Complex time-delay dynamical systems of quadratic polynomials mapping

Yuanyuan Sun · Zhixing Lu · Peng Li

Abstract Time delays widely exist in the complex dynamical systems. In this paper, time-delay complex dynamical systems are studied by analyzing the properties of corresponding Julia sets. Experimental results show that the stability of complex dynamical system changes greatly due to the influence of time delay. The variations in time-delay Julia sets are studied. In addition, the conditions holding the dynamical systems stable are found. Time delay affects the stability of complex dynamical systems, and the existence of the fixed and periodic points is also discussed.

Keywords Time delay · Complex dynamical systems · Julia sets · Fractal

1 Introduction

In the 1970s, Julia sets were obtained by mathematician Gaston Julia who studied the simple iterative map in the complex plane. In recent years, there are many developments in Julia sets, including dynamical systems [1–3], complex networks [4–6], and the fractal arts [7].

As we know, dynamical system theory began with Newton in the seventeenth century and took a radical turn with Poincare a century ago. The dynami-

cal systems occur in four different flavors [8] such as flows, cascades, iterations, and shifts. The complex dynamical systems are associated with the iteration (repeated application) of a rational mapping of the complex plane, such as Julia sets.

The existence of a delay in a system model could have several causes, for example the measure of a system variable or the physical nature of a signal transmission. Most research works on time delay were about differential dynamical systems [9–12] and neglected the problems of the complex dynamical systems. In this paper, we are concerned with the time-delay complex dynamical systems by drawing corresponding Julia sets.

The outline of this paper is as follows. We devote Sect. 2 to a discussion of governing equations and the theoretical analysis about complex time-delay dynamical systems. Definition of time-delay iteration and the numerical experiments is presented in Sect. 3, showing how the time delay affects the stability of complex dynamical systems.

2 Governing equations and theoretical analysis

2.1 Time-delay equation and theoretical analysis

For each quadratic polynomial $R(z) = az^2 + 2bz + d$, we have a polynomial $M(z) = az + b$ and a quadratic polynomial $f(z) = z^2 + c$, such that $M^{-1} \circ f \circ M = R(z)$ [13]. It is clear that $f(z)$ is a conformal mapping

Y. Sun (✉) · Z. Lu · P. Li
College of Computer Science and Technology, DaLian
University of Technology, Dalian, China
e-mail:syuan@dlut.edu.cn

of $R(z)$ and $f(z)$ has the same dynamical system as $R(z)$. In this case, the way to investigate the properties of complex quadratic polynomials is to study $f(z) = z^2 + c$.

We draw the time-delay Julia set utilizing the escape time algorithm [14], which is a common method to draw a fractal set. For $f(z) = z^2 + c$, we define f^n to be $\underbrace{f \circ f \circ \cdots \circ f}_{n}$, and $f_n(z) = x_n + iy_n, c = p + iq$, where $p, q, x_n, y_n \in R, n \in N_+$. So (x_n, y_n) can be calculated by Eq. (1) as follows:

$$\begin{cases} x_n = x_{n-1}^2 - y_{n-1}^2 + p \\ y_n = 2x_{n-1}y_{n-1} + q \end{cases} \quad (1)$$

There are two axes of coordinates for each complex iterate function, so the time delay may happen on either of them. When a time delay happens on the abscissa axis, Eq. (1) should be changed into

$$\begin{cases} x_{n+1} = x_n \\ y_{n+1} = 2x_n y_n + q \end{cases} \quad (2)$$

Equation (2) can be simplified as $y_{n+1} = 2x_n y_n + q$.

