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Strong Convergent Inertial Two-subgradient Extragradient Method for Finding Minimum-norm Solutions of Variational Inequality Problems

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Abstract

In 2012, Censor et al. (Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization 61(9):1119–1132, 2012b) proposed the two-subgradient extragradient method (TSEGM). This method does not require computing projection onto the feasible (closed and convex) set, but rather the two projections are made onto some half-space. However, the convergence of the TSEGM was puzzling and hence posted as open question. Very recently, some authors were able to provide a partial answer to the open question by establishing weak convergence result for the TSEGM though under some stringent conditions. In this paper, we propose and study an inertial two-subgradient extragradient method (ITSEGM) for solving monotone variational inequality problems (VIPs). Under more relaxed conditions than the existing results in the literature, we prove that proposed method converges strongly to a minimum-norm solution of monotone VIPs in Hilbert spaces. Unlike several of the existing methods in the literature for solving VIPs, our method does not require any line-search technique, which could be time-consuming to implement. Rather, we employ a simple but very efficient self-adaptive step size method that generates a non-monotonic sequence of step sizes. Moreover, we present several numerical experiments to demonstrate the efficiency of our proposed method in comparison with related results in the literature. Finally, we apply our result to image restoration problem. Our result in this paper improves and generalizes several of the existing results in the literature in this direction.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|| \cdot ||$. Let C be a nonempty, closed and convex subset of H, and let $A : H \to H$ be a mapping. The variational inequality problem (VIP) is formulated as finding a point $p \in C$ such that

$$\langle x - p, Ap \rangle \geq 0, \quad \forall\, x \in C. \tag{1.1}$$

We denote the solution set of the VIP (1.1) by $VI(C, A)$. In the recent years, the VIP has received great research attention due to its wide areas of applications, such as in structural analysis, economics, optimization theory (Alakoya and Mewomo 2022; Aubin and Ekeland 1984; Ogwo et al. 2022), operations research, sciences and engineering (see Baiocchi and Capelo 1984; Censor et al. 2012a; Godwin et al. 2023; Kinderlehrer and Stampacchia 2000 and the references therein). The VIP is an important mathematical model that has been widely utilized to formulate and investigate a plethora of competitive equilibrium problems in various disciplines, such as traffic network equilibrium problems, spatial price equilibrium problems, oligopolistic market equilibrium problems, financial equilibrium problems, migration equilibrium problems, environmental network and ecology problems, knowledge network problems, supply chain network equilibrium problems, internet problems, etc., see, e.g., Geunes and Pardalos (2003), Nagurney (1999), Nagurney and Dong (2002) for further examples and details. The study of VIPs in finite dimensional spaces was initiated independently by Smith (1979) and Dafermos (1980). They set up the traffic assignment problem in terms of a finite dimensional VIP. On the other hand, Lawphongpanich and Hearn (1984), and Panicucci et al. (2007) studied the traffic assignment problems based on Wardrop user equilibrium principle via a variational inequality model. Since then, several other economics related problems like Nash equilibrium problem, spatial price equilibrium problems, internet problems, dynamic financial equilibrium problems and environmental network and ecology problems have been investigated via variational inequality problem (see Aussel et al. 2016; Ciarciá and Daniele 2016; Nagurney et al. 2007; Scrimali and Mirabella 2018).

There are two common approaches to solving the VIP, namely: the regularised methods and the projection methods. In this study, our interest is in the projection methods. The earliest and simplest projection method for solving the VIP is the projected gradient method (GM), which is presented as follows:

Algorithm 1.1 *(Gradient Method (GM))*

$$x_{n+1} = P_C(x_n - \lambda A x_n), \tag{1.2}$$

for each $n \geq 1$, where P_C denotes the metric projection map. Observe that the GM requires calculating only one projection per iteration onto the feasible C. However, the method only converges when the cost operator A is α-strongly monotone and L-Lipschitz continuous, where $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$. These stringent conditions greatly limit the scope of applications of the GM (1.2).

