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Nonlinear analysis of a four-dimensional fractional hyper-chaotic system based on general Riemann–Liouville–Caputo fractal–fractional derivative

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Abstract In this study, a four-dimensional fractional hyperchaotic model is analyzed based on general Riemann–Liouville–Caputo (RLC) fractal–fractional derivative (FFD). A series of new operators are constructed using three different elements, namely, the general Mittag–Leffler function, exponential decay, and power law. The operators have two parameters: One is considered as fractional order and the other as fractal dimension. The Qi hyperchaotic fractional attractor is modeled by using these operators, and the models are solved numerically using a very efficient numerical scheme. Meanwhile, the existence and uniqueness of solutions have been investigated to justify the physical adequacy of the model and the numerical scheme proposed in the resolution. The numerical simulations for some specific fractional order and fractal dimension are presented. Furthermore, these results obtained via generalized Caputo–Fabrizio and fractal–fractional derivative show some crossover effects, which is due to non-index law property. Finally, these obtained from generalized fractal–fractional derivative show very strange and new attractors with self-similarities.

Keywords Fractal–Fractional Derivative (FFD) · Fractional-Order Hyperchaotic System · Riemann–Liouville–Caputo (RLC) · General Caputo–Fabrizio · Qi Hyper-chaotic system

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1 Introduction

Recently, chaotic phenomena in numerous natural and social systems have attracted great interest since E. Lorenz discovered the first physical chaotic attractor [1]. Chaos synchronization has also attracted a great deal of attention since Pecora and Carroll established a chaos synchronization scheme for two identical chaotic systems with different initial conditions [2]. In recent years, study on the dynamics of fractional-order differential systems has greatly attracted interest of many researchers. Doye et al. proposed a robust fractional-order proportional–integral (FOPI) observer for the synchronization of nonlinear fractional-order chaotic systems, and the proposed FOPI observer is robust against Lipschitz additive nonlinear uncertainty [3]. In addition, an adaptive observer is proposed for the joint estimation of states and parameters of a fractional nonlinear system with external perturbations [4]. Ndolane Sene et al. analyzed two types of diffusion processes obtained with the fractional diffusion equations described by the Atangana–Baleanu–Caputo (ABC) fractional derivative. Whereas, they also addressed the mathematical analysis of the fishery model in the context of the fractional derivative operator using the Caputo–Fabrizio derivative [5,6]. It is demonstrated that some fractional-order differential systems behave chaotically or hyperchaotically, such as the fractional-order Chua's system [7], the fractional Rössler system [8], the fractional modified Duffing system, fractional-order Lorenz system, Chen system, Lü system, and fractional-order Qi four-wing chaotic system [9–13].

Moreover, based on fractional transfer function approximation in frequency domain, the fractional form of a four-wing autonomous integral-order chaotic system was firstly discussed, and some chaotic attractors in the different orders of the fractional-order system were found [14]. Meanwhile, chaotic and fractal–fractional derivative applications in physics, engineering and medicine have caught much attention [15,16]. A challenging problem is that a new model on novel coronavirus disease (COVID-19) with four compartments including susceptible, exposed, infected, and recovered class with fractal–fractional derivative are proposed by Ali et al. [17]. In [18], Muhammad Altaf Khan and Abdon Atangana described the mathematical modeling and dynamics of a novel corona virus (2019-nCoV).

There are several other application schemes, such as based on the role of fractal–fractional derivative on ferromagnetic fluid via fractal Laplace transform, a first problem via fractal–fractional differential operator was presented by Abro [19,20]. There have been many prior studies that addressed fractional-order systems and fractal–fractional derivative methods in chaos and financial [21]. For example, a novel investigation for banking data through mathematical model with a novel operator known as fractal–fractional in the sense of Caputo derivative was presented by Li et al. [22]. In [23], Wanting Wang et al. investigated the dynamics of the competition between rural and commercial bank with in the framework of fractal–fractional Atangana–Baleanu derivative sense, and the parameters for the competition fractal–fractional model was estimated effectively.

The advantages of fractal–fractional operators are the memory impact and the illustrative physical properties that are conserved. Using these types of operators, more effective and up-to-date researches have been revealing over time. For example, new fractal–fractional operators that have different features can be defined and have been used extensively to model real-life problems, such as the COVID-19 coronavirus epidemic. Due to the memory effect, nonlinear models integrate all past information, making it easier to predict and translate epidemic models more accurately. Because of effective properties, fractal–fractional operators have found wide applications to model dynamics processes in many well-known fields, such as biology, physics, finance and economics, science and engineering, mechanics, and mathematical modeling, whereas some numerical and approximate solution methods and their illustrative applications have been stated in the literature [15,16,24]. In this context, fractal–fractional operators and fractional calculus theory with their illustrative applications are attracting attention all over the world day by day. Moreover, fractional-order models have been already used for modeling of electrical circuits (such as domino ladders, tree structures) and elements (coils, memristor, etc.). Fractional calculus is a very useful tool in describing the evolution of systems with memory, which typically are dissipative and to complex systems.

Based on the above analysis, few researchers proposed and conducted the advantages of fractal–fractional operators with hyperchaotic model. In this study, a four-dimensional fractal–fractional hyperchaotic model is analyzed in detail via general Riemann–Liouville–Caputo (RLC) fractal–fractional derivative (FFD), and a series of new operators are constructed using generalized Caputo–Fabrizio and the series of fractal–fractional derivative firstly. The conclusions are shown that this new concept is the future to modelling complexities with self-similarities in some complex dynamical networks and multi-agent systems.

The rest of the paper is organized as follows: In Sect. 2, the definition of fractional-order derivative and its approximation are described. In Sect. 3, the preliminaries on fractal–fractional calculus are presented. In Sect. 4, numerical schemes and examples are given in detail. Finally, in Sect. 5, conclusions are drawn.

2 Fractional-order derivative and its approximation

The Mittag–Leffler function is an entire function, which was defined by the series special function [25, 26],

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in C, \mathbb{R}(\alpha) > 0, z \in C \tag{1}$$

The more general Mittag–Leffler function was described in the following equation [25,26],

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta\in C,$$
$$\mathbb{R}(\alpha)>0,\mathbb{R}(\beta)>0,z\in C \tag{2}$$

with C being the set of complex numbers are called Mittag–Leffler functions.

Fractional calculus has been known since the early seventeenth century. It has useful applications in many fields of science like physics, engineering, mathematical biology, and finance. To discuss fractional-order chaotic system, the fractional-order differential equations need to be solved. There are three commonly used definitions of the fractional-order differential operator: Riemann–Liouville, Grunwald–Letnikov, and Caputo definitions. One of them is the Riemann–Liouville definition which is described by [27,28]:

$$D^{\alpha}f(t)=\frac{d^l}{dt^l}J^{\theta}f(t),\quad \alpha>0, \tag{3}$$

where $\theta=l-\alpha$, J^{θ} is the θ-order Riemann–Liouville integral operator which is given as follows:

$$J^{\theta}u(t)=\frac{1}{\Gamma(\theta)}\int_0^t(t-\tau)^{\theta-1}u(\tau)d(\tau),\quad \theta>0. \tag{4}$$

The best-known Riemann–Liouville definition of fractional-order, which is described by [29]:

$$\frac{d^qf(t)}{dt^q}=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t\frac{f(\tau)}{(t-\tau)^{q-n+1}}d\tau, \tag{5}$$

where n is an integer such that $n-1\le q<n$, $\Gamma(\cdot)$ is the well-known Eulers gamma function.

However, the most common definition is the Caputo definition, since it is widely used in real applications [29]. The fractional derivative of order δ, based on Caputo definition, is given by:

$$D^{\delta}f(t)=\begin{cases}\dfrac{1}{\Gamma(m-\delta)}\displaystyle\int_0^t\frac{f^m(\tau)}{(t-\tau)^{\delta-m+1}})d\tau, & m-1<\delta<m,\\ \dfrac{d^m}{dt^m}f(t), & \delta=m,\end{cases} \tag{6}$$

where $m=[\delta]$, and $\Gamma(\cdot)$ is the gamma function given by:

$$\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt,\quad \Gamma(x+1)=x\Gamma(x). \tag{7}$$

In order to solve fractional-order systems numerically with a step size h, Grünwald–Letnikov method of approximation [30] is used, which is given by:

$$D^{\delta}f(t)\approx h^{-\delta}\sum_{j=0}^{k}(-1)^j\binom{\delta}{j}f(t_{k-j}) \tag{8}$$

$$D_*^{\alpha}f(t)=J^{l-\alpha}f^l(t), \tag{9}$$

where f^l represents the lth-order derivative of $f(t)$ and $l=[\alpha]$; this means that l is the first integer which is not less than α. The operator D_*^{α} is called the Caputo differential operator of order α. Hence, the Caputo type is choose throughout this study.

3 Preliminaries on fractal–fractional calculus

Now, some definitions are presented that will be used in this study.

Definition 1 The fractal–fractional derivative (FFD) of $f(t)$ with order $\gamma-\alpha$ in the Riemann–Liouville sense is defined as follows [31]:

$$^{FF-RL}D_{0,t}^{\alpha,\gamma}\{f(t)\}=\frac{1}{\Gamma(m-\alpha)}\frac{d}{dt^{\gamma}}\int_0^t(t-s)^{m-\alpha-1}f(s)ds, \tag{10}$$

where $m-1<\alpha,\gamma\le m$, $\alpha,\gamma,m\in\mathbb{N}$, and $\frac{df(s)}{ds^{\gamma}}=\lim\limits_{t\to s}\frac{f(t)-f(s)}{t^{\gamma}-s^{\gamma}}$.

Definition 2 The fractal–fractional derivative (FFD) of $f(t)$ with order $\gamma-\alpha$ in the Liouville–Caputo sense is defined as follows [32]:

$$^{FF-C}D_{0,t}^{\alpha,\gamma}\{f(t)\}=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}\left(\frac{d}{ds^{\gamma}}f(s)\right)ds, \tag{11}$$

where $m-1<\alpha,\gamma\le m$, $\alpha,\gamma,m\in\mathbb{N}$, and $\frac{df(s)}{ds^{\gamma}}=\lim\limits_{t\to s}\frac{f(t)-f(s)}{t^{\gamma}-s^{\gamma}}$.

