

Solving systems of nonlinear difference equations by the multiple scales perturbation method

M. Rafei · W.T. Van Horssen

Received: 22 September 2011 / Accepted: 9 February 2012 / Published online: 29 February 2012
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Abstract In this paper, we apply an improved version of the multiple scales perturbation method to a system of weakly nonlinear, regularly perturbed ordinary difference equations. Such systems arise as a result of the discretization of a system of nonlinear differential equations, or as a result in the stability analysis of nonlinear oscillations. In our procedure, asymptotic approximations of the solutions of the difference equations will be constructed which are valid on long iteration scales.

Keywords Nonlinear difference equation · Multiple scales perturbation method · Difference operator

1 Introduction

For scientists and engineers, the analysis of nonlinear dynamical systems is an important field of research since the solutions of these systems can exhibit counterintuitive and sometimes unexpected behavior. To obtain useful information from these systems, the multiple scales perturbation method can play an important role.

Nowadays, the multiple time-scales perturbation method for differential equations is well developed, well accepted, and a very popular method to approximate solutions of weakly nonlinear differential equations. For difference equations, this perturbation method is recently improved by Van Horssen [1] such that it can be applied to a large class of problems. In [1], a version of the multiple scales perturbation method is presented in a complete “difference operator” setting. This method can for instance be applied to problems for systems with time-varying masses. Examples of such systems can be found in robotics, rotating crankshafts, conveyor systems, excavators, cranes, biomechanics, and in fluid-structure interaction problems [2, 3]. The oscillations of electric transmission lines and cables of cable-stayed bridges with water rivulets on the surface are also examples of time-varying dynamic systems [4]. For these mechanical constructions, the 1-mode Galerkin approximation of the continuous model will lead to a single degree-of-freedom oscillator (sdofo)-equation. These sdofo are considered to be representative models for testing numerical methods and for studying forces which are acting on the system [5]. In [6] and [7], the forced vibrations of a linear sdofo with a time-varying mass were studied. The forced vibrations are due to small masses which are periodically hitting and leaving the oscillator with different velocities. In [8], the vibrations of a damped, linear sdofo with a time-varying mass were studied, and the stability properties for the free, and for the forced vibrations (due to small masses and an

M. Rafei (✉) · W.T. Van Horssen
Delft Institute of Applied Mathematics (DIAM), Delft
University of Technology, Mekelweg 4, 2628 CD Delft,
The Netherlands
e-mail: M.Rafei@tudelft.nl

W.T. Van Horssen
e-mail: W.T.VanHorssen@tudelft.nl

external force (for instance, a windforce)) were presented for various parameter values. A system of two nonlinear ordinary difference equations (ODEs) is obtained when also windforces are included in the model. To analyze the system of ODEs, numerical methods were used. In the analysis, a small parameter ε was defined for the relative mass which is added periodically. Then, also a perturbation method can be applied. In this paper, we are going to study similar systems of ODEs. It will be shown how the improved multiple scales perturbation method can be applied to such systems of ODEs. Moreover, several bifurcation problems will be studied in detail.

2 Problem definition

In this paper, we consider a weakly nonlinear perturbed system of two first-order ODEs. A nice representation for such a system can be written as

$$\begin{cases} x_{n+1} = f_{1,0}(x_n, y_n) + \varepsilon f_{1,1}(x_n, y_n) + O(\varepsilon^2), \\ y_{n+1} = f_{2,0}(x_n, y_n) + \varepsilon f_{2,1}(x_n, y_n) + O(\varepsilon^2), \end{cases} \quad (1)$$

where

$$\begin{cases} f_{1,0}(x_n, y_n) = a_{11}x_n + a_{12}y_n, \\ f_{2,0}(x_n, y_n) = a_{21}x_n + a_{22}y_n, \\ f_{1,1}(x_n, y_n) = b_{10} + b_{11}x_n + b_{12}y_n + b_{13}x_n^2 \\ \quad + b_{14}x_ny_n + b_{15}y_n^2 + b_{16}x_n^3 \\ \quad + b_{17}x_n^2y_n + b_{18}x_ny_n^2 + b_{19}y_n^3, \\ f_{2,1}(x_n, y_n) = b_{20} + b_{21}x_n + b_{22}y_n + b_{23}x_n^2 \\ \quad + b_{24}x_ny_n + b_{25}y_n^2 + b_{26}x_n^3 \\ \quad + b_{27}x_n^2y_n + b_{28}x_ny_n^2 + b_{29}y_n^3, \end{cases} \quad (2)$$

and where a_{ij} and b_{ij} are real constants, and $0 < \varepsilon \ll 1$.

