Guaranteed cost positive consensus for multi-agent systems with multiple time-varying delays and MDADT switching

Xiangyang Cao · Chenghui Zhang ·
Daduan Zhao · Yan Li

Received: 23 June 2021 / Accepted: 27 October 2021 / Published online: 30 January 2022
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Abstract This paper investigates the guaranteed cost positive consensus for linear multi-agent systems (MASs) with multiple time-varying delays and switching topologies. First, necessary and sufficient conditions for the positivity of the focused system are given. The positive consensus criterion and switching signals are derived by combining multiple Lyapunov–Krasovskii functionals with mode-dependent average dwell time approaches. Then, the guaranteed cost positive consensus is studied via the state errors among neighbouring agents and control inputs. It realizes a trade-off between the consensus regulation performance and the control energy consumption of MASs. Particularly, the determined upper bound of the cost function depends merely on the initial conditions of the agents and is independent of the switching movements. Finally, simulation results are given to validate the proposed control scheme.

Keywords Guaranteed cost positive consensus · Multiple time-varying delays · Topology-dependent average dwell time · Multi-agent systems

1 Introduction

During the past two decades, distributed coordinative control of multi-agent systems (MASs) has attracted intensive attention owing to its wide applications in various areas, such as secondary control of microgrids [37], multiple spacecraft attitude synchronization [1], swarms of robots [31] and sensor networks [3]. As a fundamental research topic in this area, consensus problem is to design a distributed interaction protocol, such that all the agents reach an agreement asymptotically or in a finite time interval. Up to now, many interesting results have been reported on the consensus of MASs with fixed/switching topologies [9,20], constant/time-varying delays [28,34], noise/attack in transmission channels [11,19] or other features [16].

In many real-world scenarios such as emission control problem in a fleet of hybrid vehicles [10], the sensor networks for greenhouse monitoring [2] as well as the variability of the transmission rate of COVID-19 between real social networks [29], systems involve quantities that their states are intrinsically non-negative; thus, positive consensus instead of consensus should be considered. Positive consensus means that the designed protocol possesses the property of achieving the global convergence of the consensus errors and, more importantly, the capability of preserving the system state always non-negative. The pioneering theoretical works on this problem can be found in [24] and [25], where the positive consensus of homoge-

X. Cao · C. Zhang · D. Zhao (✉) · Y. Li
School of Control Science and Engineering, Shandong
University, Jinan 250061, People's Republic of China
e-mail: liyan_cse@sdu.edu.cn

Fig. 1 Data transmission network

neous MASs was considered. Recently, by defining the line graph, [22] and [26] addressed the positive edge consensus of the network systems subjected to uncertainties and input saturation, respectively. Along this research line, [21] further investigated the positive edge consensus via output feedback protocols without utilizing the information of algebraic connectivity. More recently, necessary and sufficient conditions for the positive consensus were developed in [12], and [13] further extended this results in robust and nonfragile cases via observer-based output-feedback protocols. Nevertheless, these results assumed that the underlying communication topologies are fixed.

In practice, the topology always changes due to various controllable and uncontrollable effects, such as the data communication network shown in Fig. 1. The data packets are transmitted from the control centre to the communication terminal through the interaction network and the communication topology will be switched between the busy-time and idle-time models according to a certain switching signal. The topology switching will render the system dynamics more complicated and cause significant technical challenges to the solvability of the positive consensus problem. Additionally, the techniques of unconstrained state consensus techniques and positive consensus with fixed topology are not directly applicable for the above situation. Partially because topology-independence protocols are no longer effective, and partly because the decreasence of the common Lyapunov function is not maintained as the topology changes. Some control techniques in switched systems, such as the multiple Lyapunov functions technique [23, 35] and average dwell time (ADT) approach [36], have been introduced into MASs to cope with the topology switching. Some reliable results on consensus problems with switching topologies can be found in [11, 30, 33] and the references therein.

Moreover, time delays are unavoidably encountered in modelling networked MASs and may cause undesirable dynamic behaviours such as oscillation and instability. Thus, much attention has been focused on searching delay-dependent consensus criteria [8, 17, 38]. References [17] and [38] addressed the consensus problem of MASs with constant time delays. In [8], less conservative consensus criteria for MASs with time-varying delay are proposed by taking advantages of the sliding-mode approach and the reciprocally convex combination technique. Furthermore, for the networked control system, especially for MASs, the communication networks among agents usually have a spatial extent due to the presence of a variety of bandwidths and data transfer rates. Hence, there is a distribution of propagation delays over a certain time interval. So that the distributed delays should also be taken into account. However, the consensus problem for MASs with time-varying and distributed delays is still open. Furthermore, when the positivity constraints, switching topologies and multiple time delays happen simultaneously in the MASs, the consensus problem will be more complicated and challenging in both theory and real applications.

On the other hand, when controlling practical MASs, it is desirable to design a control protocol that can achieve consensus and guarantee an adequate level of performance. One effective approach to this issue is the guaranteed cost control. It considers the joint effect of consensus regulation performance and energy consumption of the control efforts. Thus system performance degradation incurred by the uncertainties or time delays is guaranteed to be less than a derived lower bound [15]. Some results about the guaranteed cost consensus of MASs have been reported in [6, 28, 30]. In [28] and [30], two guaranteed cost consensus protocols were presented for MASs with constant time delay or switching topologies, respectively. The guaranteed cost consensus of second-order MASs was considered in [6] with hybrid impulsive control. However, to the best of our knowledge, the guaranteed cost positive consensus analysis and design problems for linear MASs with multiple time-varying delays and switching topologies remain open and important, which will be investigated in this work.

Compared with the existing related results, the main contributions of this paper are detailedly outlined as follows.

- (i) The focused topology is switching among finite undirected connected graphs. Thus, the positive consensus protocols of linear MASs in [12, 13, 24–26] under fixed topology are not applicable to our problem. Moreover, the distributed time-varying delay is considered, instead of the constant delay in [17, 28, 38].
- (ii) A mode-dependent average dwell time (MDADT) approach is introduced to construct switching signals, which permits each topology to have its own ADT, rather than the unified constraints on all topologies in [23, 36] having linear dynamics. Moreover, the positivity of the closed-loop MASs is rigorously proved for any switching signal.
- (iii) A guaranteed cost positive consensus scheme is developed to achieve the trade-off between the positive consensus regulation performances and control energy consumption. In particular, the established stability criterion and upper bound of the cost are, respectively, independent of the number of agents and the switching movements.

The remainder of this paper is organized as follows. Section 2 gives some preliminaries and formulates the control objective. The main results are presented in Sects. 3 and 4. In Sect. 3, sufficient and necessary conditions are derived for preserving the positivity of the open-loop and closed-loop linear MASs with switching topologies. Section 4 establishes a guaranteed cost positive consensus criterion, providing an upper bound of the cost function. Simulation results are provided in Sect. 5 to illustrate the theoretical results. Section 6 gives some concluding remarks.

2 Preliminaries and problem formulation

2.1 Notation and graph theory

Throughout this paper, $\mathbf{Z}_+$, $\mathbf{R}$, and $\mathbf{R}_+$ denote the set of all positive integers, real numbers and non-negative real numbers, respectively. Let $\mathbf{R}^n$ and $\mathbf{R}^{n \times m}$ be the space of the vectors of n -dimensional Euclidean space and $n \times m$ real matrix space, respectively. I_N denotes the N -dimensional identity matrix. The representation $A > 0$ (≥ 0) means that the elements of matrix A are all positive (non-negative). Let $\|\cdot\|$ denotes Euclidean norm for vectors or the spectral norm of matrices. For a right continuous function $\phi : [-\tau, 0] \rightarrow \mathbf{R}^n$, the norm is defined by $\|\phi\|_c = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$. Let

$\lambda_{\min}(A)$, $\lambda_2(A)$ and $\lambda_{\max}(A)$ denote the smallest, the minimum nonzero and the largest eigenvalues of A , respectively.

Some basic concepts for the algebraic graph theory of the agents are introduced below. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted graph, where $\mathcal{V} = \{1, 2, \dots, N\}$ represents a set of N agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N \times N}$ represents the weighted adjacency matrix. An edge from agents i to j is denoted by $(i, j) \in \mathcal{E}$. The graph is undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. If $a_{ij} > 0$, then agents i and j can directly communicate with each other, and $a_{ij} = 0$ otherwise. The set of neighbours of agent i is denoted by $\mathcal{N}_i = \{j \mid j \in \mathcal{V}, (i, j) \in \mathcal{E}\}$. The Laplacian matrix $L = [l_{ij}] \in \mathbf{R}^{N \times N}$ associated with $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1}^N a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$.

In this paper, we assume that all the topologies are connected and undirected graphs. An important property for the corresponding Laplacian matrix L is given in the following lemma, which is useful for later analysis.

Lemma 1 [11] *For a connected undirected graph $\mathcal{G}$, there holds that the Laplacian matrix L is positive semi-definite and 0 is a simple eigenvalue.*

2.2 System and control objectives

In this paper, we consider a class of linear MASs consisting of N identical agents where the dynamics of agent i is described by

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + A_d x_i(t - d_1(t)) \\ \quad + A_b \int_{t-d_2(t)}^t x_i(s) ds + Bu_i(t), \\ x_i(\theta) = \varphi_i(\theta), \quad \theta \in [-\tau, 0], \quad i \in \mathcal{V}, \end{cases} \quad (1)$$

where $x_i(t) \in \mathbf{R}^n$ and $u_i(t) \in \mathbf{R}^m$ are the system state and control input of the agent i , respectively. Matrices A , A_d , A_b and B are known constant matrices with appropriate dimensions, and (A, B) is assumed to be stabilizable. The function $d_1(t)$ denotes the time-varying delay, which satisfies $0 \leq d_1(t) \leq \tau_1$ and $\dot{d}_1(t) \leq h < 1$, $d_2(t)$ is the time-varying distributed delay satisfying $0 \leq d_2(t) \leq \tau_2$, where τ_1 , τ_2 and h are known positive constants. $\varphi_i(\theta)$ is a continuous vector-valued initial function defined on $[-\tau, 0]$, where $\tau = \max\{\tau_1, \tau_2\}$.

