Comment on „On the integrability of 2D Hamiltonian systems with variable Gaussian curvature” by A. A. Elmandouh

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Abstract In the paper [1], the author formulates in Theorem 2 necessary conditions for integrability of a certain class of Hamiltonian systems with non-constant Gaussian curvature, which depends on local coordinates. We give a counterexample to show that this theorem is not correct in general. This contradiction is explained in some extent. However, the main result of this note is our theorem that gives new simple and easy to check necessary conditions to integrability of the system considered in [1]. We present several examples, which show that the obtained conditions are effective. Moreover, we justify that our criterion can be extended to wider class of systems, which are given by non-meromorphic Hamiltonian functions.

Keywords Ordinary differential equations · Integrability · Non-integrability · Hamiltonian systems in curved spaces · Differential Galois theory

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1 Introduction

The integrability and solvability studies of Hamiltonian systems in curved spaces are currently in a great activity. Let us mention papers concerning geodesic flows in various metrics g_{ij} to g_{ij} on manifolds governed by Hamiltonian functions $H = g^{ij} p_i p_j$, see, e.g., [2–7]. Such systems describing trajectories along geodesics are very important for understanding geometry of manifolds. Topological obstructions for their integrability can be found in [8]. Recently, geodesic flows are also very popular by their physical meaning as trajectories of test mass points in various space-times, see, e.g., [9, 10]. Detection of gravity waves is related to the description of time-evolution of the space-distance between two mass-points on geodesics. This problem for the Zipoy–Voorhees metric around rotating Kerr black hole was investigated, e.g., in [11].

There is also a growing interest of integrability analysis of Hamiltonian systems in curved spaces with nonzero potentials. For instance, Yehia in [12, 13] obtained local forms of certain classes of systems with first integrals up to the fourth degree in the momenta. Valent in [14, 15] using the methods introduced in [16, 17], studied their global properties by obtaining systems defined on manifolds $\mathbb{E}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$. Ballesteros and others [18, 19] constructed analogues of the harmonic oscillator and the Hénon–Heiles model in spaces with a constant curvature. The integrability of the Hamiltonian which describes the motion of a material point on a surface with a constant Gaussian curvature

was also preformed in [20], and later enhanced in [21] to a more general form of the metric.

In the recent paper [1], the author extend the integrability analysis of Hamiltonian systems in curved spaces by a family of weight-homogeneous ones governed by the following Hamiltonian function:

$$\begin{cases} H = T + V, \\ T = \frac{1}{2}r^n \Lambda(\vartheta) \left(p_r^2 + \frac{p_\vartheta^2}{r^2} \right), \\ V = r^m U(\vartheta), \end{cases} \quad (1)$$

where $n, m \in \mathbb{Z}$ and $\Lambda(\vartheta)$, $U(\vartheta)$ are meromorphic functions. Hamiltonian (1) defines the motion of a particle moving under the influence of the homogeneous potential $V(r, \vartheta) = r^m U(\vartheta)$ on a surface with Gaussian curvature

$$\kappa(r, \vartheta) = \frac{r^{n-2}}{\Lambda(\vartheta)} \left[\Lambda(\vartheta) \Lambda''(\vartheta) - \Lambda'(\vartheta)^2 \right], \quad (2)$$

where prime denotes the derivative with respect to ϑ .

The author of paper [1] derived the necessary integrability conditions of system (1) deduced from the Morales–Ramis theory [22]. This approach is based on the analysis of the differential Galois group of variational equations of a considered system along a certain particular solution. The Morales–Ramis theory has been used recently for study integrability of various important physical and astronomical systems, see, e.g., papers [23–33]. We also mention review paper [34], in which certain examples can be found.

In most applications, the variational equations are transformed into a system with rational coefficients and, if it, or its subsystem, can be written as a one second-order equation, then the verification if the identity component of its differential Galois group is Abelian can be performed with the help of the Kovacic algorithm [35]. However, sometimes it is possible to transform the variational equations into the system of differential equations with known differential Galois group, for instance into the Riemann P -equation.

