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ORIGINAL PAPER

Estimations and Control of Julia Sets of the SIS Model Perturbed by Noise

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Abstract The estimations and control of Julia sets of the SIS(susceptible-infectious-susceptible) model under noise perturbation are studied. At first, a discrete SIS model is introduced, and the effects of additive and multiplicative noises on the fractal characteristics of the SIS model are discussed. Then, estimations of the Julia sets of the SIS model under additive and multiplicative noise perturbations are given, respectively. At last, the feedback control method is used to set appropriate controllers to realize control of the Julia set, and the influence of noise on the Julia set of the SIS model is reduced. The reason why this method is effective is also explained.

Keywords Julia sets · Additive noise · Multiplicative noise · Estimations · The feedback control

1 Introduction

With the continuous development of human civilization, the medical and health care level has been greatly improved compared with the past. However, there are still new infectious diseases posing challenges to humanity. In 2020, the outbreak of COVID-19 took many innocent lives and brought unprecedented impacts on the global economy and trade. To better prevent and control infectious diseases, many researchers should analyze them from a mathematical point of view. In fact, infectious disease models have been studied since the 20th century [1–3]. New mathematical models are built based on classical models to study the transmission mechanism and coping strategies of infectious diseases, computer viruses, and rumors in groups. Fatmawati et al. [4] considered a novel fractional model to investigate the (tuberculosis) TB model dynamics with two age groups of humans. Liu et al. [5] proposed a mechanism considering the co-evolution between information states and network topology simultaneously. Alshammari and Khan [6] established a complex SIR epidemic dynamics model based on nonlinear incidence and nonlinear recovery considering the impact of available hospital beds and reduction interventions on the spread of infectious diseases. In recent years, many scholars have focused on the stability and existence of periodic solutions, the equilibrium position of equilibrium points, and the search for bifurcation points. Pastor and Vespignani [7] studied epidemic dynamics in bounded scale-free networks with soft and hard connectivity cut-offs. Amine et al. [8] proposed the global dynamics of a SIRI epidemic model with latency and a general nonlinear incidence function. Khan et al. [9] formulated a new mathematical model for the dynamics of COVID-19 with quarantine and isolation. On account of the effect of limited treatment resources on

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the control of epidemic disease, a saturated removal rate is incorporated into Hethcote's SIR epidemiological model by Zhang and Suo [10].

Nature is full of randomness, the human society is. In some economic and physical models, stochastic perturbation exists widely, leading to the complexity and uncertainty of some factors in the models. When the influence brought by these disturbances cannot be ignored, designing the system according to the deterministic theory will make the system behavior deviate from the original requirements, which requires people to analyze the model in an uncertain sense. Zhang and Chen [11] discussed the H_∞ control problem for a class of nonlinear stochastic systems with both state- and disturbance-dependent noise. Ugrinovskii and Petersen [12] studied existence and optimality properties of so-called guaranteed cost controllers for an uncertain system subject to structured uncertainty. They consider the effects of random disturbances on the system. In addition, noise also has a significant influence on chaos and fractals. Argyris and Andreadis [13] studied the influence of noise on a mathematical model which contains the coexistence of chaotic attractors. Inspired by the study of noise of the Mandelbrot map in two parameter deformation families, Negi and Rani [14] introduced a new noise criterion and analyzed its effect on the usual and superior Mandelbrot maps. Wang et al. [15] researched on the structural characteristic and the fission-evolution law of the generalized Mandelbrot set (generalized M set in short) perturbed by composing noise of additive and multiplicative, analyzed the effect of random perturbation to the generalized M set. Wang et al. [16] researched the structural characteristic and the fission-evolution law of four different kinds of generalized Julia set (generalized J set in short) with different parameter c, analyzed the effect of random perturbation to the generalized J set, and illuminated the stability of the generalized J set. Due to the widespread existence of random disturbance, filtering is vital to observe the true value of the model better. Fridman et al. [17] considered the problem of robust H_2 estimation of a combination of states of a stationary linear system with time delays. Nkwayep et al. [18] developed an integrated Kalman filter (EnKf) method to estimate immeasurable state variables and unknown parameters in COVID-19 models. Nguang and Shi [19] considered the problem of designing a delay-dependent robust H_∞ filter for time delay Takagi-Sugeno fuzzy models. Inspired by the denoising effect of filtering in the noise-affected system, estimations of the Julia sets of the noise-affected model are given to observe better the impact of noise on the overall shape of the Julia set of the SIS(susceptible-infectious-susceptible) model.

After Mandelbrot published an epoch-making paper entitled "How Long Is the Coast of Britain?" in the 20th century [20], people realized that fractals could be used to explain some irregular or not smooth figures or sets. More and more scholars devoted themselves to the study of fractals. Gujar and Bhavsar [21] considered the generalized transformation function $z \rightarrow z^\alpha + C$ for generating fractal images. The symmetries of Julia sets of Newton's method is investigated by Yang [22]. Sun and Zhang [23] studied the forced Brusselator model from the fractal viewpoint. Zhang et al. [24] introduced a visualization of Julia sets of the complex Henon map system with two complex variables. In recent years, some scholars have tried to study the fractal characteristics of some fractional models and discuss the fractal dynamics of models from the perspective of fractional order. Sun and Liu [25] introduced the fractional Potts model on diamond-like hierarchical lattices. Wang et al. [26] investigated the structures and properties of the spatial Julia set generated by a fractional complex Lotka-Volterra system with noise. In nature and science, fractals are still a new field, and it has more functions and significance to be studied. Therefore, the fractal characteristics of the SIS model under noise disturbance are considered.

Due to the importance of the SIS model in the infectious disease model, the fractal characteristics of the SIS model under noise disturbance are discussed. Firstly, the discrete form of the SIS model is given, and the Julia set of the SIS model is made according to it. Secondly, additive and multiplicative noise effects on Julia sets of discrete-form SIS models are considered respectively. Thirdly, to more clearly observe the impact of noise on the overall shape of the Julia set of the SIS model, estimations of the Julia sets of the SIS model under the influence of noise are given. Finally, according to the matrix disturbance theory, two controllers are designed using the feedback control method to control the Julia set of the model. After control, the Julia set of the model has a larger attractive domain and becomes more stable under noise interference.