Assume that the communication topologies of system (1) vary with time and switch among finite number of undirected connected graphs. Specifically, let

the right continuous function $\sigma(t) : [0, \infty) \rightarrow \underline{S} = \{1, 2, \dots, S\}$ denote the switching signal of the communication topologies, while $\underline{S}$ is the set of S possible topologies.

The ADT and MDADT are well-used tools to cope with switched systems' analyses and syntheses [15, 36]. Since only communication topology switches for the MASs, i.e. different topologies represent corresponding modes, the MDADT strategy is employed in this paper to design the controlled switching signal.

Definition 1 For any switching signal $\sigma(t)$ and any $t_2 \geq t_1 \geq 0$, let $N_{\sigma p}(t_1, t_2)$ denote the switching numbers that the p th topology is activated over the interval $[t_1, t_2)$, and $T_p(t_1, t_2)$ denote the total running time of the p th topology over the interval $[t_1, t_2)$. If there exist $N_{0p} \geq 0$ and $T_{\alpha p} > 0$ such that

$$N_{\sigma p}(t_1, t_2) \leq N_{0p} + \frac{T_p(t_1, t_2)}{T_{\alpha p}}, \quad \forall t_2 \geq t_1 \geq 0, \forall p \in \underline{S}, \quad (2)$$

then $T_{\alpha p}$ and $N_{0p} \geq 0$ are called MDADT and mode-dependent chattering bounds, respectively. As commonly used in the literature, we choose $N_{0p} = 0$ in this paper.

The following definitions, cited from [4], give a characterization of the positivity of system (1) and will be employed in the later development.

Definition 2 System (1) is said to be positive if for any initial condition $\varphi_i(\theta) \geq 0, \theta \in [-\tau, 0], i \in \mathcal{V}$ and any switching signal $\sigma(t)$, the corresponding trajectory satisfies $x_i(t) \geq 0$ for all $t \geq t_0$.

Definition 3 A is called a Metzler matrix if the off-diagonal entries of matrix A are non-negative.

Lemma 2 [16] Let $A \in \mathbb{R}^{n \times n}$. Then $e^{At} \geq 0, \forall t \geq 0$, if and only if A is a Metzler matrix.

Next, we will give the definitions of positive consensus and guaranteed cost positive consensus for MASs (1).

Definition 4 Positive consensus: design a distributed consensus protocol $u_i(t)$ such that:

- (i) overall system (1) is positive, i.e. $x_i(t) \geq 0$ for $t \geq t_0$;
- (ii) overall system (1) reaches consensus, i.e. $\lim_{t \rightarrow +\infty} \|x_i(t) - x_j(t)\| = 0, i, j \in \mathcal{V}$, for any non-negative initial conditions.

Remark 1 It is worth emphasizing that, unlike the normal consensus problem which only requires the system to converge to the same trajectory and has no strict restrictions on the initial condition and state trajectory, the positive consensus is a stronger constraint. It not only calls for the closed-loop system to achieve consensus but also compels the trajectories of all agents to remain in the non-negative orthant, that is $x_i(t) \geq 0$ for $t \geq t_0, i \in \mathcal{V}$.

Definition 5 Consider a quadratic cost function of system (1) as follows

$$J = \sum_{i=1}^N \int_{t_0}^{+\infty} \left(\sum_{j \in \mathcal{N}_i} a_{ij}^{\sigma(t)} (x_i(t) - x_j(t))^T M_1 (x_i(t) - x_j(t)) + u_i^T(t) M_2 u_i(t) \right) dt, \quad (3)$$

where $M_1 \in \mathbb{R}^{n \times n}$ and $M_2 \in \mathbb{R}^{m \times m}$ are known positive-definite matrices.

Definition 6 Consider system (1) and cost function (3), if there exists distributed protocol $u_i(t), i \in \mathcal{V}$, and positive scalar J^* such that the corresponding closed-loop system is *positive consensus* and the cost function satisfies $J \leq J^*$, then the closed loop is called *guaranteed cost positive consensus*, where J^* is an upper bounded of the cost function and the corresponding protocol is said to be a distributed guaranteed cost positive consensus protocol.

Remark 2 The two parts of the cost function, $\int_{t_0}^{+\infty} (\sum_{j \in \mathcal{N}_i} a_{ij}^{\sigma(t)} (x_i(t) - x_j(t))^T M_1 (x_i(t) - x_j(t))) dt$ and $\int_{t_0}^{+\infty} (u_i^T(t) M_2 u_i(t)) dt$, represent the consensus regulation performance and the evaluation indexes of the control energy consumptions, respectively. The objective of guaranteed cost positive consensus is to make a trade-off between these two parts by designing an available $u_i(t)$ with two given weights matrices M_1 and M_2 . In addition, one point should be kept in mind that the classical quadratic cost function, i.e. $\int_{t_0}^{\infty} (x^T(t) M_1 x(t) + u^T(t) M_2 u(t)) dt$, involving system state and control, is no longer applicable in the guaranteed cost consensus of MASs, since the system state may intend to infinity even if the consensus is achieved.

The problems to be considered in this paper are:

- (i) for system (1), design a distributed protocol $u_i(t)$ and a switching signal $\sigma(t)$ to achieve positive consensus;

- (ii) for system (1), design a distributed protocol $u_i(t)$ and a switching signal $\sigma(t)$ to achieve guaranteed cost positive consensus.

The following definitions, cited from [14], will be used in the subsequent analysis.

Definition 7 The function $V : \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}_+$ belongs to class ζ if

- (i) the function V is continuous in each of the sets $[t_k, t_{k+1}) \times \mathbf{R}^n$ and for each $x, y \in \mathbf{R}^n$, $t \in [t_k, t_{k+1})$, $k \in \mathbf{Z}^+$, $\lim_{(t,y) \rightarrow (t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists;
 (ii) $V(t, x(t))$ is locally Lipschitzian in all $x \in \mathbf{R}^n$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 8 Given a function $V \in \zeta$, the upper right-hand derivative of V along the solution $x_i(t)$ of system (1) is defined by

$$D^+ V(t, x_i(t)) = \limsup_{\iota \rightarrow 0} \frac{1}{\iota} [V(t + \iota, x_i(t + \iota)) - V(t, x_i(t))], \quad i \in \mathcal{V}. \quad (4)$$

Lemma 3 [7] (Schur Complement) Given constant matrices S_1, S_2, S_3 with appropriate dimensions, where $S_1^T = S_1$ and $0 < S_2 = S_2^T$, then $S_1 + S_3^T S_2^{-1} S_3 < 0$ if and only if $\begin{bmatrix} S_1 & S_3^T \\ S_3 & -S_2 \end{bmatrix} < 0$.

Lemma 4 [23] For any positive-definite matrix $S \in \mathbf{R}^{n \times n}$, positive constant α and vector function $\varsigma : [-\tau, \infty) \rightarrow \mathbf{R}^n$ such that the integrations concerned are well defined, then

$$\begin{aligned} & \frac{e^{\alpha\tau} - 1}{\alpha} \int_{t-d(t)}^t e^{-\alpha_{\sigma(t)}\tau} \varsigma^T(s) S \varsigma(s) ds \\ & \geq \left(\int_{t-d(t)}^t \varsigma(s) ds \right)^T S \left(\int_{t-d(t)}^t \varsigma(s) ds \right), \end{aligned}$$

where $0 \leq d(t) \leq \tau$, $t \geq 0$.

3 Positivity analysis

In this section, the positivity property of system (1) is discussed, where sufficient and necessary conditions are derived for preserving the positivity of the open-loop and closed-loop MASs.

Firstly, a key proposition is presented to ensure the positivity of open-loop system (1) under any switching signal.

Proposition 1 System (1) with $u_i \equiv 0$ is positive if and only if A is Metzler, $A_d \geq 0$ and $A_b \geq 0$.

Proof (Sufficiency) For any $T > 0$, $t_0 = 0$, let $\{t_0, t_1, t_2, \dots, t_{N_{\sigma}(t_0, T)}\}$ denote the switching sequence of switching topologies on the interval $[t_0, T)$. From system (1) with $u_i \equiv 0$, $\forall t \in [t_0, t_1)$, $\forall i \in \mathcal{V}$, it follows,

$$\begin{aligned} & \exp\{-At\}(\dot{x}_i(t) - Ax_i(t)) \\ & = \exp\{-At\} \left(A_d x_i(t - d_1(t)) \right. \\ & \quad \left. + A_b \int_{t_0-d_2(t)}^{t_0} x_i(s) ds \right). \end{aligned}$$

Integrating both sides of the above equation over $[t_0, t)$ gives

$$\begin{aligned} \exp\{-At\}x_i(t) & = \int_{t_0}^t \exp\{-As\} \left(A_d x_i(s - d_1(s)) \right. \\ & \quad \left. + A_b \int_{t_0-d_2(t)}^{t_0} x_i(s) ds \right) ds + x_i(t_0), \end{aligned}$$

that is,

$$\begin{aligned} x_i(t) & = \int_{t_0}^t \exp\{A(t-s)\} \left(A_d x_i(s - d_1(s)) \right. \\ & \quad \left. + A_b \int_{t_0-d_2(t)}^{t_0} x_i(s) ds \right) ds + \exp\{At\}\varphi_i(0). \end{aligned}$$

By Lemma 2, if A is a Metzler matrix, $A_d \geq 0$ and $A_b \geq 0$, then we can obtain that $x_i(t) \geq 0$, $\forall t \in [t_0, t_1)$, and $x_i(t_1) = x_i(t_1^-) \geq 0$. Similar to the above process, $\forall t \in [t_1, t_2)$, we have

$$\begin{aligned} x_i(t) & = \int_{t_1}^t \exp\{A(t-s)\} \left(A_d x_i(s - d_1(s)) \right. \\ & \quad \left. + A_b \int_{t_1-d_2(t)}^{t_1} x_i(s) ds \right) ds + \exp\{At\}\varphi_i(t_1) \geq 0 \end{aligned}$$

and $x_i(t_2) = x_i(t_2^-) \geq 0$. Recursively, if A is a Metzler matrix, $A_d \geq 0$ and $A_b \geq 0$, then there holds that $x_i(T) \geq 0$, $\forall T > 0$, $i \in \mathcal{V}$.

