c)=0$ and $B_k(b,c)=0$, we obtain that $\overline{A}_k(c)=0$ and $\overline{B}_k(c)=0$. The undetermined behaviour with respect to s_2 can be used to avoid secular terms in the solutions of the $O(\varepsilon\sqrt{\varepsilon})-$ problem (39). Following the derivation of (61)–(67) together with (69), we obtain

$$\overline{A}_k(s_2)=0,\quad \overline{B}_k(s_2)=0.\tag{71}$$

- When k satisfies the conditions that there always exist p_1, p_2 s.t. $\frac{2k}{p_1}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2k}{p_2}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$, then there exist $j_1=k$, $j_2=\frac{kp_1}{p_2}=\theta k$ s.t. $\frac{2j_1}{p_1}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2j_2}{p_1}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$; and there exist $j_1=\frac{kp_2}{p_1}=\vartheta k$, $j_2=k$ s.t. $\frac{2j_1}{p_2}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2j_2}{p_2}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$. Then, the functions of $A_k(s_1,s_2)$ and $B_k(s_1,s_2)$ have to satisfy:

$$\begin{aligned}\frac{\partial A_k}{\partial s_1}&=\frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3\frac{A_{\theta k}D_{p_1(k)}-B_{\theta k}C_{p_1(k)}}{2}\\&\quad+\frac{EA\pi^3}{4mg\hat{l}}(\vartheta-1)\vartheta k^3\frac{A_{\vartheta k}D_{p_2(k)}-B_{\vartheta k}C_{p_2(k)}}{2}\\&\quad+\frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\sin\left(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}s_1^2+\psi_k(s^{(k)})\right),\end{aligned}$$

$$\frac{\partial B_k}{\partial s_1} = -\frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3 \frac{A_{\theta k}C_{p_1(k)} + B_{\theta k}(\tau)D_{p_1(k)}}{2} - \frac{EA\pi^3}{4mg\hat{l}}(\vartheta-1)\vartheta k^3 \frac{A_{\vartheta k}C_{p_2(k)} + B_{\vartheta k}D_{p_2(k)}}{2} - \frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\cos\left(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}} s_1^2 + \psi_k(s^{(k)})\right), \tag{72}$$

where

$$C_{p_1(k)} = \frac{1}{4\lambda_{p_1}}(d^4_{k,\theta k} - d^4_{\theta k,k}) \int_b^{s_1}(A_{\theta k}(\overline{\tau})B_k(\overline{\tau}) + B_{\theta k}(\overline{\tau})A_k(\overline{\tau}))d\overline{\tau},$$
$$D_{p_1(k)} = \frac{1}{4\lambda_{p_1}}(d^4_{k,\theta k} - d^4_{\theta k,k}) \int_b^{s_1}(A_{\theta k}(\overline{\tau})A_k(\overline{\tau}) - B_{\theta k}(\overline{\tau})B_k(\overline{\tau}))d\overline{\tau},$$
$$C_{p_2(k)} = \frac{1}{4\lambda_{p_2}}(d^4_{\vartheta k,k} - d^4_{k,\vartheta k}) \int_b^{s_1}(A_k(\overline{\tau})B_{\vartheta k}(\overline{\tau}) + B_k(\overline{\tau})A_{\vartheta k}(\overline{\tau}))d\overline{\tau},$$
$$D_{p_2(k)} = \frac{1}{4\lambda_{p_2}}(d^4_{\vartheta k,k} - d^4_{k,\vartheta k}) \int_b^{s_1}(A_k(\overline{\tau})A_{\vartheta k}(\overline{\tau}) - B_k(\overline{\tau})B_{\vartheta k}(\overline{\tau}))d\overline{\tau}, \tag{73}$$

and $p_1(k) = \frac{2k}{1+\sqrt{\frac{EA}{mg}}}$, $p_2(k) = \frac{2k}{\sqrt{\frac{EA}{mg}}-1}$, $\theta = \frac{\sqrt{\frac{EA}{mg}}-1}{1+\sqrt{\frac{EA}{mg}}}$, $\vartheta = \frac{\sqrt{\frac{EA}{mg}}+1}{\sqrt{\frac{EA}{mg}}-1}$.

By noting that $A_{\vartheta k} = 0$ and $B_{\vartheta k} = 0$ inside the resonance zone around $s^{(k)}$, it follows that system (72) can be written as

$$\frac{\partial A_k}{\partial s_1} = \frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3 \frac{A_{\theta k}D_{p_1(k)} - B_{\theta k}C_{p_1(k)}}{2} + \frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\sin\left(\frac{\omega_1 l_0 v_0 e^{v_0\tau_k}}{2}s_1^2 + \psi_k(s_k)\right),$$
$$\frac{\partial B_k}{\partial s_1} = -\frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3 \frac{A_{\theta k}C_{p_1(k)} + B_{\theta k}D_{p_1(k)}}{2} - \frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\cos\left(\frac{\omega_1 l_0 v_0 e^{v_0\tau_k}}{2}s_1^2 + \psi_k(s_k)\right), \tag{74}$$

where $C_{p_1(k)}(s_1, s_2)$ and $D_{p_1(k)}(s_1, s_2)$ are given by (73). For any mode k satisfying the conditions that there exist p_1, p_2 s.t. $\frac{2k}{p_1} = \sqrt{\frac{EA}{mg}} + 1$ or $\frac{2k}{p_2} = \sqrt{\frac{EA}{mg}} - 1$, we can always find k_1 (k_1 is an integer), s.t. $\theta^{n-1}k = k_1$, and $\theta^n k$ is not an integer, $n = 1, 2, ...$ From that, we get a mode sequence $(k_1, \vartheta k_1, \vartheta^2 k_1, \ldots, k, \vartheta^n k_1, \ldots)$. We firstly solve the ordinary differential equations (74) for mode k_1, which can be rewritten as (68) (here the mode k_1 is denoted by k), and it can be solved as in (69). For the mode k_2 in (74), $k = k_2$, $A_{\theta k} = A_{k_1}$ and $B_{\theta k} = B_{k_1}$, thereby inside the resonance zone around $s^{(k_2)}$, we can obtain the solutions A_{k_2} and B_{k_2} from (74). Next, by using an iterative method we can predict and obtain the functions A_k and B_k. Note that (74) is a nonlinear perturbation problem. It is hard to obtain the analytical, explicit solution, but we can find properties of A_k and B_k by the above analysis, which can be used to describe the behaviour of the solution $\tilde{T}_{k,0}(s_0, s_1, s_2)$ of the $O(\sqrt{\varepsilon})$-problem (37). Moreover, the solution of (74) can be obtained by numerical calculations. Now, inside the resonance zone around $s^{(k)}$ in (26), the solution of the $O(\sqrt{\varepsilon})$-problem (37) is given by (69) and (74).