Definition 3 The general fractal–fractional derivative (FFD) of $f(t)$ with order $\gamma-\alpha$ in the Riemann–Liouville–Caputo sense is defined as follows:

$$^{FF-RLC}D_{0,t}^{\alpha,\gamma}\{f(t)\}=\frac{1}{\Gamma(m-\alpha)}\left\{\xi_1\left[\frac{d}{dt^{\gamma}}\int_0^t(t-s)^{m-\alpha-1}f(s)ds\right]+\xi_2\left[\int_0^t(t-s)^{m-\alpha-1}\left(\frac{d}{ds^{\gamma}}f(s)\right)ds\right]\right\}, \tag{12}$$

where $m-1<\alpha,\gamma\le m$, $\alpha,\gamma,m\in\mathbb{N}$, $\xi_1\in\{0,1\}$, $\xi_2\in\{0,1\}$, and $\frac{df(s)}{ds^{\gamma}}=\lim\limits_{t\to s}\frac{f(t)-f(s)}{t^{\gamma}-s^{\gamma}}$. when $\xi_1=1$, $\xi_2=0$, it is called Riemann–Liouville fractional-order definition. when $\xi_1=0$, $\xi_2=1$, it is called Liouville–Caputo fractional-order definition.

Definition 4 The Caputo–Fabrizio FFD of $f(t)$ with order $\gamma-\alpha$ in the Riemann–Liouville sense is defined as follows [33]:

$$
\begin{aligned}
&{}^{FF-CFRL}D_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{M(\alpha)}{1-\alpha}\frac{d}{dt^{\gamma}}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f(s)ds,
\end{aligned} \tag{13}
$$

where $0 < \alpha, \gamma \le m, \alpha, \gamma, m \in \mathbb{N}$, and $M(0) = M(1) = 1$.

Definition 5 The Caputo–Fabrizio FFD of $f(t)$ with order $\gamma-\alpha$ in the Liouville–Caputo sense is defined as follows [34]:

$$
\begin{aligned}
&{}^{FF-CFLC}D_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)\left(\frac{d}{ds^{\gamma}}f(s)\right)ds,
\end{aligned} \tag{14}
$$

where $0 < \alpha, \gamma \le m, \alpha, \gamma, m \in \mathbb{N}$, and $M(0) = M(1) = 1$.

Definition 6 The general Caputo–Fabrizio FFD of $f(t)$ with order $\gamma-\alpha$ in the Riemann–Liouville–Caputo sense is defined as follows:

$$
\begin{aligned}
&{}^{FF-CFRLLC}D_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{M(\alpha)}{1-\alpha}\left\{\zeta_1\left[\frac{d}{dt^{\gamma}}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f(s)ds\right]\right. \\
&\quad \left. +\zeta_2\left[\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)\left(\frac{d}{ds^{\gamma}}f(s)\right)ds\right]\right\},
\end{aligned} \tag{15}
$$

where $0 < \alpha, \gamma \le m, \alpha, \gamma, m \in \mathbb{N}$, $\zeta_1 \in \{0, 1\}$, $\zeta_2 \in \{0, 1\}$, and $M(0) = M(1) = 1$. when $\zeta_1 = 1, \zeta_2 = 0$, it is called Caputo–Fabrizio FFD in the Riemann–Liouville sense fractional-order definition. when $\zeta_1 = 0$, $\zeta_2 = 1$, it is called Caputo–Fabrizio FFD in the Liouville–Caputo sense fractional-order definition.

Definition 7 The Yuhang FFD of $f(t)$ with order $\gamma-\alpha$ in the Riemann–Liouville sense is defined as follows:

$$
\begin{aligned}
&{}^{FF-RLY}D_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{Y(\alpha)}{1-\alpha}\frac{d}{dt^{\gamma}}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) f(s)ds,
\end{aligned} \tag{16}
$$

where $0 < \alpha, \gamma \le 1, \alpha, \gamma, m \in \mathbb{N}$, and $Y(\alpha) = (1-\alpha)^2 + \frac{\alpha}{\Gamma(\alpha)}$.

Definition 8 The Yuhang FFD of $f(t)$ with order $\gamma-\alpha$ in the Liouville–Caputo sense is defined as follows:

$$
\begin{aligned}
&{}^{FF-LCY}D_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{Y(\alpha)}{1-\alpha}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right)\left(\frac{d}{ds^{\gamma}}f(s)\right)ds,
\end{aligned} \tag{17}
$$

where $0 < \alpha, \gamma \le 1, \alpha, \gamma \in \mathbb{N}$, and $Y(\alpha) = (1-\alpha)^2 + \frac{\alpha}{\Gamma(\alpha)}$.

Definition 9 The general Yuhang FFD of $f(t)$ with order $\gamma-\alpha$ in the Riemann–Liouville–Caputo sense is defined as follows:

$$
\begin{aligned}
&{}^{FF-RLCY}D_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{Y(\alpha)}{1-\alpha}\left\{\eta_1\left[\frac{d}{dt^{\gamma}}\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) f(s)ds\right]\right. \\
&\quad \left. +\eta_2\left[\int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right)\left(\frac{d}{ds^{\gamma}}f(s)\right)ds\right]\right\},
\end{aligned} \tag{18}
$$

where $0 < \alpha, \gamma \le 1, \alpha, \gamma \in \mathbb{N}$, $\eta_1 \in \{0, 1\}$, $\eta_2 \in \{0, 1\}$, and $Y(\alpha) = (1-\alpha)^2 + \frac{\alpha}{\Gamma(\alpha)}$. when $\eta_1 = 1, \eta_2 = 0$, it is called Yuhang FFD in the Riemann–Liouville sense fractional-order definition. when $\eta_1 = 0, \eta_2 = 1$, it is called Yuhang FFD in the Liouville–Caputo sense fractional-order definition.

Definition 10 The Liouville–Caputo fractal–fractional integral of $f(t)$ with order α is defined as follows [35]:

$$
\begin{aligned}
&{}^{FF-LC}I_{0,t}^{\alpha,\gamma}\{f(t)\} \\
&= \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}f(s)ds,
\end{aligned} \tag{19}
$$

where $m-1 < \alpha, \gamma \le m, \alpha, \gamma, m \in \mathbb{N}$, and $\frac{df(s)}{ds^{\gamma}} = \lim_{t\to s}\frac{f(t)-f(s)}{t^{\gamma}-s^{\gamma}}$.

Definition 11 The Caputo–Fabrizio fractal–fractional integral of $f(t)$ with order α is defined as follows [36]:

$$
\begin{aligned}
&{}^{FF-CF}I_{0,t}^{\alpha,\gamma}\{f(t)\} = \frac{\alpha\gamma}{M(\alpha)}\int_0^t \left(s^{\alpha-1}f(s)\right)ds \\
&\quad +\frac{\gamma(1-\alpha)t^{\gamma-1}}{M(\alpha)}f(t),
\end{aligned} \tag{20}
$$

where $0 < \alpha, \gamma \le m, \alpha, \gamma, m \in \mathbb{N}$, and $M(0) = M(1) = 1$.

Definition 12 The Yuhang fractal–fractional integral of $f(t)$ with order α is defined as follows:

$$
{}^{FF-Y}I_{0,t}^{\alpha,\gamma}\{f(t)\} = \frac{\alpha\gamma}{Y(\alpha)}\int_0^t \left(s^{\alpha-1}f(s)(t-s)^{\alpha-1}\right)ds + \frac{\gamma(1-\alpha)t^{\gamma-1}f(t)}{Y(\alpha)}, \tag{21}
$$

where $0<\alpha,\gamma\leq 1, \alpha,\gamma\in\mathbb{N}$, and $Y(\alpha)=(1-\alpha)^2+\frac{\alpha}{\Gamma(\alpha)}$.

Definition 13 The general Liouville–Caputo–Fabrizio–Yuhang fractal–fractional integral of $f(t)$ with order α is defined as follows:

$$
\begin{aligned}
{}^{FF-CCFAB}I_{0,t}^{\alpha,\gamma}\{f(t)\} &= \gamma\left\{\lambda_1\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}f(s)ds\right]\right. \\
&+\lambda_2\left[\frac{\alpha}{M(\alpha)}\int_0^t\left(s^{\alpha-1}f(s)\right)ds + \frac{\gamma(1-\alpha)t^{\gamma-1}}{M(\alpha)}f(t)\right] \\
&+\lambda_3\left[\frac{\alpha}{Y(\alpha)}\int_0^t\left(s^{\alpha-1}f(s)(t-s)^{\alpha-1}\right)ds \left.+\frac{\gamma(1-\alpha)t^{\gamma-1}f(t)}{Y(\alpha)}\right]\right\},
\end{aligned} \tag{22}
$$

here, $\lambda_i\in\{0,1\}, i=1,2,3$.

when $\lambda_1=1, \lambda_2=0$ and $\lambda_3=0, m-1<\alpha, \gamma\leq m, \alpha,\gamma,m\in\mathbb{N}$, and $\frac{df(s)}{ds^\gamma}=\lim_{t\to s}\frac{f(t)-f(s)}{t^\gamma-s^\gamma}$, it is called Liouville–Caputo fractional-order definition.

when $\lambda_1=0, \lambda_2=1$ and $\lambda_3=0, 0<\alpha,\gamma\leq m, \alpha,\gamma,m\in\mathbb{N}$, and $M(0)=M(1)=1$, it is called Caputo–Fabrizio fractional-order definition.

when $\lambda_1=0, \lambda_2=0$ and $\lambda_3=1, 0<\alpha,\gamma\leq 1, \alpha,\gamma\in\mathbb{N}$, and $Y(\alpha)=(1-\alpha)^2+\frac{\alpha}{\Gamma(\alpha)}$, it is called Yuhang fractional-order definition.

4 Numerical schemes and examples

In this section, theorems and detailed proof about three numerical schemes have been designed, namely, Caputo fractal–fractional, Caputo–Fabrizio fractal–fractional, and the Yuhang FFD operators, respectively.