(Necessity) Since system (1) with $u_i \equiv 0$ is positive for any initial condition $\varphi_i(\cdot) \geq 0$, there holds that $x_i(t) \geq 0$ for all $t \geq t_0$, $i \in \mathcal{V}$. We first prove that A is Metzler via reductio ad absurdum. Suppose there exists an element $a_{qg} < 0$, $q \neq g$, where a_{qg} is in the q th row and g th column of A . From system (1) with $u_i \equiv 0$, it follows that

$$\begin{aligned}\dot{x}_{iq}(t) = & \sum_{k=1, k \neq q, k \neq g}^n a_{qk}x_{ik}(t) + a_{qq}x_{iq}(t) + a_{qg}x_{ig}(t) \\ & + \sum_{k=1}^n a_{dqk}x_{ik}(t - d_1(t)) \\ & + \sum_{k=1}^n a_{bqk} \int_{t-d_2(t)}^t x_{ik}(s)ds,\end{aligned}$$

where $x_{ik}(t)$ represents the k th element of $x_i(t)$, $i \in \mathcal{V}$. We can see that $\dot{x}_{iq}(t) < 0$ is possible if $x_{iq}(t) = 0$, $x_{ig}(t) \neq 0$ and terms a_{qk} , a_{dqk} and a_{bqk} take non-negative values small enough. Then, $x_{iq}(t^+) < 0$, which yields a contradiction with the positivity of system (1). Therefore, we get that matrix A is Metzler.

Next, we show $A_d \geq 0$. Analogously, suppose that A_d has an element $a_{dqg} < 0$, there holds

$$\begin{aligned}\dot{x}_{iq}(t) = & \sum_{k=1, k \neq q}^n a_{qk}x_{ik}(t) + a_{qq}x_{iq}(t) \\ & + \sum_{k=1, k \neq g}^n a_{dqk}x_{ik}(t - d_1(t)) \\ & + a_{dqg}x_{ig}(t - d_1(t)) \\ & + \sum_{k=1}^n a_{bqk} \int_{t-d_2(t)}^t x_{ik}(s)ds, \quad i \in \mathcal{V}.\end{aligned}$$

It can be obtained that $x_{iq}(t^+) < 0$ is possible if $x_{iq}(t) = 0$ and terms a_{qk} , a_{dqk} and a_{bqk} take non-negative values small enough, which yields a contradiction with the positivity of system (1). Thus, $A_d \geq 0$.

In the same way, if A_b has an element $a_{bqg} < 0$, then system (1) may not be positive. So, $A_d \geq 0$. Based on these three points discussed above, the necessity is obtained.

Therefore, we can conclude that system (1) with $u_i(t) \equiv 0$ is positive under any switching signals if and only if A is a Metzler matrix, $A_d \geq 0$ and $A_b \geq 0$. The proof is completed. $\square$

Next, based on a consensus protocol to be proposed, the feasibility conditions for preserving the positivity of closed-loop system (1) will be derived. Consider the following distributed protocol for agent i in (1):

$$u_i(t) = -K_{\sigma(t)} \sum_{j \in \mathcal{N}_i} a_{ij}^{\sigma(t)} (x_i(t) - x_j(t)), \quad \sigma(t) \in \underline{\mathcal{S}}, \quad (5)$$

where $K_{\sigma(t)}$ is the feedback matrix to be determined, and $a_{ij}^{\sigma(t)}$ is the time-varying weight varying with the switching signal $\sigma(t)$.

Let $x(t) = [x_1^T(t), x_2^T(t), \dots, x_N^T(t)]^T$, $x(t - d_1(t)) = [x_1^T(t - d_1(t)), x_2^T(t - d_1(t)), \dots, x_N^T(t - d_1(t))]^T$ and $u(t) = [u_1^T(t), u_2^T(t), \dots, u_N^T(t)]^T$. System (1) with (5) can be rewritten as a compact form:

$$\begin{cases} \dot{x}(t) = (I_N \otimes A)x(t) + (I_N \otimes A_d)x(t - d_1(t)) \\ \quad + (I_N \otimes A_b) \int_{t-d_2(t)}^t x(s)ds + (I_N \otimes B)u(t), \\ u(t) = -(L_{\sigma(t)} \otimes K_{\sigma(t)})x(t), \end{cases} \quad (6)$$

where $L_{\sigma(t)}$ is the Laplacian matrix of the connected undirected graph $\mathcal{G}$. By (6), we obtain

$$\begin{aligned}\dot{x}(t) = & A_{\sigma(t)}^* x(t) + (I_N \otimes A_d)x(t - d_1(t)) \\ & + (I_N \otimes A_b) \int_{t-d_2(t)}^t x(s)ds\end{aligned} \quad (7)$$

where $A^* := I_N \otimes A - (L_{\sigma(t)} \otimes B K_{\sigma(t)})$. The positive consensus problem of (1) is equivalent to that of switched system (7). As a critical basis of establishing the positive consensus, the positivity of closed-loop system (7) will be preserved by a necessary and sufficient condition in the forthcoming proposition.

Proposition 2 *The positivity of closed-loop system (7) is preserved if and only if $A - \omega_{\max}^{\sigma(t)} B K_{\sigma(t)}$, $\forall \sigma(t) \in \underline{\mathcal{S}}$, are Metzler matrices, $A_d \geq 0$, $A_b \geq 0$ and $B K_{\sigma(t)} \geq 0$, where $\omega_{\max}^{\sigma(t)} = \max\{\omega_i^{\sigma(t)}, i \in \mathcal{V}\}$ with $\omega_i^{\sigma(t)} = \sum_{j \in \mathcal{N}_i} a_{ij}^{\sigma(t)}$, $\forall \sigma(t) \in \underline{\mathcal{S}}, i \in \mathcal{V}$.*

Proof Based on Proposition 1, the positivity of system (7) can be preserved if and only if $A_{\sigma(t)}^*$, $\forall \sigma(t) \in \underline{\mathcal{S}}$, are Metzler matrices, $A_d \geq 0$ and $A_b \geq 0$. Write the matrices $A_{\sigma(t)}^*$ as the following block structure:

$$A_{\sigma(t)}^* = \begin{bmatrix} A - \sum_{j \in \mathcal{N}_1} a_{1j}^{\sigma(t)} B K_{\sigma(t)} & a_{12}^{\sigma(t)} B K_{\sigma(t)} & \cdots & a_{1N}^{\sigma(t)} B K_{\sigma(t)} \\ a_{21}^{\sigma(t)} B K_{\sigma(t)} & A - \sum_{j \in \mathcal{N}_2} a_{2j}^{\sigma(t)} B K_{\sigma(t)} & \cdots & a_{2N}^{\sigma(t)} B K_{\sigma(t)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}^{\sigma(t)} B K_{\sigma(t)} & a_{N2}^{\sigma(t)} B K_{\sigma(t)} & \cdots & A - \sum_{j \in \mathcal{N}_N} a_{Nj}^{\sigma(t)} B K_{\sigma(t)} \end{bmatrix}. \quad (8)$$

From Definition 3 and (8), we get that $A_{\sigma(t)}^*$ are Metzler matrices if and only if $\forall \sigma(t) \in \underline{\mathcal{S}}$, blocks

$A - \sum_{j \in \mathcal{N}_i} a_{ij}^{\sigma(t)} B K_{\sigma(t)}$, $i \in \mathcal{V}$, are Metzler matrices and blocks $a_{ij}^{\sigma(t)} B K_{\sigma(t)} \geq 0$. Since $a_{ij}^{\sigma(t)}$, $\forall (i, j) \in \mathcal{E}$, $\forall \sigma(t) \in \underline{S}$, are non-negative, then, matrices $a_{ij}^{\sigma(t)} B K_{\sigma(t)} \geq 0$ are equivalent to matrices $B K_{\sigma(t)} \geq 0$. Noting that $\varpi_i^{\sigma(t)} > 0$, it is not hard to derive that $A - \varpi_i^{\sigma(t)} B K_{\sigma(t)}$ are Metzler matrices if and only if $A - \varpi_{\max}^{\sigma(t)} B K_{\sigma(t)}$ are Metzler matrices.

Thus, we can conclude that the positivity of system (7) is preserved when $A - \varpi_{\max}^{\sigma(t)} B K_{\sigma(t)}$ are Metzler matrices, $A_d \geq 0$, $A_b \geq 0$ and $B K_{\sigma(t)} \geq 0$, $\sigma(t) \in \underline{S}$. This completes the proof. $\square$

Remark 3 It can be seen that the proof of Proposition 2 mainly uses the properties of Metzler matrix described in Definition 3, which is a fundamental feature of positive linear system (see [4]). For an isolated positive linear system $\dot{x} = Fx + Gu$ with $u = Hx$, the positivity is preserved if and only if $F + GH$ is a Metzler matrix. However, due to the interaction between agents, the criteria for satisfying the positivity of MASs are involved with the topological property and thus are not the same as that in an isolated system. Moreover, the topology switching makes the parameter pair (A, A_d, A_b, B, K) in our paper constrained to S pair, which is different from that of single pair in the isolated positive linear system (see [14, 15]).