2 The SIS model in discrete form and the Julia set

In the SIS model, the birth rate and death rate are not considered, and the population is divided into infected and susceptible people. Susceptible people have a certain chance to become infected after exposure to the virus, and infected people become susceptible after effective treatment. The two types of people will carry out transformation between each other [27], thus

$$\begin{cases} \frac{dx}{dt} = cxy - ux, \\ \frac{dy}{dt} = -cxy + ux, \end{cases} \tag{1}$$

where x is the infected persons, y is the susceptible persons, c is the transmission rate of infected persons contacting susceptible persons, and u is the effective cure rate of infected persons becoming susceptible persons.

Discretized the model (1) and approximate difference form of the SIS model is obtained

$$\begin{cases} \frac{x(t+\Delta t)-x(\Delta t)}{\Delta t} = cxy - ux, \\ \frac{y(t+\Delta t)-y(\Delta t)}{\Delta t} = -cxy + ux, \end{cases} \tag{2}$$

of course, the smaller Δt is, the closer (2) is to (1).

After the transformation, then we have

$$\begin{cases} x(t+\Delta t) = ax(t)y(t) - bx(t) + x(t), \\ y(t+\Delta t) = -ax(t)y(t) + bx(t) + y(t). \end{cases} \tag{3}$$

where $a = c\Delta t$ and $b = u\Delta t$.

Using x_{n+1} for $x(t+\Delta t)$, y_{n+1} for $y(t+\Delta t)$, x_n for $x(t)$, y_n for $y(t)$, then (3) becomes

$$\begin{cases} x_{n+1} = ax_n y_n - bx_n + x_n, \\ y_{n+1} = -ax_n y_n + bx_n + y_n. \end{cases}$$

Definition 2.1 [23,24] The filled-in Julia set of function f on the complex plane is defined as: $K(f) = \{z \in C \mid f^k(z) \nrightarrow \infty, k \to \infty\}$, the Julia set of the function f is defined to be the boundary of $K(f)$, i.e, $J(f) = \partial K(f)$.

The image of the Julia set of the SIS model is given in Fig.1 when $a = 0.008, b = 0.016$. In all the images that follow in this article, we always set $a = 0.008, b = 0.016$. Fig.1 is the Julia set of the SIS model, and the area surrounded by the curve is the filled-in Julia set. Suppose the initial values of the number of infected persons x and the number of susceptible persons y are in the filled-in Julia set. In this case, the infectious disease is relatively stable, and the number of both parties does not increase to infinity over time. Suppose the initial values of the number of infected persons and the number of susceptible persons exceed this area. In this case, at least one of the number of infected persons and the number of susceptible persons tends to infinity [28].

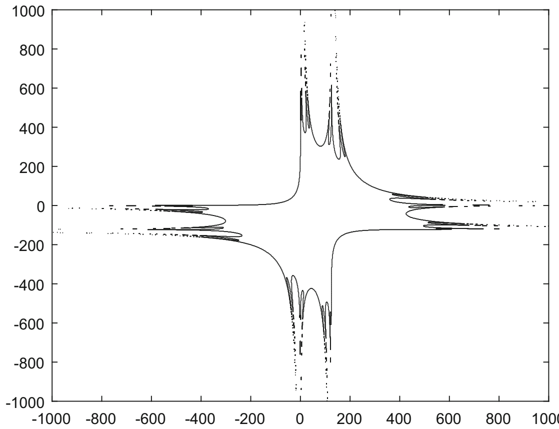


Fig. 1 Original Julia set of the SIS model when $a = 0.008$, $b = 0.016$

3 Julia sets of the SIS model perturbed by noise

In real life, there will always be noise to disrupt the model due to the presence of random interference. In some cases, the effect of noise on the model cannot be ignored. In the following discussion, the impact of different additive and multiplicative noises on the fractal characteristics of the SIS model will be observed by adjusting parameters to change the noise.

3.1 The SIS model perturbed by the additive noise

Additive noise has nothing to do with the designed signal, but always interferes with the designed signal. Noise is present whether there is a signal or not. Some random factors, such as seasonal changes, population movements, etc., may lead to random changes in the number of infected and susceptible people with the disease. The SIS model with the additive noise [15] is

$$\begin{cases} x_{n+1} = ax_n y_n - bx_n + x_n + \mu_n, \\ y_{n+1} = -ax_n y_n + bx_n + y_n + \omega_n, \end{cases} \tag{4}$$

where a and b are still described above, μ_n is the noise obeying $N(\theta_1, \sigma_1^2)$, ω_n is the noise obeying $N(\theta_2, \sigma_2^2)$. x_n and μ_n are mutually independent, y_n and ω_n are mutually independent.

To better observe the influence of additive noise on Julia sets of the SIS model, normal additive noises with different mean and variance are added to the model. Images of each Julia set of the SIS model with different additive noises are shown in Fig. 2.

It can be seen from Fig. 2 that when the model is affected by normal additive noise with mean zero and slight variance, the Julia set of the SIS model and the filled-in Julia set do not change basically (Fig. 2(a)). With the increase of noise variance, the Julia set of the SIS model under the influence of noise is no longer a series of curves and is composed of a large number of irregular points (Fig. 2(b) and 2(c)). However, when the mean value of noise is not 0 and its value is significant, the overall shape of the Julia set changes significantly (Fig. 2(e), 2(d) and 2(f)).

3.2 The SIS model perturbed by the multiplicative noise

The relationship between multiplicative noise and the signal is multiplicative. If the signal exists, it exists. If the signal does not exist, it does not exist. The randomness of multiplicative noise is considered to be caused by the time-varying or nonlinearity of the system, and it has stronger time-varying and anti-filtering properties than additive noise. Assuming that the cure rate will increase or decrease due to virus mutation or vaccine development, the multiplicative noise model [15] can be obtained as follows.

$$\begin{cases} x_{n+1} = ax_n y_n - bx_n + x_n + \alpha_n x_n, \\ y_{n+1} = -ax_n y_n + bx_n + y_n - \alpha_n x_n, \end{cases} \tag{5}$$

where α_n is the noise obeying $N(\theta_3, \sigma_3^2)$, which is independent of x_n and y_n.