Theorem 1 *Consider the following differential equations in the fractal–fractional Liouville–Caputo sense:*

$$
{}^{FF-LC}D_{0,t}^{\alpha,\gamma}\{x(t)\} = f(t,x(t),y(t),z(t)). \tag{23}
$$

Let the equation (23) be converted to the Volterra case, and the numerical scheme of this system using a Caputo fractal–fractional approach at t_{n+1} is given by:

$$
x(t_{n+1}) = x(0)+\frac{\gamma}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-s)^{\alpha-1}s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})ds. \tag{24}
$$

Then Liouville–Caputo FFD has been obtained, where

$$
\begin{aligned}
x(t_{n+1}) &= x(0)-\frac{(\Delta t)^\alpha\gamma}{\Gamma(\alpha+2)}\sum_{j=0}^{n}\left\{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})\right. \\
&\left[(n-j)^\alpha(\alpha+1+n-j) -(n+1-j)^{\alpha+1}\right]+t_j^{\gamma-1}f(t_j,x_j,y_j,z_j) \\
&\left.\left[(n+1-j)^\alpha(n-j-\alpha)-(n-j)^{\alpha+1}\right]\right\}.
\end{aligned} \tag{25}
$$

Proof The above integral can be approximated as:

$$
x(t_{n+1}) = x(0)+\frac{\gamma}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})ds. \tag{26}
$$

Within the finite interval $[t_j,t_{j+1}]$, it should be approximated that the function $s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})$ using the Lagrangian piecewise interpolation such that:

$$
\begin{aligned}
\Psi_j(s) &= s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1}) \\
&= \frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j,x_j,y_j,z_j) + \frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1}).
\end{aligned} \tag{27}
$$

Thus, (26) is given by:

$$
x(t_{n+1}) = x(0)+\frac{\gamma}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\Psi_j(s)ds. \tag{28}
$$

Substituting Eqs. (27) into (28), which has:

$$
\begin{aligned}
x(t_{n+1}) &= x(0)+\frac{\gamma}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha-1}\left[\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\right. \\
&\left.+\frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})\right]ds.
\end{aligned} \tag{29}
$$

Solving the integral of the right-hand side, the following numerical scheme is obtained:

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\alpha\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left[\frac{t_{j+1}-s}{t_{j+1}-t_j} t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j} t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \right] d(t_{n+1}-s)^{\alpha}. \tag{30}$$

Integrating over $(t_{n+1}-s)^{\alpha}$, (31) is given by:

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\alpha\Gamma(\alpha)} \sum_{j=0}^{n} \Bigg\{ \left[\frac{t_{j+1}-s}{t_{j+1}-t_j} t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j} t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \right] (t_{n+1}-s)^{\alpha} \Big|_{t_j}^{t_{j+1}} - \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} d \left[\frac{t_{j+1}-s}{t_{j+1}-t_j} t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j} t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \right] \Bigg\}. \tag{31}$$

Which has

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\alpha\Gamma(\alpha)} \sum_{j=0}^{n} \Bigg\{ \left[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \right] (t_{n+1} - t_{j+1})^{\alpha} - \left[t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \right] (t_{n+1}-t_j)^{\alpha} - \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} d \left[\frac{t_{j+1}-s}{t_{j+1}-t_j} t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j} t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \right] \Bigg\}. \tag{32}$$

and it follows from (32) that

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\alpha\Gamma(\alpha)} \sum_{j=0}^{n} \Bigg\{ \left[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1} z_{j+1}) \right] (t_{n+1} - t_{j+1})^{\alpha} - \left[t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \right] (t_{n+1}-t_j)^{\alpha} - \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} d \left[\frac{t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)}{t_{j+1}-t_j} \right] s \Bigg\}. \tag{33}$$

Which has

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\alpha\Gamma(\alpha)} \sum_{j=0}^{n} \Bigg\{ \left[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1} z_{j+1}) \right] (t_{n+1} - t_{j+1})^{\alpha} - \left[t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \right] (t_{n+1}-t_j)^{\alpha} - \left[\frac{t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)}{t_{j+1}-t_j} \right] \int_{t_j}^{t_{j+1}} (t_{n+1}-s)^{\alpha} ds \Bigg\}. \tag{34}$$

and thus,

$$x(t_{n+1})=x(0)-\frac{\gamma}{\alpha\Gamma(\alpha)}\sum_{j=0}^{n}\Bigg\{\Big[t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1}z_{j+1})\Big](t_{n+1}-t_{j+1})^{\alpha}$$
$$-\Big[t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\Big](t_{n+1}-t_j)^{\alpha} \tag{35}$$
$$+\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}\right]\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha}d(t_{n+1}-s)\Bigg\}.$$

It can be observed by Eq. (35) which obtains

$$x(t_{n+1})=x(0)-\frac{\gamma}{\alpha\Gamma(\alpha)}\sum_{j=0}^{n}\Bigg\{\Big[t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1}z_{j+1})\Big](t_{n+1}-t_{j+1})^{\alpha}$$
$$-\Big[t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\Big](t_{n+1}-t_j)^{\alpha} \tag{36}$$
$$+\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}\right]\frac{(t_{n+1}-s)^{\alpha+1}}{\alpha+1}\Big|_{t_j}^{t_{j+1}}\Bigg\}.$$

and thus,

$$x(t_{n+1})=x(0)-\frac{\gamma}{\alpha\Gamma(\alpha)}\sum_{j=0}^{n}\Bigg\{\Big[t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1}z_{j+1})\Big](t_{n+1}-t_{j+1})^{\alpha}$$
$$-\Big[t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\Big](t_{n+1}-t_j)^{\alpha}+\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}\right] \tag{37}$$
$$\times\frac{(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}}{\alpha+1}\Bigg\}.$$

It follows from (37) that

$$x(t_{n+1})=x(0)-\frac{\gamma}{\Gamma(\alpha+1)}\sum_{j=0}^{n}\Bigg\{\Big[t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1}z_{j+1})\Big](t_{n+1}-t_{j+1})^{\alpha}$$
$$-\Big[t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\Big](t_{n+1}-t_j)^{\alpha}+\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}\right] \tag{38}$$
$$\times\frac{(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}}{\alpha+1}\Bigg\}.$$

Which has

$$x(t_{n+1})=x(0)-\frac{\gamma}{\Gamma(\alpha+1)}\sum_{j=0}^{n}\Bigg\{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1}z_{j+1})(t_{n+1}-t_{j+1})^{\alpha}$$
$$-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)(t_{n+1}-t_j)^{\alpha}+\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j} \tag{39}$$
$$\times\frac{(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}}{\alpha+1}\Bigg\}.$$

Assuming that $\Delta t = t_{j+1} - t_j$, then equation (39) can be reduced to

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1} z_{j+1})(n-j)^{\alpha} (\Delta t)^{\alpha} - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha}(\Delta t)^{\alpha} + \frac{t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)}{\Delta t} \times \frac{(n-j)^{\alpha+1}(\Delta t)^{\alpha+1} - (n+1-j)^{\alpha+1}(\Delta t)^{\alpha+1}}{\alpha+1} \Big\}. \tag{40}$$

For simplicity of notation, it can be written as:

$$x(t_{n+1}) = x(0) - \frac{\gamma}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha}(\Delta t)^{\alpha} - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha}(\Delta t)^{\alpha} + \Big[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \Big] \times \frac{(n-j)^{\alpha+1}(\Delta t)^{\alpha} - (n+1-j)^{\alpha+1}(\Delta t)^{\alpha}}{\alpha+1} \Big\}. \tag{41}$$

and thus,

$$x(t_{n+1}) = x(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha} - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha} + \Big[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \Big] \frac{(n-j)^{\alpha+1} - (n+1-j)^{\alpha+1}}{\alpha+1} \Big\}. \tag{42}$$

It can be replaced by Eq. (42) which obtains

$$x(t_{n+1}) = x(0) - \frac{(\Delta t)^{\alpha}\gamma}{(\alpha+1)\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha}(\alpha+1) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha}(\alpha+1) + \Big[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \Big] \Big[(n-j)^{\alpha+1} - (n+1-j)^{\alpha+1} \Big] \Big\}. \tag{43}$$

It follows from (43) that

$$x(t_{n+1}) = x(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha}(\alpha+1) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha}(\alpha+1) + \Big[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \Big] \Big[(n-j)^{\alpha+1} - (n+1-j)^{\alpha+1} \Big] \Big\}. \tag{44}$$

Which has

$$x(t_{n+1}) = x(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha}(\alpha+1) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha}(\alpha+1) + t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha+1} - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n-j)^{\alpha+1} - t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n+1-j)^{\alpha+1} + t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha+1} \Big\}. \tag{45}$$

Therefore, it can be observed by Eq. (45) which obtains

$$x(t_{n+1}) = x(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \big[(n-j)^{\alpha}(\alpha+1+n-j)$$

$$-(n+1-j)^{\alpha+1}\Big]+t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\,[(n+1-j)^{\alpha}(n-j-\alpha)-(n-j)^{\alpha+1}\Big]\Big\}. \tag{46}$$

□

This completes the proof of the Theorem.