Now, a complex dynamical system problem is changed into the problem about a real dynamical system. Then, we carry out the following theorem:

Theorem 1 *Let (x, y) be a point under the iteration of Eq. (2), (x, y) can hold on being stable if and only if the absolute value of x is less than or equal to half.*

Proof According to mathematical induction, the result of (x, y) iterated on Eq. (2) is as follows:

$$2xy + q, (2x)^2 y + q(2x + 1), \ldots, (2x)^n y$$
$$+ [(2x)^{n-1} + \cdots + 2x + 1]q.$$

Let y_n be the value of $(2x)^n y + [(2x)^{n-1} + \cdots + 2x + 1]q$; we can get the result of y_n after n iterations by the property of geometric sequence,

$$y_n = (2x)^{n-1}\left(2xy - \frac{q}{1-x}\right) + \frac{q}{1-x}.$$

It is easy to see that

$$\lim_{n \to +\infty} y_n = Y = \begin{cases} \frac{q}{1-x} & |x| < 1/2 \\ 2xy & |x| = 1/2 \\ +\infty & |x| > 1/2 \end{cases}$$

If the absolute value of x is less than or equal to half, the result is convergent.

The proof is completed. □

When time delay happens on the ordinate axis, Eq. (1) should be changed into

$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 + p \\ y_{n+1} = y_n \end{cases} \quad (3)$$

We study the iterate function $f(x) = x^2 + t$, where $t = p - y_n^2, x_n = x$. Then, we can carry out Theorem 2.

Theorem 2 *For iterate function $f(x) = x^2 + t$, if $t > \frac{1}{4}$, $f(x)$ does not have fixed point and period-2 points; if $t \leq \frac{1}{4}$, $f(x)$ has two fixed points; if $t \leq -\frac{3}{4}$, $f(x)$ has two period-2 points.*

Proof Let $x^2 + t = x$; it is clear that if t is greater than one-quarter, there is no solution for the equation, so there is no fixed point for the iterate function $f(x)$. If t is less than or equal to a quarter, the fixed points are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - t}$.

Let $\left(x^2 + t\right)^2 + t = x$; the equation can be changed as $(x^2 - x + t)(x^2 + x + 1 + t) = 0$. It is easy to know that if $t > -\frac{3}{4}$, there is no period-2 points in the dynamical systems; if $t \leq -\frac{3}{4}$, there are two period-2 points,

$$x_1 = -\frac{1}{2} + \sqrt{-\frac{3}{4} - t} \text{ and } x_2 = -\frac{1}{2} - \sqrt{-\frac{3}{4} - t}.$$

The proof is completed. □

In order to analyze properties of the period-2 points, let

$$q_c^{\pm} = -\frac{1}{2} \pm \sqrt{-\frac{3}{4} - t},$$

Then, we can get

$$\left|(f_t^2)'(q_t^{\pm})\right| = \left|f_t'(q_t^-)\right| \cdot \left|f_t'(q_t^+)\right|$$
$$= \left|2 \times \left(-\frac{1}{2} - \sqrt{-\frac{3}{4} - t}\right)\right| \cdot \left|2 \times \left(-\frac{1}{2} + \sqrt{-\frac{3}{4} - t}\right)\right|$$
$$= 4|1 + t|$$

As we know, for an iterative function $f(q_c^{\pm})$, if

$$\begin{array}{ll} \left|f(q_c^{\pm})'\right| < 1, & q_c^{\pm} \text{ are attractive,} \\ \left|f(q_c^{\pm})'\right| > 1, & q_c^{\pm} \text{ are repelling,} \\ \left|f(q_c^{\pm})'\right| = 1, & q_c^{\pm} \text{ are neuter [15].} \end{array}$$

According to Theorem 2 and [15], we can get a consequence as:

Consequence 1 If $-\frac{5}{4} < t < -\frac{3}{4}$, the periods are attractive;
if $t < -\frac{5}{4}$, the periods are repelling;

if $t = -\frac{3}{4}$ or $t = -\frac{5}{4}$, the periods are neuter.

The proof is easy to be obtained.

For short-lived time-delay Julia sets, if the points are in Julia sets after time-delay iterations, they will converge to the periodic points finally. Their initial points are in the Julia sets. On the contrary, if the points are out of Julia sets after time-delay iterations, they will escape in the end. Therefore, their initial points are not in the Julia sets.

For the sustained time-delay Julia sets, the abscissa axis and ordinate axis time delays are considered separately. When the time delay happens on the ordinate axis, the convergence of the iterative system is determined by the absolute value of abscissa, which is the initial value of time-delay iteration. If it is not greater than half, the iterative system is convergent, otherwise it is divergent. When the time delay happens on the abscissa axis, we should analyze the properties of the iterative system periodic points.