Normal multiplicative noise with zero mean and different variance is added to the model to better observe the effect of multiplicative noise on the Julia set of the SIS model. Images of each Julia set of the SIS model with different multiplicative noises are shown in Fig. 3.

It can be seen from the images that under the influence of multiplicative noise, the Julia set of the SIS model changes to some extent, and some randomly distributed points show an unstable state of the Julia set of the SIS model. It's not difficult to see that the noise has a more significant impact on the Julia set near the $x - axis$ than the Julia set in the rest of the image (Fig. 3(a) and 3(b)) because the multiplicative noise random terms is related to x_n. When the variance of multiplicative noise is gradually increased, similar to the influence of additive noise on the model, the Julia set of the SIS model becomes more and more unstable (Fig. 3(a), 3(b), 3(c) and 3(d)).

In general, with the increase of variance of both additive noise and multiplicative noise, the Julia set of the SIS model tends to expand outward and squeeze inward, indicating that the influence of noise will make the Julia set of the SIS model become "unstable". When the added noise variance is slight and the mean is zero, the overall shape of the Julia set will not change much. When the noise variance is further increased, the Julia set is composed of many points instead of smooth curves.

We will analyze the effect of the Julia set perturbation on epidemics from the fractal perspective. The filled-in Julia set can be thought of as stable domains within which epidemics do not suddenly become uncontrollable. The Julia set can be seen as the "threshold" at which the epidemic will not break out, but once this "threshold" is exceeded, the epidemic will get out of control. The Julia set after noise perturbation is in a "random" state, and the "threshold" becomes uncertain, which is very unfavorable for our prediction and control.

4 Estimations of Julia Sets of the SIS Model under Noise Perturbation

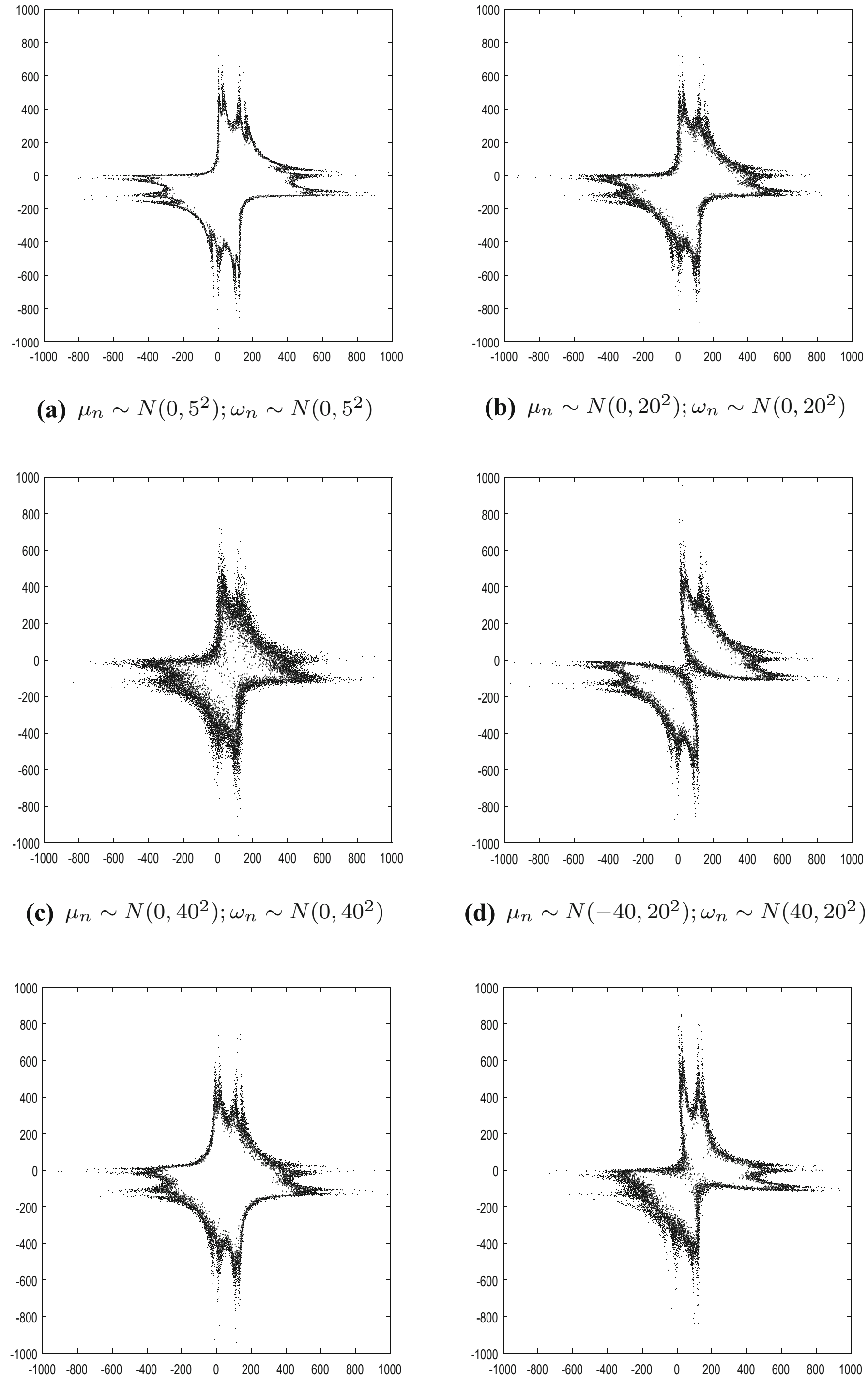
Because in the iterative process of the initial value points of the SIS model, the output is affected by the input and the noise disturbance, which makes the output Julia set cannot be accurately observed. To better observe the Julia set, we suppress the noise signal and increase the smoothness of the Julia set of the model perturbed by noise, thereby realizing the estimations of the Julia sets of the SIS model disturbed by noise.