Theorem 2 *Consider the following differential equations in the Caputo–Fabrizio FFD in the Liouville–Caputo sense:*

$$^{FF-CFC}D_{0,t}^{\alpha,\gamma}\{x(t)\}=f(t,x(t),y(t),z(t)). \tag{47}$$

Applying the Caputo–Fabrizio integral, which obtains

$$x(t)=x(0)+\frac{\alpha\gamma}{M(\alpha)}\int_0^t\left(s^{\gamma-1}f(s,x,y,z)\right)ds+\frac{\gamma(1-\alpha)t^{\gamma-1}}{M(\alpha)}f(t,x,y,z). \tag{48}$$

Here, the detailed derivation of the numerical scheme is presented. Thus, at t_{n+1}, more precisely,

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{M(\alpha)}\int_0^{t_{n+1}}\left(s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})\right)ds+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{49}$$

Then the Caputo–Fabrizio–Caputo FFD is obtained, where

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{2M(\alpha)}\sum_{j=0}^{n}\Big[t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\Big](t_{j+1}+t_j)+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{50}$$

Proof The above integral (49) can be approximated to:

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{M(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left(s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})\right)ds+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{51}$$

Approximating $s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})$ in $[t_j,t_{j+1}]$, that is, within the finite interval $[t_j,t_{j+1}]$, using the Lagrangian piecewise interpolation, the function $s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})$ is approximated to equation (52):

$$\begin{aligned}\Psi_j(s)&=s^{\gamma-1}f(s,x_{n+1},y_{n+1},z_{n+1})\\&=\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)\\&\quad+\frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1}).\end{aligned} \tag{52}$$

Thus, which obtains

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{M(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\Big[\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}\times f(t_j,x_j,y_j,z_j)+\frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})\Big]ds+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{53}$$

It can be observed by Eq. (53) which obtains

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{M(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}s\right]ds+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{54}$$

It follows from (54) that

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{M(\alpha)}\sum_{j=0}^{n}\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}\right]\int_{t_j}^{t_{j+1}}s\,ds+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{55}$$

and thus,

$$x(t_{n+1})=x(0)+\frac{\alpha\gamma}{M(\alpha)}\sum_{j=0}^{n}\left[\frac{t_{j+1}^{\gamma-1}f(t_{j+1},x_{j+1},y_{j+1},z_{j+1})-t_j^{\gamma-1}f(t_j,x_j,y_j,z_j)}{t_{j+1}-t_j}\right]\frac{s^2}{2}\Big|_{t_j}^{t_{j+1}}+\frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)}f(t_{n+1},x_{n+1},y_{n+1},z_{n+1}). \tag{56}$$

which has

$$x(t_{n+1}) = x(0) + \frac{\alpha\gamma}{M(\alpha)} \sum_{j=0}^{n} \left[\frac{t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)}{t_{j+1} - t_j} \right] \frac{t_{j+1}^2 - t_j^2}{2} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{57}$$

and hence,

$$x(t_{n+1}) = x(0) + \frac{\alpha\gamma}{2M(\alpha)} \sum_{j=0}^{n} \left[t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \right] (t_{j+1} + t_j) + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{58}$$

□

This shows that the identities in (50) are satisfied.

Theorem 3 *Consider the following differential equations in the Yuhang FFD in the Liouville–Caputo sense:*

$${}^{FF-LCY}D_{0,t}^{\alpha,\gamma}\{x(t)\} = f(t, x(t), y(t), z(t)). \tag{59}$$

Applying the Yuhang integral of Definition 7, which has

$$x(t) = x(0) + \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha)} \int_0^t \left(s^{\gamma-1}(t-s)^{\alpha-1} f(s, x, y, z) \right) ds + \frac{\gamma(1-\alpha)t^{\gamma-1}}{Y(\alpha)} f(t, x, y, z). \tag{60}$$

If $x(t_{n+1})$ is equal to the following equation at t_{n+1}:

$$x(t_{n+1}) = x(0) + \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} \left(s^{\gamma-1}(t_{n+1}-s)^{\alpha-1} f(s, x_{n+1}, y_{n+1}, z_{n+1}) \right) ds + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{61}$$

Then the Caputo–Yuhang FFD is obtained, where

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma(\Delta t)^\alpha}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \left[(\alpha+2)(n-j)^\alpha - (n+1-j)^{\alpha+1} \right] - t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \left[(\alpha-n+j)(n+1-j)^\alpha + (n-j)^{\alpha+1} \right] \Big\} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{62}$$

Proof The above system (61) can be approximate and expressed as

$$x(t_{n+1}) = x(0) + \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(s^{\gamma-1}(t_{n+1}-s)^{\alpha-1} f(s, x_{n+1}, y_{n+1}, z_{n+1}) \right) ds + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{63}$$

Approximating $s^{\gamma-1} f(s, x_{n+1}, y_{n+1}, z_{n+1})$ in $[t_j, t_{j+1}]$, that is, within the finite interval $[t_j, t_{j+1}]$, using the Lagrangian piecewise interpolation, the function $s^{\gamma-1} f(s, x_{n+1}, y_{n+1}, z_{n+1})$ is approximated to:

$$\Psi_j(s) = s^{\gamma-1} f(s, x_{n+1}, y_{n+1}, z_{n+1}) = \frac{t_{j+1}-s}{t_{j+1}-t_j} t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j} t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}). \tag{64}$$

Thus, which is given by

$$x(t_{n+1}) = x(0) + \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left(\left[\frac{t_{j+1}-s}{t_{j+1}-t_j} t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j} t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1}) \right] (t_{n+1}-s)^{\alpha-1} \right) ds + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{65}$$

The following numerical scheme is obtained:

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma}{Y(\alpha)\alpha\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left(\left[\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})\right]\right)d(t_{n+1}-s)^{\alpha} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{66}$$

More precisely,

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha+1)}\sum_{j=0}^{n}\left\{\left[\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})\right](t_{n+1}-s)^{\alpha}\Big|_{t_j}^{t_{j+1}} - \int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha}d\left[\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})\right]\right\} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{67}$$

That is

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha+1)}\sum_{j=0}^{n}\left\{\left[t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(t_{n+1}-t_{j+1})^{\alpha} - t_j^{\gamma-1}f(t_j, x_j, y_j, z_j)(t_{n+1}-t_j)^{\alpha}\right] - \int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha}d\left[\frac{t_{j+1}-s}{t_{j+1}-t_j}t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + \frac{s-t_j}{t_{j+1}-t_j}t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})\right]\right\} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{68}$$

and thus,

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha+1)}\sum_{j=0}^{n}\left\{\left[t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(t_{n+1}-t_{j+1})^{\alpha} - t_j^{\gamma-1}f(t_j, x_j, y_j, z_j)(t_{n+1}-t_j)^{\alpha}\right] - \frac{-t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})}{t_{j+1}-t_j}\int_{t_j}^{t_{j+1}}(t_{n+1}-s)^{\alpha}ds\right\} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{69}$$

That is

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha+1)}\sum_{j=0}^{n}\left\{\left[t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(t_{n+1}-t_{j+1})^{\alpha} - t_j^{\gamma-1}f(t_j, x_j, y_j, z_j)(t_{n+1}-t_j)^{\alpha}\right] + \frac{-t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})}{t_{j+1}-t_j} \times \frac{(t_{n+1}-t_{j+1})^{\alpha+1}-(t_{n+1}-t_j)^{\alpha+1}}{\alpha+1}\right\} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{70}$$

Assuming that $\Delta t = t_{j+1} - t_j$, then equation (70) can be reduced to

$$x(t_{n+1}) = x(0) - \frac{\alpha\gamma}{Y(\alpha)\Gamma(\alpha+1)}\sum_{j=0}^{n}\left\{\left[t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha}(\Delta t)^{\alpha} - t_j^{\gamma-1}f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha}(\Delta t)^{\alpha}\right] + \frac{-t_j^{\gamma-1}f(t_j, x_j, y_j, z_j) + t_{j+1}^{\gamma-1}f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})}{\Delta t} \times \frac{(n-j)^{\alpha+1}(\Delta t)^{\alpha+1}-(n+1-j)^{\alpha+1}(\Delta t)^{\alpha+1}}{\alpha+1}\right\} + \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}). \tag{71}$$

Incidentally, it can be replaced by Eq. (71) which obtains

$$
\begin{aligned}
x(t_{n+1}) &= x(0) \\
&- \frac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big\{ \Big[t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha} \\
&- t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha} \Big] \\
&+ \Big[-t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1}) \Big] \\
&\times \frac{(n-j)^{\alpha+1} - (n+1-j)^{\alpha+1}}{\alpha+1} \Big\} \\
&+ \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}).
\end{aligned} \tag{72}
$$

which yields

$$
\begin{aligned}
x(t_{n+1}) &= x(0) \\
&- \frac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)(\alpha+1)\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big\{ (\alpha+1) \Big[t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha} \\
&- t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha} \Big] \\
&+ \Big[-t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) + t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1}) \Big] \\
&\times \Big[(n-j)^{\alpha+1} - (n+1-j)^{\alpha+1} \Big] \Big\} \\
&+ \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}).
\end{aligned} \tag{73}
$$

Which has

$$
\begin{aligned}
x(t_{n+1}) &= x(0) \\
&- \frac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ (\alpha+1) t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha} \\
&- (\alpha+1) t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n+1-j)^{\alpha} \\
&- t_j^{\gamma-1} f(t_j, x_j, y_j, z_j)(n-j)^{\alpha+1} + t_j^{\gamma-1} f(t_j \\
&, x_j, y_j, z_j)(n+1-j)^{\alpha+1} \\
&+ t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n-j)^{\alpha+1} \\
&- t_{j+1}^{\gamma-1} f(t_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})(n+1-j)^{\alpha+1} \Big\} \\
&+ \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}).
\end{aligned} \tag{74}
$$

That is

$$
\begin{aligned}
x(t_{n+1}) &= x(0) \\
&- \frac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1}) \Big[(\alpha+1)(n-j)^{\alpha} \\
&+ (n-j)^{\alpha+1} - (n+1-j)^{\alpha+1} \Big] \\
&- t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \Big[(\alpha+1)(n+1-j)^{\alpha} \\
&+ (n-j)^{\alpha+1} - (n+1-j)^{\alpha+1} \Big] \Big\} \\
&+ \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}).
\end{aligned} \tag{75}
$$

Therefore,

$$
\begin{aligned}
x(t_{n+1}) &= x(0) \\
&- \frac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} f(t_{j+1} \\
&, x_{j+1}, y_{j+1}, z_{j+1}) \Big[(\alpha+2)(n-j)^{\alpha} \\
&- (n+1-j)^{\alpha+1} \Big] \\
&- t_j^{\gamma-1} f(t_j, x_j, y_j, z_j) \Big[(\alpha-n+j)(n+1-j)^{\alpha} \\
&+ (n-j)^{\alpha+1} \Big] \Big\} \\
&+ \frac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)} f(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}).
\end{aligned} \tag{76}
$$

□

This completes the proof of the Theorem.