Based on the technique notes discussed above, the guaranteed cost positive consensus of MASs (1) will be analysed in the next section.

4 Guaranteed cost positive consensus

This section discusses the guaranteed cost positive consensus of linear system (1) under switching topology, particularly to derive a delay-dependent criterion for reaching the guaranteed cost positive consensus, design an available switching signal and determine an upper bound of the cost function. Specifically, Sect. 4.1 establishes a positive consensus criterion, and Sect. 4.2 further investigates the guaranteed cost positive consensus problem.

4.1 Positive consensus analysis

We are going to propose a positive consensus analysis scheme for closed-loop system (7) under switch-

ing topology, in which the consensus criterion is derived by the multiple Lyapunov–Krasovskii functionals (MLKFs) approach and the corresponding switching signal is constructed by MDADT switching strategy.

For notational brevity, define the consensus error as $\delta_i = x_i - x^*$ with $x^* = \frac{1}{N} \sum_{i=1}^N x_i$, $i \in \mathcal{V}$, and let $\delta = [\delta_1^T, \delta_2^T, \dots, \delta_N^T]^T$. By system (1) with (5), we obtain that,

$$\begin{aligned} \dot{\delta}(t) = & (I_N \otimes A - L_{\sigma(t)} \otimes B K_{\sigma(t)}) \delta(t) + (I_N \otimes A_d) \\ & \times \delta(t - d_1(t)) + (I_N \otimes A_b) \int_{t-d_2(t)}^t \delta(s) ds. \end{aligned} \quad (9)$$

It can be seen that the positive consensus issue of system (7) is transformed into the asymptotic stabilization problem of delayed switched system (9). Some sufficient conditions in the form of LMI and a class of controlled switching signal will be presented in the following theorem for reaching the positive consensus.

Theorem 1 Under the conditions of Proposition 2, if there exist matrices $X_p \in \mathbf{R}^{n \times n}$ and $Y_p \in \mathbf{R}^{m \times n}$, with X_p being positive-definite, such that

$$\begin{bmatrix} \Omega_p & X_p & A_d & A_b \\ * & -I_n & 0 & 0 \\ * & * & -\frac{1-h}{e^{\alpha_p \tau_1}} I_n & 0 \\ * & * & * & -\frac{\alpha_p}{e^{\alpha_p \tau_2} - 1} I_n \end{bmatrix} < 0, \quad (10)$$

where $\Omega_p := X_p A^T + A X_p + \alpha_p X_p - \lambda_2(L_p)(Y_p^T B^T + B Y_p)$, $\alpha_p > 0$ and $\mu_p(\mu_p > 1)$ are given scalars; then, choosing $K_p = Y_p X_p^{-1}$, $p \in \underline{S}$, system (1) with distributed protocol (5) can achieve positive consensus for any switching signal with MDADT satisfying

$$T_{\alpha p} > T_{\alpha p}^* = \frac{\ln \mu_p}{\alpha_p}. \quad (11)$$

Proof The positivity of system (1) with (5) is preserved in Proposition 2. Next, we pursue the conditions for achieving positive consensus. Consider the following MLKFs for system (9):

$$\begin{aligned} V_{\sigma(t)}(t, \delta(t)) = & V_{\sigma(t)1}(t, \delta(t)) + V_{\sigma(t)2}(t, \delta(t)) \\ & + V_{\sigma(t)3}(t, \delta(t)), \forall p \in \underline{S} \end{aligned} \quad (12)$$

where

$$V_{\sigma(t)1}(t, \delta(t)) = \delta^T(t) (I_N \otimes P_{\sigma(t)}) \delta(t),$$

$$V_{\sigma(t)2}(t, \delta(t)) = \int_{t-d_1(t)}^t e^{-\alpha_{\sigma(t)}(t-s)} \|\delta(s)\|^2 ds,$$

$$V_{\sigma(t)3}(t, \delta(t)) = \int_{-\tau}^0 \int_{t+\theta}^t e^{-\alpha_{\sigma(t)}(t-s)} \|\delta(s)\|^2 ds d\theta,$$

and $P_{\sigma(t)}$ is a symmetric positive definite matrix in accordance with the piecewise constant switching signal $\sigma(t)$. Writing $V_{\sigma(t)}(t, \delta(t))$ as $V_{\sigma(t)}(\delta(t))$ for the sake of simplicity. By Definition 8, the upper right-hand derivative of $V_{\sigma(t)}(\delta(t))$ along the solution of system (9) can be given as

$$\begin{aligned} D^+ V_{\sigma(t)1}(\delta(t)) &= \delta^T(t) (I_N \otimes (A^T P_{\sigma(t)} + P_{\sigma(t)} A)) \delta(t) \\ &\quad - \delta^T(t) (L_{\sigma(t)} \otimes (K_{\sigma(t)}^T B^T P_{\sigma(t)} + P_{\sigma(t)} B K_{\sigma(t)})) \delta(t) \\ &\quad + \delta^T(t - d_1(t)) (I_N \otimes A_d^T P_{\sigma(t)}) \delta(t) \\ &\quad + \delta^T(t) (I_N \otimes P_{\sigma(t)} A_d) \delta(t - d_1(t)) \\ &\quad + \int_{t-d_2(t)}^t \delta^T(s) ds (I_N \otimes A_b^T P) \delta(t) \end{aligned} \quad (13)$$

$$\begin{aligned} &+ \delta^T(t) (I_N \otimes P A_b) \int_{t-d_2(t)}^t \delta(s) ds, \\ D^+ V_{\sigma(t)2}(\delta(t)) &\leq -\alpha_{\sigma(t)} V_{\sigma(t)2}(\delta(t)) + \|\delta(t)\|^2 \\ &\quad - (1-h)e^{-\alpha_{\sigma(t)}\tau_1} \|\delta(t - d_1(t))\|^2, \end{aligned} \quad (14)$$

$$\begin{aligned} D^+ V_{\sigma(t)3}(\delta(t)) &\leq -\alpha_{\sigma(t)} V_{\sigma(t)3}(\delta(t)) + \tau \|\delta(t)\|^2 \\ &\quad - \int_{t-\tau}^t e^{-\alpha_{\sigma(t)}(t-s)} \|\delta(s)\|^2 ds. \end{aligned} \quad (15)$$

Consider the second term on the right side of equation (13). Let $0 = \lambda_1(L_{\sigma(t)}) \leq \lambda_2(L_{\sigma(t)}) \leq \dots \leq \lambda_N(L_{\sigma(t)})$ denote the eigenvalues of Laplace matrix $L_{\sigma(t)}$, $\sigma(t) \in \underline{S}$. There exist orthogonal matrix $Z_{\sigma(t)} \in \mathbf{R}^{N \times N}$ and diagonal matrix $\Lambda_{\sigma(t)} \in \mathbf{R}^{N \times N}$ satisfying $Z_{\sigma(t)}^T L_{\sigma(t)} Z_{\sigma(t)} = \Lambda_{\sigma(t)}$; then, it gives rise to

$$\begin{aligned} &-\delta^T(t) (L_{\sigma(t)} \otimes M_{\sigma(t)}) \delta(t) \\ &= -\delta^T(t) \left((Z_{\sigma(t)} \Lambda_{\sigma(t)} Z_{\sigma(t)}^{-1}) \otimes M_{\sigma(t)} \right) \delta(t) \\ &= -\delta^T(t) (Z_{\sigma(t)} \otimes I_n) (\Lambda_{\sigma(t)} \otimes M_{\sigma(t)}) (Z_{\sigma(t)}^T \otimes I_n) \delta(t), \end{aligned} \quad (16)$$

where $M_{\sigma(t)} := K_{\sigma(t)}^T B^T P_{\sigma(t)} + P_{\sigma(t)} B K_{\sigma(t)} > 0$. Choosing $Z_{\sigma(t)} = [\mathbf{1}_N / \sqrt{N}, z_2^{\sigma(t)}, \dots, z_N^{\sigma(t)}]$, where $z_i^{\sigma(t)} \in \mathbf{R}^n$ are eigenvectors of $L_{\sigma(t)}$ associated with eigenvalues $\lambda_i(L_{\sigma(t)})$. Then, there holds $\Lambda_{\sigma(t)} = \text{diag}(0, \lambda_2(L_{\sigma(t)}), \dots, \lambda_N(L_{\sigma(t)}))$. Define $\hat{\delta}(t) = [\hat{\delta}_1^T(t), \dots, \hat{\delta}_N^T(t)]^T = (Z_{\sigma(t)}^T \otimes I_n) \delta(t)$ with $\hat{\delta}_1^T(t) = (\mathbf{1}_N^T / \sqrt{N}) \otimes I_n \delta(t) = 0$. It then follows from (16) that

$$\begin{aligned} &-\delta^T(t) (L_{\sigma(t)} \otimes M_{\sigma(t)}) \delta(t) \\ &= -\hat{\delta}^T(t) (\Lambda_{\sigma(t)} \otimes M_{\sigma(t)}) \hat{\delta}(t) \end{aligned}$$

$$\begin{aligned} &= -\sum_{i=2}^N \lambda_i (L_{\sigma(t)}) \hat{\delta}_i^T(t) M_{\sigma(t)} \hat{\delta}_i(t) \\ &\leq -\lambda_2(L_{\sigma(t)}) \delta^T(t) (I_N \otimes M_{\sigma(t)}) \delta(t). \end{aligned} \quad (17)$$

In addition, noting that $\tau = \max\{\tau_1, \tau_2\}$, the last term of (15) becomes

$$\begin{aligned} &-\int_{t-\tau}^t e^{-\alpha_{\sigma(t)}(t-s)} \|\delta(s)\|^2 ds \\ &\leq -\int_{t-\tau}^t e^{-\alpha_{\sigma(t)}\tau} \|\delta(s)\|^2 ds \\ &\leq -\int_{t-d_2(t)}^t e^{-\alpha_{\sigma(t)}\tau_2} \|\delta(s)\|^2 ds. \end{aligned} \quad (18)$$