4.1 Estimations of Julia Sets of the SIS Model under Additive Noise Perturbation

Write (4) in vector form

$$\varphi_{n+1} = F(\varphi_n) + \xi_n,$$

Fig. 2 Julia sets under different additive noise perturbations



(a) $\mu_n \sim N(0, 5^2); \omega_n \sim N(0, 5^2)$

(b) $\mu_n \sim N(0, 20^2); \omega_n \sim N(0, 20^2)$

(c) $\mu_n \sim N(0, 40^2); \omega_n \sim N(0, 40^2)$

(d) $\mu_n \sim N(-40, 20^2); \omega_n \sim N(40, 20^2)$

(e) $\mu_n \sim N(40, 20^2); \omega_n \sim N(-40, 20^2)$

(f) $\mu_n \sim N(-50, 20^2); \omega_n \sim N(0, 20^2)$

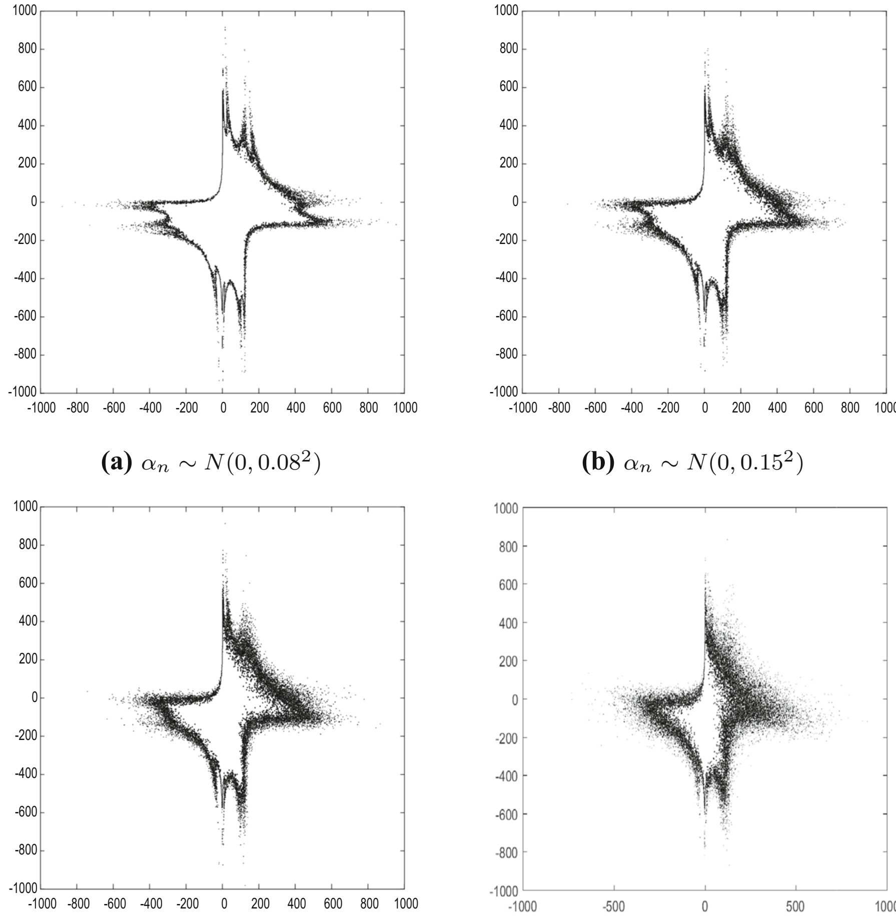


Fig. 3 Julia sets of under different multiplicative noise perturbations

(a) $\alpha_n \sim N(0, 0.08^2)$ **(b)** $\alpha_n \sim N(0, 0.15^2)$

(c) $\alpha_n \sim N(0, 0.25^2)$ **(d)** $\alpha_n \sim N(0, 0.50^2)$

where $\varphi_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, $F(\varphi_n) = \begin{bmatrix} ax_ny_n - bx_n + x_n \\ -ax_ny_n + bx_n + y_n \end{bmatrix}$, $\xi_n = \begin{bmatrix} \mu_n \\ \omega_n \end{bmatrix}$. In this subsection, let μ_n be the noise obeying $N(0, \sigma_1^2)$, ω_n be the noise obeying $N(0, \sigma_2^2)$.

Referring to Hu's construction in [29], we design

$$\begin{cases} \varphi_{k+1|k} = F(\varphi_{k|k}), \\ \varphi_{k+1|k+1} = \varphi_{k+1|k} + g(\frac{\sum_{i=1}^{n} \varphi_{k+1}^{i}}{N} - \varphi_{k+1|k}), \end{cases}$$

where $\varphi_{k+1|k}$ is the one-step prediction of φ_k at moment k and $\varphi_{k|k}$ is the estimation of φ_k at moment k, N is the number of runs of the whole model and φ_{k+1}^{i} is the i the run of the whole model at φ_{k+1}, g is a constant with $0 < g < 1$.

The explanation is given below. It is natural to use $F(\varphi_{k|k})$ to get the one-step prediction of $\varphi_{k+1|k}$ at moment k according to [29]. Next we explain how to get $\varphi_{k+1|k+1}$. For each Julia set to be iterated at the initial point, namely $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. In this case, to make sure it's unbiased, we directly take the value of $\varphi_{1|1}$ here ($\varphi_{1|1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$). By adding noise, the observed value $\varphi_2, \varphi_3, \cdots, \varphi_n$ with noise can be obtained. Run the whole model N times at φ_1 to get $\varphi_2^k, \varphi_3^k, \cdots, \varphi_n^k$, where $k = 1, 2, \cdots, N$. According to Gaussian's least squares estimate, the most likely value of $\varphi_{n|n}$ is $\frac{\sum_{i=1}^{n} \varphi_{k+1}^{i}}{N}$. Although the value of $\varphi_{n|n}$ as $\frac{\sum_{i=1}^{n} \varphi_{k+1}^{i}}{N}$ can minimize the error between the estimated value and the observed value, the value of $\varphi_{n|n}$ as $\frac{\sum_{i=1}^{n} \varphi_{k+1}^{i}}{N}$ still has great inaccuracy since φ_n^k is the observed value containing noise. $\varphi_{k+1|k}$ is the one-step prediction of φ_k at moment k. $\varphi_{k+1|k}$ contains the information of $\varphi_{k|k}$, in order to better estimate the real value of the model