This is the case of [1]. The author calculated the variational equations of Hamilton's equations governed by Hamiltonian (1) along a particular solution lying on the manifold

$$\mathcal{N} = \left\{ (r, p_r, \vartheta, p_\vartheta) \in \mathbb{C}^4 \mid \vartheta = \vartheta_0, p_\vartheta = 0 \right\}. \quad (3)$$

Next, thanks to the change of the independent variable

$$t \rightarrow z = \frac{U(\vartheta_0)}{h} r^m, \quad \text{with } m \neq 0, \quad (4)$$

where h is the energy of the chosen particular solutions, the author transformed the variational equations into the Riemann P -equation

$$\begin{aligned} \Theta''(z) + \left(\frac{1 - \alpha - \alpha'}{z} + \frac{1 - \beta - \beta'}{z - 1} \right) \Theta'(z) \\ + \left(\frac{\alpha\alpha'}{z^2} + \frac{\beta\beta'}{(z - 1)^2} + \frac{\gamma\gamma' - \alpha\alpha' - \beta\beta'}{z(z - 1)} \right) \Theta(z) = 0, \end{aligned} \quad (5)$$

where Θ denotes the variation of ϑ . Then, he introduced the new parameters

$$\lambda_1 := m^2 \frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)}, \quad \lambda_2 := 1 + \frac{U''(\vartheta_0)}{(m - n)U(\vartheta_0)}, \quad (6)$$

and he wrote the differences of the exponents at the singularities $z = 0$, $z = 1$, $z = \infty$, in the form

$$\begin{aligned} \rho = \alpha - \alpha' &= \frac{\sqrt{(n - 2)^2 - 8\lambda_1}}{2m}, \\ \sigma = \beta - \beta' &= \frac{1}{2}, \\ \tau = \gamma - \gamma' &= \frac{\sqrt{(m - n - 2)^2 + 8(m - n)\lambda_2 - 8\lambda_1}}{2m}. \end{aligned} \quad (7)$$

The author of [1] derived the necessary integrability conditions of system (1) using the Kimura theorem [36], see also Appendix. This theorem gives the necessary and sufficient conditions, which guarantee that the identity component of the Riemann P -equation is solvable. The main result of paper [1] is the following theorem.

Theorem 1 (Elmandouh [1]) *Suppose the two functions $\Lambda(\vartheta)$ and $U(\vartheta)$ are two meromorphic functions, and assume there is $\vartheta_0 \in \mathbb{C}$ such that*

$$\Lambda'(\vartheta_0) = 0 = U'(\vartheta_0), \quad \text{with } \Lambda(\vartheta_0)U(\vartheta_0) \neq 0. \quad (8)$$

If the Hamiltonian system (1) is Liouville integrable, then the number

$$\begin{aligned} \lambda &:= \lambda_2 - \frac{\lambda_1}{m - n} \\ &= 1 + \frac{1}{m - n} \left[\frac{U''(\vartheta_0)}{U(\vartheta_0)} - m^2 \frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)} \right], \end{aligned} \quad (9)$$

belongs to the sets $\mathcal{J}_i(m, n)$, which are listed in Table 1.

We claim that this theorem is in general not correct. Let us exemplify this by the Hamiltonian function

$$H = \frac{1}{2 \sin^2 \vartheta} r^6 \left(p_r^2 + \frac{p_\vartheta^2}{r^2} \right) + \left(\frac{r}{\sin \vartheta} \right)^m, \quad (10)$$

where $m \in \mathbb{Z}^*$. For this Hamiltonian, we have

$$n = 6, \quad \Lambda(\vartheta) = \frac{1}{\sin^2 \vartheta}, \quad U(\vartheta) = \frac{1}{\sin^m \vartheta}. \quad (11)$$

Both Λ and U are meromorphic functions so the first statement of Theorem 1 is satisfied.

We take point $\vartheta_0 = \frac{\pi}{2}$ at which condition (8) is fulfilled. Then, value of λ defined in (9) at ϑ_0 is

$$\lambda = 1 + \frac{m}{m-6} (1-2m). \quad (12)$$

Comparing this value with the forms of the sets $\mathcal{J}_i$ defined in Table 1 in [1], we conclude that only for $m = \pm 1$, the necessary integrability condition is satisfied. It is even better visible when we look at the differences of the exponents (7). In this case they took the values

$$\rho = \frac{2\sqrt{1-m^2}}{m}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{\sqrt{16-15m^2}}{2m}. \quad (13)$$

Now it is evident that for $m \neq \pm 1$ neither case A nor case B of the Kimura theorem can be fulfilled because ρ and τ are both complex.