To summarize, the solution $\hat{w}(\xi, s)$ of Eq. (13) readily follows:

$$\hat{w}(\xi, s) = \sum_{n=1}^{\infty}[A_n(\sqrt{\varepsilon}(s-s^{(n)}), \varepsilon(s-s^{(n)})) \cos(n\pi(s-s^{(n)})) + B_n(\sqrt{\varepsilon}(s-s^{(n)}), \varepsilon(s-s^{(n)})) \sin(n\pi(s-s^{(n)}))]sin(n\pi\xi) + O(\varepsilon), \tag{75}$$

where $s^{(n)}$ is given by (26), and A_n and B_n are given by (60), (67), (69), and (74).

In the longitudinal direction, according to the analysis of (44)–(57), $\tilde{Y}_{k,0}$ in Eq. (44) can be approximated as:

$$\tilde{Y}_{k,0}(s_0, s_1, s_2) = C_k(s_1, s_2)\cos(\lambda_k s_0) + D_k(s_1, s_2)\sin(\lambda_k s_0). \tag{76}$$

- When k does not satisfy the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$C_k(s_1, s_2) = D_k(s_1, s_2) = 0, \tag{77}$$

which follows from (47) and (56).

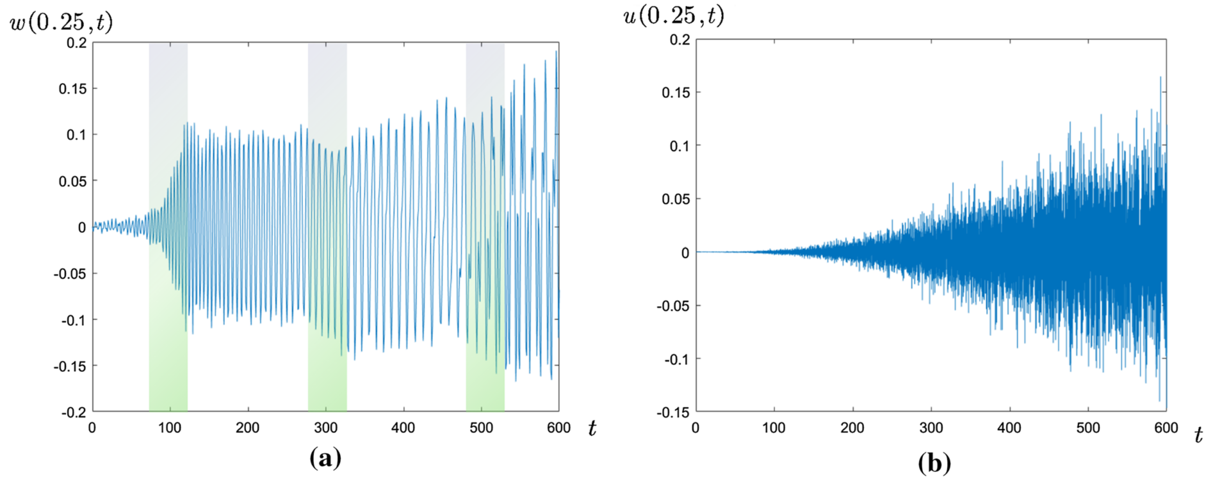

Fig. 2 **a** Transverse displacements $w(0.25, t)$. **b** Longitudinal displacements $u(0.25, t)$

- When k satisfies the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$C_k(s_1, s_2) = \int_b^{s_1} -\frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}B_{j_1} + B_{j_1-k}A_{j_1}) + \frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{j_2}B_{j_2+k} + B_{j_2}A_{j_2+k})d\bar{\tau},$$
$$D_k(s_1, s_2) = \int_b^{s_1} \frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}A_{j_1} - B_{j_1-k}B_{j_1}) - \frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{k+j_2}A_{j_2} - B_{k+j_2}B_{j_2})d\bar{\tau}, \tag{78}$$

which follows from (49) and (56). And inside the resonance zone around $s^{(k)}$, A_k and B_k are given by (69) and (74); outside the resonance zone, A_k and B_k are given by (60) and (67).

The solution $\hat{u}(\xi, s)$ of Eq. (14) readily follows:

$$\hat{u}(\xi, s) = \sum_{n=1}^{\infty}[C_n(\sqrt{\varepsilon}(s - s^{(n)}), \varepsilon(s - s^{(n)})) \cos(n\pi(s - s^{(n)})) + D_n(\sqrt{\varepsilon}(s - s^{(n)}), \varepsilon(s - s^{(n)})) \sin(n\pi(s - s^{(n)}))]sin(n\pi\xi) + O(\varepsilon), \tag{79}$$

where $s^{(n)}$ is given by (26), C_n and D_n are given by (77) and (78).

By the three-timescales perturbation method, we obtained that for special frequencies in the boundary excitations and for certain parameter values of the longitudinal stiffness and the conveyance mass, the transverse solution $\hat{w}(\xi, s)$ of Eq. (13) jumps up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the longitudinal solution $\hat{u}(\xi, s)$ of Eq. (14) jumps up from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$. We cannot (always) construct formal approximations of the solutions, but we can get properties and predictions of solution behaviours analytically on timescales of $O(\frac{1}{\varepsilon})$. Based on the properties and equations in the analysis, the approximated solutions for transverse and longitudinal motions will be computed by using an iterative method as well as by using a numerical method in the next section. Also the approximations will be computed by using a central finite difference scheme in the next section to verify the analytical results in this section.

