Nonlinear Dyn (2014) 77:521–527
DOI 10.1007/s11071-014-1314-4

ORIGINAL PAPER

Conformal invariance and conserved quantity of Mei symmetry for Appell equations in a nonholonomic system of Chetaev's type

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Received: 19 November 2013 / Accepted: 13 February 2014 / Published online: 12 March 2014

Abstract For a nonholonomic system of Chetaev's type, the conformal invariance and the conserved quantity of Mei symmetry for Appell equations are investigated. First, under the infinitesimal one-parameter transformations of group and the infinitesimal generator vectors, Mei symmetry and conformal invariance of differential equations of motion for the system are defined, and the determining equation of Mei symmetry and conformal invariance for the system are given. Then, by means of the structure equation to which the gauge function is satisfied, the Mei-conserved quantity corresponding to the system is derived. Finally, an example is given to illustrate the application of the result.

Keywords Appell equation · Mei-conserved quantity · Conformal invariance · Nonholonomic system of Chetaev's type

1 Introduction

Seeking for symmetries and conserved quantities of dynamical systems is one of the ways by which people studies the laws of motion of dynamical systems to a much deeper levels. Research of symmetries and conserved quantities of constrained dynamical systems plays an important role in modern mechanical and mathematical sciences, and it is also a developing direction of modern mathematics, mechanics and physics [1–3]. Fruitful achievements have been gained in looking for conserved quantities by means of Noether symmetry, Lie symmetry and Mei symmetry [4–21]. Theories of the conformal invariance are classified into the gauge field theories in 1960s and 1970s, particularly are a hot topic of gravitational gauge field [22,23]. In 1997, Galiullin et al. had proposed concepts of conformal invariance and conformal factor of Birkhoff equations in the study of dynamics in Birkhoffian systems [23], and discussed the relationships between the invariance and the conformal invariance, Lie symmetry and the conformal invariance under the infinitesimal transformations of Pfaff action. Since entering the twenty-first century, Chinese scholars have carried out certain new researches and have gained some achievements in the study of symmetries and conserved quantities for mechanical systems by means of theories of the conformal invariance [24–35]. However, for a long time, there are fewer results to Appell equations. Especially at present there is no papers to present publicly the conformal invariance and conserved quantity of Mei symmetry of Appell equations expressed directly from Appell functions in nonholonomic systems of Chetaev's type. Reference [34] studied the conformal invariance and conserved quantity of Mei

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symmetry of Lagrange equations in nonholonomic systems of Chetaev's type. Reference [35] studied the conformal invariance and conserved quantity of Mei symmetry of Appell equations in holonomic systems. This paper will examine conformal invariance and Mei-conserved quantity of Mei symmetry of Appell equations in nonholonomic systems of Chetaev's type. First, Appell equations in nonholonomic systems of Chetaev's type are given; second, the infinitesimal one-parameter transformations of group and their generator vectors are introduced, the Mei symmetry and conformal invariance of dynamical equations in nonholonomic system of Chetaev's type are defined, and the determining equations of the conformal invariance of Mei symmetry in the system are given; finally, the corresponding Mei-conserved quantity is derived from the structure equation to which the gauge functions will be satisfied.

2 Appell equation and differential equations of motion for a nonholonomic system of Chetaev's type

Suppose that a nonholonomic mechanic system consists of N particles with masses m_i and its position vector r_i, respectively. And let the configuration of the system be determined by n generalised coordinates $q_s (s = 1, 2, \ldots, n)$, its motion is subject to the g ideal bilateral Chetaev nonholonomic constraints

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (\beta = 1, 2, \ldots, g), \tag{1}$$

and the restriction condition of constrains (1) imposed on virtual displacement and the energy of acceleration are, respectively,

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0 \quad (\beta = 1, 2, \ldots, g), \tag{2}$$

$$S = S(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \frac{1}{2} m_i \ddot{r}_i^2, \tag{3}$$

the uniform subscripts denote the Einstein summation convention [15] in (2) and (3) as well as in the following text. The Appell equations of the system can be expressed as follows:

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} = Q_s + \Gamma_s \quad (s = 1, 2, \ldots, n), \tag{4}$$

where λ_β is the βth multiplier corresponding to the constraints. Before the integration of differential equations of motion, the multipliers can be expressed as a function of t, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$: $\lambda_\beta = \lambda_\beta (t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the generalised forces corresponding to the sth generalised coordinate q_s. Γ_s is the constraint force corresponding to the sth generalised coordinate q_s

$$\Gamma_s = \Gamma_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s = 1, 2, \ldots, n), \tag{5}$$

let

$$\Lambda_s = \Lambda_s (t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = Q_s + \Gamma_s \quad (s = 1, 2, \ldots, n), \tag{6}$$