Since the necessary integrability condition is not satisfied due to $\lambda \notin \mathcal{J}_i$, then, according to Theorem 1, it means that for $m \neq \pm 1$ system (10) is not integrable. However, this conclusion is actually wrong because Hamiltonian (10) possesses the additional first integral for arbitrary m , and it has the form:

$$I = r^2 \cos \vartheta p_r + r \sin \vartheta p_\vartheta, \quad (14)$$

see [37]. As I is functionally independent with H implies that system (10) is, in fact, integrable.

We discuss this contradiction in the last section of this paper. At first, we give the correct formulation of the integrability obstructions for system (1). However, we define the main integrability theorem in the form different as in [37]. It is simply, effective and easy for applications.

2 Integrability analysis

We state the following theorem.

Theorem 2 Assume that $\Lambda(\vartheta)$ and $U(\vartheta)$ are complex meromorphic functions of variable ϑ and there exist a point $\vartheta_0 \in \mathbb{C}$, such that

$$\Lambda'(\vartheta_0) = 0 = U'(\vartheta_0), \quad \text{with } \Lambda(\vartheta_0)U(\vartheta_0) \neq 0. \quad (15)$$

If the Hamiltonian system defined by Hamiltonian (1) is integrable in the sense of Liouville, then the numbers

$$\begin{aligned} \lambda_1 &:= \frac{1}{2m} \sqrt{(n-2)^2 - 8 \frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)}}, \\ \lambda_2 &:= \frac{1}{2m} \sqrt{(m-n+2)^2 + 8 \left(\frac{U''(\vartheta_0)}{U(\vartheta_0)} - \frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)} \right)}, \end{aligned} \quad (16)$$

belong to one item of the following list

$Lp.$	λ_1	λ_2
(i)	arbitrary	$\frac{1}{2} + 2p \pm \lambda_1, \frac{3}{2} + 2p \pm \lambda_1, \frac{1}{2} + p$
(ii)	$\frac{1}{2} + s$	arbitrary
(iii)	$\pm \frac{1}{3} + s$	$\pm \frac{1}{3} + p, \pm \frac{1}{4} + p, \pm \frac{1}{5} + p, \pm \frac{2}{5} + p$
(iv)	$\pm \frac{1}{4} + s$	$\pm \frac{1}{3} + p$
(v)	$\pm \frac{1}{5} + s$	$\pm \frac{1}{3} + p, \pm \frac{2}{5} + p$
(vi)	$\pm \frac{2}{5} + s$	$\pm \frac{1}{3} + p, \pm \frac{1}{5} + p$

(17)

where $m, n, s, p \in \mathbb{Z}$ and $m \neq 0$.

Proof Since the variational equations given in Eq. (18) in [1] are incorrectly calculated, we make the integrability analysis of system (1) starting from the very beginning.

The Hamiltonian equations $\dot{\mathbf{x}} = \mathbf{X}_H(\mathbf{x})$ governed by Hamiltonian (1) are as follows:

$$\begin{cases} \dot{r} = r^n \Lambda(\vartheta) p_r, \\ \dot{p}_r = -\frac{n}{2} r^{n-1} \Lambda(\vartheta) p_r^2 + \frac{2-n}{2} r^{n-3} \Lambda(\vartheta) p_\vartheta^2 - m r^{m-1} U(\vartheta), \\ \dot{\vartheta} = r^{n-2} \Lambda(\vartheta) p_\vartheta, \\ \dot{p}_\vartheta = -\frac{1}{2} r^n \Lambda'(\vartheta) (p_r^2 + r^{-2} p_\vartheta^2) - r^m U'(\vartheta). \end{cases} \quad (18)$$

If we assume that there exist a point $\vartheta_0 \in \mathbb{C}$ such that $U'(\vartheta_0) = 0 = \Lambda'(\vartheta_0)$, then system (18) has two-dimensional invariant manifold (3). Restricting the right-hand sides of (18) to $\mathcal{N}$, we obtain

$$\begin{cases} \dot{r} = r^n \Lambda(\vartheta_0) p_r, \\ \dot{p}_r = -\frac{n}{2} r^{n-1} \Lambda(\vartheta_0) p_r^2 - m r^{m-1} U(\vartheta_0), \\ \dot{\vartheta} = 0, \\ \dot{p}_\vartheta = 0. \end{cases} \quad (19)$$

Solution of (19) determines our particular solution $\mathbb{C} \ni t \rightarrow \varphi_{\mathcal{N}}(t) = (r(t), p_r(t), 0, 0)$. In fact, we have the whole family of particular solutions lying on a fixed energy level

$$H_{\mathcal{N}} = \frac{1}{2} r^n \Lambda(\vartheta_0) p_r^2 + r^m U(\vartheta_0) = h. \quad (20)$$