Substituting $d_2(t) \leq \tau_2$ into (18) and using Lemma 4 yields

$$\begin{aligned} &-\int_{t-d_2(t)}^t e^{-\alpha_{\sigma(t)}\tau_2} \|\delta(s)\|^2 ds \\ &\leq -\frac{\alpha_{\sigma(t)}}{e^{\alpha_{\sigma(t)}\tau_2} - 1} \left\| \int_{t-d_2(t)}^t \delta(s) ds \right\|^2. \end{aligned} \quad (19)$$

Combining (13)–(15) with (17)–(19) leads to

$$\begin{aligned} D^+ V_{\sigma(t)}(\delta(t)) &= D^+ V_{\sigma(t)1}(\delta(t)) + D^+ V_{\sigma(t)2}(\delta(t)) \\ &\quad + D^+ V_{\sigma(t)3}(\delta(t)) \\ &\leq -\alpha_{\sigma(t)} V_{\sigma(t)}(\delta(t)) + \delta^T(t) \left(I_N \otimes \left(A^T P_{\sigma(t)} \right. \right. \\ &\quad \left. \left. + P_{\sigma(t)} A - \lambda_2(L_{\sigma(t)}) (K_{\sigma(t)}^T B^T P_{\sigma(t)} \right. \right. \\ &\quad \left. \left. + P_{\sigma(t)} B K_{\sigma(t)}) + \alpha_{\sigma(t)} P_{\sigma(t)} + (\tau + 1) I_n \right) \right) \delta(t) \\ &\quad - (1-h)e^{-\alpha_{\sigma(t)}\tau_1} \|\delta(t - d_1(t))\|^2 \\ &\quad - \frac{\alpha_{\sigma(t)}}{e^{\alpha_{\sigma(t)}\tau_2} - 1} \left\| \int_{t-d_2(t)}^t \delta(s) ds \right\|^2 \\ &\quad + \delta^T(t - d_1(t)) (I_N \otimes A_d^T P_{\sigma(t)}) \delta(t) \\ &\quad + \delta^T(t) (I_N \otimes P_{\sigma(t)} A_d) \delta(t - d_1(t)) \\ &\quad + \int_{t-d_2(t)}^t \delta^T(s) ds (I_N \otimes A_b^T P) \delta(t) \\ &\quad + \delta^T(t) (I_N \otimes P A_b) \int_{t-d_2(t)}^t \delta(s) ds. \end{aligned} \quad (20)$$

Denote $\eta(t) = [\delta^T(t), \delta^T(t - d_1(t)), \int_{t-d_2(t)}^t \delta^T(s) ds]^T$, (20) can be written as

$$\begin{aligned} D^+ V_{\sigma(t)}(\delta(t)) &+ \alpha_{\sigma(t)} V_{\sigma(t)}(\delta(t)) \\ &\leq \eta^T(t) (I_N \otimes \Psi_{\sigma(t)}) \eta(t), \end{aligned} \quad (21)$$

where

$$\Psi_{\sigma(t)} := \begin{bmatrix} \Theta_{\sigma(t)} & P_{\sigma(t)} A_d & P_{\sigma(t)} A_b \\ * & -(1-h)e^{-\alpha_{\sigma(t)} \tau_1} I_n & 0 \\ * & * & -\frac{\alpha_{\sigma(t)}}{e^{\alpha_{\sigma(t)} \tau_2} - 1} I_n \end{bmatrix} \quad (22)$$

with $\Theta_{\sigma(t)} := A^T P_{\sigma(t)} + P_{\sigma(t)} A + \alpha_{\sigma(t)} P_{\sigma(t)} + (\tau + 1)I_n - \lambda_2(L_{\sigma(t)})(K_{\sigma(t)}^T B^T P_{\sigma(t)} + P_{\sigma(t)} B K_{\sigma(t)})$. By Lemma 3, we know linear matrix inequality (10) is equivalent to the inequality below

$$\begin{bmatrix} \Gamma_{\sigma(t)} & A_d & A_b \\ * & -(1-h)e^{-\alpha_{\sigma(t)} \tau_1} I_n & 0 \\ * & * & -\frac{\alpha_{\sigma(t)}}{e^{\alpha_{\sigma(t)} \tau_2} - 1} I_n \end{bmatrix} < 0, \quad (23)$$

where $\Gamma_{\sigma(t)} := X_{\sigma(t)} A^T + A X_{\sigma(t)} + \alpha_{\sigma(t)} X_{\sigma(t)} + (\tau + 1)X_{\sigma(t)}^T X_{\sigma(t)} - \lambda_2(L_{\sigma(t)})(Y_{\sigma(t)}^T B^T + B Y_{\sigma(t)})$. Multiplying $\text{diag}\{X_{\sigma(t)}^{-1}, I_n, I_n\}$ from the left hand and then from the right hand of (23), and writing $P_{\sigma(t)} = X_{\sigma(t)}^{-1}$ and $Y_{\sigma(t)} = K_{\sigma(t)} X_{\sigma(t)}$, $\sigma(t) \in \underline{S}$, we obtain matrix inequality (23) is equivalent to the $\Psi_{\sigma(t)} < 0$ in (22). From this and (21), we get

$$D^+ V_{\sigma(t)}(\delta(t)) \leq -\alpha_{\sigma(t)} V_{\sigma(t)}(\delta(t)). \quad (24)$$

Let $t_1 < t_2 < \dots < t_{k-1} < t_k$, $k \in \mathbf{Z}_+$, denote the switching instants of the switching signal $\sigma(t)$ over the interval $[t_0, t)$. Integrating both sides of (24) during the period $[t_k, t)$ for $t \in [t_k, t_{k+1})$ gives rise to

$$V_{\sigma(t)}(\delta(t)) \leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}(\delta(t_k)). \quad (25)$$

For any switching instant t_k , $k \in \mathbf{Z}_+$, denote $\sigma(t_k) = p$, $\sigma(t_k^-) = q$, $\forall p, q \in \underline{S}$, and choose $\mu_p \geq 1$ such that

$$P_p \leq \mu_p P_q, \forall (\sigma(t_k) = p, \sigma(t_k^-) = q) \in \underline{S} \times \underline{S}, \quad p \neq q, \quad (26)$$

then, it follows from (12) and (26) that

$$\begin{aligned} V_{\sigma(t_k)}(\delta(t_k)) &= \delta^T(t_k)(I_N \otimes P_{\sigma(t_k)})\delta(t_k) \\ &+ \int_{t_k-d_1(t_k)}^{t_k} e^{-\alpha_{\sigma(t_k)}(t_k-s)} \|\delta(s)\|^2 ds \\ &+ \int_{-\tau}^0 \int_{t_k+\theta}^{t_k} e^{-\alpha_{\sigma(t_k)}(t_k-s)} \|\delta(s)\|^2 ds d\theta \\ &\leq \mu_{\sigma(t_k)} \delta^T(t_k^-)(I_N \otimes P_{\sigma(t_k^-)})\delta(t_k^-) \\ &+ \mu_{\sigma(t_k)} \int_{t_k^- - d_1(t_k^-)}^{t_k^-} e^{-\alpha_{\sigma(t_k^-)}(t_k^- - s)} \|\delta(s)\|^2 ds \\ &+ \mu_{\sigma(t_k)} \int_{-\tau}^0 \int_{t_k^- + \theta}^{t_k^-} e^{-\alpha_{\sigma(t_k^-)}(t_k^- - s)} \|\delta(s)\|^2 ds d\theta \\ &= \mu_{\sigma(t_k)} V_{\sigma(t_k^-)}(\delta(t_k^-)), \end{aligned} \quad (27)$$

where $\delta(t_k^-) = \lim_{s \rightarrow 0^-} \delta(t_k + s)$, which together with (25) results in

$$V_{\sigma(t)}(\delta(t)) \leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} \mu_{\sigma(t_k)} V_{\sigma(t_k^-)}(\delta(t_k^-)). \quad (28)$$

Combining (27) with (28) yields that

$$\begin{aligned} V_{\sigma(t)}(\delta(t)) &\leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} e^{-\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})} \times \\ &\mu_{\sigma(t_k)} V_{\sigma(t_{k-1})}(\delta(t_{k-1})) \\ &\leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} e^{-\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})} \mu_{\sigma(t_k)} \mu_{\sigma(t_{k-1})} \times \\ &V_{\sigma(t_{k-1}^-)}(\delta(t_{k-1}^-)) \\ &\leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} e^{-\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})} \dots e^{-\alpha_{\sigma(t_1-t_0)} \times} \\ &\mu_{\sigma(t_k)} \mu_{\sigma(t_{k-1})} \dots \mu_{\sigma(t_1)} V_{\sigma(t_0)}(\delta(t_0)) \\ &= \left(\prod_{i=1}^k \mu_{\sigma(t_i)} \right) e^{-\alpha_{\sigma(t_k)}(t-t_k)} \times \\ &e^{-\sum_{i=1}^k \alpha_{\sigma(t_{i-1})}(t_i-t_{i-1})} V_{\sigma(t_0)}(\delta(t_0)). \end{aligned} \quad (29)$$