Λ_s is called the generalised force corresponding to the sth generalised coordinate q_s. Therefore, (4) can be rewritten simply as

$$\frac{\partial S}{\partial \ddot{q}_s} = \Lambda_s \quad (s = 1, 2, \ldots, n). \tag{7}$$

(7) is the Appell equations of a holonomic system corresponding to (1) and (4) for nonholonomic systems of Chetaev's type. If the initial condition of motion satisfies (1), the solution of (7) will give the orbit of motion of the system. By virtue of (7), one can solve all generalised accelerations, namely

$$\ddot{q}_s = \alpha_s (t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, 2, \ldots, n). \tag{8}$$

3 Conformal invariance of Mei symmetry for Appell equations in nonholonomic systems of Chetaev's type

Introducing the infinitesimal transformations of group of time and generalised coordinates

$$t^* = t + \Delta t,\ q_s^*(t^*) = q_s(t) + \Delta q_s \quad (s = 1, 2, \ldots, n), \tag{9}$$

and their expansions

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\ q_s^*(t^*) = q_s(t) + \varepsilon\xi_s (t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, 2, \ldots, n), \tag{10}$$

where ε is an infinitesimal parameter and ξ_0 and ξ_s are the generating functions of the infinitesimal transformations. Introducing a generating vector of the infinitesimal transformations

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}, \tag{11}$$

as well as its first expansion and its second expansion

$$\tilde{X}^{(1)} = X^{(0)} + \left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}_s},$$

$$\tilde{X}^{(2)} = \tilde{X}^{(1)} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \ddot{q}_s \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial \ddot{q}_s}, \tag{12}$$

where the total derivative along the trajectory of motion of the system with respect to t is

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial \mathrm{t}} + \dot{q}_s \frac{\partial}{\partial \mathrm{q}_s} + \alpha_s \frac{\partial}{\partial \dot{\mathrm{q}}_s} + \dot{\alpha}_s \frac{\partial}{\partial \ddot{\mathrm{q}}_s}. \tag{13}$$

From (10), we obtain

$$\begin{aligned} \frac{\mathrm{d}q_s^*}{\mathrm{d}t^*} &= \frac{\mathrm{d}q_s + \varepsilon \mathrm{d}\xi_s}{\mathrm{d}t + \varepsilon \mathrm{d}\xi_0} = \dot{q}_s + \varepsilon\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right) + O(\varepsilon^2), \\ \frac{\mathrm{d}^2 q_s^*}{\mathrm{d}t^{*2}} &= \ddot{q}_s + \varepsilon\left[\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)^{\cdot} - \ddot{q}_s\dot{\xi}_0\right] + O(\varepsilon^2). \end{aligned} \tag{14}$$

Suppose that after undergoing the infinitesimal transformations (10), the dynamic functions S, Λ_s and f_α of the system become S^*, Λ_s^* and f_α^*, respectively, note (7) and then take the Taylor expansions of S^*, Λ_s^* and f_α^* at the point of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, where the total derivative along the trajectory of the system with respect to t is expressed as (14), we have

$$\begin{aligned} S^* &= S\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right) = S(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) \\ &\quad + \varepsilon\left\{\frac{\partial S}{\partial t}\xi_0 + \frac{\partial S}{\partial q_s}\xi_s + \frac{\partial S}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right. \\ &\quad \left. + \frac{\partial S}{\partial \ddot{q}_s}[(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)^{\cdot} - \ddot{q}_s\dot{\xi}_0]\right\} + O(\varepsilon^2), \end{aligned}$$

namely

$$S^* = S(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(2)}(S) + O(\varepsilon^2), \tag{15}$$

$$\begin{aligned} \Lambda_s^* &= \Lambda_s^*\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(1)}(\Lambda_s) \\ &\quad + O(\varepsilon^2) \quad (s = 1, 2, \ldots, n), \end{aligned} \tag{16}$$

$$\begin{aligned} f_\beta^* &= f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = f_\beta\left(t, q, \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}\right) + \varepsilon \tilde{X}^{(1)}(f_\beta) \\ &\quad + O(\varepsilon^2) \quad (\beta = 1, 2, \ldots, g). \end{aligned} \tag{17}$$

Definition 1 If the form of Appell equations (7) for the system keeps invariant when the dynamical functions S and Λ_s are replaced by S^* and Λ_s^*, respectively, under the infinitesimal transformations (10), namely

$$\frac{\partial S^*}{\partial \ddot{q}_s} = \Lambda_s^* \quad (s = 1, 2, \ldots, n), \tag{18}$$

then the invariance is called the Mei symmetry of Appell equations (7) of a holonomic system corresponding to Appell equations (1) and (4) for the nonholonomic system of Chetaev's type.