3 Algorithms design and numerical experiments

3.1 Time-delay iteration

Because the time delay may happen on all the iterations, without the loss of generality, we suppose it happens when the dynamical system is in initial, unstable, and stable statuses. Let k is the time of the iteration, n is the time when the time delay occurs. In our experiments, the time limit of the escape time algorithm is 300. The k is a loop variable of the algorithm; therefore, the range of k is [1, 300]. According to the experimental results, almost all the convergent initial points in complex plane will converge into Julia sets after 200 iterations, so we consider the dynamical system is stable at this moment. It is regarded as an unstable status when the graph is similar with a Julia set and only a part of it is fractal. The experimental results show that when n is in [15, 30], most of the dynamical systems are unstable. For the Julia set of $c = -0.08 + 0.68i$ or $c = 0.305 + 0.5475i$, the range of n is [20, 25]. In our experiments, the dynamical system is unstable when $n = 20$ and is initial when $n = 1$. For each status, we design the iteration as follows:

(1) Short-lived time delay

$k < n$, using Eq. (1) to calculate the value of (x_n, y_n).
$n \le k < n + 5$, using Eqs. (2) or (3).
$n \ge n + 5$, using Eq. (1).

(2) Sustained time delay

$k < n$, using Eq. (1) to calculate the value of (x_n, y_n).
$k \ge n$, using Eqs. (2) or (3).

3.2 Numerical experiments and analysis

For $f(z) = z^2 + c$, let $c = -0.08 + 0.68i$ or $c = 0.305 + 0.5475i$, R is the escape radius, and $R \ge \max(2, |c|)$. Figure 1 shows no time-delay Julia sets. There are a short-lived and a sustained time-delay Julia set on each couple of graphs (as shown in Figs. 2, 3, 4, 5). For short-lived time delay, the duration of the

Fig. 1 No time-delay Julia sets **a** $c = -0.08 + 0.68i$. **b** $c = 0.305 + 0.5475i$
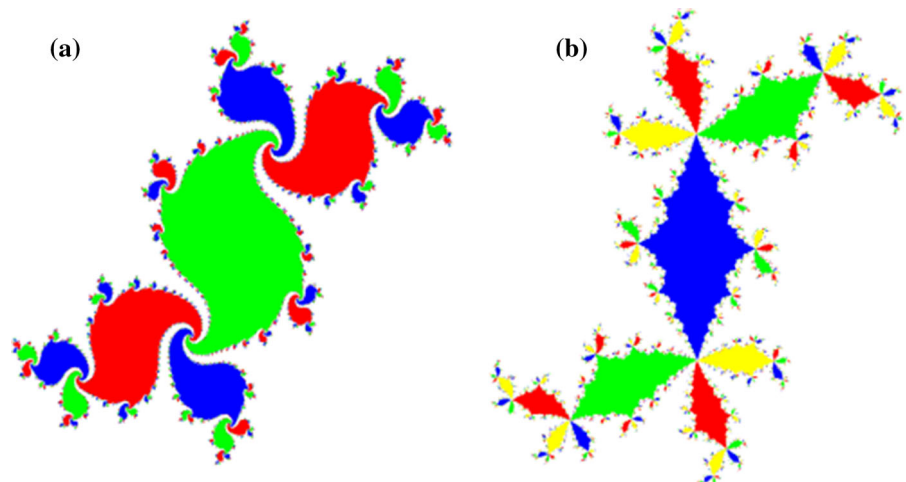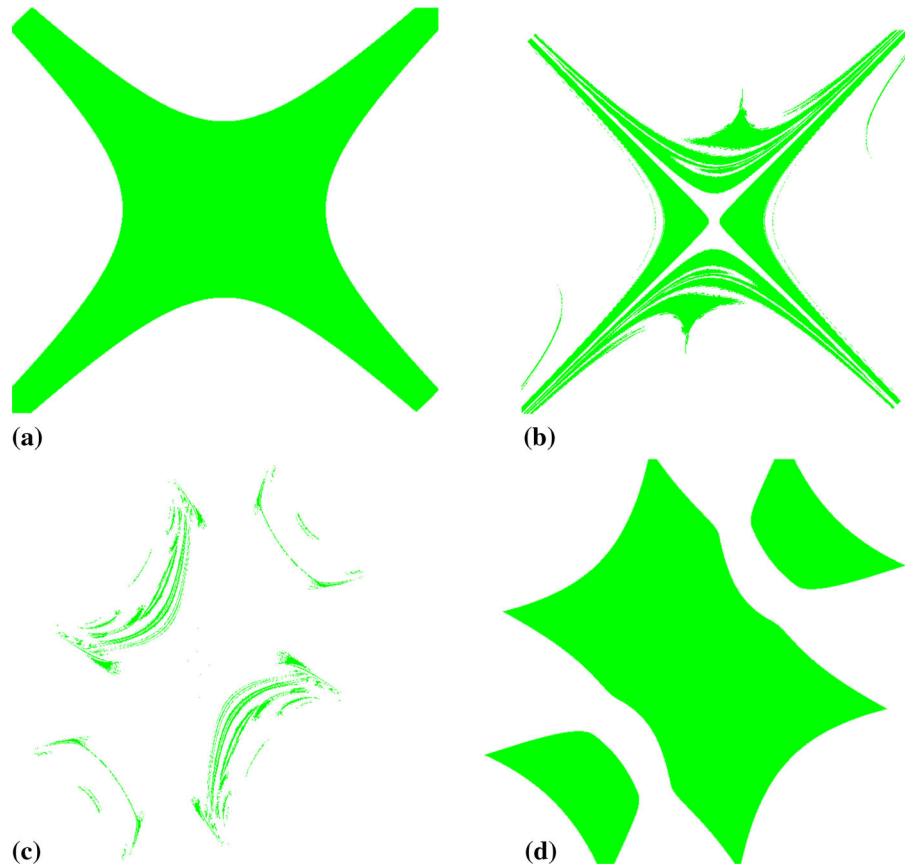
(a)