From Definition 1, $\forall p \in \underline{S}$, it follows that

$$\begin{aligned} V_{\sigma(t)}(\delta(t)) &\leq \left(\prod_{p=1}^S \mu_p^{N_{\sigma p}(t_0, t)} \right) e^{-\sum_{p=1}^S \alpha_p T_p(t_0, t)} V_{\sigma(t_0)}(\delta(t_0)) \\ &\leq \left(\prod_{p=1}^S \mu_p^{\frac{T_p(t_0, t)}{T_{\alpha p}}} \right) e^{-\sum_{p=1}^S \alpha_p T_p(t_0, t)} V_{\sigma(t_0)}(\delta(t_0)) \\ &= e^{\sum_{p=1}^S \left(\frac{\ln \mu_p}{T_{\alpha p}} - \alpha_p \right) T_p(t_0, t)} V_{\sigma(t_0)}(\delta(t_0)). \end{aligned} \quad (30)$$

Using (11) and choosing $\alpha = \min_{p \in \underline{S}} \left\{ \alpha_p - \frac{\ln \mu_p}{T_{\alpha p}} \right\} < 0$, we then arrive at

$$V_{\sigma(t)}(\delta(t)) \leq e^{-\alpha(t-t_0)} V_{\sigma(t_0)}(\delta(t_0)). \quad (31)$$

By simple calculation, we get

$$\begin{aligned} V_{\sigma(t_0)}(\delta(t_0)) &\leq \lambda_{\max}(P_p) \|\delta(t_0)\|_c^2 + \|\delta(t_0)\|_c^2 \int_{-\tau_1}^0 e^{\alpha_p s} ds \\ &+ \|\delta(t_0)\|_c^2 \int_{-\tau}^0 \int_{\theta}^0 e^{\alpha_p s} ds d\theta \\ &\leq (\lambda_{\max}(P_p) + \alpha_p^{-1}(1 + \tau_1) + \alpha_p^{-2} e^{-\alpha_p \tau}) \\ &\times \|\delta(t_0)\|_c^2 \\ &= \xi_{p1} \|\delta(t_0)\|_c^2, \end{aligned} \quad (32)$$

where $\|\delta(t_0)\|_c = \sup_{-\tau \leq \theta \leq 0} \{\|\delta(\theta)\|\}$. Moreover, from (12), it follows that

$$V_p(\delta(t)) \geq \xi_{p2} \|\delta(t)\|^2, \quad \forall \delta(t) \in \mathbb{R}^{Nn}, \quad \forall p \in \underline{S}, \quad (33)$$

where $\xi_{p2} = \lambda_{\min}(P_p)$. Let $\xi_1 = \max_{p \in \underline{S}} \{\xi_{p1}\}$ and $\xi_2 = \min_{p \in \underline{S}} \{\xi_{p2}\}$. Substituting (32) and (33) into (31) yields

$$\|\delta(t)\|^2 \leq \frac{\xi_1}{\xi_2} e^{-\alpha(t-t_0)} \|\delta(t_0)\|_c^2. \quad (34)$$

Therefore, the positive consensus of system (1) with (5) is achieved. This completes the proof. $\square$

Remark 4 It should be pointed out that the ADT approach guarantees that the number of topology switching in any finite time interval is bounded and the average time between two consecutive switching instants is not less than a common constant. However, the individual properties of each topology are neglected since all topologies have the same dwell-time constraint, whereas MDADT approach allows every topology to have its own ADT and compensates for the deficiency of ADT by determining the infimum of average time among the intervals associated with the i -th topology. Moreover, if $T_{\alpha p} = T_{\alpha q}$, $\forall p, q \in \underline{S}$, the MDADT is turned into ADT. In addition, as an extreme case, if there holds $T_{\alpha p} \rightarrow 0$ in (2), then the switching signal can be arbitrary.

Remark 5 Since the positivity of linear MASs with communication delays can not be preserved, this paper does not involve such delays. Specifically, for general linear MASs $\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$ with distributed protocol $u_i(t) = -K \sum_{j \in \mathcal{N}_i} a_{ij}(x_i(t-d(t)) - x_j(t-d(t)))$ having bounded communication delay $d(t)$. The compact form of the corresponding closed-loop system is $\dot{x}(t) = (I_N \otimes A)x(t) - (L \otimes BK)x(t-d(t))$. As discussed in [5] and [18], this system is positive if and only if A is a Metzler matrix and $-(L \otimes BK) \geq 0$. From the properties of Laplacian matrix L and Kronecker product, it follows that the sum of the elements in any row of the nonzero matrix $L \otimes BK$ is 0. Thus, condition $-(L \otimes BK) \geq 0$ cannot be satisfied, which means that the positivity of the above closed-loop MASs cannot be preserved.

4.2 Guaranteed cost positive consensus analysis

Based on Sect. 4.1, this subsection is devoted to establishing the guaranteed cost positive consensus criterion and determining an upper bound of cost function (3). The corresponding technical result is summarized in the following theorem.

Theorem 2 Under the conditions of Proposition 2, if there exist matrices $X_p \in \mathbb{R}^{n \times n}$ and $Y_p \in \mathbb{R}^{m \times n}$, $\forall p \in \underline{S}$, with X being positive-definite, such that the following condition holds

$$\begin{bmatrix} \Omega_p & X_p & X_p M_1 & Y_p^T M_2 & A_d & A_b \\ * & -(1+\tau)I_n & 0 & 0 & 0 & 0 \\ * & * & \frac{-M_1}{2\lambda_{\max}(L_p)} & 0 & 0 & 0 \\ * & * & * & \frac{-M_2}{\lambda_{\max}^2(L_p)} & 0 & 0 \\ * & * & * & * & \frac{-(1-h)I_n}{e^{\alpha_p \tau_1}} & 0 \\ * & * & * & * & * & \frac{-\alpha_p I_n}{e^{\alpha_p \tau_2} - 1} \end{bmatrix} < 0, \quad (35)$$

where $\alpha_p > 0$ and $\mu_p (\mu_p > 1)$ are given scalars, then, choosing $K_p = Y_p X_p^{-1}$, system (1) with distributed protocol (5) can achieve guaranteed cost positive consensus for any switching signal with MDADT satisfying (11), and the cost function has an upper bound

$$J^* = \xi_1 \|\delta(t_0)\|_c^2. \quad (36)$$

Proof By (3) and (5), it is not hard to show that

$$\begin{aligned} & \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \left(a_{ij}^{\sigma(t)} (x_i(t) - x_j(t))^T M_1 (x_i(t) - x_j(t)) \right) \\ &= 2\delta^T(t) (L_{\sigma(t)} \otimes M_1) \delta(t), \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=1}^N (u_i^T(t) M_2 u_i(t)) \\ &= \delta^T(t) (L_{\sigma(t)}^2 \otimes K_{\sigma(t)}^T M_2 K_{\sigma(t)}) \delta(t). \end{aligned}$$

Then, cost function (3) can be written as

$$\begin{aligned} J &= \int_{t_0}^{+\infty} \delta^T(t) \left(2L_{\sigma(t)} \otimes M_1 \right. \\ &\quad \left. + L_{\sigma(t)}^2 \otimes K_{\sigma(t)}^T M_2 K_{\sigma(t)} \right) \delta(t) dt. \end{aligned} \quad (37)$$

Taking the same way as (13)–(20) and invoking (37), we deduce

$$D^+ V_{\sigma(t)}(\delta(t)) + \alpha_{\sigma(t)} V_{\sigma(t)}(\delta(t))$$

$$\begin{aligned}
 & + \delta(t)(2L_{\sigma(t)} \otimes M_1 + L_{\sigma(t)}^2 \otimes K_{\sigma(t)}^T M_2 K_{\sigma(t)}) \delta(t) \\
 & \leq \eta^T(t)(I_N \otimes \Psi_{\sigma(t)}) \eta(t) + \delta^T(t) \left(2I_N \otimes \right. \\
 & \quad \left. (\lambda_{\max}(L_{\sigma(t)}) M_1 + \lambda_{\max}^2(L_{\sigma(t)}) K_{\sigma(t)}^T M_2 K_{\sigma(t)}) \right) \delta(t) \\
 & \leq \eta^T(t)(I_N \otimes \Phi_{\sigma(t)}) \eta(t), \quad (38)
 \end{aligned}$$

where

$$\Phi_{\sigma(t)} = \begin{bmatrix} \mathcal{V}_{\sigma(t)} & P_{\sigma(t)} A_d & P_{\sigma(t)} A_b \\ * & -(1-h)e^{-\alpha_{\sigma(t)} \tau_1} I_n & 0 \\ * & * & -\frac{\alpha_{\sigma(t)}}{e^{\alpha_{\sigma(t)} \tau_2} - 1} I_n \end{bmatrix} \quad (39)$$

with $\mathcal{V}_{\sigma(t)} := A^T P_{\sigma(t)} + P_{\sigma(t)} A + \alpha_{\sigma(t)} P_{\sigma(t)} + (1+\tau) I_n - \lambda_2(L_{\sigma(t)}) (K_{\sigma(t)}^T B^T P_{\sigma(t)} + P_{\sigma(t)} B K_{\sigma(t)}) + 2\lambda_{\max}(L_{\sigma(t)}) M_1 + \lambda_{\max}^2(L_{\sigma(t)}) K_{\sigma(t)}^T M_2 K_{\sigma(t)}$. From Lemma 3, it can be seen that (35) is equivalent to the following inequality

$$\begin{bmatrix} \Upsilon_{\sigma(t)} & A_d & A_b \\ * & -(1-h)e^{-\alpha_{\sigma(t)} \tau_1} I_n & 0 \\ * & * & -\frac{\alpha_{\sigma(t)}}{e^{\alpha_{\sigma(t)} \tau_2} - 1} I_n \end{bmatrix} < 0, \quad (40)$$

where $\Upsilon_{\sigma(t)} := X_{\sigma(t)} A^T + A X_{\sigma(t)} + \alpha_{\sigma(t)} X_{\sigma(t)} + (1+\tau) X_{\sigma(t)}^T X_{\sigma(t)} - \lambda_2(L_{\sigma(t)}) (Y_{\sigma(t)}^T B^T + B Y_{\sigma(t)}) + 2\lambda_{\max}(L_{\sigma(t)}) X_{\sigma(t)}^T M_1 X_{\sigma(t)} + \lambda_{\max}^2(L_{\sigma(t)}) Y_{\sigma(t)}^T M_2 Y_{\sigma(t)}$. As with the processing of (23), we know inequality (40) is equivalent to the $\Phi_{\sigma(t)} < 0$ in (39), from which it can be obtained that

$$\begin{aligned}
 & D^+ V_{\sigma(t)}(\delta(t)) + \delta(t)(2L_{\sigma(t)} \otimes M_1 \\
 & \quad + L_{\sigma(t)}^2 \otimes K_{\sigma(t)}^T M_2 K_{\sigma(t)}) \delta(t) \\
 & \leq -\alpha_{\sigma(t)} V_{\sigma(t)}(\delta(t)). \quad (41)
 \end{aligned}$$

Noting that $2L_{\sigma(t)} \otimes M_1 + L_{\sigma(t)}^2 \otimes K_{\sigma(t)}^T M_2 K_{\sigma(t)} \geq 0$. Inequality (10) is a sufficient condition for (35) and (41) implies (24). Then, similar to the process of (24)–(34), we know that the positive consensus is achieved for system (1) with controller (5).