Let $\mathbf{y} := [R, P_R, \Theta, P_{\Theta}]^T$ define the variations of $\mathbf{x} := [r, p_r, \vartheta, p_{\vartheta}]^T$. Then, the first-order variational equations along the particular solution $\varphi_{\mathcal{N}}(t)$ are given by

$$\dot{\mathbf{y}} = A(t) \cdot \mathbf{y}, \quad A(t) := \frac{\partial X_H}{\partial \mathbf{x}}(\varphi_{\mathcal{N}}(t)). \quad (21)$$

The explicit form of the Jacobian matrix $A(t)$ is as follows:

$$A = \begin{pmatrix} nr^{n-1} \Lambda(\vartheta_0) p_r & r^n \Lambda(\vartheta_0) & 0 & 0 \\ a(r, p_r) & -nr^{n-1} \Lambda(\vartheta_0) p_r & 0 & 0 \\ 0 & 0 & 0 & r^{n-2} \Lambda(\vartheta_0) \\ 0 & 0 & b(r, p_r) & 0 \end{pmatrix},$$

where

$$a = \frac{n(1-n)}{2} r^{n-2} \Lambda(\vartheta_0) p_r^2 + m(1-m) r^{m-2} U(\vartheta_0), \quad (22)$$

$$b = -\frac{1}{2} r^n \Lambda''(\vartheta_0) p_r^2 - r^m U''(\vartheta_0).$$

Since the motion takes place on (r, p_r) plane, the equations for $\dot{\Theta}$ and $\dot{P}_{\Theta}$ form a closed subsystem of normal variational equations. For simplicity, we write them as a one second-order differential equation

$$\ddot{\Theta} + P(r, p_r) \dot{\Theta} + Q(r, p_r) \Theta = 0, \quad (23)$$

where the coefficients P and Q are given by

$$P = (2-n) r^{n-1} \Lambda(\vartheta_0) p_r, \quad (24)$$

$$Q = \frac{1}{2} r^{n-2} \Lambda(\vartheta_0) [r^n \Lambda''(\vartheta_0) p_r^2 + 2r^m U''(\vartheta_0)].$$

Next, we perform the change of the independent variable (4) by using the energy first integral (20). Hence, we transform the normal variational equation (23) into the system with rational coefficients

$$\Theta''(z) + \frac{1}{2} \left(\frac{1}{z-1} + \frac{2m-n+2}{mz} \right) \Theta'(z) + \frac{1}{2m^2} \left(\frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)z} - \frac{U''(\vartheta_0)}{U(\vartheta_0)z(z-1)} \right) \Theta(z) = 0. \quad (25)$$

This equation is the Riemann P -equation (5), with the differences of the exponents at singularities $z = 0$, $z = 1$ and $z = \infty$ defined simply by

$$\rho = \lambda_1, \quad \sigma = \frac{1}{2}, \quad \tau = \lambda_2, \quad (26)$$

where the quantities λ_1 and λ_2 are postulated in (16). The proof of Theorem 2 consists of the direct application of the Kimura theorem 3 to the obtained Riemann P -equation (25) with the differences of the exponents (26). Indeed, the first four elements given in item (i) of the list (17) were deduced from Case A of Theorem 3, while the remaining entries of (17) come from Case B of this theorem. $\square$

Remark 1 Let us underline that in our analysis we assumed that $m \neq 0$. This is because for $m = 0$ the normal variational equations (21) are solvable. Thus, in this case there is no obstacle for the integrability of the considered system. In order, to define the integrability obstructions for the system higher-order variational equations or a different approach must be used.

From the form of integrability list (17), we can deduce the following corollary.

Corollary 1 *Let us assume that the quantities*

$$\frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)} \in \mathbb{Q}, \quad \frac{U''(\vartheta_0)}{U(\vartheta_0)} \in \mathbb{Q}. \quad (27)$$

If λ_1 and λ_2 defined in (16) are both irrational, then system (1) is not integrable in the Liouville sense.