In our work, by using the multiple scales perturbation method in terms of difference operators, we will obtain secular-free solutions of system (1) which can be used to analyze, for example, systems with time-varying masses. Furthermore, our analysis will give a complete bifurcation analysis of system (1) when the eigenvalues of matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad (3)$$

are complex valued. This analysis strongly depends on the eigenvalues of the problem when $\varepsilon = 0$. The most interesting case from the applicational point of view is the case when the eigenvalues of matrix A are complex valued with nonzero real and imaginary parts, and are in modulus smaller than or equal to 1. The analysis in this paper will be restricted to this case.

3 The multiple scales perturbation method for ODEs

In this section, the multiple scales perturbation method for ODEs will be presented in a complete “difference operator” setting. Before introducing this method, several operators have to be defined (and motivated). The well-known shift operator E , the difference operator Δ , and the identity operator I are defined as follows:

$$\begin{aligned} Ex_n &= x_{n+1}, \quad \Delta x_n = x_{n+1} - x_n, \quad \text{and} \\ Ix_n &= x_n. \end{aligned} \quad (4)$$

The relationship between these operators easily follows from (4):

$$E = \Delta + I \quad \Leftrightarrow \quad \Delta = E - I. \quad (5)$$

The solution of a weakly perturbed ODE usually contains a rapidly changing part in n , and a slowly changing part in n . This is usually referred to as multiple scales behavior. It should be observed that these notations are similar to the ones used in the multiple timescales perturbation method for ODEs. Now it is assumed that $x_n = x(n, \varepsilon n)$. This assumption implies that the solution of the ODE depends on two variables. So, the ODE actually becomes a partial difference equation. For that reason also, partial shift operators and partial difference operators have to be defined. The following definitions are proposed [1]:

$$\begin{aligned} E_1 x(n, \varepsilon n) &= x(n+1, \varepsilon n), \\ E_\varepsilon x(n, \varepsilon n) &= x(n, \varepsilon(n+1)), \\ \Delta_1 x(n, \varepsilon n) &= x(n+1, \varepsilon n) - x(n, \varepsilon n) \\ &= (E_1 - I)x(n, \varepsilon n), \\ \Delta_\varepsilon x(n, \varepsilon n) &= x(n, \varepsilon(n+1)) - x(n, \varepsilon n) \\ &= (E_\varepsilon - I)x(n, \varepsilon n). \end{aligned} \quad (6)$$

From (4), (5), and (6), it follows that (assuming $x_n = x(n, \varepsilon n)$):

$$\begin{aligned}\Delta x_n &= x_{n+1} - x_n = x(n+1, \varepsilon(n+1)) - x(n, \varepsilon n) \\ &= E_1 E_\varepsilon x(n, \varepsilon n) - Ix(n, \varepsilon n) \\ &= (\Delta_1 + I)(\Delta_\varepsilon + I)x(n, \varepsilon n) - Ix(n, \varepsilon n) \\ &= (\Delta_1 + \Delta_\varepsilon + \Delta_1 \Delta_\varepsilon)x(n, \varepsilon n).\end{aligned}\quad (7)$$

And so, it follows that

$$\Delta = \Delta_1 + \Delta_\varepsilon + \Delta_1 \Delta_\varepsilon, \quad \text{and} \quad E = E_1 E_\varepsilon. \quad (8)$$

Furthermore, for the partial difference operators Δ_1 and Δ_ε it is assumed that [1]:

$$\begin{aligned}\Delta_1 x(n, \varepsilon n) &= O(x(n, \varepsilon n)), \quad \text{and} \\ \Delta_\varepsilon x(n, \varepsilon n) &= O(\varepsilon x(n, \varepsilon n)).\end{aligned}\quad (9)$$

In fact, this assumption (9) implies that the variation in the dependent variable $x(n, \varepsilon n)$ with respect to one of the independent variables is proportional to the product of the absolute value of the dependent variable and the variation in that particular independent variable.

When x_n depends on $m+1$ scales, the given definitions can readily be generalized, yielding (for $j = 0, 1, \dots, m$):

$$\begin{aligned}x_n &= x(n, \varepsilon n, \varepsilon^2 n, \dots, \varepsilon^m n), \\ E_{\varepsilon^j} x(n, \dots, \varepsilon^m n) &= x(n, \varepsilon n, \dots, \varepsilon^j(n+1), \dots, \varepsilon^m n), \\ \Delta_{\varepsilon^j} x(n, \dots, \varepsilon^m n) &= (E_{\varepsilon^j} - I)x(n, \dots, \varepsilon^m n), \quad (10) \\ E &= E_1 E_\varepsilon E_{\varepsilon^2} \cdots E_{\varepsilon^m}, \\ \Delta &= (\Delta_1 + I)(\Delta_\varepsilon + I) \cdots (\Delta_{\varepsilon^m} + I) - I, \\ \Delta_{\varepsilon^j} x(n, \dots, \varepsilon^m n) &= O(\varepsilon^j x(n, \dots, \varepsilon^m n)).\end{aligned}$$