5 Numerical results

Since the initial boundary value problem (1) for the transverse vibration and the initial boundary value problem (2) for the longitudinal vibration are complicated with nonlinear and coupled terms and a lot of parameters, we cannot construct formal explicit approximations of the solutions. To make the problems easier to analyse and simulate, we transform the problems (1)–(2) to the problems (13)–(14) by putting the problems dimensionless, by converting the problems to

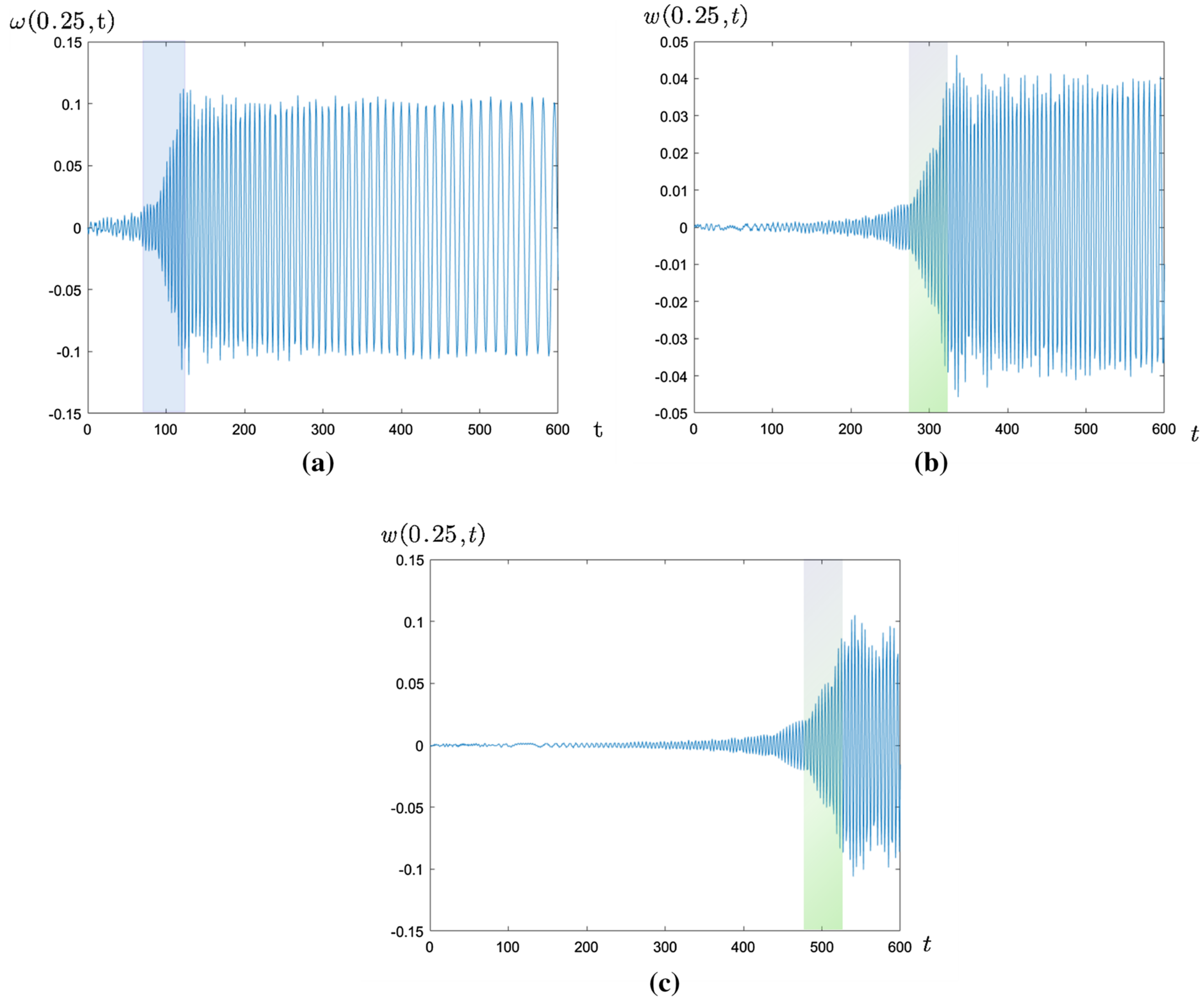

Fig. 3 **a** The first mode displacement, **b** the second mode displacement, and **c** the third mode displacement as functions of time for $\xi = 0.25$

a fixed spatial domain, and by Liouville–Green transformation. So, on the one hand, we can compute the transverse and the longitudinal motions of the cable for (13) and (14) by computing numerically the solutions of the ordinary differential Eq. (75) and (79). On the other hand, we can compute numerically the solutions of the problem (13) and the problem (14) straight-forwardly by applying a finite difference method. By comparing the results (the displacements and the energy) in the above two different methods, the obtained analytical results in the last section can be verified.

5.1 Analytical approximations

The numerical results simulating the transverse and the longitudinal vibration responses are computed based on the analytical expression (75) for $\hat{w}(\xi, s)$ and the expression (79) for $\hat{u}(\xi, s)$. The computations are performed by using the following parameters:

$$\begin{aligned} &\varepsilon = 0.01, \ \frac{EA}{mg} = 9, \ \mu_0 = 1, \\ &c_{u,0} = 1, \ c_{1,0} = 1, \ c_{2,0} = 1, \\ &\beta_{1,0} = 1, \ \beta_{2,0} = 1, \\ &\omega_1 = 0.5\pi, \ \omega_2 = 0.6\pi, \ l_0 = 1, v_0 = 1, \end{aligned} \tag{80}$$

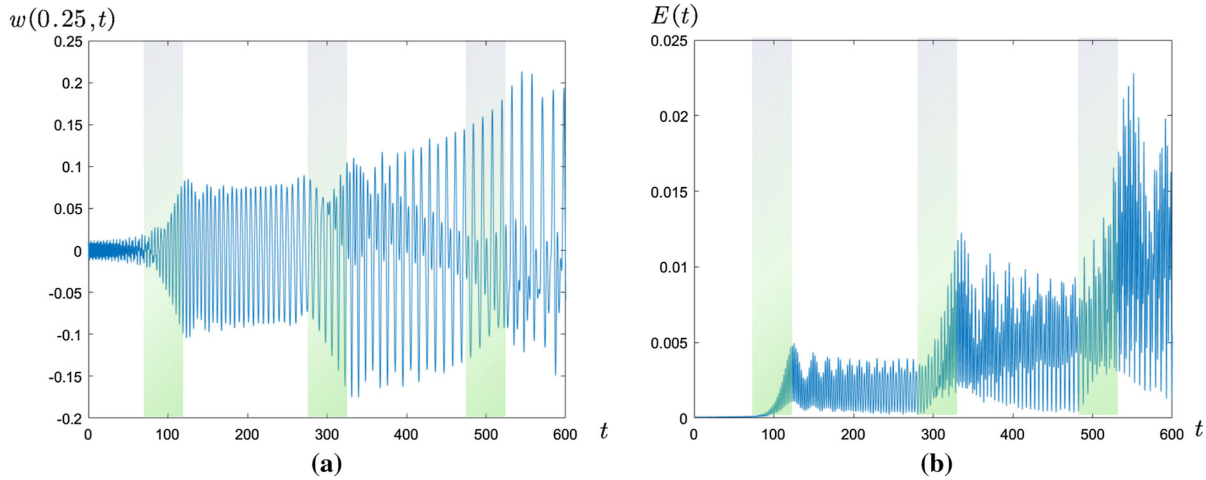

Fig. 4 **a** Transverse displacements $w(0.25, t)$, $\varepsilon = 0.01$. **b** Transverse vibratory energy