Proof Suppose quantities (16) have the form

$$\lambda_1 = \frac{\sqrt{r}}{2m}, \quad \lambda_2 = \frac{\sqrt{s}}{2m}, \quad r, s \in \mathbb{Q} \quad (28)$$

and they are irrational. If system (1) is integrable, then according to Theorem 2, one of two possibilities

$$\sqrt{s} = k + \sqrt{r}, \quad \text{or} \quad \sqrt{s} = k - \sqrt{r}, \quad k \in \mathbb{Z}^*, \quad (29)$$

holds true. However, squaring (29), we get

$$s = k^2 + 2k\sqrt{r} + r, \quad \text{or} \quad s = k^2 - 2k\sqrt{r} + r, \quad (30)$$

which is the contradiction because $r, s \in \mathbb{Q}$, while the right-hand sides of (30) are irrational. This ends the proof. $\square$

3 Applications of Theorem 2

In this section, we present the advantage of Theorem 2 by applying it on some simple examples. For these models, the metrics have nonzero curvature.

Example 1 As the first example, we reconsider the Hamiltonian function (10). We take point $\vartheta_0 = \frac{\pi}{2}$

at which condition (15) is fulfilled. Taking this into account, we have the following

$$n = 6, \quad \frac{\Lambda''(\frac{\pi}{2})}{\Lambda(\frac{\pi}{2})} = 2, \quad \frac{U''(\frac{\pi}{2})}{U(\frac{\pi}{2})} = m. \quad (31)$$

Then, from definition (16) we get

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2} \operatorname{sgn}(m), \quad (32)$$

It is easy to see that these values of λ_1 and λ_2 belong to the first item (i) of integrability list (17). Thus, the necessary integrability condition is satisfied for arbitrary $m \in \mathbb{Z}^*$, as it should be since we have already met that Hamiltonian (10) is integrable with the first integral (14).

Example 2 We have defined λ_1 and λ_2 in such a way that they equal to the differences of the exponents at $z = 0$ and $z = \infty$, see (26). With this choice checking conditions is straightforward and we avoid to derive and to check cumbersome form of sets $\mathcal{J}_i(m, n)$ listed in Table 1 of [1]. For instance, if λ_1 and λ_2 are both irrational and condition (27) is satisfied, then according to Corollary 1, the system is not integrable. Let us exemplify this by the following two-parameter family of Hamiltonian systems generated by the Hamiltonian function

$$H = \frac{1}{2} r^n \cos \vartheta \left(p_r^2 + \frac{p_\vartheta^2}{r^2} \right) + r^m \cos \vartheta. \quad (33)$$

Hence, functions Λ , U standing in (1), are as follows:

$$\Lambda(\vartheta) = \cos \frac{\vartheta}{3}, \quad U(\vartheta) = \cos \vartheta. \quad (34)$$

At point $\vartheta_0 = 0$, quantities (27) read

$$\frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)} = -\frac{1}{9}, \quad \frac{U''(\vartheta_0)}{U(\vartheta_0)} = -1, \quad (35)$$

and the integrability coefficients (16) are given by

$$\lambda_1 = \frac{\sqrt{9(n-2)^2 + 8}}{6m}, \quad \lambda_2 = \frac{\sqrt{9(m-n+2)^2 - 64}}{6m}. \quad (36)$$

It can be proved by simple arguments that λ_1 and λ_2 are both irrational for all values of the parameters $n, m \in \mathbb{Z}$ with $m \neq 0$. Thus, according to Corollary 1, two-parameter family of Hamiltonian systems governed by (33) is not integrable in the sense of Liouville.

Claim 1 *The next example shows that in many cases we need to modify assumptions of Theorem 2. In practice, we have to consider systems for which the Hamiltonian is not meromorphic. The best known example is the N -body problem. Nevertheless, the Morales–Ramis theorem was used successfully to prove the non-integrability. The methods to cope with algebraic potentials are described in [38] and [39]. Here, we claim that in Theorem 2 we can assume that numbers n and m are rational. Moreover, we can assume that functions $\Lambda(\vartheta)$ and $U(\vartheta)$ are algebraic functions of $\cos(\vartheta)$ or $\sin(\vartheta)$. Then, the thesis of this theorem is valid with a small modification. To prove this we introduce additional variables in order to extend the system to a higher-dimensional space. The key point is to make this extension in such a way that the right-hand sides of the extended system are rational. For the consider system, it can be done in the following way:*