Next, we are going to determine the cost of system (1) with protocol (5). Denoting $\Delta(t) := \delta(t)(2L_{\sigma(t)} \otimes M_1 + L_{\sigma(t)}^2 \otimes K_{\sigma(t)}^T M_2 K_{\sigma(t)}) \delta(t)$ and integrating both sides of (41) from t_k to t for $t \in [t_k, t_{k+1})$ yields that

$$\begin{aligned}
 V_{\sigma(t)}(t)(\delta(t)) & \leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}(\delta(t_k)) \\
 & \quad - \int_{t_k}^t e^{-\alpha_{\sigma(t_k)}(t-s)} \Delta(s) ds. \quad (42)
 \end{aligned}$$

Substituting (26) into (42) gives

$$\begin{aligned}
 V_{\sigma(t)}(t)(\delta(t)) & \leq \mu_{\sigma(t_k)} e^{-\alpha_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}(\delta(t_k^-)) \\
 & \quad - \int_{t_k}^t e^{-\alpha_{\sigma(t_k)}(t-s)} \Delta(s) ds \\
 & \leq \left(\prod_{i=1}^k \mu_{\sigma(t_i)} \right) e^{-\alpha_{\sigma(t_k)}(t-t_k)} e^{\sum_{i=1}^{k-1} \alpha_{\sigma(t_i)}(t_i-t_{i-1})} \times \\
 & \quad V_{\sigma(t_0)}(t_0) - \left(\prod_{i=1}^k \mu_{\sigma(t_i)} \right) e^{-\alpha_{\sigma(t_k)}(t-t_k)} \times \\
 & \quad e^{-\sum_{i=2}^{k-1} \alpha_{\sigma(t_i)}(t_i-t_{i-1})} \int_{t_0}^{t_1} e^{-\alpha_{\sigma(t_0)}(t_1-s)} \Delta(s) ds \\
 & \quad - \dots - \mu_{\sigma(t_k)} e^{-\alpha_{\sigma(t_k)}(t-t_k)} \times \\
 & \quad \int_{t_{k-1}}^{t_k} e^{-\alpha_{\sigma(t_{k-1})}(t_k-s)} \Delta(s) ds \\
 & \quad - \int_{t_k}^t e^{-\alpha_{\sigma(t_k)}(t-s)} \Delta(s) ds \\
 & = \left(\prod_{p=1}^S \mu_p^{N_{\sigma p}(t_0, t)} \right) e^{-\sum_{p=1}^S \alpha_p T_p(t_0, t)} V_{\sigma(t_0)}(\delta(t_0)) \\
 & \quad - \int_{t_0}^t \left(\prod_{p=1}^S \mu_p^{N_{\sigma p}(s, t)} \right) e^{-\sum_{p=1}^S \alpha_p T_p(s, t)} \Delta(s) ds. \quad (43)
 \end{aligned}$$

Noting that $V_{\sigma(t)}(\delta(t)) \geq 0$, one has

$$\begin{aligned}
 & \int_{t_0}^t \left(\prod_{p=1}^S \mu_p^{N_{\sigma p}(s, t)} \right) e^{-\sum_{p=1}^S \alpha_p T_p(s, t)} \Delta(s) ds \\
 & \leq \left(\prod_{p=1}^S \mu_p^{N_{\sigma p}(t_0, t)} \right) e^{-\sum_{p=1}^S \alpha_p T_p(t_0, t)} V_{\sigma(t_0)}(\delta(t_0)). \quad (44)
 \end{aligned}$$

Multiplying both sides of (44) by $\left(\prod_{p=1}^S \mu_p^{-N_{\sigma p}(t_0, t)} \right) \times e^{\sum_{p=1}^S \alpha_p T_p(t_0, t)}$ leads to

$$\begin{aligned}
 & \int_{t_0}^t \left(\prod_{p=1}^S \mu_p^{-N_{\sigma p}(t_0, s)} \right) e^{\sum_{p=1}^S \alpha_p T_p(t_0, s)} \Delta(s) ds \\
 & \leq V_{\sigma(t_0)}(\delta(t_0)). \quad (45)
 \end{aligned}$$

Combining (11) with (45) yields

$$\begin{aligned}
 V_{\sigma(t_0)}(\delta(t_0)) & \geq \int_{t_0}^t e^{\sum_{p=1}^S \left(\alpha_p - \frac{\ln \mu_p}{T_{\alpha p}} \right) T_p(t_0, s)} \Delta(s) ds \\
 & \geq \int_{t_0}^t \Delta(s) ds = J. \quad (46)
 \end{aligned}$$

Thus, the cost of system (1) satisfies

$$J \leq J^* = \xi_1 \|\delta(t_0)\|_c^2. \quad (47)$$

This completes the proof. $\square$

Remark 6 Through the above analysis, the performance of system (1) is under the joint effect of the consensus protocols, the dynamics of the agent, the switching topologies and the time-varying delays. Although the positive consensus problem of (1) is transformed into that of switched delay system (7), the relevant methods in the isolated switched system (see [15, 23, 35, 36] and the references therein) cannot be directly used to deal with the problem we focused as a result of the interaction between agents.

Remark 7 Many available results on the consensus of MASs with switching topologies have been reported (see [11, 33, 38] and the references therein), while the consensus regulation performance and the energy consumption of control effects have been rarely considered simultaneously. This paper realizes a trade-off between the consensus regulation performance and the energy consumption. It is worth pointing out that the established guaranteed cost positive consensus criteria only depend on the eigenvalues of the Laplacian matrices, i.e. $\lambda_2(L_p)$ and $\lambda_{\max}(L_p)$ do not depend on the number of agents. Moreover, the determined upper bound of the cost function only depends on the initial conditions of the agents and is independent of the switching movements.

Remark 8 As can be seen from condition (35), the second small and maximum eigenvalues of the Laplacian matrices of every topology, i.e. $\lambda_2(L_p)$ and $\lambda_{\max}(L_p)$, $p \in \underline{S}$, play a critical role in solving the guaranteed cost positive consensus problem. When the number of the agent is large, these parameters may be not easy to obtain. Aiming at this issue, many methods have been proposed to estimate the eigenvalues of the matrix. For example, the method in [32] and the Geršgorin disc theorem in [7] have been used to estimate the eigenvalues $\lambda_2(L_p)$ and $\lambda_{\max}(L_p)$, respectively. Therefore, it is not necessary to discuss the exact values of these two parameters in this paper.

Remark 9 Theorem 2 presents the sufficient condition for designing the guaranteed cost positive consensus controller of system (1). Recalling (32) and (46), if τ_1 , τ_2 and α_p are given, the value of J only depends on

matrix P_p , $p \in \underline{S}$. Define an admissible set $\mathfrak{N}_{(X_p, Y_p)} = \{(X_p, Y_p) : X_p \in \mathbf{R}^{n \times n}, Y_p \in \mathbf{R}^{m \times n}, p \in \underline{S} \text{ such that Proposition 2 and (35) hold}\}$. Then we can obtain an optimal cost value J^* and optimal guaranteed cost positive consensus control law (5) with $K_p = Y_p^{*-1} X_p^*$, $p \in \underline{S}$, where the triplet (J^*, X^*, Y^*) is a solution of the optimization problem formulated by

$$\min_{J, X_p, Y_p} J \text{ subject to } (X_p, Y_p) \in \mathfrak{N}_{(X_p, Y_p)}, p \in \underline{S}.$$

Theorem 2 confirms the feasibility of guaranteed cost positive consensus of system (1) under the proposed scheme, but does not explicitly specify the parameters involved, i.e. the feedback gain matrices K_p in controller (5), and the MDADT $T_{\alpha p}^*$ in constructing switching signals. Thus, a summary of the steps for determining the K_p and $T_{\alpha p}^*$, $p \in \underline{S}$, will be presented in the forthcoming algorithm.

Algorithm 1

- Step 1. Input matrices A , $A_d \geq 0$, $A_b \geq 0$, B , M_1 , M_2 , positive constants τ_1 , τ_2 , h , and adjustable parameters α_p , $p \in \underline{S}$.
- Step 2. Solve (35) in Theorem 2 via LMI toolbox to obtain matrices X_p and Y_p .
- Step 3. Compute $K_p = Y_p X_p^{-1}$ and $P_p = X_p^{-1}$, and verify Proposition 2. If Proposition 2 is not satisfied, return to Step 1 and reselect α_p ; Otherwise, go to Step 4.
- Step 4. Calculate (26) to get μ_p , and substitute μ_p and α_p into (11) to obtain $T_{\alpha p}^*$.
- Step 5. Output K_p and $T_{\alpha p}^*$, $p \in \underline{S}$.