(b)

Fig. 2 Julia sets of
time-delay iterations which
happen on the initial status
($n = 1$) (*SLTD* short-lived
time delay, *STTD* sustained
time delay). **a** SLTD on
ordinate axis. **b** STTD on
ordinate axis. **c** SLTD on
abscissa axis. **d** STTD on
abscissa axis

(a)

(b)

(c)

(d)

time delay is five because most escaped points will go out of the Julia set after five iterations. However, if the delay time is too short, most of the points will be still in the Julia set and the time delay may have little impact on the Julia set.

Figure 2 shows the Julia sets of time-delay iterations which happen on the initial status. Note that they are not fractals. Figure 2a, b is about ordinate axis time-delay Julia sets; Fig. 2a is the Julia set with short-lived time-delay iterations; Fig. 2b is the sustained time-delay Julia set. Figure 2c, d is the abscissa axis time-delay Julia sets. When the time-delay iteration happens, the complex dynamical system changes into a real dynamical system and is greatly destroyed by the initial time-delay iterations. For the short-lived time delay, although it only sustains for five iterations at initial status of the dynamical system, it leads to a largely impact.

Figure 3 shows the Julia sets when time-delay iterations happen on the twentieth iteration. The results are similar to the traditional Julia sets. Some points in the

Julia sets escape periodically, because only a part of initial points converge to the periods after twenty iterations. Each part of the Julia set which is escaped after time-delay iterations is connected. There is no point which is escaped but whose any neighborhood point is convergent in the Julia sets. This is due to the fact that the points in Julia sets are dense. Dense point sets are still dense under one-dimensional real dynamical systems. There is no isolated point in time-delay Julia sets. Time-delay iteration destroys the fractal feature of the Julia set but does not change its density.

Figures 4 and 5 show the time-delay Julia sets at stable status on the ordinate and abscissa axis, respectively. The Julia sets in these pictures are fractals. Only periodic points are affected by the time-delay iteration after the dynamical system is stable. If a periodic point in the Julia set is escaped under the time-delay iteration, then all the initial points converging to the periodic point will escape.