1. We introduce new variables $c = \cos \vartheta$ and $s = \sin \vartheta$. Of course, $f_1(c, s) = c^2 + s^2 - 1 = 0$.
2. Then, $v(c, s) = \Lambda(\vartheta)$ and $u(c, s) = U(\vartheta)$ are algebraic functions of (c, s) . Let $f_2(v, c, s)$ and $f_3(u, c, s)$ be their respective minimal polynomials.
3. Finally, let $\rho_1 = r^n$ and $\rho_2 = r^m$ and let $f_4(\rho_1, r)$ and $f_5(\rho_2, r)$ be their respective minimal polynomials.
4. Now we take variables $\mathbf{z} = (r, p_r, p_\vartheta, c, s, u, v, \rho_1, \rho_2)$ and calculate their time derivatives using the original Hamilton's equations, standard rules of differentiation of compositions and implicit functions.
5. One can check that obtained system $\dot{\mathbf{z}} = \mathbf{Z}(\mathbf{z})$ has rational right-hand sides. Moreover, the Hamiltonian, which in new variables reads

$$H = \frac{1}{2} \rho_1 v \left(p_r^2 + \frac{p_\vartheta^2}{r^2} \right) + \rho_2 u, \quad (37)$$

has first integrals $f_1, \dots, f_5$.

6. On the common level $f_i(\mathbf{z}) = 0$ for $i = 1, \dots, 5$, the extended system coincides with the original one.

After the described extension, we can apply the Morales–Ramis theory. In practice, however, it is allowed to perform all the calculations in the original variables in the way we show in the example below.

Example 3 We consider one-parameter family of Hamiltonian systems governed by a Hamiltonian function of the form:

$$H = \frac{1}{2}r^{2-\frac{m}{2}} \cos^{\frac{3}{2}}\left(\frac{3m}{2}\vartheta\right) \left(p_r^2 + \frac{p_\vartheta^2}{r^2}\right) + r^m \cos^{\frac{3}{2}}\left(\frac{3m}{2}\vartheta\right). \quad (38)$$

We take point $\vartheta_0 = 0$ at which condition (15) is fulfilled. Taking this into account, we have the following

$$n = 2 - \frac{m}{2}, \quad \frac{\Lambda''(0)}{\Lambda(0)} = -\frac{3m^2}{2} = \frac{U''(0)}{U(0)}. \quad (39)$$

Thus, the integrability coefficients (16) take the values

$$\lambda_1 = \frac{7}{4} \operatorname{sgn}(m), \quad \lambda_2 = \frac{3}{4} \operatorname{sgn}(m). \quad (40)$$

It is easy to verify that these values of λ_1 and λ_2 belong to the first item (i) of integrability list (17). Thus, the necessary integrability condition is satisfied for arbitrary $m \in \mathbb{Q}^*$. Indeed, the Hamiltonian system defined by Hamiltonian (38) is integrable. The additional first integral, functionally independent with (38), is defined by

$$\begin{aligned} I = & r^{2-\frac{3m}{2}} \cos\left(\frac{3m}{2}\vartheta\right) \left(3p_r^2 + \frac{p_\vartheta^2}{r^2}\right) p_\vartheta \\ & + 2r^{3-\frac{3m}{2}} \sin\left(\frac{3m}{2}\vartheta\right) p_r^3 \\ & + 4r \sin\left(\frac{3m}{2}\vartheta\right) p_r + 4 \cos\left(\frac{3m}{2}\vartheta\right) p_\vartheta. \end{aligned} \quad (41)$$

According to our knowledge, this is the new integrable system with the non-constant Gaussian curvature of the form:

$$\kappa(r, \vartheta) = -\frac{3m^2}{2} r^{-\frac{m}{2}} \left[\cos\left(\frac{3m}{2}\vartheta\right) \right]^{-\frac{4}{3}}. \quad (42)$$

4 Remarks and conclusions

In this comment, we pointed out that Theorem 2 postulated in paper [1] is in general incorrect. We give the example of integrable system, which is according to Theorem 2 not integrable. It is rather difficult to find a unique source of failure in Theorem 2. First of all the differences of the exponents (7) of the Riemann P equation (5) were incorrectly calculated. Equivalently, the parameter λ_1 , which appears in ρ and τ was wrongly

defined. This implies that the forms of sets $\mathcal{J}_i$, given in Eq. (29) in work [1], are also wrongly defined because $\Delta_1 = \sqrt{(n-2)^2 - 8\lambda_1}$ is incorrect.