Definition 2 If the form of Appell equations (7) for the system and the form of the constraint equations (1) keep invariant when the dynamical functions S and Λ_s are replaced by S^* and Λ_s^*, respectively, under the infinitesimal transformations (10), namely

$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = 0 \quad (\beta = 1, \ldots, g). \tag{19}$$

If (19) and (18) are both tenable, the symmetry is called the weak Mei symmetry of Appell equations (7) of a holonomic system corresponding to Appell equations (1) and (4) for the nonholonomic system of Chetaev's type.

Neglecting the terms of ε^2 and the higher order terms for (17), and using (1) and (19), the restriction equations of Mei symmetry for the nonholonomic constraint equations (1) under the infinitesimal transformations (10) are easily obtained as follows:

$$\tilde{X}^{(1)}\left\{f_\beta\left(t, \boldsymbol{q}, \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}\right)\right\} = 0 \quad (\beta = 1, \ldots, g). \tag{20}$$

If considering that the restriction on which the Chetaev condition equations (2) impose on the generating functions ξ_0 and ξ_s of the infinitesimal transformations, we get

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0 \quad (\beta = 1, \ldots, g;\ s = 1, \ldots, n), \tag{21}$$

(21) is called the additional restriction equations.

Definition 3 If the form of Appell equations (7) for the system and the form of the constraint equations (1) keep invariant when the dynamical functions S and Λ_s are replaced by S^* and Λ_s^*, respectively, under the infinitesimal transformations (10), and it requires that the generating functions ξ_0 and ξ_s of the infinitesimal transformations satisfy the restriction equations (20) and the additional restriction equations (21), then the symmetry is called the strict Mei symmetry of Appell equations (7) of a holonomic system corresponding to Appell equations (1) and (4) for the nonholonomic system of Chetaev's type.

Substituting (15) and (16) into (18), and neglecting the terms of ε^2 and the higher order terms, by virtue of Eq. (7), the determining equations of Mei symmetry of Appell equations (7) for the nonholonomic system of Chetaev's type result

$$\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right] - \tilde{X}^{(1)}(\Lambda_s) = 0. \tag{22}$$

Definition 4 For Appell equations (7) in the nonholonomic system of Chetaev's type, if there exists a matrix M_s^k to satisfy the following equations

$$\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_s) = M_s^k\left(\frac{\partial S}{\partial \ddot{q}_k}-\Lambda_k\right) \quad (s,k=1,2,\ldots,n). \tag{23}$$

Then Eq. (7) has the conformal invariance of Mei symmetry under the infinitesimal transformations (10). Eq. (23) is the determining equations for satisfying the conformal invariance of Mei symmetry, where M_s^k is the conformal factor.

Proposition 1 *If Eq. (7) has Mei symmetry under the infinitesimal transformations (10) and there exists a matrix M_s^k to satisfy*

$$\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_s) - \left\{\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_s)\right\}\Bigg|_{\frac{\partial S}{\partial \ddot{q}_s}=\Lambda_s} = \Gamma_s^k\left(\frac{\partial S}{\partial \ddot{q}_k}-\Lambda_k\right) \quad (s,k=1,2,\ldots,n). \tag{24}$$

Then the necessary and sufficient condition to which Eq. (7) has both the conformal invariance and the Mei symmetry under the infinitesimal transformations (10) is expressed as follows: $M_s^k=\Gamma_s^k$.

Proof Since the Mei symmetry of Eq. (7) satisfies Eq. (22), if there exists a matrix Γ_s^k satisfying (24), then (24) becomes

$$\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_s) = \Gamma_s^k\left(\frac{\partial S}{\partial \ddot{q}_k}-\Lambda_k\right) \quad (s,k=1,2,\ldots,n). \tag{25}$$

From the definition equation (23), the conformal factor of the system is $M_s^k=\Gamma_s^k$.

Vice versa, from the determining equations (23) and (24), it is easy to verify

$$\left(M_s^k-\Gamma_s^k\right)\left(\frac{\partial S}{\partial \ddot{q}_k}-\Lambda_k\right) = \left\{\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_s)\right\}\Bigg|_{\frac{\partial S}{\partial \ddot{q}_s}=\Lambda_s} \quad (s,k=1,2,\ldots,n). \tag{26}$$

If $M_s^k=\Gamma_s^k$, it is easily to obtain Eq. (22), hence the system has Mei symmetry.