For the abscissa axis time-delay Julia sets, according to Theorem 1, if the abscissa absolute value of a

Fig. 3 Julia sets of time-delay iterations which happen on unstable status ($n = 20$). **a** SLTD on ordinate axis. **b** STTD on ordinate axis. **c** SLTD on abscissa axis. **d** STTD on abscissa axis

(a)

(b)

(c)

(d)
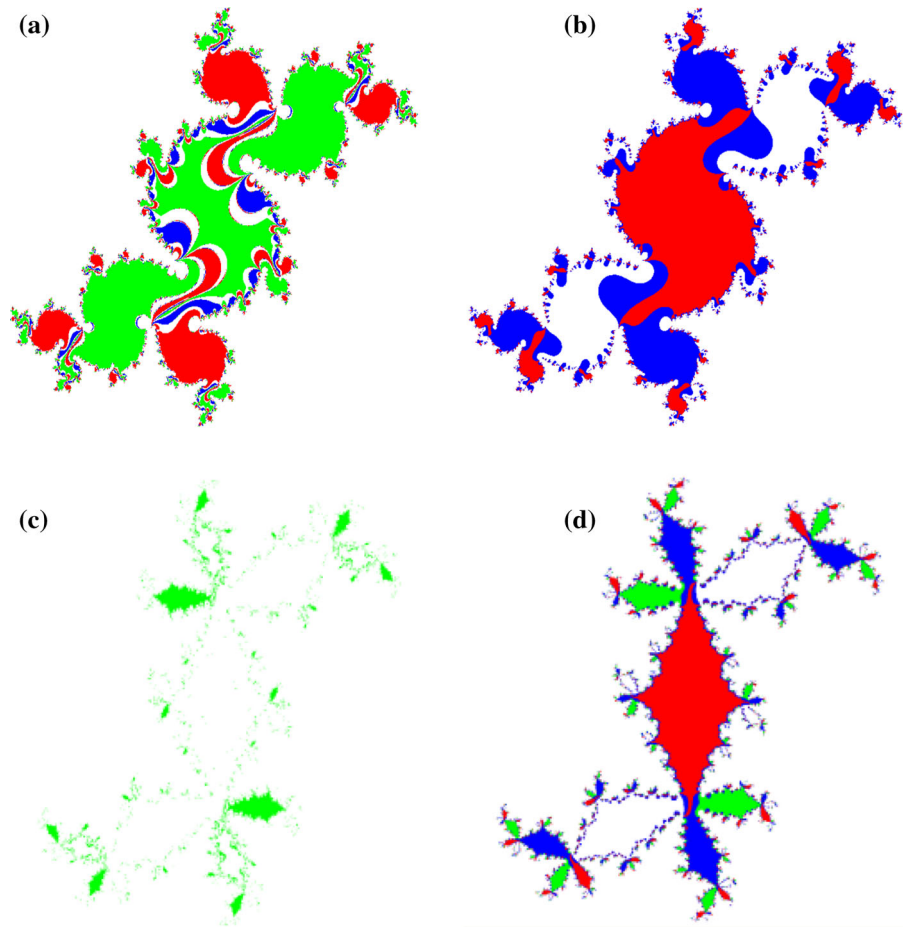
point is less than or equal to half, then the point will be convergent. When $c = -0.08 + 0.68i$, the periodic points are

$P_1(-0.045858, 0.668557)$,

$P_2(-0.524866, 0.618683)$,

$P_3(-0.187284, 0.030549)$.

The iterative functions starting with P_1 and P_3 are convergent. P_1 is much faster than P_3 when they converge to the stable points. P_2 is divergent under the sustained time-delay iteration. Because 0.524866 is near to a half, P_2 diverges slowly. In our experiment, the value of P_2 is 548,419.176875 after 300 iterations.

The time when an initial point converges to each periodic point of the Julia set is different [16]. For example, if a point converges to P_1 after 200 iterations, it will converges to P_2 after two 201 iterations, converges to P_3 after 202 iterations, and converges to P_1 after 203 iterations. Because the time-delay iterative

system which starts with P_1 or P_3 is converge, the point is convergent when the time delay happens on 200th or 202th iteration, it is divergent when the time delay happens on 201th iteration. That is why the petals of the Julia set are lost by periodicity.