Now it will be shown how these operators can be used for system (1). Using (4) and (5), it follows that (1) can be rewritten in:

$$\begin{cases} Ex_n = f_{1,0}(x_n, y_n) + \varepsilon f_{1,1}(x_n, y_n) + O(\varepsilon^2), \\ Ey_n = f_{2,0}(x_n, y_n) + \varepsilon f_{2,1}(x_n, y_n) + O(\varepsilon^2). \end{cases} \quad (11)$$

Assuming that x_n and y_n depend on two scales (a fast scale n , and a slow scale εn), it follows that $x_n = x(n, \varepsilon n)$ and $y_n = y(n, \varepsilon n)$. By using (8), (11)

becomes

$$\begin{cases} (\Delta_1 + \Delta_\varepsilon + \Delta_1 \Delta_\varepsilon + I)x_n \\ \quad = f_{1,0}(x_n, y_n) + \varepsilon f_{1,1}(x_n, y_n) + O(\varepsilon^2), \\ (\Delta_1 + \Delta_\varepsilon + \Delta_1 \Delta_\varepsilon + I)y_n \\ \quad = f_{2,0}(x_n, y_n) + \varepsilon f_{2,1}(x_n, y_n) + O(\varepsilon^2). \end{cases} \quad (12)$$

To construct an approximation for x_n and y_n , one now has to substitute into (12) a formal power series (in ε) for x_n and y_n , that is,

$$\begin{cases} x_n = x(n, \varepsilon n) = x_0(n, \varepsilon n) + \varepsilon x_1(n, \varepsilon n) \\ \quad + \varepsilon^2 x_2(n, \varepsilon n) + \cdots, \\ y_n = y(n, \varepsilon n) = y_0(n, \varepsilon n) + \varepsilon y_1(n, \varepsilon n) \\ \quad + \varepsilon^2 y_2(n, \varepsilon n) + \cdots. \end{cases} \quad (13)$$

Then, by taking together those terms of equal powers in ε , one obtains as $O(1)$ -problem

$$\begin{cases} (\Delta_1 + I)x_0 = f_{1,0}(x_0, y_0), \\ (\Delta_1 + I)y_0 = f_{2,0}(x_0, y_0), \end{cases} \quad (14)$$

where $x_0 = x_0(n, \varepsilon n)$ and $y_0 = y_0(n, \varepsilon n)$, and as $O(\varepsilon)$ -problem

$$\begin{cases} \varepsilon((\Delta_1 + I)x_1 - f_{1,0}(x_1, y_1)) \\ \quad = \varepsilon f_{1,1}(x_0, y_0) - (\Delta_\varepsilon + \Delta_1 \Delta_\varepsilon)x_0, \\ \varepsilon((\Delta_1 + I)y_1 - f_{2,0}(x_1, y_1)) \\ \quad = \varepsilon f_{2,1}(x_0, y_0) - (\Delta_\varepsilon + \Delta_1 \Delta_\varepsilon)y_0, \end{cases} \quad (15)$$

where $x_1 = x_1(n, \varepsilon n)$, $y_1 = y_1(n, \varepsilon n)$.

4 Two complex eigenvalues

When the $O(1)$ -problem has two complex valued eigenvalues (namely, λ_1 and λ_2), we restrict ourselves to two cases for r : $0 < r < 1$, or $r = 1$, where r is the absolute value of the complex eigenvalues. It follows from (14) that

$$\begin{pmatrix} x_0(n, \varepsilon n) \\ y_0(n, \varepsilon n) \end{pmatrix} = g_0(\varepsilon n) \begin{pmatrix} c_1 \\ 1 \end{pmatrix} \lambda_1^n + h_0(\varepsilon n) \begin{pmatrix} c_2 \\ 1 \end{pmatrix} \lambda_2^n, \quad (16)$$

where $c_1 = \frac{\lambda_1 - a_{22}}{a_{21}}$ and $c_2 = \frac{\lambda_2 - a_{22}}{a_{21}}$ are complex conjugates, and where $g_0(\varepsilon n)$ and $h_0(\varepsilon n)$ are still arbitrary functions which can be used to avoid unbounded

behavior in $x_1(n, \varepsilon n)$ and $y_1(n, \varepsilon n)$ on the $O(\frac{1}{\varepsilon})$ iteration scale. Then, by substituting (16) into the $O(\varepsilon)$ -

problem (15), and after rearranging terms, one finally obtains as $O(\varepsilon)$ -problem