Example 1 Consider the four-dimensional hyper-Qi chaotic system involving the FFD in the Liouville–Caputo sense [37].

$$
\begin{cases}
{}^{FF-LC}D_{0,t}^{\alpha,\gamma}\{x(t)\} = a(y(t)-x(t)) + y(t)z(t)u(t), \\
{}^{FF-LC}D_{0,t}^{\alpha,\gamma}\{y(t)\} = b(x(t)+y(t)) - x(t)z(t)u(t), \\
{}^{FF-LC}D_{0,t}^{\alpha,\gamma}\{z(t)\} = -cz(t) + x(t)y(t)u(t), \\
{}^{FF-LC}D_{0,t}^{\alpha,\gamma}\{u(t)\} = -du(t) + x(t)y(t)z(t).
\end{cases} \tag{77}
$$

which has the strange chaotic attractor in three-dimensional space is shown in Fig. 1a–d when $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 1 \times 10^{-5}$, simulation time $t = 200s$, and initial conditions $x(0) = 2$, $y(0) = 0.2$, $z(0) = 1$, and $u(0) = 2$ for $\alpha = 1$ and $\gamma = 1$.

The four-dimensional hyper-Qi chaotic system (77) in two-dimensional plane is shown in Fig. 2a–f when $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 1 \times 10^{-5}$, simulation time $t = 200s$, and initial conditions $x(0) = 2$, $y(0) = 0.2$, $z(0) = 1$, and $u(0) = 2$ for $\alpha = 1$ and $\gamma = 1$.

Based on Theorem 1, the numerical scheme of the four-dimensional hyper-Qi chaotic system (77) is given by:

$$\begin{cases} x(t_{n+1}) = x(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \left[a(y(t_{j+1}) - x(t_{j+1})) + y(t_{j+1})z(t_{j+1})u(t_{j+1}) \right] \\ \qquad \times \left[(n-j)^{\alpha}(\alpha+1+n-j) - (n+1-j)^{\alpha+1} \right] \\ \qquad + t_j^{\gamma-1} \left[a(y(t_j) - x(t_j)) + y(t_j)z(t_j)u(t_j) \right] \\ \qquad \times \left[(n+1-j)^{\alpha}(n-j-\alpha) - (n-j)^{\alpha+1} \right] \Big\}, \\ y(t_{n+1}) = y(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \left[b(x(t_{j+1}) + y(t_{j+1})) - x(t_{j+1})z(t_{j+1})u(t_{j+1}) \right] \\ \qquad \times \left[(n-j)^{\alpha}(\alpha+1+n-j) - (n+1-j)^{\alpha+1} \right] \\ \qquad + t_j^{\gamma-1} \left[b(x(t_j) + y(t_j)) - x(t_j)z(t_j)u(t_j) \right] \\ \qquad \times \left[(n+1-j)^{\alpha}(n-j-\alpha) - (n-j)^{\alpha+1} \right] \Big\}, \\ z(t_{n+1}) = z(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \left[-cz(t_{j+1}) + x(t_{j+1})y(t_{j+1})u(t_{j+1}) \right] \\ \qquad \times \left[(n-j)^{\alpha}(\alpha+1+n-j) - (n+1-j)^{\alpha+1} \right] \\ \qquad + t_j^{\gamma-1} \left[-cz(t_j) + x(t_j)y(t_j)u(t_j) \right] \\ \qquad \times \left[(n+1-j)^{\alpha}(n-j-\alpha) - (n-j)^{\alpha+1} \right] \Big\}, \\ u(t_{n+1}) = u(0) - \frac{(\Delta t)^{\alpha}\gamma}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \left[-du(t_{j+1}) + x(t_{j+1})y(t_{j+1})z(t_{j+1}) \right] \\ \qquad \times \left[(n-j)^{\alpha}(\alpha+1+n-j) - (n+1-j)^{\alpha+1} \right] \\ \qquad + t_j^{\gamma-1} \left[-du(t_j) + x(t_j)y(t_j)z(t_j) \right] \\ \qquad \times \left[(n+1-j)^{\alpha}(n-j-\alpha) - (n-j)^{\alpha+1} \right] \Big\}. \end{cases} \tag{78}$$

Without loss of generality, numerical simulations for hyper-Qi chaotic system are shown in Liouville–Caputo sense equation (77), which are considered the different values of α and γ chosen arbitrarily.

From a 3-D perspective: Considering the FFD in the Liouville–Caputo sense and the numerical scheme given by Eq. (77), Numerical results for hyper-Qi chaotic system (78) in three-dimensional space are shown in Fig. 3a–d for $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 5 \times 10^{-3}$, simulation time $t = 200s$, and initial conditions $x(0) = 2$, $y(0) = 0.2$, $z(0) = 1$, and $u(0) = 2$ for $\alpha = 0.95$ and $\gamma = 0.95$, respectively. It can be observed, Fig. 3 a is a three-dimensional image located at three-dimensional coordinate system $x(t) - y(t) - z(t)$; Fig. 3b is a three-dimensional image located at three-dimensional coor-

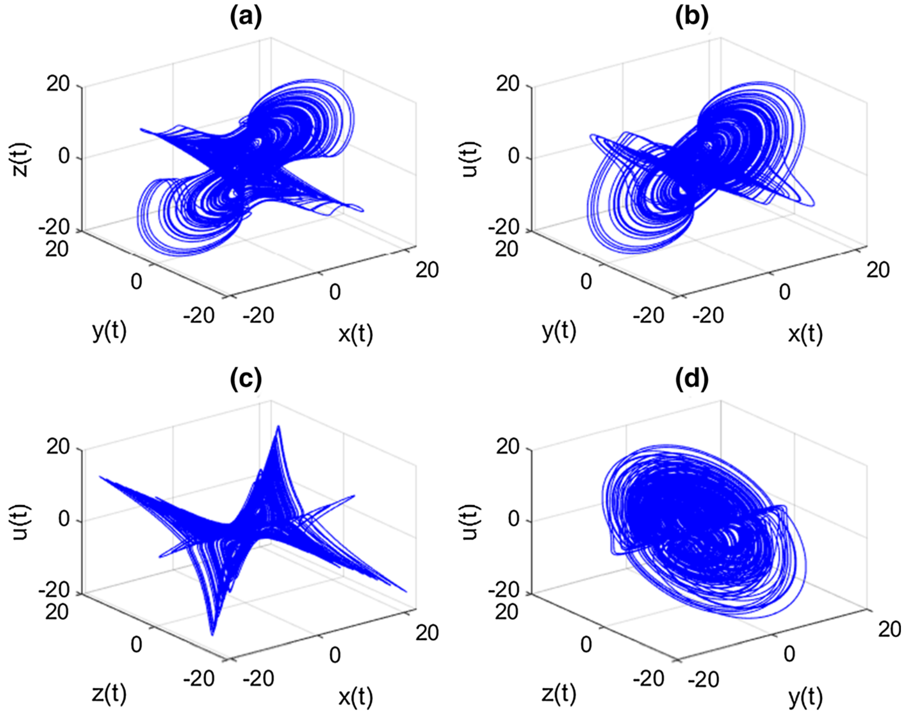

Fig. 1 Phase portraits of the hyper-Qi chaotic system in (**a**) the x(t)-y(t)-z(t) space; (**b**) the x(t)-y(t)-u(t) space, (**c**) the x(t)-z(t)-u(t) space, and (**d**) the y(t)-z(t)-u(t) space

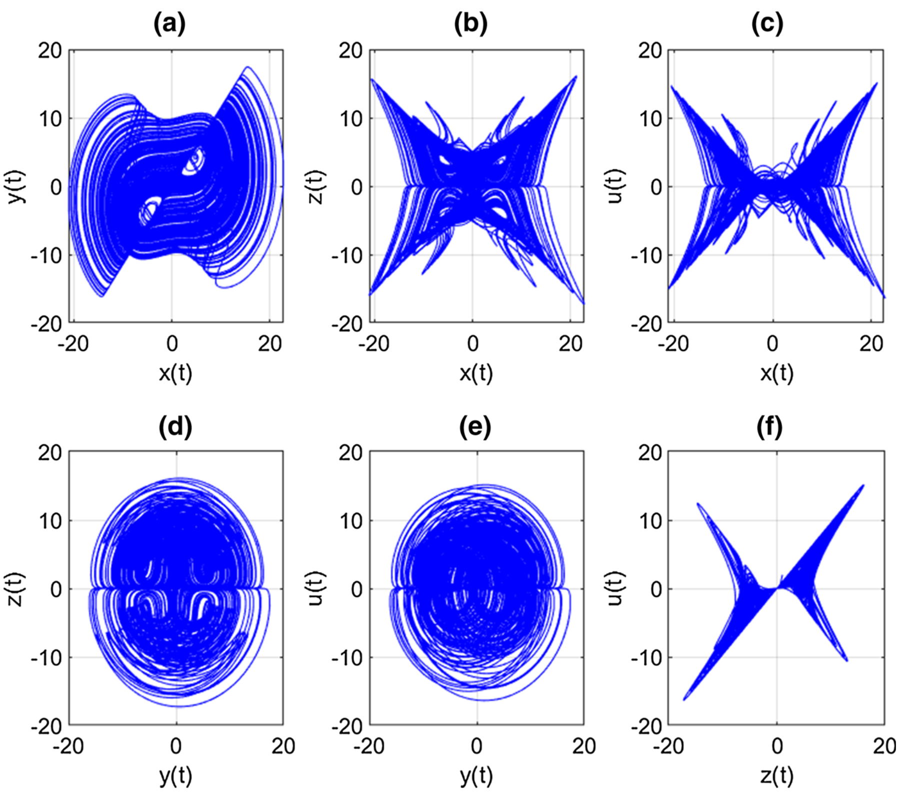

Fig. 2 Phase portraits of the hyper-Qi chaotic system and projected on (**a**) the x(t)-y(t) plane; (**b**) the x(t)-z(t) plane; (**c**) the x(t)-u(t) plane; (**d**) the y(t)-z(t) plane; (**e**) the y(t)-u(t) plane; and (**f**) the z(t)-u(t) plane