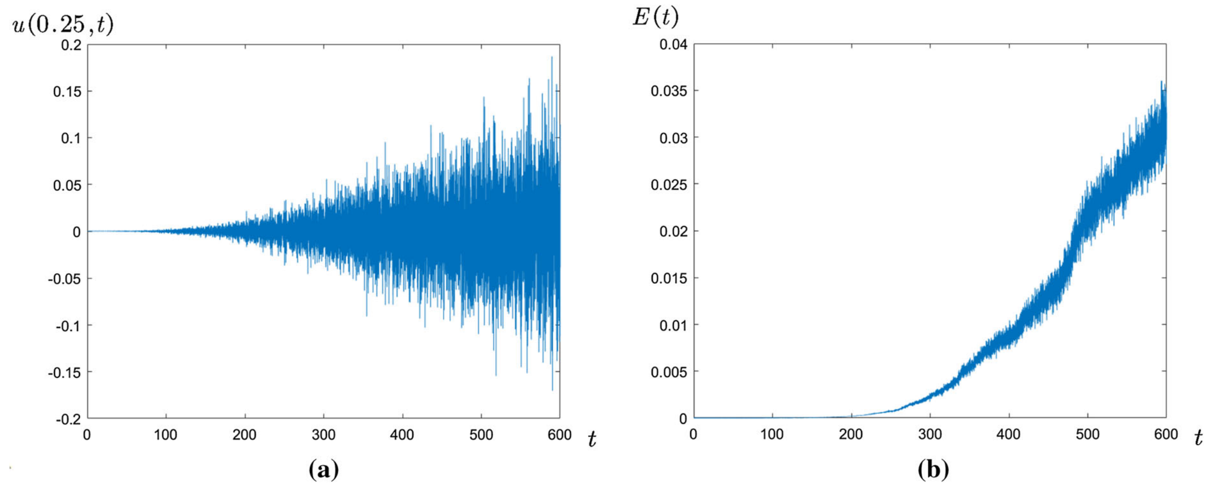

Fig. 5 **a** Longitudinal displacements $u(0.25, t)$, $\varepsilon = 0.01$. **b** Longitudinal vibratory energy

and the initial conditions are taken to be:

$$\hat{w}(\xi, 0) = 0.01 \sin(1.5\xi), \quad \hat{w}_s(\xi, 0) = 0,$$
$$\hat{u}(\xi, 0) = 0.0001 \sin(1.5\xi),$$
$$\hat{u}_s(\xi, 0) = 0, 0 \le \xi \le 1. \tag{81}$$

By using the Liouville–Green transformation with $\frac{ds}{dt} = \frac{1}{l(t)} = \frac{1}{l_0 + vt}$, we have

$$t = \frac{l_0 e^{vs} - l_0}{v}. \tag{82}$$

It follows from (82) and the resonance times given by (26) that the resonance zones (in the transverse direction) are located around the times

$$t_k = \frac{k\pi}{\omega_1 v} - \frac{l_0}{v}, \tag{83}$$

where the resonance time depends on the mode number k. For the first three oscillation modes of transverse motions, resonance emerges at times $t_1 \approx 100$, $t_2 \approx 300$, $t_3 \approx 500$. The displacements of the first and third mode are given by (69) and (75), the displacements of the second mode are given by (74) and (75). Around the first resonance time t_1, the displacement amplitudes jump up from initial states $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Around the second resonance time t_2 and the third resonance time t_3, the amplitudes jump up again from the $O(\varepsilon)$ level to the $O(\sqrt{\varepsilon})$ level. They are all illustrated in Fig. 2a.

The displacements of the longitudinal motion are given by (79), which are illustrated in Fig. 2b.

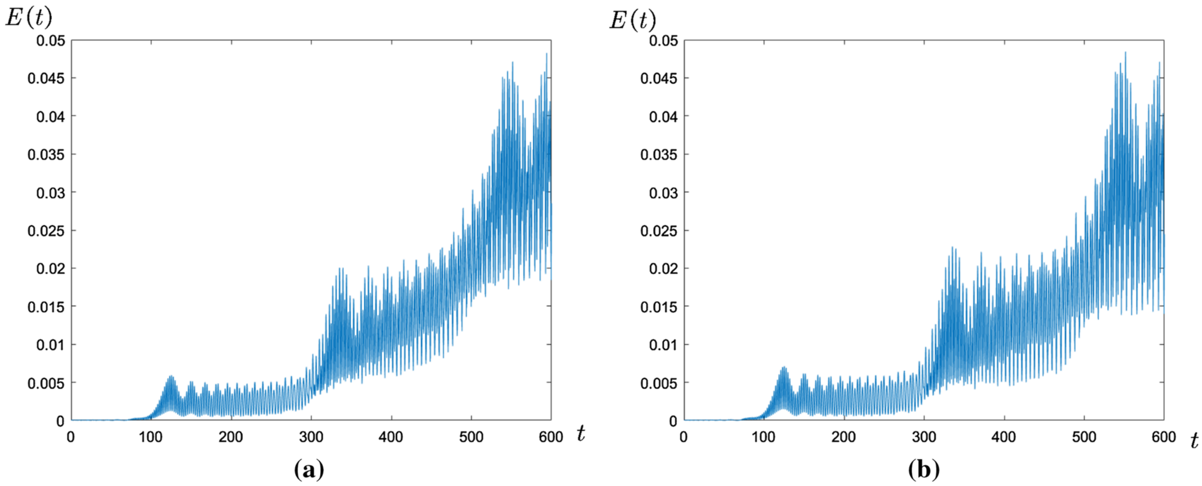

Fig. 6 **a** The total mechanical energy based on analytical results. **b** Analytical approximations The total mechanical energy based on numerical results