For the ordinate axis time delay, the convergences of the iterative functions depend on the ordinate axis values when time-delay iterations start. For the Julia set with $c = 0.305 + 0.5475i$, its periodic points are

$Q_1(-0.093552, 0.87272)$,

$Q_2(-0.447889, 0.71079)$,

$Q_3(0.000382, -0.089209)$,

$Q_4(0.297042, 0.547432)$.

For $t_n = p - y_n^2, n = 1, 2, 3, 4$, then $t_1 = -0.456640$, $t_2 = -0.200222$, $t_3 = 0.2970417$, $t_4 = 0.005318$. t_1, t_2, and t_4 are $<1/4$. According to Consequence 1, Q_1, Q_2, and Q_4 converge to stable points. Since t_3 is $>1/4$, Q_3 is divergent.
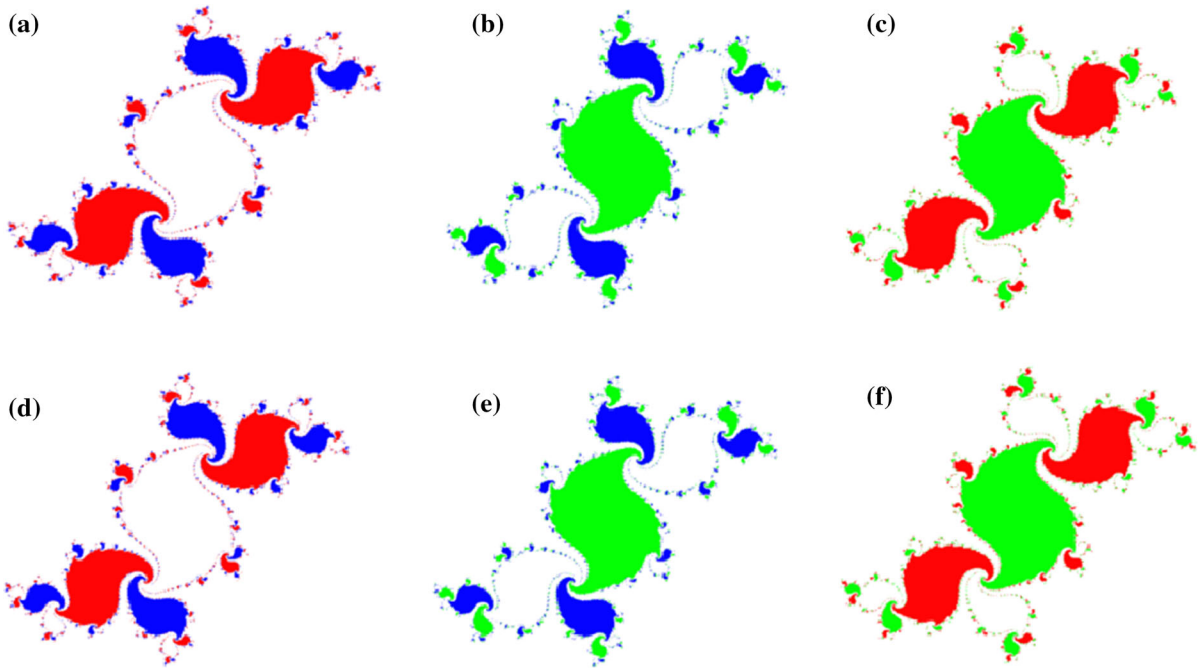
Fig. 4 Time-delay Julia sets on the ordinate axis at stable status ($n = 200$). **a** SLTD on 200th iteration. **b** SLTD on 201th iteration. **c** SLTD on 202th iteration. **d** STTD from 200th iteration. **e** STTD from 201th iteration. **f** STTD from 202th iteration
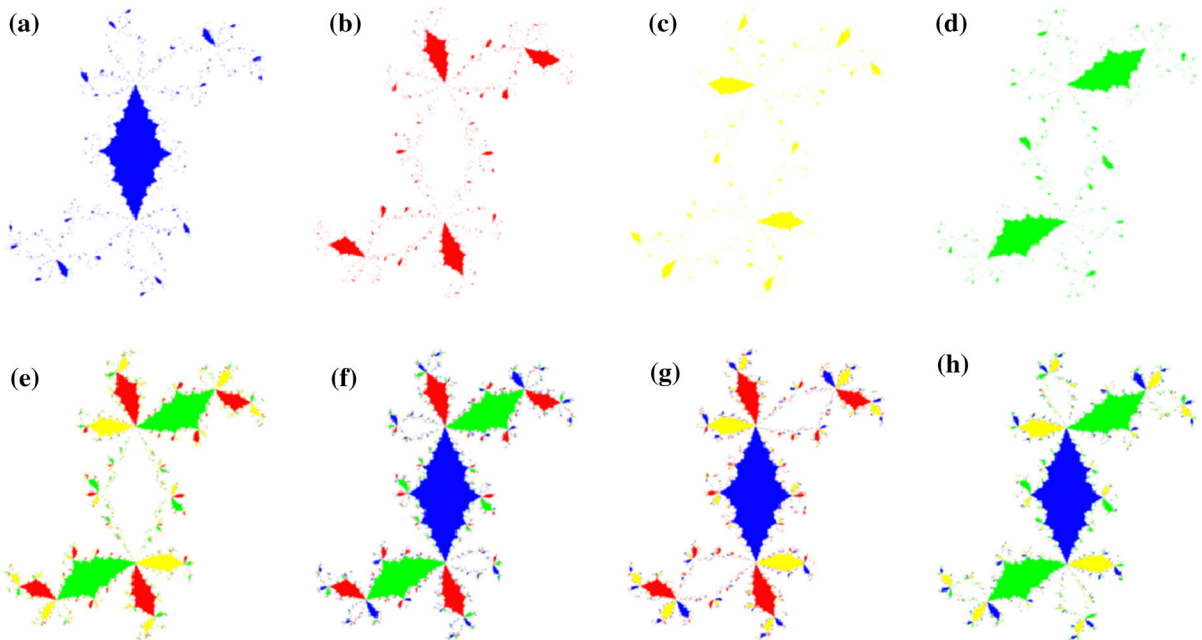