In order to relax the conditions for the convergence of the GM to a solution of the VIP, Korpelevich (1976) and Antipin (1976) independently proposed the following extragradient method (EGM) in finite-dimensional Euclidean space:

Algorithm 1.2 *(Extragradient Method (EGM))*

$$\begin{cases} x_1 \in C \\ y_n = P_C(x_n - \lambda Ax_n) \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$, $A : \mathbb{R}^n \to \mathbb{R}^n$ is monotone and L-Lipschitz continuous. If the solution set $VI(C, A)$ is nonempty, the EGM (1.3) generates a sequence that converges to a solution of the VIP.

Observe that the EGM needs to compute two projections onto the feasible set C and two evaluations of the operator A per iteration. In general, computing projection onto an arbitrary closed and convex set C is complicated. This limitation can affect the efficiency of the EGM. In the recent years, the EGM has attracted the attention of researchers, who improved it in various ways (see, e.g., Ceng et al. 2021; Duong and Gibali 2019; Godwin et al. 2022; He et al. 2019) and the references therein. One of the major areas of improvement of the method is to minimize the number of projections onto the feasible set C per iteration (Thong et al. 2020; Uzor et al. 2022). Censor et al. (2011) initiated an attempt in this direction by modifying the EGM and replacing the second projection with a projection onto a half-space. The resulting method requires only one projection onto the feasible set C and is known as the subgradient extragradient method (SEGM). The SEGM is presented as follows:

Algorithm 1.3 *(Subgradient Extragradient Method (SEGM))*

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda Ax_n), \\ T_n = \{z \in H : \langle x_n - \lambda Ax_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n). \end{cases} \tag{1.4}$$

Censor et al. (2011) obtained weak convergence result for the SEGM (1.4) under the same assumptions as the EGM (1.3). Since there is an explicit formula to calculate projection onto an half-space, the SEGM can be considered as an improvement over the EGM. However, we observe that the SEGM still requires computing one projection onto the closed convex set C per iteration. This can still be a great barrier to the implementation of the SEGM.

In order to address this limitation, Censor et al. (2012b) also proposed the following method called the two-subgradient method (TSEGM):

Algorithm 1.4 *(Two-Subgradient Extragradient Method (TSEGM))*

$$\begin{cases} x_1 \in H, \\ y_n = P_{C_n}(x_n - \lambda Ax_n), \\ C_n := \{x \in H : c(w_n) + \langle \zeta_n, x - w_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(x_n - \lambda Ay_n), \end{cases} \tag{1.5}$$

where $\zeta_n \in \partial c(x_n)$. Here, $\partial c(x)$ denotes the sub-differential of the convex function $c(\cdot)$ at x defined in (2.2).

The idea behind the TSEGM is in that any closed and convex set C can be expressed as

$$C = \{x \in H \mid c(x) \leq 0\}, \tag{1.6}$$

where $c : H \to \mathbb{R}$ is a convex function. For instance, we can let $c(x) := \text{dist}(x, C)$, where "dist" is the distance function. We observe that the two projections in the TSEGM (1.5) are made onto an half-space, which makes it easier to implement. However, the convergence of the TSEGM (1.5) was puzzling and was therefore posted as an open question by Censor et al. (2012b).

At this point, we briefly discuss the inertial technique. The inertial algorithm is based on a discrete version of the second-order dissipative dynamical system, which was first proposed by Polyak (1964). The main feature of the inertial-algorithm is that the method uses the previous two iterates to generate the next iterate. It is worth mentioning that this small change can greatly improve the convergence rate of an iterative method. In the recent years, many researchers have constructed very fast iterative methods by employing the inertial technique, see, e.g., Alakoya and Mewomo (2022), Alakoya et al. (2022), Gibali et al. (2020), Godwin et al. (2023), Wickramasinghe et al. (2023) and the references therein.