$$\begin{aligned} \begin{pmatrix} \varepsilon((\Delta_1 + I)x_1 - f_{1,0}(x_1, y_1)) \\ \varepsilon((\Delta_1 + I)y_1 - f_{2,0}(x_1, y_1)) \end{pmatrix} &= \begin{pmatrix} M_{10} \\ M_{20} \end{pmatrix} + \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \lambda_1^n + \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix} \lambda_2^n + \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix} \lambda_1^{2n} + \begin{pmatrix} M_{14} \\ M_{24} \end{pmatrix} \lambda_1^n \lambda_2^n \\ &+ \begin{pmatrix} M_{15} \\ M_{25} \end{pmatrix} \lambda_2^{2n} + \begin{pmatrix} M_{16} \\ M_{26} \end{pmatrix} \lambda_1^{3n} + \begin{pmatrix} M_{17} \\ M_{27} \end{pmatrix} \lambda_1^{2n} \lambda_2^n + \begin{pmatrix} M_{18} \\ M_{28} \end{pmatrix} \lambda_1^n \lambda_2^{2n} + \begin{pmatrix} M_{19} \\ M_{29} \end{pmatrix} \lambda_2^{3n}, \end{aligned} \quad (17)$$

where M_{ij} (for $i = 1, 2$, and $j = 0, \dots, 9$) are given by

$$\begin{aligned} \begin{pmatrix} M_{10} \\ M_{20} \end{pmatrix} &= \varepsilon \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix}, \\ \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} &= \begin{pmatrix} \varepsilon(b_{11}c_1 + b_{12})g_0 - c_1\lambda_1\Delta_\varepsilon g_0(\varepsilon n) \\ \varepsilon(b_{21}c_1 + b_{22})g_0 - \lambda_1\Delta_\varepsilon g_0(\varepsilon n) \end{pmatrix}, \\ \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix} &= \begin{pmatrix} \varepsilon(b_{11}c_2 + b_{12})h_0 - c_2\lambda_2\Delta_\varepsilon h_0(\varepsilon n) \\ \varepsilon(b_{21}c_2 + b_{22})h_0 - \lambda_2\Delta_\varepsilon h_0(\varepsilon n) \end{pmatrix}, \\ \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix} &= \varepsilon \begin{pmatrix} b_{13}c_1^2 + b_{14}c_1 + b_{15} \\ b_{23}c_1^2 + b_{24}c_1 + b_{25} \end{pmatrix} g_0^2, \\ \begin{pmatrix} M_{14} \\ M_{24} \end{pmatrix} &= \varepsilon \begin{pmatrix} 2b_{13}c_1c_2 + b_{14}(c_1 + c_2) + 2b_{15} \\ 2b_{23}c_1c_2 + b_{24}(c_1 + c_2) + 2b_{25} \end{pmatrix} g_0h_0, \\ \begin{pmatrix} M_{15} \\ M_{25} \end{pmatrix} &= \varepsilon \begin{pmatrix} b_{13}c_2^2 + b_{14}c_2 + b_{15} \\ b_{23}c_2^2 + b_{24}c_2 + b_{25} \end{pmatrix} h_0^2, \\ \begin{pmatrix} M_{16} \\ M_{26} \end{pmatrix} &= \varepsilon \begin{pmatrix} b_{16}c_1^3 + b_{17}c_1^2 + b_{18}c_1 + b_{19} \\ b_{26}c_1^3 + b_{27}c_1^2 + b_{28}c_1 + b_{29} \end{pmatrix} g_0^3, \\ \begin{pmatrix} M_{17} \\ M_{27} \end{pmatrix} &= \varepsilon \begin{pmatrix} 3b_{16}c_1^2c_2 + b_{17}c_1(c_1 + 2c_2) + b_{18}(2c_1 + c_2) + 3b_{19} \\ 3b_{26}c_1^2c_2 + b_{27}c_1(c_1 + 2c_2) + b_{28}(2c_1 + c_2) + 3b_{29} \end{pmatrix} g_0^2h_0, \\ \begin{pmatrix} M_{18} \\ M_{28} \end{pmatrix} &= \varepsilon \begin{pmatrix} 3b_{16}c_1c_2^2 + b_{17}c_2(2c_1 + c_2) + b_{18}(c_1 + 2c_2) + 3b_{19} \\ 3b_{26}c_1c_2^2 + b_{27}c_2(2c_1 + c_2) + b_{28}(c_1 + 2c_2) + 3b_{29} \end{pmatrix} g_0h_0^2, \\ \begin{pmatrix} M_{19} \\ M_{29} \end{pmatrix} &= \varepsilon \begin{pmatrix} b_{16}c_2^3 + b_{17}c_2^2 + b_{18}c_2 + b_{19} \\ b_{26}c_2^3 + b_{27}c_2^2 + b_{28}c_2 + b_{29} \end{pmatrix} h_0^3, \end{aligned} \quad (18)$$