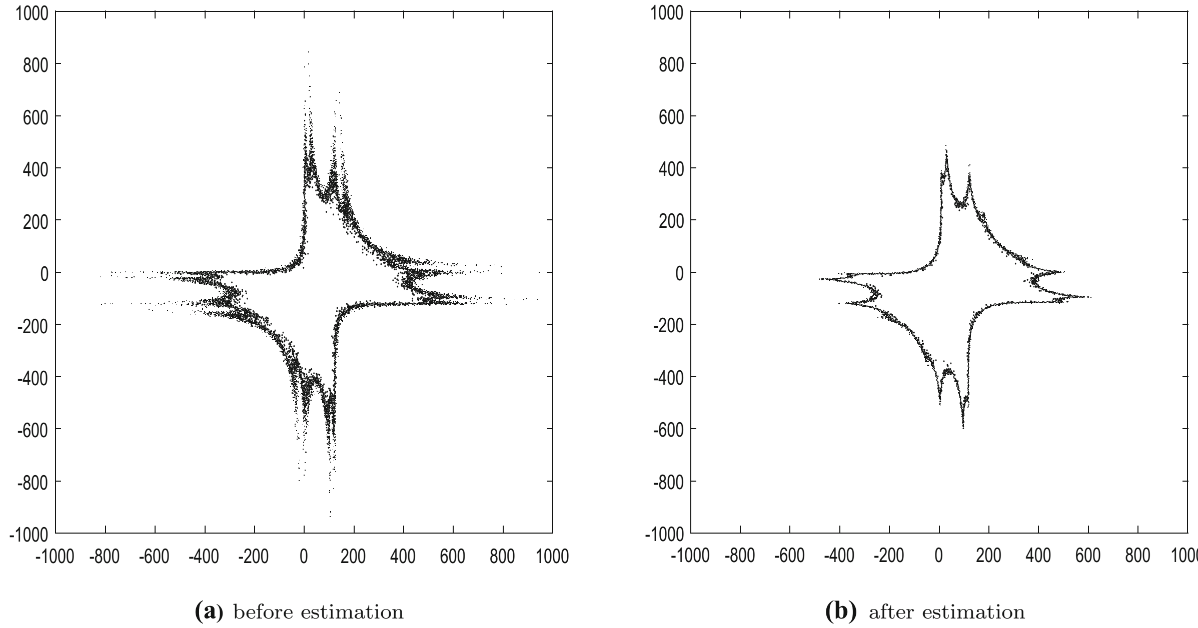


Fig. 4 Comparison of Julia sets of SIS model affected by additive noise before and after estimation when $\mu_n \sim N(0, 15^2)$; $\omega_n \sim N(0, 15^2)$

at moment $k+1$, $\varphi_{k+1|k}$ is used to construct $\varphi_{k+1|k+1}$. One way is to carry out weighted processing, then we have

$$\varphi_{k+1|k+1} = (1-g)\varphi_{k+1|k} + g\frac{\sum_{i=1}^{n}\varphi_{k+1}^{i}}{N} = \varphi_{k+1|k} + g(\frac{\sum_{i=1}^{n}\varphi_{k+1}^{i}}{N} - \varphi_{k+1|k}).$$

When $\mu_n \sim N(0, 15^2)$ and $\omega_n \sim N(0, 15^2)$, the Julia sets of SIS model affected by additive noise before and after estimation are given in Fig. 4.

4.2 Estimations of Julia Sets of the SIS Model under Multiplicative Noise Perturbation

Write (5) in vector form

$$\varphi_{n+1} = F(\varphi_n) + \xi_n,$$

where $\varphi_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, $F(\varphi_n) = \begin{bmatrix} ax_ny_n - bx_n + x_n \\ -ax_ny_n + bx_n + y_n \end{bmatrix}$, $\xi_n = \begin{bmatrix} \alpha_n x_n \\ -\alpha_n x_n \end{bmatrix}$. In this subsection, let α_n be the noise obeying $N(0, \sigma_3^2)$.

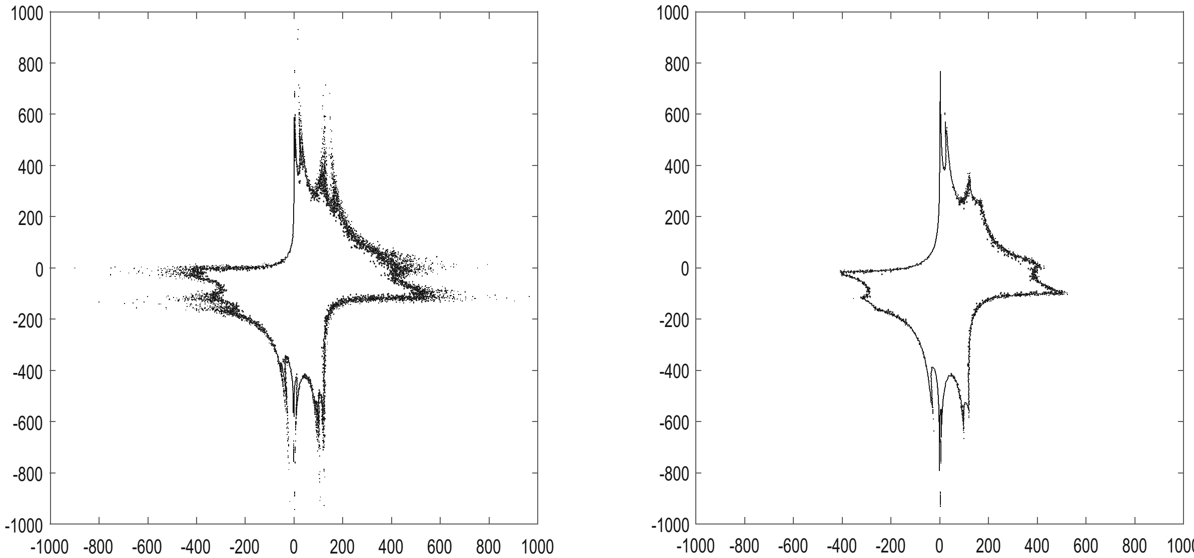
Referring to Hu's construction in [29], we design

$$\begin{cases} \varphi_{k+1|k} = F(\varphi_{k|k}), \\ \varphi_{k+1|k+1} = \varphi_{k+1|k} + g\left(\frac{\sum_{i=1}^{n}\varphi_{k+1}^{i}}{N} - \varphi_{k+1|k}\right), \end{cases}$$

where $\varphi_{k+1|k}$ is the one-step prediction of φ_k at moment k and $\varphi_{k|k}$ is the estimation of φ_k at moment k, N is the number of runs of the whole model and φ_{k+1}^{i} is the i th run of the whole model at φ_{k+1}, g is a constant with $0 < g < 1$.