In 2019, Cao and Guo (2020) partially answered the open question posted by Censor et al. (2012b). by combining the inertial technique with the TSEGM and obtained a weak convergence result for the proposed algorithm (Algorithm 7.1) under the assumptions that the cost operator A is monotone, Lipschitz continuous, the convex function $c : H \to \mathbb{R}$ in (1.6) is continuously differentiable and the Gâteaux differential $c'(\cdot)$ is Lipschitz continuous.

We need to point out at this point that all the above methods are not applicable when the Lipschitz constant of the cost operator is unknown because the step size of the algorithms depends on prior knowledge of the Lipschitz constant of the cost operator. We also note that the ITSEGM proposed by Cao and Guo (2020) requires prior knowledge of the Lipschitz constant of the Gâteaux differential $c'(\cdot)$ of $c(\cdot)$. In most cases, the Lipschitz constants of these operators are unknown or difficult to calculate. All of these drawbacks may hinder the implementation of these algorithms. Moreover, all the above methods only give weak convergence results under these stringent conditions.

Bauschke and Combettes (2001) pointed out that in solving optimization problems, strong convergent iterative methods are more applicable, and hence more desirable than their weak convergent counterparts. Thus, it is important to develop algorithms that generate strong convergence sequence when solving optimization problems.

Very recently, Ma and Wang (2022) tried to improve on the results of Cao and Guo (2020) by proposing a new TSEGM (Algorithm 7.2), which uses a self-adaptive step size such that the implementation of their algorithm does not require prior knowledge of the Lipschitz constant of the cost operator. However, we note that the implementation of their result also requires knowledge of the Lipschitz constant of the Gâteaux differential $c'(\cdot)$ of $c(\cdot)$. Moreover, the authors were also only able to obtain weak convergence result for their proposed algorithm.

Considering the above review, it is pertinent to ask the following research questions:

Can we construct a new inertial two-subgradient extragradient method, which is applicable when the Lipschitz constant of the cost operator A and/or when the Lipschitz constant of the Gâteaux differential $c'(\cdot)$ of $c(\cdot)$ are unknown? Can we obtain a strong convergence result for this method?

In this paper, we provide affirmative answers to the above questions. More precisely, we introduce a new inertial two-subgradient extragradient method which does not require knowledge of the Lipschitz constant of the cost operator nor knowledge of the Lipschitz constant of the Gâteaux differential $c'(\cdot)$ of $c(\cdot)$. This makes our results applicable to a larger class of problems. Moreover, we prove that the sequence generated by our proposed algorithm converges strongly to a minimum-norm solution of the VIP. In many practical problems, finding the minimum-norm solution is very important and useful. All of the above highlighted properties are some of the improvements of our proposed method over the results of Ma and Wang (2022), and Cao and Guo (2020). In addition, the proof of our strong convergence theorem does not rely on the usual "two cases approach" widely employed by authors to prove strong convergence results. We also point out that unlike several of the existing results in the literature, our method does not involve any linesearch technique which could be computationally expensive to implement (e.g., see Cai et al. 2022; Peeyada et al. 2020; Suantai et al. 2020) nor does it require evaluating any inner product function which is not easily evaluated unlike the norm function (see Muangchoo et al. 2021, Corollary 4.4). Rather, we employ a simple but very efficient self-adaptive step size technique, which generates a non-monotonic sequence of step sizes with less dependency on the initial step size. This makes our method more efficient and less expensive to implement. Moreover, we present several numerical experiments to demonstrate the computational advantage of our proposed method over the existing methods in the literature. Finally, we apply our result to image restoration problem. The results of the numerical experiments show that our method is more efficient than several of the existing methods in the literature. Clearly, our proposed method is economically viable and our results improve and generalize several of the existing results in the literature in this direction.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and lemmas employed in the paper. In Section 3, we present our proposed algorithm and highlight some of its features. Convergence analysis of the proposed method is discussed in Section 4. In Section 5 we present some numerical experiments and apply our result to image restoration problem. Finally, in Section 6 we give a concluding remark.