where $g_0 = g_0(\varepsilon n)$ and $h_0 = h_0(\varepsilon n)$. In system (17) for $x_1(n, \varepsilon n)$ and $y_1(n, \varepsilon n)$, it is obvious that the right-hand side contains terms (i.e., multiples of $(c_1, 1)^T \lambda_1^n$ and of $(c_2, 1)^T \lambda_2^n$, where T refers to the transpose of the matrix), which are solutions of the homogeneous system. For the case $0 < r < 1$, it follows from (17) that only two vectors $(M_{11}, M_{21})^T \lambda_1^n$ and $(M_{12}, M_{22})^T \lambda_2^n$ contain secular, and also nonsecular

terms. Therefore, we decompose the sum of these vectors into two linearly independent directions to separate secular and nonsecular terms,

$$\begin{aligned} &\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \lambda_1^n + \begin{pmatrix} M_3 \\ M_4 \end{pmatrix} \lambda_2^n \\ &= \left\{ m_1 \begin{pmatrix} c_1 \\ 1 \end{pmatrix} + m_2 \begin{pmatrix} c_2 \\ 1 \end{pmatrix} \right\} \lambda_1^n \end{aligned}$$

Table 1 Values of M_k (for $k = 1, \dots, 4$) in (19) for some representations of the absolute value of the eigenvalues λ_1 and λ_2

r	$M_1 =$	$M_2 =$	$M_3 =$	$M_4 =$
$0 < r < 1$	M_{11}	M_{21}	M_{12}	M_{22}
$r = 1$	$M_{11} + M_{17}$	$M_{21} + M_{27}$	$M_{12} + M_{18}$	$M_{22} + M_{28}$

$$+ \left\{ m_3 \begin{pmatrix} c_1 \\ 1 \end{pmatrix} + m_4 \begin{pmatrix} c_2 \\ 1 \end{pmatrix} \right\} \lambda_2^n, \quad (19)$$

where M_k (for $k = 1, \dots, 4$) are obtained based on the relationship between the eigenvalues λ_1 and λ_2 , see Table 1. From (19), we obtain

$$\begin{cases} m_1 = \frac{c_2 M_2 - M_1}{c_2 - c_1}, \\ m_2 = \frac{M_1 - c_1 M_2}{c_2 - c_1}, \\ m_3 = \frac{c_2 M_4 - M_3}{c_2 - c_1}, \\ m_4 = \frac{M_3 - c_1 M_4}{c_2 - c_1}. \end{cases} \quad (20)$$

To avoid secular behavior in $x_1(n, \varepsilon n)$ and $y_1(n, \varepsilon n)$, it follows that $m_1 = 0$ and $m_4 = 0$, that is,

$$M_1 = c_2 M_2, \quad \text{and} \quad M_3 = c_1 M_4. \quad (21)$$

If we solve system (21) for $\Delta_\varepsilon g_0(\varepsilon n)$ and $\Delta_\varepsilon h_0(\varepsilon n)$, for the case $0 < r < 1$ and based on Table 1, it follows that

$$\begin{cases} \Delta_\varepsilon g_0(\varepsilon n) = \varepsilon k_1 g_0(\varepsilon n), \\ \Delta_\varepsilon h_0(\varepsilon n) = \varepsilon k_2 h_0(\varepsilon n), \end{cases} \quad (22)$$

where

$$\begin{aligned} k_1 &= \frac{c_1 b_{11} + b_{12} - c_1 c_2 b_{21} - c_2 b_{22}}{\lambda_1 (c_1 - c_2)}, \quad \text{and} \\ k_2 &= -\frac{c_2 b_{11} + b_{12} - c_1 c_2 b_{21} - c_1 b_{22}}{\lambda_2 (c_1 - c_2)}. \end{aligned} \quad (23)$$

Since, for the complex eigenvalues, c_1 and c_2 are complex conjugates, and so k_1 and k_2 are, and since the solutions $x_0(n, \varepsilon n)$ and $y_0(\varepsilon n)$ need to be real, it then follows that $h_0(\varepsilon n) = \overline{g_0(\varepsilon n)}$, where the overline refers to complex conjugates. Therefore, system (22) can be reduced to a single equation

$$\Delta_\varepsilon g_0(\varepsilon n) = \varepsilon k_1 g_0(\varepsilon n), \quad (24)$$

which has as a solution

$$g_0(\varepsilon n) = g_0(0)(1 + \varepsilon k_1)^n. \quad (25)$$

From (24), it then follows that if $|1 + \varepsilon k_1|$ is less than 1, then the equilibrium point $(x(n, \varepsilon n), y(n, \varepsilon n)) = (0 +$