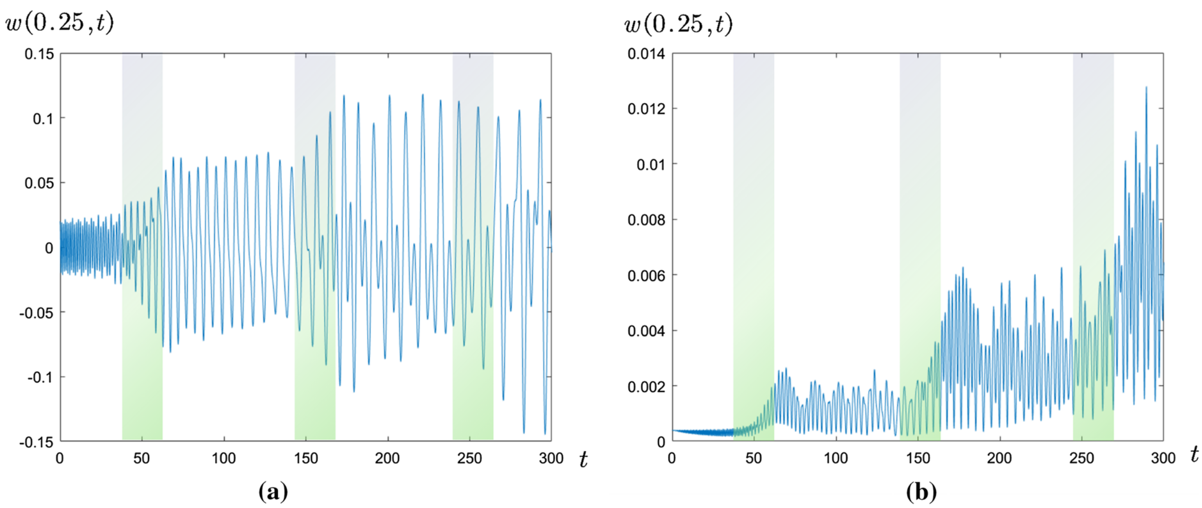

Fig. 7 **a** Transverse displacements $w(0.25, t)$, $\varepsilon = 0.02$, $\omega_1 = 0.5$. **b** Transverse vibratory energy

Also, to observe the displacement amplitudes of transverse motion clearly, Fig. 3 shows the close-up of the fragments of Fig. 2a. Figure 3a, b, and c shows the first mode displacement, the second mode displacement, and the third mode displacement of the transverse motions, respectively.

5.2 Numerical approximations

In this subsection, the finite difference method is applied in both the time and the space domain for both PDEs and boundary conditions in (6) and (7) with space grid size $d\xi = 5 \times 10^{-2}$, and time step $dt = 5 \times 10^{-3}$. We rewrite the so-obtained discretized Eq. (6) and (7) in matrix forms and use as numerical time integration method, the Crank–Nicolson method (see Appendix B). Note that the same parameter values as for the analytic approximations in Sect. 5.1 are used here for the computations.

In Fig. 4, the transverse displacements and the vibratory energy of the cable on timescales up to t = 600 are presented. In Fig. 4, one can see that the transverse resonances emerge around times $t_1 = 100$, $t_2 = 300$ and $t_3 = 500$. In the resonance zones, the displacements

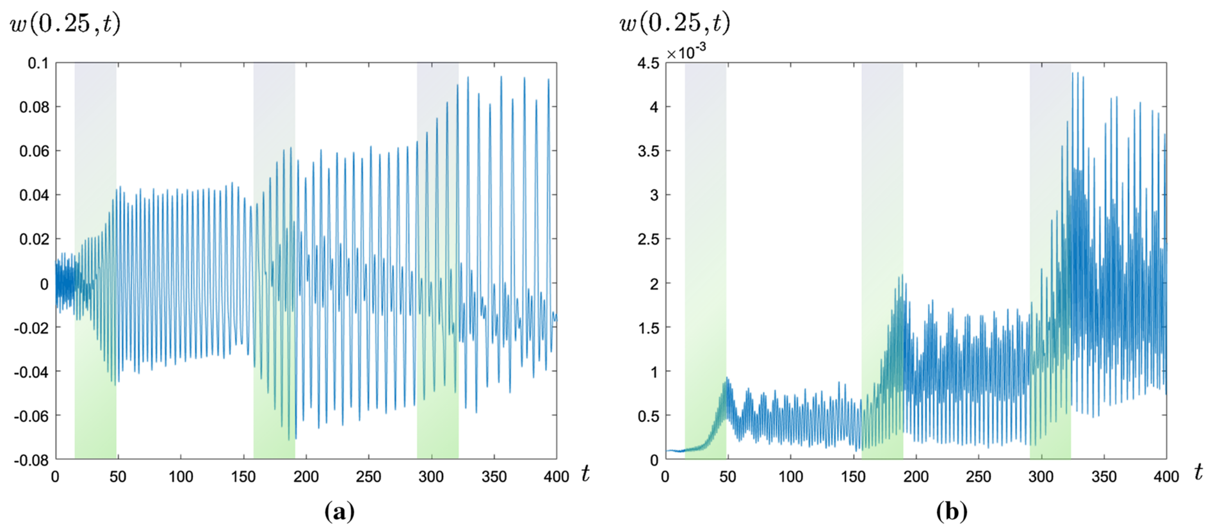

Fig. 8 **a** Transverse displacements $w(0.25, t)$, $\varepsilon = 0.01$, $\omega_1 = 0.75\pi$. **b** Transverse vibratory energy

and the energy increase, and in between these zones, stay constant (approximately). Around the first resonance time t_1, the displacement amplitudes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Around the second resonance time t_2 and the third resonance time t_3, the amplitudes change again at the $O(\sqrt{\varepsilon})$ level, where ε is a small parameter with $\varepsilon = 0.01$.

In Fig. 5, the longitudinal displacements and the vibratory energy of the cable on timescales up to t = 600 are given. In Fig. 5, one can see that the longitudinal displacements increase from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$, and that the vibratory energy increases from $O(\varepsilon^4)$ to $O(\varepsilon)$. In Fig. 6, the total mechanical energy (see also Appendix C for definitions) based on the analytical results and the total energy based on the numerical results can be compared. Based on the Figs. 2, 4, 5, and 6, we can draw the conclusion that the general dynamic behaviour of the solution as approximated by direct numerical integration of the problem is in agreement with the analytic approximations as obtained by applying perturbation methods.