2 Preliminaries

In what follows, we assume that C is a nonempty, closed and convex subset of a real Hilbert space H. We denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively and $w_\omega(x_n)$ denotes set of weak limits of $\{x_n\}$, that is,

$$w_\omega(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

Let H be a real Hilbert space, for a nonempty closed and convex subset C of H, the *metric projection* $P_C : H \to C$ (Taiwo et al. 2021) is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = \inf\{||x - z|| : z \in C\}.$$

It is known that P_C is nonexpansive and has the following properties (Alakoya and Mewomo 2022; Uzor et al. 2022):

1. $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y \rangle$ for all $x, y \in C$;
2. for any $x \in H$ and $z \in C, z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C; \tag{2.1}$$

3. for any $x \in H$ and $y \in C$,

$$||P_C x - y||^2 + ||x - P_C x||^2 \leq ||x - y||^2;$$

4. for any $x, y \in H$ with $y \neq 0$, let $Q = \{z \in H : \langle y, z - x \rangle \leq 0\}$. Then, for all $u \in H$, $P_Q(u)$ is given by

$$P_Q(u) = u - \max\left\{0, \frac{\langle y, u - x \rangle}{||y||^2}\right\} y,$$

which gives an explicit formula for calculating the projection of any given point onto a half-space.

Lemma 2.1 *Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\delta \in \mathbb{R}$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$;
(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2$;
(iii) $||\delta x + (1 - \delta)y||^2 = \delta||x||^2 + (1 - \delta)||y||^2 - \delta(1 - \delta)||x - y||^2$.

Definition 2.2 An operator $A : H \to H$ is said to be

(i) *α-strongly monotone*, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha||x - y||^2, \quad \forall x, y \in H;$$

(ii) *α-inverse strongly monotone (α-cocoercive)*, if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha||Ax - Ay||^2, \quad \forall x, y \in H;$$

(iii) *monotone*, if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in H;$$

(iv) L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y||, \quad \forall\, x, y \in H;$$

It is known that if A is α-strongly monotone and L-Lipschitz continuous, then A is $\frac{\alpha}{L^2}$-inverse strongly monotone. Furthermore, α-inverse strongly monotone operators are $\frac{1}{\alpha}$-Lipschitz continuous and monotone but the converse is not true.

Definition 2.3 Bauschke and Combettes (2017) A function $c : H \to \mathbb{R}$ is said to be Gâteaux differentiable at $x \in H$, if there exists an element denoted by $c'(x) \in H$ such that

$$\lim_{h \to 0} \frac{c(x + hv) - c(x)}{h} = \langle v, c'(x) \rangle, \quad \forall\, v \in H, \ h \in [0, 1],$$

where $c'(x)$ is called the Gâteaux differential of c at x. Recall that if for each $x \in H$, c is Gâteaux differentiable at x, then c is Gâteaux differentiable on H.

Definition 2.4 Bauschke and Combettes (2017) A convex set $c : H \to \mathbb{R}$ is said to be subdifferentiable at a point $x \in H$ if the set

$$\partial c(x) = \{\zeta \in H \mid c(y) \geq c(x) + \langle \zeta, y - x \rangle, \quad \forall y \in H\} \tag{2.2}$$

is nonempty. Each element in $\partial c(x)$ is called a subgradient of c at x. We note that if c is subdifferentiable at each $x \in H$, then c is subdifferentiable on H. It is also known that if c is Gâteaux differentiable at x, then c is subdifferentiable at x and $\partial c(x) = \{c'(x)\}$.

Definition 2.5 Let H be a real Hilbert space. A function $c : H \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly lower semi-continuous (w-lsc) at $x \in H$, if

$$c(x) \leq \liminf_{n \to \infty} c(x_n)$$

holds for every sequence $\{x_n\}$ in H satisfying $x_n \rightharpoonup x$.