$O(\varepsilon), 0 + O(\varepsilon))$ of system (1) is an asymptotically stable focus, if $|1 + \varepsilon k_1|$ is bigger than 1, then the equilibrium point $(x(n, \varepsilon n), y(n, \varepsilon n)) = (0 + O(\varepsilon), 0 + O(\varepsilon))$ of system (1) is an unstable focus, and if k_1 is zero, then the equilibrium point $(x(n, \varepsilon n), y(n, \varepsilon n)) = (0 + O(\varepsilon), 0 + O(\varepsilon))$ is a higher singularity and the $O(\varepsilon^2)$ -problem for $x_2(n, \varepsilon n)$ and $y_2(n, \varepsilon n)$ has to be studied (which is outside the scope of this paper).

For the case $r = 1$ and when the eigenvalues of matrix A in (3) are complex with nonzero real and imaginary parts, it follows from (17) that only four vectors $(M_{11}, M_{21})^T \lambda_1^n$, $(M_{12}, M_{22})^T \lambda_2^n$, $(M_{17}, M_{27})^T \lambda_1^{2n} \lambda_2^n$, and $(M_{18}, M_{28})^T \lambda_1^n \lambda_2^{2n}$ contain secular, and also non-secular terms. Therefore, we decompose the sum of these vectors into two linearly independent directions to separate secular and nonsecular terms which follow from the same equation as (19), for the case $r = 1$, where M_k (for $k = 1, \dots, 4$) are obtained based on the relationship $\lambda_1 \lambda_2 = 1$, see Table 1. If we solve system (21) for $\Delta_\varepsilon g_0(\varepsilon n)$ and $\Delta_\varepsilon h_0(\varepsilon n)$, where $h_0(\varepsilon n) = \overline{g_0(\varepsilon n)}$, for the case $r = 1$, and based on Table 1, we have again a single equation

$$\Delta_\varepsilon g_0(\varepsilon n) = \varepsilon g_0(\varepsilon n) (k_1 + k_3 g_0(\varepsilon n) \overline{g_0(\varepsilon n)}), \quad (26)$$

where

$$\begin{aligned} k_3 &= (3c_1^2 c_2 (b_{16} - c_2 b_{26}) + c_1 (c_1 + 2c_2) (b_{17} - c_2 b_{27}) \\ &\quad + (2c_1 + c_2) (b_{18} - c_2 b_{28}) + 3(b_{19} - c_2 b_{29})) \\ &\quad / \lambda_1 (c_1 - c_2). \end{aligned} \quad (27)$$

Consider

$$\begin{aligned} g_0(\varepsilon n) &= g_{0,1}(\varepsilon n) + i g_{0,2}(\varepsilon n), \\ k_1 &= k_{11} + i k_{12}, \quad \text{and} \\ k_3 &= k_{31} + i k_{32}, \end{aligned} \quad (28)$$

where $g_{0,1}(\varepsilon n)$ and $g_{0,2}(\varepsilon n)$ are real functions, and where k_{11}, k_{12}, k_{31} , and k_{32} are real constants. Then

(26) becomes

$$\begin{cases} \Delta_\varepsilon g_{0,1} = \varepsilon \{ [k_{11} + k_{31}(g_{0,1}^2 + g_{0,2}^2)]g_{0,1} \\ \quad - [k_{12} + k_{32}(g_{0,1}^2 + g_{0,2}^2)]g_{0,2} \}, \\ \Delta_\varepsilon g_{0,2} = \varepsilon \{ [k_{11} + k_{31}(g_{0,1}^2 + g_{0,2}^2)]g_{0,2} \\ \quad + [k_{12} + k_{32}(g_{0,1}^2 + g_{0,2}^2)]g_{0,1} \}, \end{cases} \quad (29)$$

where $g_{0,1} = g_{0,1}(\varepsilon n)$ and $g_{0,2} = g_{0,2}(\varepsilon n)$. As far as we know, there are no exact solutions available for system (29). However system (29) has always an equilibrium point in $(g_{0,1}, g_{0,2}) = (0, 0)$, and an equilibrium “circle”

$$g_{0,1}^2 + g_{0,2}^2 = -\frac{k_{11}k_{31} + k_{12}k_{32}}{k_{31}^2 + k_{32}^2}, \quad (30)$$

when $k_{31}k_{12} = k_{32}k_{11}$.