5 Simulation results

In this section, simulations are performed to verify the proposed control scheme. For MASs (1) with five agents labelled from 1 to 5, the system parameters are given as follows:

$$A = \begin{bmatrix} -1 & 2 \\ 1.1 & -1 \end{bmatrix}, A_d = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, A_b = \begin{bmatrix} 0.1 & 0.4 \\ 0.1 & 0.2 \end{bmatrix}, \\ B = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix}, M_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}, M_2 = 0.2.$$

Assume that the underlying interaction topology switches among four connected undirected graphs, which is shown in Fig. 2. The activation time of each graph will be determined later.

Fig. 2 Switching topology graphs

In order to achieve guaranteed cost positive consensus of system (1) under switching topologies, we design the distributed consensus protocol of form (5) and establish some sufficient conditions in Theorem 2. Let $\alpha_1 = \alpha_2 = 0.4$, $\alpha_3 = 0.72$, $\alpha_4 = 1$, $d_1(t) = 0.1 + 0.1\sin(t)$ and $d_2(t) = 0.02 + 0.1\cos(t)$, then we get $h = 0.1$, $\tau_2 = 0.12$ and $\tau_1 = 0.2$. Solving inequalities (35) by MATLAB, we obtain

$$\begin{aligned} X_1 &= \begin{bmatrix} 1.8741 & 0.3515 \\ 0.3515 & 1.6935 \end{bmatrix}, X_2 = \begin{bmatrix} 1.8860 & 0.3217 \\ 0.3217 & 1.7186 \end{bmatrix}, \\ X_3 &= \begin{bmatrix} 1.9009 & 0.2696 \\ 0.2696 & 1.7575 \end{bmatrix}, X_4 = \begin{bmatrix} 1.9064 & 0.2192 \\ 0.2192 & 1.7872 \end{bmatrix}, \\ Y_1 &= \begin{bmatrix} 10.6544 \\ 12.1623 \end{bmatrix}^T, Y_2 = \begin{bmatrix} 7.5914 \\ 8.6697 \end{bmatrix}^T, \\ Y_3 &= \begin{bmatrix} 7.9091 \\ 9.0609 \end{bmatrix}^T, Y_4 = \begin{bmatrix} 4.4395 \\ 5.1125 \end{bmatrix}^T, \end{aligned}$$

then, the matrices P_p , K_p , $A - \varpi_{\max}^p BK_p$ and parameters μ_p , $p = 1, 2, 3, 4$, can be calculated as

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.5552 & -0.1153 \\ -0.1153 & 0.6144 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 0.5477 & -0.1025 \\ -0.1025 & 0.6011 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 0.5378 & -0.0825 \\ -0.0825 & 0.5817 \end{bmatrix}, \\ P_4 &= \begin{bmatrix} 0.5321 & -0.0653 \\ -0.0653 & 0.5675 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} 4.5137 \\ 6.2450 \end{bmatrix}^T, K_2 = \begin{bmatrix} 3.2689 \\ 4.4326 \end{bmatrix}^T, \\ K_3 &= \begin{bmatrix} 3.5059 \\ 4.6179 \end{bmatrix}^T, K_4 = \begin{bmatrix} 2.0285 \\ 2.6118 \end{bmatrix}^T, \\ A - \varpi_{\max}^1 BK_1 &= \begin{bmatrix} -2.0833 & 0.5012 \\ 0.1973 & -2.2490 \end{bmatrix}, \end{aligned}$$

$$A - \varpi_{\max}^2 BK_2 = \begin{bmatrix} -2.1768 & 0.4043 \\ 0.1193 & -2.3298 \end{bmatrix},$$

$$A - \varpi_{\max}^3 BK_3 = \begin{bmatrix} -2.2621 & 0.3376 \\ 0.0482 & -2.3854 \end{bmatrix},$$

$$A - \varpi_{\max}^4 BK_4 = \begin{bmatrix} -1.9737 & 0.7464 \\ 0.2886 & -2.0447 \end{bmatrix},$$

$$\mu_1 = 1.1271, \mu_2 = 1.0546, \mu_3 = 1.1803, \mu_4 = 1.1399.$$

It is clear that $A - \varpi_{\max}^p BK_p$ are Metzler matrices and $BK_p \geq 0$, $p = 1, 2, 3, 4$. From (11), we have $T_{\alpha 1}^* = 0.2991$, $T_{\alpha 2}^* = 0.1329$, $T_{\alpha 3}^* = 0.2302$ and $T_{\alpha 4}^* = 0.1654$. By simple calculation, we get an upper bound of the cost value: $J^* = \xi_1 \|\delta(t_0)\|_c^2 = 20.9591$.

Let $T_{\alpha 1} = 0.3$, $T_{\alpha 2} = 0.14$, $T_{\alpha 3} = 0.24$, $T_{\alpha 4} = 0.17$ and choose the initial condition of system (1) as $x_1(0) = [2, 0.2]^T$, $x_2(0) = [1, 0.6]^T$, $x_3(0) = [3, 0.4]^T$, $x_4(0) = [1.2, 0.3]^T$ and $x_5(0) = [2.5, 0.5]^T$. By simulation, the state trajectories of the resulting closed-loop system with MDADT are shown in Fig. 3a, and it can be seen that the states of all agents are always non-negative and achieve consensus. The evolution of control inputs and the random switching signal satisfying condition (11) are depicted in Figs. 3b and 3c, respectively. The trajectory of the cost function J is given in Fig. 3d, which shows that the relationship between the cost bound J^* and the actual cost J satisfies $J < J^*$. Furthermore, by repeating this simulation with different MDADT switching signals, we find that the corresponding actual cost values are less than 20.9591. This indicates that switching movements do not affect the determined upper bound J^* . Thus, the desired *guaranteed cost positive consensus* of MASs (1) is achieved.

Moreover, in order to assess the comprehensive performance of the proposed method, pulse disturbance and random noise are, respectively, considered in the previous simulation. First, to evaluate the effect of pulse disturbance, suppose that two pulse signals with periods of 0.6s and 1s and amplitude of 3 are imposed on the two components of the system states (i.e. x_{i1} and x_{i2} , $i = 1, \dots, 5$), respectively. The corresponding simulation results are given in Fig. 4a, and we can see that the proposed control strategy can effectively dominate the pulse disturbances. Meanwhile, the actual cost value does not exceed J^* . On the other hand, for the case of random noise, assume that the two compo-

Fig. 3 Simulation results under MDADT switching

Fig. 4 Simulation results under pulse disturbance and additive random noise

nents of the system state are, respectively, subjected to two additive random noise with variances of 20 and 30 and means of 0. The evolutions of the state and cost of system (1) are shown in Fig. 4b. These two cases show that the proposed distributed control strategy possesses certain disturbance rejection performance.

Furthermore, for the sake of comparison, simulated studies are conducted between the proposed MDADT approach, the well-known ADT approach and the switching method in reference [27]. Taking the same simulated system, initial states and sampling rate as the previous simulation, the corresponding switching signals, state trajectories and cost values are exhibited in Fig. 5. The relevant parameters and computation results under three different switching schemes are listed in Table 1.

As shown in Fig. 3d, 5b, d, all of the ADT switching, the aperiodic switching in reference [27] and the con-

Fig. 5 Simulation results under the ADT approach and the approach in reference [27]

cerned MDADT approach are effective for our control issue, while the actual cost of the last one is lower. Moreover, from Table 1 and Fig. 3c, we know that the MDADT approach allows $T_{\alpha p}^*$ to be different for $p = 1, 2, 3, 4$, and the switching numbers of various topologies are different as well. One special case of MDADT approach is $T_{\alpha}^* = 0.3039$ by setting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.6$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1.2$, which corresponds to the ADT switching approach. Moreover, the ADT approach with periodic switching is the scheme used in reference [27]. Accordingly, the switching number of the MDADT approach can be more than that of ADT and the approach employed in [27] (this fact can also be seen from Fig. 3c, 5a, c). That is, the designed MDADT switching is more general and less conservative, which is consistent with the intuitive analysis in Remark 4 and the existing conclusion in the literature.

6 Conclusion

The paper has first introduced the concept of guaranteed cost positive consensus in linear MASs with multiple time-varying delays and switching topologies. Based on the MLKFs and MDADT approaches, two criteria have been established to achieve positive consensus

Table 1 Comparison of three switching schemes

Switching schemes Features	MDADT Aperiodic	ADT Aperiodic	Scheme in [27] Periodic
Parameters	$\alpha_1 = 0.4, \mu_1 = 1.1271$ $\alpha_2 = 0.4, \mu_2 = 1.0546$ $\alpha_3 = 0.72, \mu_3 = 1.1803$ $\alpha_4 = 1, \mu_4 = 1.1399$	$\alpha = 0.6$ $\mu = 1.2$	period=0.5
Switching signals	$T_{\alpha 1}^* = 0.2991$ $T_{\alpha 2}^* = 0.1329$ $T_{\alpha 3}^* = 0.2302$ $T_{\alpha 4}^* = 0.1654$	$T_{\alpha}^* = 0.3039$	switch with period 0.5

and guaranteed cost positive consensus, respectively. The designed switching signal permits each topology to have its own ADT. It is shown that the derived stability conditions are independent of the number of agents, which ensures the scalability of MASs. Meanwhile, the determined upper bound of the cost function is independent of the switching behaviour and is only related to the initial conditions of all agents. For networked MASs, the disturbances or attacks inevitably occur in transmission channels [11, 19]. Further work will consider the resilient guaranteed cost consensus issue for MASs with transmission delay and disturbances/attacks.

Acknowledgements This work is supported by the National Natural Science Foundation of China (61973193, U1964207), the Key Program of the National Natural Science Foundation of China (62133008) and the Innovative Research Groups of National Natural Science Foundation of China (61821004).

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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