When $\alpha_n \sim N(0, 0.08^2)$, the Julia sets of SIS model affected by multiplicative noise before and after estimation are given in Fig. 5.

As can be seen from the image comparison (Fig. 4(a) and 4(b), Fig. 5(a) and 5(b)), the number of some random points in the photo is significantly reduced after estimation, which indicates that the noise signal is suppressed. In addition, the smoothness of the Julia set of the model perturbed by noise is improved, and we can more clearly observe the effect of additive noise and multiplicative noise on the overall shape of the model, respectively.



(a) before estimation

(b) after estimation

Fig. 5 Comparison of Julia sets of SIS model affected by multiplicative noise before and after estimation when $\alpha_n \sim N(0, 0.08^2)$

5 Control of Julia set of the SIS model under noise disturbance

In a chaotic system, feedback control can effectively control the chaotic system to the unstable equilibrium point or periodic solution, and the control effect has strong robustness under weak noise interference [30]. In the following, we apply the feedback control method to the noise-affected system to better control the Julia set.

The original model of the system is

$$\begin{cases} x_{n+1} = a x_n y_n - b x_n + x_n, \\ y_{n+1} = -a x_n y_n + b x_n + y_n. \end{cases} \tag{6}$$

Let

$$f(x_n, y_n) = a x_n y_n - b x_n. \tag{7}$$

Substitute (7) into the original system (6) and we get

$$\begin{cases} x_{n+1} = f(x_n, y_n) + x_n, \\ y_{n+1} = -f(x_n, y_n) + y_n. \end{cases} \tag{8}$$

(x^*, y^*) is a equilibrium point of the system if and only if

$$\begin{cases} x^* = f(x^*, y^*) + x^*, \\ y^* = -f(x^*, y^*) + y^*. \end{cases}$$

Then we have

$$f(x^*, y^*) = 0.$$

The equilibrium points of the system are

$$\{(x, \frac{b}{a}) \mid \forall x \in R\}, \{(0, y) \mid \forall y \in R\}.$$

The following first considers the control of the original model, then adds noise, and uses images to show whether the Julia set of the controlled model has good robustness.

5.1 Linear feedback control

Here, we only consider the case where $a \neq 0$, $b \neq 0$ and the equilibrium point $(x^*, y^*) = (0, \frac{b}{a})$ is considered.

In the system, the first coordinate of the equilibrium point is set to 0 in the hope that the number of infected persons will approach 0 as far as possible under control, and the second coordinate is set to $\frac{b}{a}$ for the convenience of subsequent eigenvalues calculation.

Add controllers to the system (8), then we get

$$\begin{cases} x_{n+1} = f(x_n, y_n) + x_n + u_1, \\ y_{n+1} = -f(x_n, y_n) + y_n + u_2. \end{cases} \tag{9}$$

Using linear feedback controllers

$$\begin{cases} u_1 = -k(x_n - x^*), \\ u_2 = -k(y_n - y^*). \end{cases} \tag{10}$$

Then we get

$$\begin{cases} x_{n+1} = f(x_n, y_n) + (1-k)x_n + kx^*, \\ y_{n+1} = -f(x_n, y_n) + (1-k)y_n + ky^*. \end{cases} \quad (11)$$

Consider the following mapping

$$\begin{aligned} B(x, y) =&(f(x, y) + (1-k)x + kx^*, \\ &- f(x, y) + (1-k)y + ky^*) \\ =&(axy + (1-k-b)x + kx^*, \\ &- axy + bx + (1-k)y + ky^*). \end{aligned} \quad (12)$$

The Jacobian matrix of (12) is

$$J(B) = \begin{bmatrix} ay + (1-k-b) & ax \\ -ay + b & -ax + (1-k) \end{bmatrix}.$$

At the equilibrium point $(x^*, y^*) = (0, \frac{b}{a})$, the Jacobian matrix of (12) becomes

$$J(B) = \begin{bmatrix} 1-k & 0 \\ 0 & 1-k \end{bmatrix}. \quad (13)$$

The eigenvalues of the matrix(13) are

$\lambda_{1,2} = 1 - k.$

It is well known that $|\lambda_{1,2}| < 1$ is one of the conditions to guarantee the stability of the equilibrium.Then $0 < k < 2$.

Under linear feedback control, the Julia sets of the SIS model affected by noise are given in Fig. 6. Among them, Fig. 6(a) 6(b) are the effect images of the controlled system perturbed by the additive noise when $\mu_n \sim N(0, 15^2)$; $\omega_n \sim N(0, 15^2)$. Fig. 6(c) 6(d) are the effect images of the controlled system perturbed by the multiplicative noise when $\alpha_n \sim N(0, 0.08^2)$.

5.2 Nonlinear feedback control

Here, the equilibrium point $(x^*, y^*) = (0, \frac{b}{a})$ is considered and $a \neq 0, b \neq 0$

The controlled system is

$$\begin{cases} x_{n+1} = f(x_n, y_n) + x_n + u_1, \\ y_{n+1} = -f(x_n, y_n) + y_n + u_2. \end{cases} \quad (14)$$

Using nonlinear feedback control, the controllers are designed as

$$\begin{cases} u_1 = -k(f(x_n, y_n) + x_n - x^*), \\ u_2 = k(f(x_n, y_n) - y_n + y^*). \end{cases} \quad (15)$$

Put (15) into (14) and we obtain

$$\begin{cases} x_{n+1} = (1-k)f(x_n, y_n) + (1-k)x_n + kx^*, \\ y_{n+1} = (-1+k)f(x_n, y_n) + (1-k)y_n + ky^*. \end{cases} \quad (16)$$

Consider the following mapping

$$\begin{aligned} B(x, y) = &((1-k)f(x, y) + (1-k)x \\ &+ kx^*, (-1+k)f(x, y) + (1-k)y + ky^*) \\ &= ((1-k)(axy - bx) + (1-k)x \\ &+ kx^*, (-1+k)(axy - bx) + (1-k)y + ky^*). \end{aligned} \quad (17)$$