Fig. 5 Time-delay Julia sets on abscissa axis at stable status ($n = 200$). **a** SLTD on 200th iteration. **b** SLTD on 201th iteration. **c** SLTD on 202th iteration. **d** SLTD on 203th iteration. **e** STTD from 200th iteration. **f** STTD from 201th iteration. **g** STTD from 202th iteration. **h** STTD from 203th iteration

There is an interesting phenomenon, the lost petals change clockwise. It is because that the periodic points change counterclockwise. Our experiments have proved that the argument of each periodic point becomes larger under the complex iteration. So Q_3 follows by Q_2, does not follow by Q_1.

4 Conclusions

In the foregoing numerical analysis, time-delay Julia set is discussed. Time-delay iterations can destroy fractal feature of Julia sets when it happens on initial and unstable status. In some cases, the branches of the time-delay Julia sets may disappear in turn, which also shows some kind of periodicity.

For the abscissa axis time delay, the system can maintain stability if and only if the absolute value of abscissa is less than or equal to half.

For the ordinate axis time delay, if $p - y_n^2$ is on $(-\frac{5}{4}, \frac{3}{4})$, points are attractive; if $p - y_n^2 = -\frac{3}{4}$ or $p - y_n^2 = -\frac{5}{4}$ periods are neuter; if $p - y_n^2$ is on $(-\infty, -\frac{5}{4})$ and $(-\frac{3}{4}, +\infty)$, periods are repelling.

Although the time-delay iteration in this study is based on the mapping of a quadratic polynomial, the research methods can also be applied to other mappings, such as high-order polynomials. Not only does our study shed light on an example for the studies of the complex time-delay dynamical systems, it also provides a template for the analysis of this type of dynamical systems. Future work will focus on the mappings of general polynomials and more complex time-delay conditions.

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