Moreover, in Fig. 7, we make different choices for ε in the numerical approximations to see the influence of the values of ε on the system dynamics. For $\varepsilon = 0.02$, the first three resonance times become $t_1 = 50$, $t_2 = 150$, and $t_3 = 250$, and the displacement amplitudes also change. In Fig. 8, we make different choices for the frequencies of boundary excitations in the numerical approximations. For $\omega_1 = 0.75\pi$, the first three resonance times become $t_1 = 33$, $t_2 = 167$, and $t_3 = 300$, and the displacement amplitudes also change.

6 Conclusion

In this paper, we studied the coupled transverse and longitudinal vibrations and associated resonances induced by boundary excitations in a hoisting system. The problem is described by nonlinear coupled partial differential equations on a time-varying spatial interval with small harmonic disturbances at one end and a moving nonclassical boundary condition at the other end. Assuming that the transverse harmonic boundary disturbances and the corresponding initial values are of order ε, and the longitudinal harmonic boundary disturbances and the corresponding initial values are of order ε^2, it is shown in this paper that for special frequencies in the boundary excitations and that for certain parameter values of the longitudinal stiffness and the conveyance mass, many large oscillations arise in transverse and longitudinal directions. The oscillation modes for transverse motion jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the oscillation modes for longitudinal motion jump up from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$. To obtain these results, the method of separation of variables is presented, and perturbation methods (such as averaging methods, and singular perturbation techniques) are used. Furthermore, since the initial-boundary value problems for the transverse motion and the longitudinal motion are nonlinearly coupled, we cannot (always) construct formal approximations of the solutions but we can get properties and predictions of solution behaviour analytically on timescales of order ε^{-1}. Furthermore, approximations of the solutions are computed by using an iterative method as well as by using a numerical method. Also approximations of the solutions of the initial-boundary value problems are computed by using a central finite difference scheme. The numerical approximations are in agreement with the analytically obtained approximations. The analytical scheme in this problem can be extended to study other, and more complicated types of moving cable systems and also to other types of gyroscopic systems, where transverse and longitudinal motions are both involved and are governed by coupled differential equations with in time slowly varying coefficients.

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Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: The derivation of Eqs. (1) and (2)

According to Fig. 1, the partial differential equation (PDE) can be derived by the extended Hamilton's principle:

$$\int_{t_1}^{t_2} (\delta E_k(t) - \delta E_p(t) + \delta W_c(t))dt = 0. \tag{A1}$$

The kinetic energy $E_k(t)$ can be represented as

$$\begin{aligned} E_k(t) = \frac{1}{2}\rho \Bigg[&\int_{e_u(t)}^{l(t)} \left(\frac{Du}{Dt} + v\right)^2 dx \\ &+ \int_{e_u(t)}^{l(t)} \left(\frac{Dw}{Dt}\right)^2 dx \Bigg] \\ &+ \frac{1}{2}m \Bigg[\left(\frac{Du}{Dt} + v\right)^2 |_{x=l(t)} \\ &+ \left(\frac{Dw}{Dt}\right)^2 |_{x=l(t)} \Bigg], \end{aligned} \tag{A2}$$

where the operator $\frac{Du}{Dt}$ is defined as $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = u_t + vu_x$, and the operator $\frac{Dw}{Dt}$ is defined as $\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x} = w_t + vw_x$. The potential energy $E_p(t)$ can be expressed as

$$\begin{aligned} E_p(t) = \frac{1}{2}EA \int_{e_u(t)}^{l(t)} z^2 dx + \int_{e_u(t)}^{l(t)} Tz dx \\ + E_{gs} - \int_{e_u(t)}^{l(t)} \rho g u dx - mgu |_{x=l(t)}, \end{aligned} \tag{A3}$$

where $z = u_x + \frac{1}{2}w_x^2$, and

$$\begin{aligned} &\delta E_k(t) - \delta E_p(t) \\ &= \rho \int_{e_u(t)}^{l(t)} \left(\frac{Du}{Dt} + v\right) \delta\frac{Du}{Dt} dx \\ &+ m \left(\frac{Du}{Dt} + v\right) \delta\frac{Du}{Dt} |_{x=l(t)} \\ &+ \rho \int_{e_u(t)}^{l(t)} \left(\frac{Dw}{Dt}\right) \delta\frac{Dw}{Dt} dx \\ &+ m \left(\frac{Dw}{Dt}\right) \delta\frac{Dw}{Dt} |_{x=l(t)} \\ &- \Bigg[EA \int_{e_u(t)}^{l(t)} z\delta z dx + \int_{e_u(t)}^{l(t)} T\delta z dx \\ &- \int_{e_u(t)}^{l(t)} \rho g \delta u dx - mg\delta u |_{x=l(t)} \Bigg]. \end{aligned} \tag{A4}$$

The virtual work δW_c done by the distributed and the lumped damping force is given as

$$\begin{aligned} \delta W_c(t) = &- \int_{e_u(t)}^{l(t)} c_2 \frac{Du}{Dt} \delta u dx \\ &- \int_{e_u(t)}^{l(t)} c_1 \frac{Dw}{Dt} \delta w dx - c_u \frac{Du}{Dt} \delta u |_{x=l(t)} . \end{aligned} \tag{A5}$$

By substituting equation (A2)–(A5) into (A1), we obtain

$$\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)}\rho\left(\frac{Du}{Dt}+v\right)\delta\frac{Du}{Dt}dxdt + \int_{t_1}^{t_2} m\left(\frac{Du}{Dt}+v\right)\delta\frac{Du}{Dt}\mid_{x=l(t)} dt + \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)}\rho\left(\frac{Dw}{Dt}\right)\delta\frac{Dw}{Dt}dxdt + \int_{t_1}^{t_2} m\left(\frac{Dw}{Dt}\right)\delta\frac{Dw}{Dt}\mid_{x=l(t)} dt - EA\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} z\delta z dxdt - \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} T\delta z dxdt + \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)}\rho g\delta u dxdt + \int_{t_1}^{t_2} mg\delta u\mid_{x=l(t)} dt - \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} c_2\frac{Du}{Dt}\delta u dxdt - \int_{t_1}^{t_2} c_u\frac{Du}{Dt}\delta u\mid_{x=l(t)} dt - \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} c_1\frac{Dw}{Dt}\delta w dxdt = 0. \quad \text{(A6)}$$