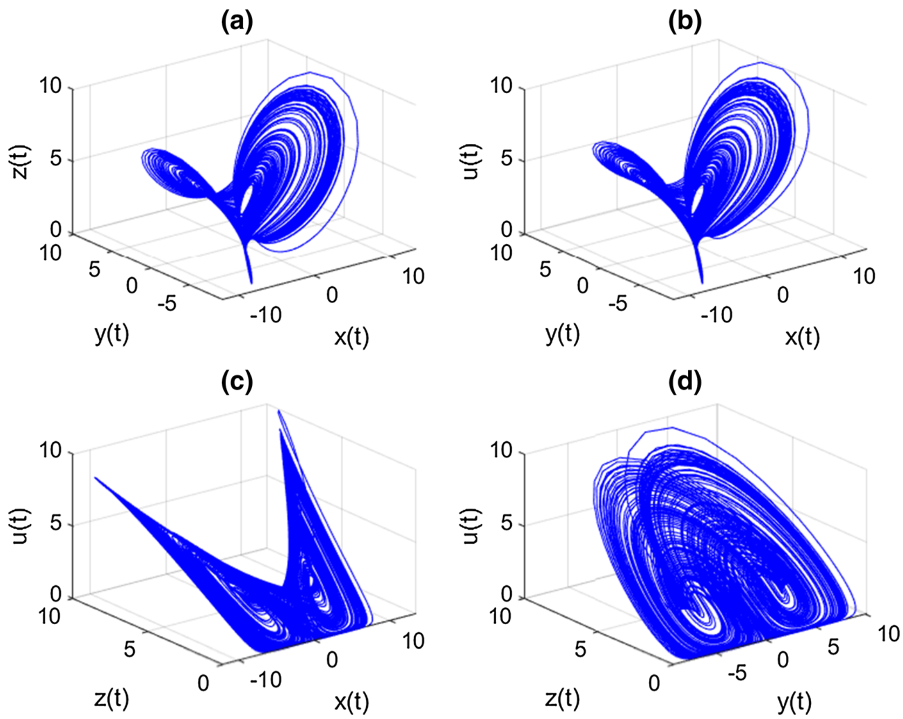

Fig. 3 Phase portraits of the hyper-Qi chaotic system FFD in (**a**) the x(t)-y(t)-z(t) space; (**b**) the x(t)-y(t)-u(t) space, (**c**) the x(t)-z(t)-u(t) space, and (**d**) the y(t)-z(t)-u(t) space

dinate system $x(t) - y(t) - u(t)$; These two sub-pictures are alike in appearance. Figure 3c shows a three-dimensional image located at three-dimensional coordinate system $x(t) - z(t) - u(t)$; Fig. 3d is a three-dimensional image located at three-dimensional coordinate system $y(t) - z(t) - u(t)$; These two sub-pictures are alike in appearance with double scrolls.

From a 2-D perspective: Considering the FFD in the Liouville–Caputo sense and the numerical scheme given by Eq. (77), Numerical results for hyper-Qi chaotic system (78) in two-dimensional space are shown in Fig. 4a–f for $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 5 \times 10^{-3}$, simulation time $t = 200s$, and initial conditions $x(0) = 2$, $y(0) = 0.2$, $z(0) = 1$, and $u(0) = 2$ for $\alpha = 0.95$ and $\gamma = 0.95$, respectively. As can be seen, Fig. 4a is like a double scroll; Fig. 4b and c are like double butterflies state with front or back view, respectively; Fig. 4d and e are like double butterflies state with top view or bottom view, respectively; Fig. 4f is like a disc on the side.

Involving the FFD in the Caputo–Fabrizio–Caputo sense, which has

$$\begin{cases} {}^{FF-CFC}D_{0,t}^{\alpha,\gamma}\{x(t)\} = a(y(t) - x(t)) + y(t)z(t)u(t), \\ {}^{FF-CFC}D_{0,t}^{\alpha,\gamma}\{y(t)\} = b(x(t) + y(t)) - x(t)z(t)u(t), \\ {}^{FF-CFC}D_{0,t}^{\alpha,\gamma}\{z(t)\} = -cz(t) + x(t)y(t)u(t), \\ {}^{FF-CFC}D_{0,t}^{\alpha,\gamma}\{u(t)\} = -du(t) + x(t)y(t)z(t). \end{cases} \tag{79}$$

Based on Theorem 2, the numerical scheme of the four-dimensional hyper-Qi chaotic system (79) is given by:

$$\begin{cases} x(t_{n+1}) = x(0) + \dfrac{\alpha\gamma}{2M(\alpha)} \displaystyle\sum_{j=0}^{n} \Big[t_{j+1}^{\gamma-1} \left[a(y(t_{j+1}) - x(t_{j+1})) + y(t_{j+1})z(t_{j+1})u(t_{j+1}) \right] \\ \qquad -t_j^{\gamma-1} \left[a(y(t_j) - x(t_j)) + y(t_j)z(t_j)u(t_j) \right] \Big] (t_{j+1} + t_j) \\ \qquad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)} \left[a(y(t_{n+1}) - x(t_{n+1})) + y(t_{n+1})z(t_{n+1})u(t_{n+1}) \right]. \\ y(t_{n+1}) = y(0) + \dfrac{\alpha\gamma}{2M(\alpha)} \displaystyle\sum_{j=0}^{n} \Big[t_{j+1}^{\gamma-1} \left[b(x(t_{j+1}) + y(t_{j+1})) - x(t_{j+1})z(t_{j+1})u(t_{j+1}) \right] \\ \qquad -t_j^{\gamma-1} \left[b(x(t_j) + y(t_j)) - x(t_j)z(t_j)u(t_j) \right] \Big] (t_{j+1} + t_j) \\ \qquad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)} \left[b(x(t_{n+1}) + y(t_{n+1})) - x(t_{n+1})z(t_{n+1})u(t_{n+1}) \right]. \\ z(t_{n+1}) = z(0) + \dfrac{\alpha\gamma}{2M(\alpha)} \displaystyle\sum_{j=0}^{n} \Big[t_{j+1}^{\gamma-1} \left[-cz(t_{j+1}) + x(t_{j+1})y(t_{j+1})u(t_{j+1}) \right] \\ \qquad -t_j^{\gamma-1} \left[-cz(t_j) + x(t_j)y(t_j)u(t_j) \right] \Big] (t_{j+1} + t_j) \\ \qquad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)} \left[-cz(t_{n+1}) + x(t_{n+1})y(t_{n+1})u(t_{n+1}) \right]. \\ u(t_{n+1}) = u(0) + \dfrac{\alpha\gamma}{2M(\alpha)} \displaystyle\sum_{j=0}^{n} \Big[t_{j+1}^{\gamma-1} \left[-du(t_{j+1}) + x(t_{j+1})y(t_{j+1})z(t_{j+1}) \right] \\ \qquad -t_j^{\gamma-1} \left[-du(t_j) + x(t_j)y(t_j)z(t_j) \right] \Big] (t_{j+1} + t_j) \\ \qquad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{M(\alpha)} \left[-du(t_{n+1}) + x(t_{n+1})y(t_{n+1})z(t_{n+1}) \right]. \end{cases} \tag{80}$$

From a 3-D perspective: Considering the FFD in the Caputo–Fabrizio–Caputo sense and the numerical scheme given by equation (79), Numerical results for hyper-Qi chaotic system (80) in three-dimensional space are shown in Fig. 5a–d for $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 3 \times 10^{-3}$, simulation time $t = 200s$, and initial conditions $x(0) = 0.1$, $y(0) = 0.5$, $z(0) = 1$, and $u(0) = 2$ for $\alpha = 0.90$ and $\gamma = 0.90$, respectively. These sub-pictures are similar to double scrolls in appearance with compact state.

From a 2-D perspective: Considering the FFD in the Caputo–Fabrizio–Caputo sense and the numerical scheme given by equation (79), Numerical results for hyper-Qi chaotic system (80) in two-dimensional space are shown in Fig. 6a–f for $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 3 \times 10^{-3}$, simulation time $t = 200s$, and initial conditions $x(0) = 0.1$, $y(0) = 0.5$, $z(0) = 1$, and $u(0) = 2$ for $\alpha = 0.90$ and $\gamma = 0.90$, respectively. As can be seen, Fig. 6a is like a double scroll with compact state; Fig. 6b and c are like double butterflies with compact state in front or back view, respectively; Fig. 6d and e is like double butterflies with compact state in top view or bottom view, respectively; Fig. 6f is like a disc on the side with compact state.

Involving the FFD in the Liouville–Caputo–Yuhang sense, which has

$$\begin{cases} {}^{FF-LCY}D_{0,t}^{\alpha,\gamma}\{x(t)\} = a(y(t) - x(t)) + y(t)z(t)u(t), \\ {}^{FF-LCY}D_{0,t}^{\alpha,\gamma}\{y(t)\} = b(x(t) + y(t)) - x(t)z(t)u(t), \\ {}^{FF-LCY}D_{0,t}^{\alpha,\gamma}\{z(t)\} = -cz(t) + x(t)y(t)u(t), \\ {}^{FF-LCY}D_{0,t}^{\alpha,\gamma}\{u(t)\} = -du(t) + x(t)y(t)z(t). \end{cases} \tag{81}$$

Based on Theorem 3, the numerical scheme of the four-dimensional hyper-Qi chaotic system (81) is given by:

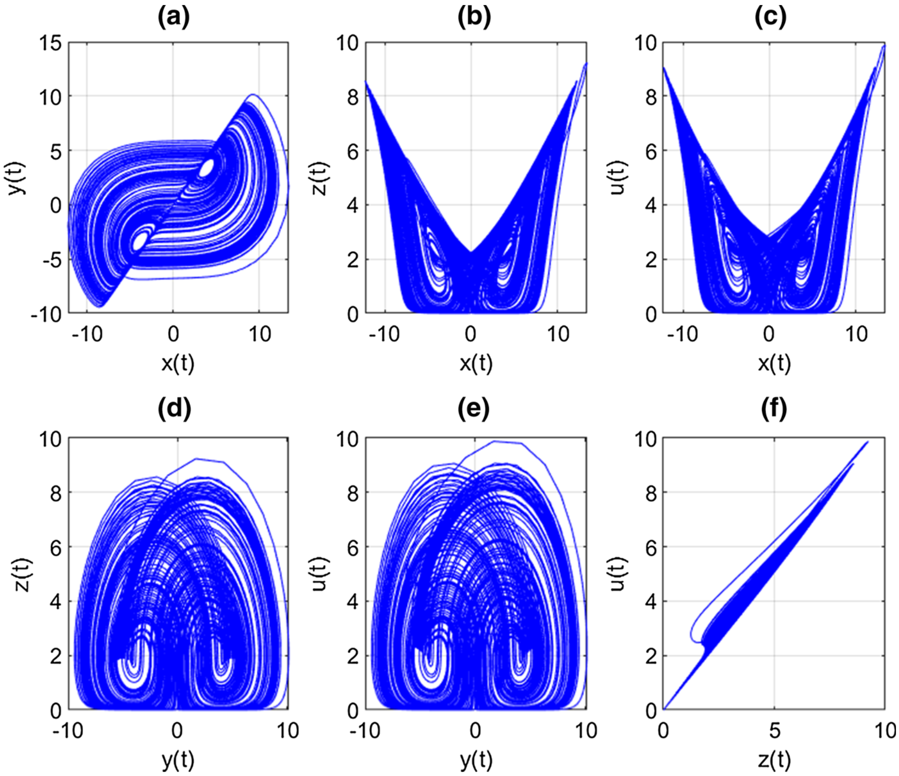

Fig. 4 Phase portraits of the hyper-Qi chaotic system FFD and projected on (**a**) the x(t)-y(t) plane; (**b**) the x(t)-z(t) plane; (**c**) the x(t)-u(t) plane; (**d**) the y(t)-z(t) plane; (**e**) the y(t)-u(t) plane; and (**f**) the z(t)-u(t) plane

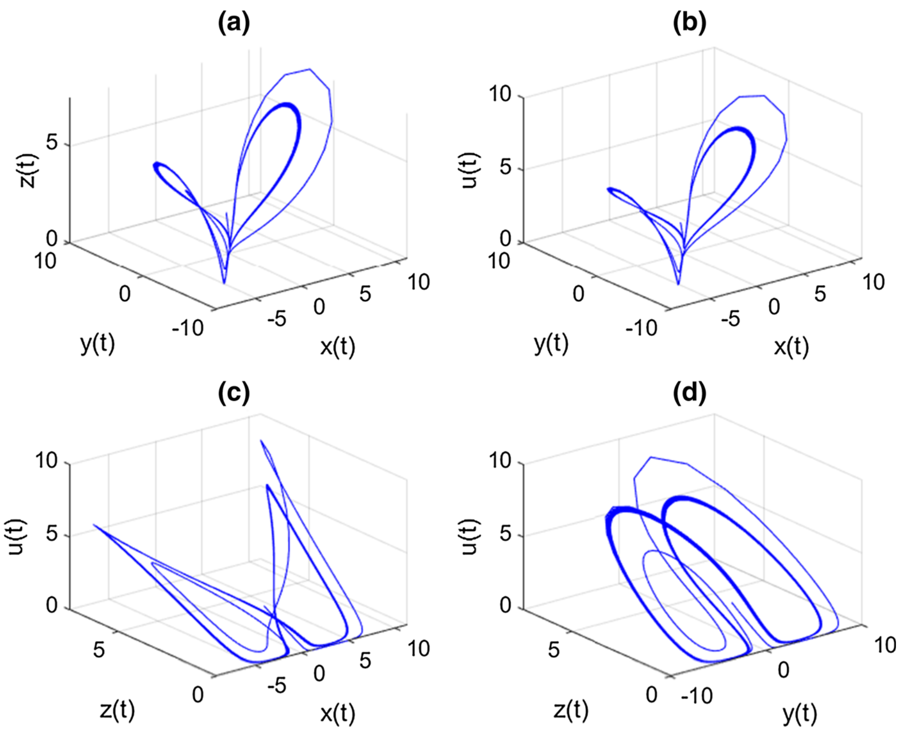

Fig. 5 Phase portraits of the hyper-Qi chaotic system FFD in (**a**) the x(t)-y(t)-z(t) space; (**b**) the x(t)-y(t)-u(t) space, (**c**) the x(t)-z(t)-u(t) space, and (**d**) the y(t)-z(t)-u(t) space

$$\begin{cases} x(t_{n+1}) = x(0) - \dfrac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \big[a(y(t_{j+1}) - x(t_{j+1})) \\ \quad +y(t_{j+1})z(t_{j+1})u(t_{j+1})\big] \Big[(\alpha+2)(n-j)^{\alpha} - (n+1-j)^{\alpha+1}\Big] \\ \quad -t_j^{\gamma-1}\big[a(y(t_j)-x(t_j)) + y(t_j)z(t_j)u(t_j)\big]\Big[(\alpha-n+j)(n+1-j)^{\alpha} + (n-j)^{\alpha+1}\Big]\Big\} \\ \quad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}\big[a(y(t_{n+1}) - x(t_{n+1})) + y(t_{n+1})z(t_{n+1})u(t_{n+1})\big]. \\ y(t_{n+1}) = y(0) - \dfrac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \big[b(x(t_{j+1}) + y(t_{j+1})) \\ \quad -x(t_{j+1})z(t_{j+1})u(t_{j+1})\big] \Big[(\alpha+2)(n-j)^{\alpha} - (n+1-j)^{\alpha+1}\Big] \\ \quad -t_j^{\gamma-1}\big[b(x(t_j)+y(t_j)) - x(t_j)z(t_j)u(t_j)\big]\Big[(\alpha-n+j)(n+1-j)^{\alpha} + (n-j)^{\alpha+1}\Big]\Big\} \\ \quad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}\big[b(x(t_{n+1}) + y(t_{n+1})) - x(t_{n+1})z(t_{n+1})u(t_{n+1})\big]. \\ z(t_{n+1}) = z(0) - \dfrac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \big[-cz(t_{j+1}) + x(t_{j+1})y(t_{j+1})u(t_{j+1})\big] \\ \quad \times \Big[(\alpha+2)(n-j)^{\alpha} - (n+1-j)^{\alpha+1}\Big] \\ \quad -t_j^{\gamma-1}\big[-cz(t_j) + x(t_j)y(t_j)u(t_j)\big]\Big[(\alpha-n+j)(n+1-j)^{\alpha} + (n-j)^{\alpha+1}\Big]\Big\} \\ \quad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}\big[-cz(t_{n+1}) + x(t_{n+1})y(t_{n+1})u(t_{n+1})\big]. \\ u(t_{n+1}) = u(0) - \dfrac{\alpha\gamma(\Delta t)^{\alpha}}{Y(\alpha)\Gamma(\alpha+2)} \sum_{j=0}^{n} \Big\{ t_{j+1}^{\gamma-1} \big[-du(t_{j+1}) + x(t_{j+1})y(t_{j+1})z(t_{j+1})\big] \\ \quad \Big[(\alpha+2)(n-j)^{\alpha} - (n+1-j)^{\alpha+1}\Big] \\ \quad -t_j^{\gamma-1}\big[-du(t_j) + x(t_j)y(t_j)z(t_j)\big]\Big[(\alpha-n+j)(n+1-j)^{\alpha} + (n-j)^{\alpha+1}\Big]\Big\} \\ \quad + \dfrac{\gamma(1-\alpha)t_{n+1}^{\gamma-1}}{Y(\alpha)}\big[-du(t_{n+1}) + x(t_{n+1})y(t_{n+1})z(t_{n+1})\big]. \end{cases} \tag{82}$$

From a 3-D perspective: Considering the FFD in the Liouville–Caputo Yuhang sense and the numerical scheme given by Eq. (81), Numerical results for hyper-Qi chaotic system (82) in three-dimensional space are shown in Fig. 7a–d for $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 5 \times 10^{-3}$, simulation time $t = 200s$, and initial conditions $x(0) = 4$, $y(0) = 4$, $z(0) = 2$, and $u(0) = 2$ for $\alpha = 0.85$ and $\gamma = 0.85$, respectively. For Fig. 7a–d, its sub-image looks like a three-dimensional single scroll in the state of a disc.

From a 2-D perspective: Considering the FFD in the Liouville–Caputo Yuhang sense and the numerical scheme given by Eq. (81), Numerical results for hyper-Qi chaotic system (82) in two-dimensional space are shown in Fig. 8a–f for $a = 25$, $b = 2$, $c = 15$, and $d = 35$, step size $h = 5 \times 10^{-3}$, simulation time $t = 200s$, and initial conditions $x(0) = 4$, $y(0) = 4$, $z(0) = 2$, and $u(0) = 2$ for $\alpha = 0.85$ and $\gamma = 0.85$, respectively. For Fig. 8a–f, its sub-image looks like planetary orbits in the solar system, respectively.

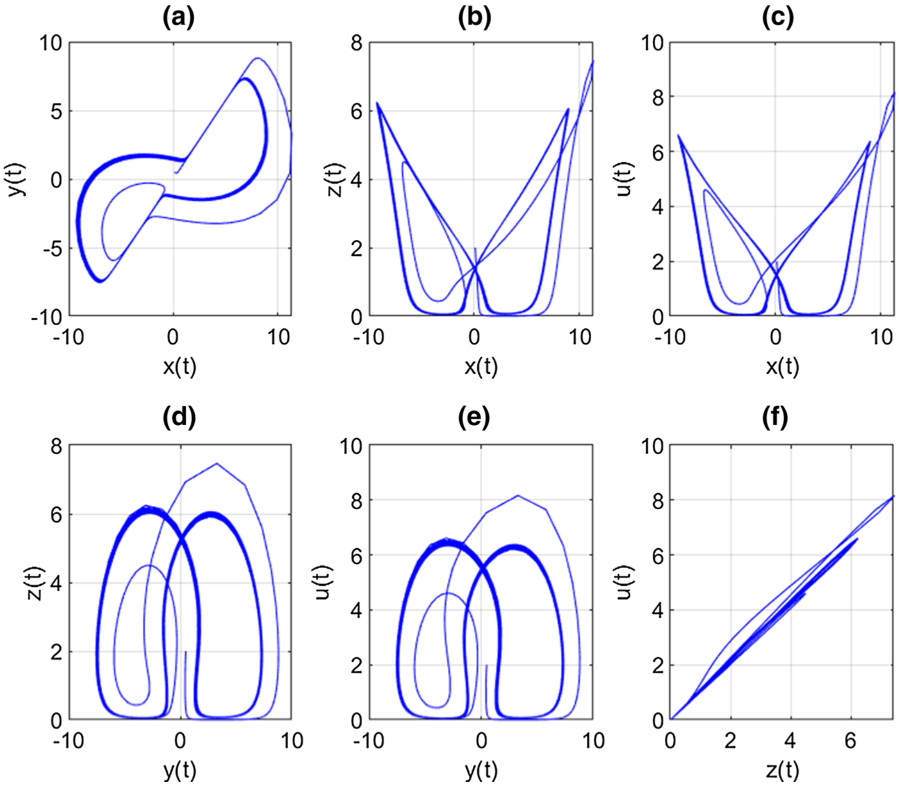

Fig. 6 Phase portraits of the hyper-Qi chaotic system FFD and projected on (**a**) the x(t)-y(t) plane; (**b**) the x(t)-z(t) plane; (**c**) the x(t)-u(t) plane; (**d**) the y(t)-z(t) plane; (**e**) the y(t)-u(t) plane; and (**f**) the z(t)-u(t) plane