For an additional analysis, we define a new variable R , where $R^2 = g_{0,1}^2 + g_{0,2}^2$. After doing some computations, it follows from (29) that

$$\begin{aligned} \Delta_\varepsilon(R^2) &= 2\varepsilon(k_{11} + k_{31}R^2)R^2 + \varepsilon^2[k_{11}^2 + k_{12}^2 \\ &\quad + 2(k_{11}k_{31} + k_{12}k_{32})R^2 \\ &\quad + (k_{31}^2 + k_{32}^2)R^4]R^2. \end{aligned} \quad (31)$$

Since the dynamics of the system is mostly influenced by the $O(\varepsilon)$ -problem rather than $O(\varepsilon^2)$ -problem we consider $O(\varepsilon)$ terms in (31) for the analysis. Then it follows that if k_{11} and k_{31} have different signs, and if k_{11} is positive, there is a stable limit cycle, and if k_{11} is negative then the limit cycle is unstable. To compute the radius of this limit cycle, the right-hand side of (31) is set equal to zero, that is, $\Delta_\varepsilon(R^{*2}) = 0$. To construct an approximation for R^{*2} , one now has to substitute into the right-hand side of (31) a formal power series (in ε) for R^{*2} , that is,

$$R^{*2} = R_0^* + \varepsilon R_1^* + \varepsilon^2 R_2^* + O(\varepsilon^3), \quad (32)$$

One solution of $\Delta_\varepsilon(R^{*2}) = 0$ is the origin, that is $R^* = 0$. To find a nontrivial approximation, by taking together those terms of equal powers in ε , one obtains as $O(1)$ -problem

$$k_{11} + k_{31}R_0^* = 0, \quad (33)$$

which has as a nontrivial solution

$$R_0^* = -\frac{k_{11}}{k_{31}}, \quad (34)$$

when $-\frac{k_{11}}{k_{31}} > 0$, and, as $O(\varepsilon)$ -problem

$$\begin{aligned} 2k_{31}R_1^* + k_{11}^2 + k_{12}^2 + 2(k_{11}k_{31} + k_{12}k_{32})R_0^* \\ + (k_{31}^2 + k_{32}^2)R_0^{*2} = 0, \end{aligned} \quad (35)$$

which has as a solution

$$R_1^* = -\frac{(k_{31}k_{12} - k_{11}k_{32})^2}{2k_{31}^3}, \quad (36)$$

and, as $O(\varepsilon^2)$ -problem

$$\begin{aligned} 2k_{31}R_2^* + 2(k_{11}k_{31} + k_{12}k_{32})R_1^* \\ + 2(k_{31}^2 + k_{32}^2)R_0^*R_1^* = 0, \end{aligned} \quad (37)$$

which has as a solution

$$R_2^* = \frac{k_{32}(k_{31}k_{12} - k_{11}k_{32})^3}{2k_{31}^5}, \quad (38)$$

and so on. So, nontrivial equilibrium points follow from

$$\begin{cases} g_{0,1}^{*2} + g_{0,2}^{*2} = R^{*2}, \\ (k_{11} + k_{31}R^{*2})g_{0,1}^* - (k_{12} + k_{32}R^{*2})g_{0,2}^* = 0, \\ (k_{11} + k_{31}R^{*2})g_{0,2}^* + (k_{12} + k_{32}R^{*2})g_{0,1}^* = 0, \end{cases} \quad (39)$$

where R^{*2} is defined in (32). Now, we expand a formal power series (in ε) for $g_{0,1}^*$ and $g_{0,2}^*$, that is,

$$\begin{cases} g_{0,1}^* = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + O(\varepsilon^3), \\ g_{0,2}^* = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + O(\varepsilon^3). \end{cases} \quad (40)$$

Then, we substitute (40) into (39) and take together those terms of equal powers in ε to find $a_0, b_0, a_1, b_1, \dots$. One obtains as $O(1)$ -problem

$$\begin{cases} a_0^2 + b_0^2 = R_0^*, \\ a_0(k_{11} + k_{31}R_0^*) - b_0(k_{12} + k_{32}R_0^*) = 0, \\ b_0(k_{11} + k_{31}R_0^*) + a_0(k_{12} + k_{32}R_0^*) = 0, \end{cases} \quad (41)$$

where R_0^* is defined in (34). It follows from (34) that (41) has nontrivial solutions for a_0 and b_0 if and only if $k_{11}k_{32} = k_{12}k_{31}$ and $a_0^2 + b_0^2 = -\frac{k_{11}}{k_{31}}$. And, as $O(\varepsilon)$ -problem

$$\begin{cases} 2a_0a_1 + 2b_0b_1 = R_1^*, \\ (a_0k_{31} - b_0k_{32})R_1^* + a_1(k_{11} + k_{31}R_0^*) \\ \quad - b_1(k_{12} + k_{32}R_0^*) = 0, \\ (b_0k_{31} + a_0k_{32})R_1^* + b_1(k_{11} + k_{31}R_0^*) \\ \quad + a_1(k_{12} + k_{32}R_0^*) = 0, \end{cases} \quad (42)$$

where R_1^* is defined in (36). Then it follows that, if $k_{11}k_{32} = k_{12}k_{31}$, (42) has nontrivial solution for a_1 and b_1 if and only if $a_0a_1 + b_0b_1 = 0$.