4 Mei-conserved quantity deduced from the Mei symmetry in the system

According to the theory of Mei symmetry of Appell equation for a nonholonomic system of Chetaev's type, when the conformal invariance of Mei symmetry meets certain conditions, corresponding conserved quantities can also result.

Proposition 2 *If the infinitesimal generators ξ_0,ξ_s and the gauge function $G_M=G_M(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of the determining equation (22) of the Mei symmetry of Appell equations (7) of a holonomic system corresponding to Appell equations (1) and (4) for the nonholonomic system of Chetaev's type satisfy the following structure equation:*

$$\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right]+(\xi_s-\dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{X}^{(2)}(S)\right]+\xi_0\left[\tilde{X}^{(1)}(\Lambda_s)\right]\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}=0. \tag{27}$$

Then the Mei-conserved quantities deduced from Mei symmetry of Eq. (7) may be expressed as follows:

$$I_M=\xi_0\tilde{X}^{(2)}(S)+\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\mathrm{M}}=\text{const.} \tag{28}$$

Proof Using Eq. (7) and the determining equation (23) of the conformal invariance of Mei symmetry, we obtain

$$\frac{\partial}{\partial \ddot{q}_s}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_s)=M_s^k\left(\frac{\partial S}{\partial \ddot{q}_k}-\Lambda_k\right)=0. \tag{29}$$

Therefore, we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t} &= \left[\frac{\partial\tilde{X}^{(2)}(S)}{\partial t}+\dot{q}_s\frac{\partial\tilde{X}^{(2)}(S)}{\partial q_s}+\alpha_s\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}\right.\\ &\quad \left.+\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s}\frac{\bar{d}\alpha_s}{dt}\right]\xi_0+\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\\ &\quad +\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}\right](\xi_s-\dot{q}_s\xi_0)+\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}\\ &\quad \times\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\alpha_s\xi_0-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)+\frac{\bar{\mathrm{d}}G_M}{\mathrm{d}t}.\end{aligned} \tag{30}$$

Note that

$$\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right]=\xi_0\frac{\partial\tilde{X}^{(2)}(S)}{\partial t}+\xi_s\frac{\partial\tilde{X}^{(2)}(S)}{\partial q_s}+\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}.$$

Then (30) becomes

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t}&=\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right]-\xi_s\frac{\partial\tilde{X}^{(2)}(S)}{\partial q_s}\\&\quad+\xi_0\dot{q}_s\frac{\partial\tilde{X}^{(2)}(S)}{\partial q_s}+\xi_0\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{dt}\\&\quad+\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}\right]\\&\quad(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}\\&=\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right]-(\xi_s-\dot{q}_s\xi_0)\frac{\partial\tilde{X}^{(2)}(S)}{\partial q_s}\\&\quad+\xi_0\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}+\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\\&\quad+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{X}^{(2)}(S)}{\partial\dot{q}_s}\right](\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}.\end{aligned}\tag{31}$$

Substituting (27) into (32), we obtain

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_M}{\mathrm{d}t}&=\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right]+(\xi_s-\dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{X}^{(2)}(S)\right]\\&\quad+\xi_0\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}+\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}+\frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}\\&=\xi_0\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\left[\frac{\partial\tilde{X}^{(2)}(S)}{\partial\ddot{q}_s}-\tilde{X}^{(1)}(\Lambda_s)\right]=0.\end{aligned}\tag{32}$$

5 Conformal invariance of Mei symmetry of Appell equations for a two-dimensional nonholonomic system of Chetaev's type

Assume that the acceleration energy, the constraint equations and the generalised forces for a two-dimensional nonholonomic system of Chetaev's type are, respectively,

$$S=\frac{1}{2}\left(\ddot{q}_1^2+\ddot{q}_2^2\right),\tag{33}$$

$$f=\dot{q}_1+t\dot{q}_2-q_2+t^2=0,\tag{34}$$

$$Q_1=Q_2=0.\tag{35}$$

Study the conformal invariance and the conserved quantity of Mei symmetry of Appell equations for the system.