The Jacobian matrix of the system (17) is

$$J(B) = \begin{bmatrix} (1-k)(ay-b) + (1-k) & (1-k)ax \\ (-1+k)(ay-b) & (-1+k)ax + (1-k) \end{bmatrix}.$$

At the equilibrium, the Jacobian matrix of the system (17) becomes

$$J(B) = \begin{bmatrix} 1-k & 0 \\ 0 & 1-k \end{bmatrix}. \quad (18)$$

The eigenvalues of the matrix (18) are

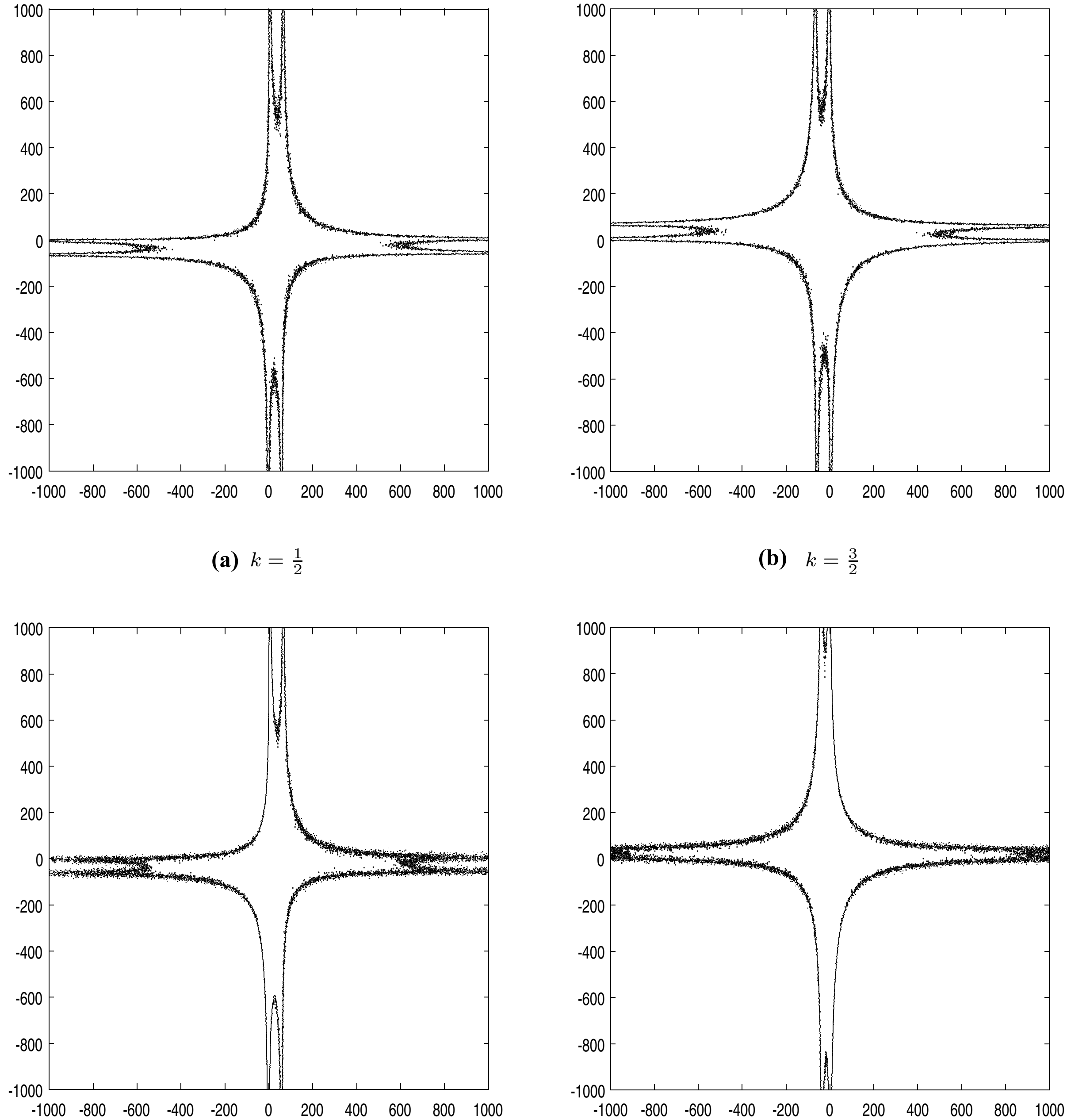
$\lambda_{1,2} = 1 - k.$

Then we get $|\lambda_{1,2}| < 1$, that is $0 < k < 2$ to guarantee the equilibrium to be stable.

Under nonlinear feedback control, the Julia sets of the SIS model affected by noise are given in Fig. 7. Among them, Fig. 7(a) 7(b) are the effect images of the controlled system perturbed by the additive noise when $\mu_n \sim N(0, 15^2)$; $\omega_n \sim N(0, 15^2)$, Fig. 7(c) and 7(d) are the effect images of the controlled system perturbed by the multiplicative noise when $\alpha_n \sim N(0, 0.08^2)$.

Fig. 6(b), 6(d) and Fig. 7(b) 7(d) show that some random points in the Julia set of the SIS model perturbed by noise are significantly reduced after feedback control, indicating that the Julia set of the system has better robustness after control. In addition, the filled-in Julia set becomes larger. The points in the filled-in Julia set will not tend to infinity under iteration, which means that after control, there is a larger stable region for infectious diseases. In this region, the number of infected people is relatively stable and will not rapidly become uncontrollable. In conclusion, the controllers adopted achieve the desired control effect.