If we define $g_{0,1} = R \cos(\phi)$ and $g_{0,2} = R \sin(\phi)$, after doing some computations, it follows from (29)

that

$$\Delta_\varepsilon \left(\frac{g_{0,2}}{g_{0,1}} \right) = \frac{\varepsilon(k_{12} + k_{32}R^2)R^2}{g_{0,1}(\varepsilon n)g_{0,1}(\varepsilon(n+1))}. \quad (43)$$

or, after expanding the right-hand side with respect to ε , it follows that

$$\Delta_\varepsilon \left(\frac{g_{0,2}}{g_{0,1}} \right) = \frac{\varepsilon(k_{12} + k_{32}R^2)R^2}{g_{0,1}^2} \left[1 - \frac{\varepsilon(k_{11}g_{0,1} - k_{12}g_{0,2} + (g_{0,1}k_{31} - g_{0,2}k_{32})R^2)}{g_{0,1}} \sum_{m=0}^{\infty} \left(\frac{-\varepsilon}{g_{0,1}} \right)^m \right], \quad (44)$$

where $g_{0,1} = g_{0,1}(\varepsilon n)$ and $g_{0,2} = g_{0,2}(\varepsilon n)$. It follows from (31) and (44) that if $k_{11}k_{32} = k_{12}k_{31}$ there are infinitely many equilibrium points and as a result singularity exists, therefore, for this case we need higher order terms when we compute secular terms. But if $k_{11}k_{32} \neq k_{12}k_{31}$, then the system has only one equilibrium point (which is located in the origin). And, by linearization of (29) around origin, it follows that if $|1 + \varepsilon k_1|$ is less than 1, then the equilibrium point $(x(n, \varepsilon n), y(n, \varepsilon n)) = (0 + O(\varepsilon), 0 +$

$O(\varepsilon))$ of system (1) is an asymptotically stable focus, if $|1 + \varepsilon k_1|$ is bigger than 1, then the equilibrium point $(x(n, \varepsilon n), y(n, \varepsilon n)) = (0 + O(\varepsilon), 0 + O(\varepsilon))$ of system (1) is an unstable focus, and if k_1 is zero, then the equilibrium point $(x(n, \varepsilon n), y(n, \varepsilon n)) = (0 + O(\varepsilon), 0 + O(\varepsilon))$ is a higher singularity and the $O(\varepsilon^2)$ -problem for $x_2(n, \varepsilon n)$ and $y_2(n, \varepsilon n)$ has to be studied (which is outside the scope of this paper). Then it also follows from (16) that the real solutions for $x_0(n, \varepsilon n)$ and $y_0(n, \varepsilon n)$, for the case when $r = 1$, are as follows:

$$\begin{pmatrix} x_0(n, \varepsilon n) \\ y_0(n, \varepsilon n) \end{pmatrix} = 2R(\varepsilon n) \begin{pmatrix} \frac{1}{a_{21}} [\cos(\phi(\varepsilon n) + (n+1)\theta) - a_{22} \cos(\phi(\varepsilon n) + n\theta)] \\ \cos(\phi(\varepsilon n) + n\theta) \end{pmatrix}, \quad (45)$$

where $\lambda_1 = \cos(\theta) + i \sin(\theta)$ and $\lambda_2 = \overline{\lambda_1}$.

5 Conclusions and remarks

In this paper, by applying the multiple scales perturbation method to a general system of two first-order ordinary difference equations, including linear, quadratic, and cubic terms, we obtain approximations of the solutions which are valid on long iteration scales. We considered two cases for which the eigenvalues are complex with nonzero real and imaginary parts, and the modulus is less than or equal to 1. For the case when the modulus is smaller than 1, we found conditions for which the solutions are stable, and for the case when the modulus is equal to 1, and for some special values of the constants, we encounter limit cycles, and a circle of equilibrium points. Our results are

in nice agreement with numerical results in [9] when ε is considered to be small. The stable and unstable limit cycles in the phase plane shown by [9] show this agreement.

The methodology that we used in this paper can also be expanded in the same way to the other cases such as two distinct real eigenvalues, and two coinciding real eigenvalues, and based on the values for the constants a_{ij} and b_{ij} we can have the bifurcation diagrams as well. The obtained results help us in a better understanding of the behavior of nonlinear oscillations. In particular, all kinds of bifurcations can be studied in detail.

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