Expanding (4), and taking notice of (7) and (33), we have

$$\begin{aligned}\ddot{q}_1&=\Lambda_1=\lambda\\\ddot{q}_2&=\Lambda_2=\lambda t\end{aligned}\tag{36}$$

Solve simultaneous Eqs. (34) and (36), we obtain

$$\lambda=-\frac{2t}{1+t^2}\tag{37}$$

Noticing (8), and substituting (37) into (36), we get

$$\begin{aligned}\ddot{q}_1&=\alpha_1=\Lambda_1=-\frac{2t}{1+t^2},\\\ddot{q}_2&=\alpha_2=\Lambda_2=-\frac{2t^2}{1+t^2}.\end{aligned}\tag{38}$$

(38) may also be rewritten as

$$\begin{aligned}\frac{\partial S}{\partial\ddot{q}_1}-\Lambda_1&=\ddot{q}_1+\frac{2t}{1+t^2}=0,\\\frac{\partial S}{\partial\ddot{q}_2}-\Lambda_2&=\ddot{q}_2+\frac{2t^2}{1+t^2}=0.\end{aligned}\tag{39}$$

Taking infinitesimal transformation generators as follows:

$$\begin{aligned}&\xi_0=0,\quad\xi_1=-t\dot{q}_1+\dot{q}_2+q_1+t,\\&\xi_2=t\dot{q}_2+\dot{q}_1-q_2+t^2.\end{aligned}\tag{40}$$

Taking the calculation, we have

$$\begin{aligned}\frac{\partial}{\partial\ddot{q}_1}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_1)&=-\ddot{q}_1-\frac{2t}{1+t^2},\\\frac{\partial}{\partial\ddot{q}_2}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_2)&=\ddot{q}_2+\frac{2t^2}{1+t^2}.\end{aligned}\tag{41}$$

Hence, from (23), (39) and (41), the conformal factor is obtained as follows:

$$\Gamma=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

Substituting (38) into (41), we obtain

$$\begin{aligned}\frac{\partial}{\partial\ddot{q}_1}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_1)&=0,\\\frac{\partial}{\partial\ddot{q}_2}\left[\tilde{X}^{(2)}(S)\right]-\tilde{X}^{(1)}(\Lambda_2)&=0.\end{aligned}\tag{42}$$

Thus, the system satisfies the Mei symmetry. The system is both the conformal invariance and the Mei symmetry in this case.

$$\tilde{X}^{(2)}(S) = \ddot{q}_1 \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t}\right) + \ddot{q}_2 \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t}\right)$$
$$= -\ddot{q}_1^2 + \ddot{q}_2^2 - \ddot{q}_1 \frac{2t}{1+t^2} + \ddot{q}_2 \frac{2t^2}{1+t^2}, \quad (43)$$

$$\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right] = 0, \quad (44)$$

$$\tilde{E}_1\left[\tilde{X}^{(2)}(S)\right] = 0, \quad (45)$$

$$\tilde{E}_2\left[\tilde{X}^{(2)}(S)\right] = 0. \quad (46)$$

By using (43)–(46), it is easy to verify the determining equations (22) and the constraint equation (20) are tenable; substituting (40) and (34) into (21) shows that the additional restrictions equation (21) holds. Hence, from definitions 1–3, the infinitesimal transformation generators expressed by (40) are the infinitesimal transformation generators of Mei symmetry and the strict Mei symmetry for the system. Hence, the system has the strict Mei symmetry.

From the structure equation (27), we get

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} G_{\mathrm{M}} = 0. \quad (47)$$

By using (38), we have

$$G_{\mathrm{M}} = \dot{q}_1 + \dot{q}_2 + \ln(1+t^2) + 2t - 2\arctan t = \text{const}. \quad (48)$$

By using (28), the Mei-conserved quantity deduced directly from conformal invariance of Mei symmetry of the system gives

$$I_{\mathrm{M}} = G_{\mathrm{M}} = \dot{q}_1 + \dot{q}_2 + \ln(1+t^2) + 2t - 2\arctan t = \text{const}. \quad (49)$$

6 Conclusion

Conformal invariance is a kind of symmetry which has universal significance and a wider range of application. This paper presents the determining equations of conformal invariance of Mei symmetry of Appell equations for nonholonomic systems of Chetaev's type, and the Mei-conserved quantity of the system is derived by using gauge functions. The conclusions of this paper not only enrich the theory of symmetry and conserved quantity of Appell equations, but also for the first time the theory of conformal invariance and conserved quantity of Mei symmetry of Appell equations for nonholonomic systems of Chetaev's type expressed directly by Appell functions is obtained.

Acknowledgments Project was supported by the National Natural Science Foundation of China (Grant No. 11142014) and Project was supported by the scientific research and innovation plan for College Graduates of Jiangsu province (Grant No. CXLX12_0720).

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