Lemma 2.6 *Bauschke and Combettes (2017) Let $c : H \to \mathbb{R} \cup \{+\infty\}$ be convex. Then the following are equivalent*:

(i)   *c is weakly sequential lower semi-continuous*;
(ii) *c is lower semi-continuous.*

Lemma 2.7 *He and Xu (2013) Assume that the solution set* VI (C, A) *of the VIP (1.1) is nonempty, and C is defined as $C := \{x \in H \mid c(x) \leq 0\}$, where $c : H \to \mathbb{R}$ is a continuously differentiable convex function. Given $p \in C$. Then $p \in$ VI (C, A) if and only if either*

1. $Ap = 0, or$

2. $p \in \partial C$ and there exists $\eta_p > 0$ such that $Ap = -\eta_p c'(p)$, where ∂C denotes the boundary of C.

Lemma 2.8 *Dong et al.* (2018) *Let C be a nonempty closed and convex subset of H. Let $A : C \to H$ be a continuous, monotone mapping and $z \in C$, then*

$$z \in VI(C, A) \iff \langle Ax, x - z \rangle \geq 0, \quad \forall\, x \in C.$$

Lemma 2.9 *Tan and Xu* (1993) *Suppose $\{\lambda_n\}$ and $\{\phi_n\}$ are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \phi_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \phi_n < +\infty$, then $\lim_{n \to \infty} \lambda_n$ exists.

Lemma 2.10 *Saejung and Yotkaew* (2012) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ with the condition: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k \to \infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition:

$$\liminf_{k \to \infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim_{k \to \infty} a_n = 0$.

3 Proposed Method

In this section, we present our proposed method and discuss some of its important features. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 3.1 Suppose that the following conditions hold:

1. The set C is defined by

$$C = \{x \in H \mid c(x) \leq 0\}; \tag{3.1}$$

   where $c : H \to \mathbb{R}$ is a continuously differentiable convex function such that $c'(\cdot)$ is L_1-Lipschitz continuous (however, prior knowledge of the Lipschitz constant is not required).

2. (a) $A : H \to H$ is monotone and L_2-Lipschitz continuous (however, prior knowledge of the Lipschitz constant is not needed).
   (b) There exists $K > 0$ such that $\|Ax\| \leq K\|c'(x)\|$ for all $x \in \partial C$.
   (c) The solution set $VI(C, A)$ is nonempty.

3. $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\xi_n\}_{n=1}^{\infty}$ are non-negative sequences satisfying the following conditions:

(a) $\alpha_n \in (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \dfrac{\xi_n}{\alpha_n} = 0, \theta > 0, \lambda_1 > 0$.

(b) $\{\beta_n\} \subset [a,b] \subset (0, 1-\alpha_n), \delta \in \left(0, -K + \sqrt{1+K^2}\right)$.

(c) Let $\{\phi_n\}$ be a nonnegative sequence such that $\sum_{n=1}^{\infty} \phi_n < +\infty$.

Algorithm 3.2

Step 0. Select two arbitrary initial points $x_0, x_1 \in H$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\xi_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \qquad (3.2)$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

Step 3. Construct the half-space

$$C_n = \{x \in H : c(w_n) + \langle c'(w_n), x - w_n \rangle \leq 0\},$$

and compute

$$y_n = P_{C_n}(w_n - \lambda_n A w_n)$$

$$z_n = P_{C_n}(w_n - \lambda_n A y_n)$$

If $c(y_n) \leq 0$ and either $w_n - y_n = 0$ or $Ay_n = 0$, then stop and y_n is a solution of the VIP. Otherwise, go to **Step 4**.

Step 4. Compute

$$x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \beta_n z_n.$$

Step 5. Compute

$$\lambda_{n+1} = \begin{cases} \min \left\{ \dfrac{\delta \|w_n - y_n\|}{\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\|}, \ \lambda_n + \phi_n \right\}, & \text{if } \|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\| \neq 0, \\ \lambda_n + \phi_n, & \text{otherwise .} \end{cases}$$

(3.3)

Set $n = n + 1$ and go back to **Step 1**.