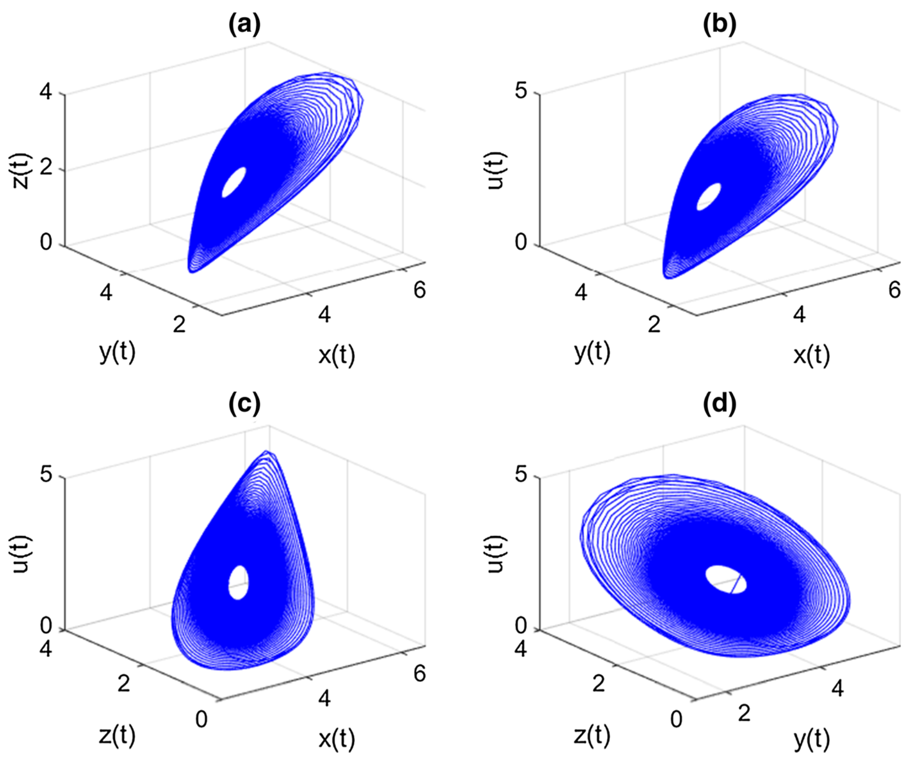

Fig. 7 Phase portraits of the hyper-Qi chaotic system FFD in (**a**) the x(t)-y(t)-z(t) space; (**b**) the x(t)-y(t)-u(t) space, (**c**) the x(t)-z(t)-u(t) space, and (**d**) the y(t)-z(t)-u(t) space

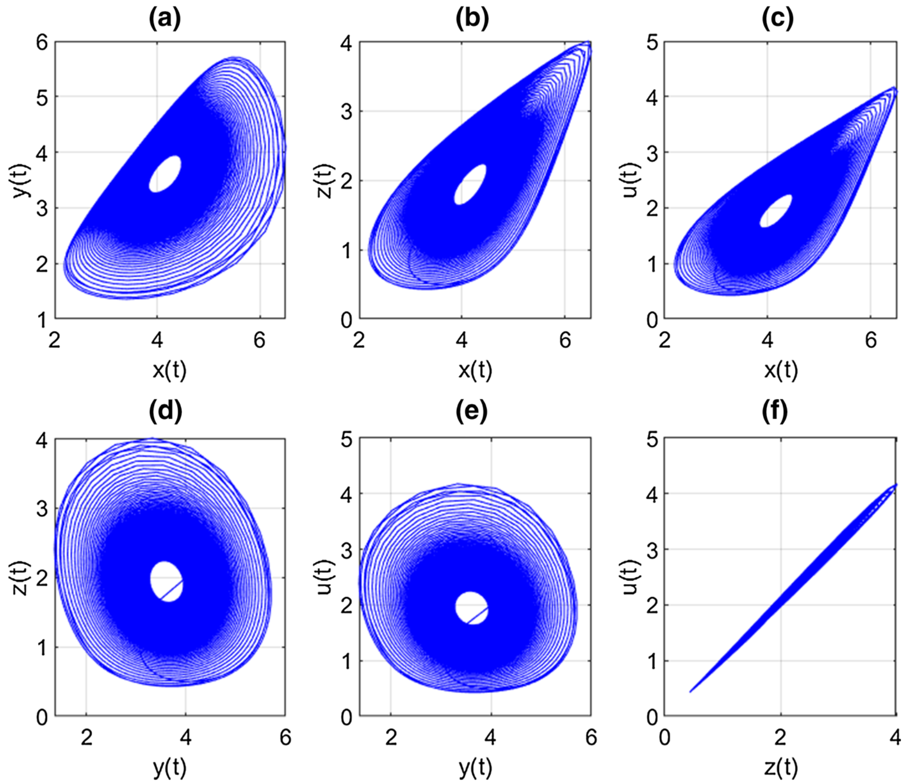

Fig. 8 Phase portraits of the hyper-Qi chaotic system FFD and projected on (**a**) the x(t)-y(t) plane; (**b**) the x(t)-z(t) plane; (**c**) the x(t)-u(t) plane; (**d**) the y(t)-z(t) plane; (**e**) the y(t)-u(t) plane; and (**f**) the z(t)-u(t) plane

In summary, based on different theorems, such as Theorems 1, 2, 3 and in view of different systems, such as systems (78), (80) and (82), Figs. 3, 4, 5, 6, 7 and 8 show different numerical simulations for the Hyper Qi chaotic system in Liouville–Caputo sense, Caputo–Fabrizio–Caputo sense, and Liouville–Caputo Yuhang sense considering different values of α and γ chosen arbitrarily.

Remark 1 The time complexity of an algorithm quantifies the amount of time taken by an algorithm to run as a function of the length of the input. Note that the time to run is a function of the length of the input and not the actual execution time of the machine on which the algorithm is running on. In this work, based on different theorems, such as Theorems 1, 2, 3 and in view of different systems, such as systems (78), (80) and (82), Figs. 3, 4, 5, 6, 7 and 8 show different numerical simulations for the Hyper Qi chaotic system with step size $h = 5 \times 10^{-3}$ and simulation time $t = 200s$. Meanwhile, the space complexity of an algorithm quantifies the amount of space taken by an algorithm to run as a function of the length of the input. In this work, based on different theorems, such as Theorems 1, 2, 3 and in view of different systems, such as systems (78), (80) and (82), Figs. 3, 4, 5, 6, 7 and 8 show different numerical simulations for the Hyper Qi chaotic system. From the perspective of space, the space taken by these subfigures are alike in appearance with double scrolls, butterflies, disc, and planetary orbits in the solar system.

Remark 2 A fractional-order four-dimensional hyper-Qi system has been introduced and analyzed as an example. Chaotic behaviors were observed with different fractional orders as a function of the systems parameters in Eq. (77). Based on different theorems, such as Theorems 1, 2, 3 and in view of systems parameters, such as systems (78), (80) and (82), Figs. 3, 4, 5, 6, 7 and 8 show different numerical simulations for the hyper-Qi chaotic system with different derivative orders $\alpha = 0.95$, $\alpha = 0.90$ and $\alpha = 0.85$, respectively. Moreover, from the comparisons about the attractors of the system with different derivative orders, the simulation figures are shown in appearance with double scrolls, butterflies, disc, and planetary orbits in the solar system.

Remark 3 At present, there are also many other methods which can be used to solve the fractional-order systems. For example, Adomian decomposition method

(ADM) [38]. ADM has the advantages of fast convergence and high precision in dealing with the fractional-order chaotic differential equations. It is of great value to the study of the fractional-order chaotic system. The proposed systems in this study can be analyzed and solved by using ADM.

5 Conclusion

This study analyzed a four-dimensional fractional hyperchaotic model in detail based on general Riemann–Liouville–Caputo fractal–fractional derivative. A series of new operators are constructed using three different elements. Moreover, the new operators have two parameters with fractional order and fractal dimension. The Qi hyperchaotic fractional attractor is modeled by using these new operators, and the models are solved numerically using a new and very efficient numerical scheme. Meanwhile, the solutions have been investigated to justify the physical adequacy of the model and the numerical scheme proposed in the resolution. The numerical simulations for some specific fractional-order dynamics and fractal dimension are presented. Nevertheless, these obtained via generalized Caputo–Fabrizio and the series of fractal–fractional derivative show some crossover effects, which is due to non-index law property. Furthermore, the obtained from generalized fractal–fractional derivative, in particular, those with the Mittag–Leffler kernel, show very strange and new attractors with self-similarities. Finally, the conclusions are shown that these new operators are the future to modelling complexities with self-similarities in some complex dynamical networks and and multi-agent systems. In addition to capture more complexities, new investigations and applications can be explored with some positive and new outcomes in various fields of nonlinear science, engineering and modern technology, cryptography, signal processing, and control process. These new explorations will be presented in future research work being processed by author of the present paper.

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Declarations

Conflict of interest The author declares that he has no conflict of interest about the publication of this paper.

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