The effectiveness of the feedback control method used in this model is explained below. It is easy to calculate that the eigenvalues at the fixed point $(0, \frac{b}{a})$ are 1. Under noise disturbance, the modulus of the eigenvalues of the jacobian matrix at (x^*, y^*) may be greater than or less than 1. Whether the fixed point is attractive or not may change in the iteration process. Still, the Julia set is the boundary of the attractive domain for attractive fixed points, which finally leads to the



(a) $k = \frac{1}{2}$

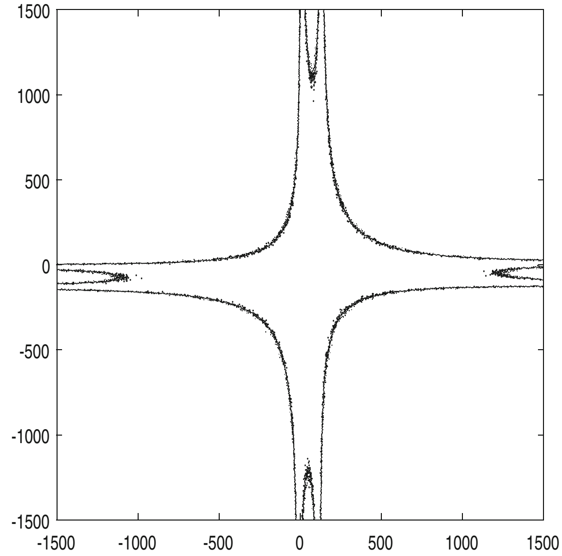
(b) $k = \frac{3}{2}$

(c) $k = \frac{1}{2}$

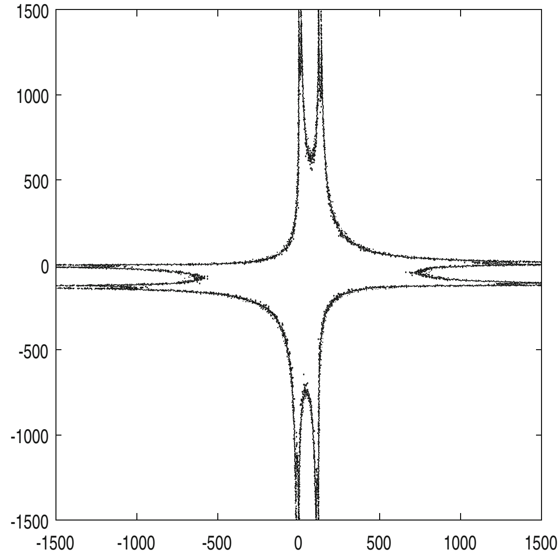
(d) $k = 1.3$

Fig. 6 The Julia sets of SIS model perturbed by noise after the linear control

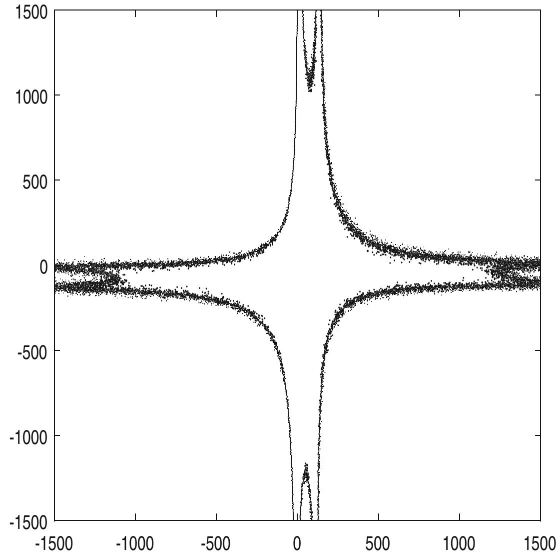
unstable state of the Julia set. By the feedback control method, the control coefficient *k* can be reasonably taken so that the modulus of the eigenvalues of the controlled system is less than 1. According to matrix perturbation theory, the system's eigenvalues will slightly change when subjected to small perturbations. The system's stability will not change when its equilibrium point is of a non-central type. In other words, in this



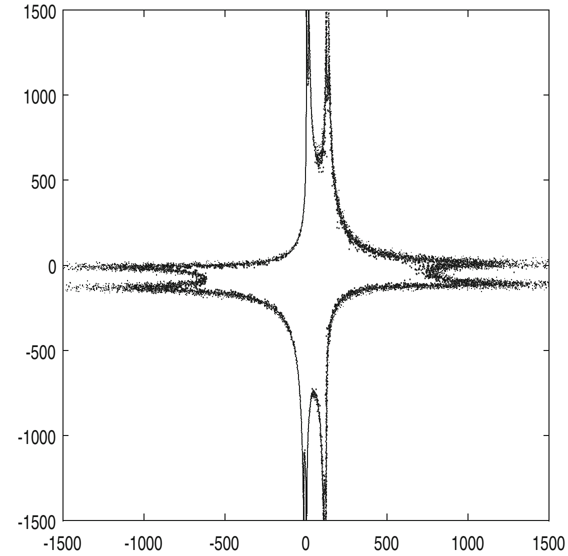
(a) $k = \frac{1}{2}$



(b) $k = \frac{1}{3}$



(c) $k = \frac{1}{2}$



(d) $k = \frac{1}{3}$

Fig. 7 The Julia sets of SIS model perturbed by noise after nonlinear control

system, the feedback control method is used to make the modulus of the eigenvalues of the controlled system less than 1, so that (x^*, y^*) is the attractive equilibrium point to reduce the interference of noise on the Julia set of SIS model.

From the comparison of images(Fig. 6(b) and 6(d), Fig. 7(b) and 7(d)), it can be seen that after feedback control, the control effect of additive noise is better than that of multiplicative noise. This is because the random terms of additive noise have nothing to do with x_n and y_n and thus have little effect on the Jacobian of the system. Unlike additive noise, the random term selected when multiplicative noise is added is related to x_n, so the Jacobian matrix of the system will also be strongly influenced by the random term. In this way, the modulus of the eigenvalues of the controlled system may be greater than 1, so the equilibrium point is not stable.

6 Conclusion

This paper mainly introduces the estimations and control of the Julia set of the SIS model under noise disturbance. It is very realistic and meaningful to present the SIS model in discrete form and discuss the perturbation of noise on its fractal characteristics. To observe the effect of noise on the overall shape of the Julia set of the SIS model, we present estimates for the Julia set of the SIS model perturbed by noise. In addition, two kinds of controllers are set up according to the feedback control method. The image results show that the controllers can effectively control the fractal characteristics of the model and increase the anti-interference of the model. Finally, the reasons why the feedback control method is effective for the model are explained.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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