Remark 3.3

- Observe that unlike the results of Cao and Guo (2020) and Ma and Wang (2022) knowledge of the Lipschitz constant of the cost operator A and knowledge of the Lipschitz constant of the Gâteaux differential $c'(\cdot)$ of $c(\cdot)$ are not required to implement our algorithm.
- Moreover, we need to point out that our algorithm does not require any line-search technique, rather we employ a more efficient step size rule in (3.3) which generates a non-monotonic sequence of step sizes. The step size is constructed such that it reduces the dependence of the algorithm on the initial step size λ_1.
- We also remark that our proposed algorithm generates a strong convergence sequence, which converges to a minimum-norm solution of the VIP.

Remark 3.4

(i) Observe that by the definition of C in (3.1) and the construction of C_n, we have $C \subset C_n$.

(ii) By Assumption 3.1 3(a), it can easily be verified from (3.2) that

$$\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$$

Remark 3.5 Observe that by (3.1) and Lemma 4.5 together with the formulation of the variational inequality problem, it is clear that if $c(y_n) \leq 0$ and either $w_n - y_n = 0$ or $Ay_n = 0$, then y_n is a solution of the VIP.

4 Convergence Analysis

First, we establish some lemmas which will be needed to prove our strong convergence theorem for the proposed algorithm.

Lemma 4.1 *Let $\{\lambda_n\}$ be a sequence generated by Algorithm* 3.2. *Then, we have* $\lim_{n\to\infty} \lambda_n = \lambda$, *where* $\lambda \in \left[\min\{\frac{\delta}{L_2+L_1}, \lambda_1\}, \lambda_1 + \Phi\right]$ *for some positive constant M and* $\Phi = \sum_{n=1}^{\infty} \phi_n$.

Proof Since A is L_2-Lipschitz continuous and $c'(\cdot)$ is L_1-Lipschitz continuous, then for the case $\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\| \neq 0$ for all $n \geq 1$ we have

$$\frac{\delta\|w_n - y_n\|}{\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\|} \geq \frac{\delta\|w_n - y_n\|}{L_2\|w_n - y_n\| + L_1\|w_n - y_n\|}$$
$$= \frac{\delta}{L_2 + L_1}.$$

Thus, by the definition of λ_{n+1}, the sequence $\{\lambda_n\}$ has lower bound $\min\{\frac{\delta}{L_2+L_1}, \lambda_1\}$ and upper bound $\lambda_1 + \Phi$. By Lemma 2.9, we have that $\lim_{n\to\infty} \lambda_n$ exists and denoted by $\lambda = \lim_{n\to\infty} \lambda_n$. It is clear that $\lambda \in \left[\min\{\frac{\delta}{L_2+L_1}, \lambda_1\}, \lambda_1 + \Phi\right]$. $\square$

Lemma 4.2 *Let $\{x_n\}$ be a sequence generated by Algorithm* 3.2 *under Assumption* 3.1. *Then, the following inequality holds for all $p \in VI(C,A)$:*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left[1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K \frac{\lambda_n}{\lambda_{n+1}}\right]\|w_n - y_n\|^2. \tag{4.1}$$

Proof From (3.3), we have

$$\lambda_{n+1} = \min\left\{\frac{\delta\|w_n - y_n\|}{\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\|}, \lambda_n + \phi_n\right\}$$
$$\leq \frac{\delta\|w_n - y_n\|}{\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\|},$$

which implies that

$$\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\| \leq \frac{\delta}{\lambda_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 1. \tag{4.2}$$

Let $p \in VI(C,A)$. For convenience, we set $v_n = w_n - \lambda_n Ay_n$ and we have $z_n = P_{C_n}(v_n)$. Since $p \in C \subset C_n$, then by applying (2.1) we obtain

$$\|z_n - p\|^2 = \|P_{C_n}(v_n) - p\|^2 \leq \|v_n - p\|^2 - \|v_n - P_{C_n}(v_n)\|^2. \tag{4.3}$$