To get correct forms of the differences of exponents (7), one has to define $\lambda_1 := \frac{\Lambda''(\vartheta_0)}{\Lambda(\vartheta_0)}$, not as postulated in Eq. (6). This permanently repeated mistake causes Theorem 1 to be generally incorrect, i.e., it is only valid for $m = \pm 1$. Unfortunately, in its applications the author of [1] has chosen examples in such a way that all of them have $m = 1$. This gave to the author as well as to the reviewers false belief in the correctness of the preformed calculations. Nonetheless, the above described redefinition of λ_1 does not make Theorem 1 fully correct. Moreover, the author restricts conditions given by Case B of the Kimura theorem and does not explicitly warn the reader about this. For example, in the proof of Case 1 on p. 937 Eq. (31) restriction $\omega \in \mathbb{Z}$ is too strong. Condition $n \in \mathbb{Z}$ is fulfilled if and only if $\omega = l/m$ for a certain $l \in \mathbb{Z}$. The same thing holds in the remaining cases. This implies that restriction for m to be a multiplicity of 2, 3, 5, see Table 1 in [1], seems to be unnecessary.

However, our aim in this note was to formulate theorem, which gives new form of the necessary integrability conditions of system (1). The key point is that we avoided complications connected with clumsy form of sets $\mathcal{J}_i(m, n)$. We give three, nontrivial examples which show that the presented formulation is easy and effective in applications. Moreover, we show also how we can extend our criterion to wider class of systems, which are given by non-meromorphic Hamiltonian functions.

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Table 1 Schwarz's table. Here $r, q, p \in \mathbb{Z}$

1	$1/2 + r$	$1/2 + q$	arbitrary	
2	$1/2 + r$	$1/3 + q$	$1/3 + p$	
3	$2/3 + r$	$1/3 + q$	$1/3 + p$	$r + q + p$ even
4	$1/2 + r$	$1/3 + q$	$1/4 + p$	
5	$2/3 + r$	$1/4 + q$	$1/4 + p$	$r + q + p$ even
6	$1/2 + r$	$1/3 + q$	$1/5 + p$	
7	$2/5 + r$	$1/3 + q$	$1/3 + p$	$r + q + p$ even
8	$2/3 + r$	$1/5 + q$	$1/5 + p$	$r + q + p$ even
9	$1/2 + r$	$2/5 + q$	$1/5 + p$	
10	$3/5 + r$	$1/3 + q$	$1/5 + p$	$r + q + p$ even
11	$2/5 + r$	$2/5 + q$	$2/5 + p$	$r + q + p$ even
12	$2/3 + r$	$1/3 + q$	$1/5 + p$	$r + q + p$ even
13	$4/5 + r$	$1/5 + q$	$1/5 + p$	$r + q + p$ even
14	$1/2 + r$	$2/5 + q$	$1/3 + p$	
15	$3/5 + r$	$2/5 + q$	$1/3 + p$	$r + q + p$ even

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Appendix. The Riemann P -equation

The Riemann P -equation is the most general second-order differential equation with three regular singularities. If we place the singularities at $z = 0$, $z = 1$ and $z = \infty$, it is given by

$$\Theta''(z) + \left(\frac{1 - \alpha - \alpha'}{z} + \frac{1 - \beta - \beta'}{z - 1} \right) \Theta'(z) + \left(\frac{\alpha\alpha'}{z^2} + \frac{\beta\beta'}{(z - 1)^2} + \frac{\gamma\gamma' - \alpha\alpha' - \beta\beta'}{z(z - 1)} \right) \Theta(z) = 0, \quad (43)$$

where (α, α') , (β, β') and (γ, γ') are the exponents at the respective singular points, see, for instance, Refs. [40, 41]. These exponents satisfy the Fuchs relation

$$\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1.$$

We denote the differences of the exponents by

$$\rho = \alpha - \alpha', \quad \sigma = \beta - \beta', \quad \tau = \gamma - \gamma'.$$

Necessary and sufficient conditions for solvability of the identity component of the differential Galois group of (43) are given by the following theorem due to Kimura [36].

Theorem 3 *The identity component of the differential Galois group of Riemann P -equation (43) is solvable iff*

- A. *at least one of the four numbers $\rho + \sigma + \tau$, $-\rho + \sigma + \tau$, $\rho + \sigma - \tau$, $\rho - \sigma + \tau$ is an odd integer, or*
- B. *the numbers ρ or $-\rho$ and σ or $-\sigma$ and τ or $-\tau$ belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz's Table 1.*

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