By integrating by parts, it follows from (A6) that

$$\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)}[-\rho(u_{tt}+2vu_{xt}+v^2u_{xx}+\hat{a}u_x+\hat{a}) + EAz_x + T_x + \rho g - c_2(u_t+vu_x)]\delta u dxdt + \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)}[-\rho(w_{tt}+2vw_{xt}+v^2w_{xx}+\hat{a}w_x) + EA(zw_x)_x + (Tw_x)_x - c_1(w_t+vw_x)]\delta w dxdt + \int_{t_1}^{t_2}[-m(u_{tt}+2vu_{xt}+v^2u_{xx}+\hat{a}u_x+\hat{a}) - EAz - T + mg - c_u(u_t+vu_x)]\delta u\mid_{x=l(t)} dt + \int_{t_1}^{t_2}[-m(w_{tt}+2vw_{xt}+v^2w_{xx}+\hat{a}w_x) - EAzw_x - Tw_x]\delta w\mid_{x=l(t)} dt + \int_{t_1}^{t_2}[-\rho v(u_t+vu_x+v)+EAz + T]\delta u\mid_{x=e_u(t)} dt + \int_{t_1}^{t_2}\dot{e_u}(t)\rho(u_t+vu_x+v)\delta u\mid_{x=e_u(t)} dt + \int_{t_1}^{t_2}[-\rho v(w_t+vw_x)+EAzw_x + Tw_x]\delta w\mid_{x=e_u(t)} dt + \int_{t_1}^{t_2}\dot{e_u}(t)\rho(w_t+vw_x)\delta w\mid_{x=e_u(t)} dt = 0.$$

So, the governing equations of motion are given by

$$\rho(u_{tt}+2vu_{xt}+v^2u_{xx}+\hat{a}u_x+\hat{a}) - EAz_x - T_x - \rho g + c_2(u_t+vu_x) = 0, \quad e_u(t) < x < l(t),\ t>0. \quad \text{(A7)}$$

$$\rho(w_{tt}+2vw_{xt}+v^2w_{xx}+\hat{a}w_x) - EA(zw_x)_x - (Tw_x)_x + c_1(w_t+vw_x) = 0, \quad e_u(t) < x < l(t),\ t>0. \quad \text{(A8)}$$

The corresponding boundary conditions on the upper end at $x = e_u(t)$ are given by:

$$[-\rho v(u_t+vu_x+v)+EAz + T + \dot{e_u}(t)\rho(u_t+vu_x+v)]\mid_{x=e_u(t)} = 0,$$
$$[-\rho v(w_t+vw_x)+EAzw_x + Tw_x + \dot{e_u}(t)\rho(w_t+vw_x)]\mid_{x=e_u(t)} = 0, \quad t\ge 0, \quad \text{(A9)}$$

and the boundary conditions at $x = l(t)$ are given by:

$$[m(u_{tt}+2vu_{xt}+v^2u_{xx}+\hat{a}u_x+\hat{a} + EAz + T - mg + c_u(u_t+vu_x)]\mid_{x=l(t)} = 0,$$
$$[m(w_{tt}+2vw_{xt}+v^2w_{xx}+\hat{a}w_x) + EAzw_x + Tw_x]\mid_{x=l(t)} = 0, \quad t\ge 0. \quad \text{(A10)}$$

Note that (A9) and (A10) are the natural boundary conditions. But (A9) is not appropriate for our problem, since the string is excited at the top boundary with the fundamental excitations $e_u(t)$ and $e_w(t)$. Thus, the correct boundary condition at the upper end are:

$$u(e_u(t),t) = e_u(t), \quad w(e_u(t),t) = e_w(t), \quad t\ge 0, \quad \text{(A11)}$$

where $e_u(t)$ and $e_w(t)$ are given in Nomenclature in section 2.1, and at the bottom boundary, the string is assumed to be fixed in horizontal direction. Thus, the corresponding transverse boundary condition at $x = l(t)$ is:

$$w(l(t),t) = 0, \quad t\ge 0. \quad \text{(A12)}$$

Considering

$$T(x,t) = [m+\rho(l(t)-x)]g, \quad e_u(t)\le x\le l(t), \quad \text{(A13)}$$

together with the governing equations given in (A7) and in (A8), the boundary excitations conditions given in

(A10), (A11), and (A12), we obtain the initial boundary value problem (1) for the transverse vibration and (2) for the longitudinal vibration. The reader is also referred to the papers [21–24] for detailed derivations of the dynamic models for strings, beams, plates, and membranes.

Appendix B: Discretization and time integration

To solve (13) numerically, it is convenient to rewrite the second order partial differential equation as a system of two coupled, first-order partial differential equations:

$$\begin{aligned}\check{w}_t &= \check{\zeta},\\ \check{\zeta}_t &= \frac{1}{l^2}[1+\mu l(1-\xi)]\check{w}_{\xi\xi}\\ &\quad+\frac{2v}{l}(\xi-1)\check{\zeta}_\xi - c_1\check{\zeta} - \frac{\mu}{l}\check{w}_\xi\\ &\quad+(1-\xi)\omega_2^2\beta_2\cos(\omega_2 t+\alpha).\end{aligned} \tag{B14}$$

Next, let us use the mesh grids $\xi_j = (j-1)\Delta\xi$ for $j = 1, 2, ldots, n, n+1$ with $n\Delta\xi = 1$. By introducing the differences, $\check{w}_{\xi\xi}(\xi_j, t) = \frac{\check{w}_{j+1}-2\check{w}_j+\check{w}_{j-1}}{(\Delta\xi)^2} + O((\Delta\xi)^2)$, $\check{\zeta}_\xi(\xi_j, t) = \frac{\check{\zeta}_{j+1}-\check{\zeta}_{j-1}}{2\Delta\xi} + O((\Delta\xi)^2)$, it follows how system (B14) can be discretized, yielding:

$$\begin{cases}\frac{d\check{w}}{dt}(\xi_j,t) = \check{\zeta}_j,\\ \frac{d\check{\zeta}}{dt}(\xi_j,t) = r_j(\check{w}_{j+1}-2\check{w}_j+\check{w}_{j-1})\\ \qquad +q_j(\check{\zeta}_{j+1}-\check{\zeta}_{j-1}) - c_1\zeta_j\\ \qquad -p(\check{w}_{j+1}-\check{w}_{j-1}) + (1-\xi)\omega_2^2\beta_2\\ \qquad \times\cos(\omega_2 t+\alpha),\end{cases}$$

where $r_j = \frac{1+\mu l(1-\xi_j)}{l^2(\Delta\xi)^2}$, $q_j = \frac{v(\xi_j-1)}{l\Delta\xi}$, $p = \frac{\mu}{2l\Delta\xi}$ for j=1,2,..,n. Further,

$$R = \begin{pmatrix} -2r_1 & r_1-p & 0 & \cdots & \cdots & 0\\ r_2+p & -2r_2 & r_2-p & \cdots & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & r_{n-1}+p & -2r_{n-1} & r_{n-1}-p\\ 0 & \cdots & \cdots & 0 & r_n+p & -2r_n\end{pmatrix} \in \mathbb{R}^{n\times n}, \text{ and}$$

$$P = \begin{pmatrix} -c_1 & q_1 & 0 & \cdots & \cdots & 0\\ -q_2 & -c_1 & q_2 & \cdots & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & -q_{n-1} & -c_1 & q_{n-1}\\ 0 & \cdots & \cdots & 0 & -q_n & -c_1\end{pmatrix} \in \mathbb{R}^{n\times n},$$

The four matrices $\emptyset$, I, R, and P compose the system matrix M:

$$M = \begin{pmatrix}\emptyset & I\\ R & P\end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

where $\emptyset$ is the zero matrix, and I is the identity matrix. In addition, let us introduce the following vectors: $w = (w_1(\xi_1,t), w_2(\xi_2,t), \ldots, w_n(\xi_n,t), \zeta_1(\xi_1,t), \zeta_2(\xi_2,t), \ldots, \zeta_n(\xi_n,t))^T$, $s = (\underbrace{0,0,\ldots,0}_{\text{n times}}, \underbrace{\bar{s}_1,\bar{s}_2,\ldots,\bar{s}_n}_{\text{n times}})^T$, where $\bar{s}_i = (1-\xi_i)\omega_2^2\beta_2\cos(\omega_2 t+\alpha)$. So, system (B14) can be written in the following matrix form: $\frac{dw}{dt} = Mw + s$. In order to perform a time integration, we apply the Crank–Nicolson method. Introducing the mesh grid in time, $t_k = k\Delta t$ for k=1,2,...,n, we obtain

$$\begin{aligned}w^{k+1} &= Dw^k\\ &\quad+\frac{\Delta t}{2}\left(I - \frac{\Delta t}{2}M^{k+1}\right)^{-1}(s^{k+1}+s^k),\end{aligned} \tag{B15}$$

where $w^k = (w_1(\xi_1,t_k), w_2(\xi_2,t_k), \ldots, w_n(\xi_n,t_k), \zeta_1(\xi_1,t_k), \zeta_2(\xi_2,t_k), \ldots, \zeta_n(\xi_n,t_k))^T$, $s^k = (\underbrace{0,0,\ldots,0}_{\text{n times}}, \underbrace{\bar{s}_1^k,\bar{s}_2^k,\ldots,\bar{s}_n^k}_{\text{n times}})^T$ with $\bar{s}_i^k = (1-\xi_i)\omega_2^2\beta_2\cos(\omega_2 t_k+\alpha)$, I is the identity matrix and $I \in \mathbb{R}^{2n\times 2n}$, and $D = (I-\frac{\Delta t}{2}M^{k+1})^{-1}(I+\frac{\Delta t}{2}M^k)$. Similarly, we can also directly integrate problem (7) with the above numerical scheme.

Appendix C: Energy

The mechanical energy of the initial-boundary value problem (2) related to the transverse motion is given by

$$E_1(t) = \frac{1}{2}\int_0^{l(t)} [\rho(w_t+vw_x)^2 + Tw_x^2]dx,$$

where T is given by (3). Using the dimensionless quantities, we rewrite the energy in a dimensionless form:

$$\begin{aligned}E_1(t) &= \frac{1}{2}\int_0^{l(t)} [(w_t+vw_x)^2\\ &\quad+(1+\mu(l(t)-x))w_x^2]dx.\end{aligned}$$

In order to define the energy on the interval (0,1), we change the variables by using the following transformation $\xi = \frac{x}{l(t)}$:

$$\begin{aligned}E_1(t) &= \frac{1}{2l(t)}\int_0^1 [(l(t)\tilde{w}_t + (1-\xi)v\tilde{w}_\xi)^2\\ &\quad+(1+l(t)\mu(1-\xi))\tilde{w}_\xi^2]d\xi.\end{aligned} \tag{C16}$$

The mechanical energy of the initial-boundary value problem (1) related to the longitudinal motion is given by

$$E_2(t) = \frac{1}{2}\int_0^{l(t)} [\rho(u_t + vu_x)^2 + EAu_x^2]dx + \frac{m}{2}[u_t(l(t), t) + vu_x(l(t), t)]^2.$$

Using the dimensionless quantities, we rewrite the energy in a dimensionless form:

$$E_2(t) = \frac{1}{2}EAL\int_0^{l(t)} [(u_t + vu_x)^2 + u_x^2]dx + \frac{EAm}{2\rho}[u_t(l(t), t) + vu_x(l(t), t)]^2.$$

In order to define the energy on the interval (0,1), we change the variables by using the following transformation $\xi = \frac{x}{l(t)}$:

$$E_2(t) = \frac{EAL}{2l(t)}\int_0^1 [(l(t)\tilde{u}_t + (1-\xi)v\tilde{u}_\xi)^2 + \tilde{u}_\xi^2]d\xi + \frac{EAm}{2\rho l^2(t)}[l(t)\tilde{u}_t(1, t) + (1-\xi)v\tilde{u}_\xi(1, t)]^2. \quad \text{(C17)}$$

The total mechanical energy is now given by

$$E(t) = \frac{1}{2l(t)}\int_0^1 [(l(t)\tilde{w}_t + (1-\xi)v\tilde{w}_\xi)^2 + (1 + l(t)\mu(1-\xi))\tilde{w}_\xi^2]d\xi + \frac{EAL}{2l(t)}\int_0^1 [(l(t)\tilde{u}_t + (1-\xi)v\tilde{u}_\xi)^2 + \tilde{u}_\xi^2]d\xi + \frac{EAm}{2\rho l^2(t)}[l(t)\tilde{u}_t(1, t) + (1-\xi)v\tilde{u}_\xi(1, t)]^2. \quad \text{(C18)}$$

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