Observe that

$$\|v_n - p\|^2 - \|v_n - P_{C_n}(v_n)\|^2$$
$$= \|(w_n - p) - \lambda_n Ay_n\|^2 - \|(w_n - z_n) - \lambda_n Ay_n\|^2$$
$$= \|w_n - p\|^2 - 2\lambda_n \langle w_n - p, Ay_n \rangle$$
$$\quad - \|w_n - z_n\|^2 + 2\lambda_n \langle w_n - z_n, Ay_n \rangle$$
$$= \|w_n - p\|^2 - \|w_n - z_n\|^2 + 2\lambda_n \langle p - z_n, Ay_n \rangle$$
$$= \|w_n - p\|^2 - \left[\|w_n - y_n\|^2 + \|y_n - z_n\|^2 \right.$$
$$\quad \left. + 2\langle w_n - y_n, y_n - z_n \rangle \right] + 2\lambda_n \langle p - y_n, Ay_n \rangle \qquad (4.4)$$
$$\quad + 2\lambda_n \langle y_n - z_n, Ay_n \rangle$$
$$= \|w_n - p\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2$$
$$\quad + 2\lambda_n \langle p - y_n, Ay_n \rangle + 2\langle z_n - y_n, w_n - \lambda_n Ay_n - y_n \rangle$$
$$= \|w_n - p\|^2 - \|w_n - y_n\|^2 - \|y_n - z_n\|^2$$
$$\quad + 2\lambda_n \langle p - y_n, Ay_n \rangle + 2\langle z_n - y_n, w_n - \lambda_n Aw_n - y_n \rangle$$
$$\quad + 2\lambda_n \langle z_n - y_n, Aw_n - Ay_n \rangle.$$

Since $y_n = P_{C_n}(w_n - \lambda_n Aw_n)$ and $z_n \in C_n$, it follows from (2.1) that

$$\langle w_n - \lambda_n Aw_n - y_n, z_n - y_n \rangle \le 0. \qquad (4.5)$$

Applying Young's inequality and (4.2), we get

$$2\lambda_n \langle z_n - y_n, Aw_n - Ay_n \rangle \le \|z_n - y_n\|^2 + \lambda_n^2 \|Aw_n - Ay_n\|^2$$
$$\le \|z_n - y_n\|^2 + \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \|w_n - y_n\|^2. \qquad (4.6)$$

Furthermore, by the monotonicity of A, we get

$$\langle p - y_n, Ay_n \rangle \le \langle p - y_n, Ap \rangle \qquad (4.7)$$

Now, applying (4.4)-(4.7) in (4.3) we get

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \right) \|w_n - y_n\|^2 + 2\lambda_n \langle p - y_n, Ap \rangle. \qquad (4.8)$$

Now, we consider the following two cases:

Case 1 : $Ap = 0$. If $Ap = 0$, then from (4.8) the desired inequality (4.1) follows.

Case 2 : $Ap \neq 0$. By Lemma 2.7, $p \in \partial C$ and there exists $\eta_p > 0$ such that $Ap = -\eta_p c'(p)$. Since $p \in \partial C$, then $c(p) = 0$. By the sub-differential inequality (2.2), we have

$$c(y_n) \geq c(p) + \langle c'(p), y_n - p \rangle$$
$$= \frac{-1}{\eta_p} \langle Ap, y_n - p \rangle.$$

From the last inequality we obtain

$$\langle p - y_n, Ap \rangle \leq \eta_p c(y_n). \tag{4.9}$$

Since $y_n \in C_n$, we obtain

$$c(w_n) + \langle c'(w_n), y_n - w_n \rangle \leq 0. \tag{4.10}$$

Again, by the sub-differential inequality (2.2), we get

$$c(y_n) + \langle c'(y_n), w_n - y_n \rangle \leq c(w_n). \tag{4.11}$$

Adding (4.10) and (4.11), we have

$$c(y_n) \leq \langle c'(y_n) - c'(w_n), y_n - w_n \rangle. \tag{4.12}$$

From (4.9) and (4.12), we get

$$\langle p - y_n, Ap \rangle \leq \eta_p \langle c'(y_n) - c'(w_n), y_n - w_n \rangle. \tag{4.13}$$

Observe that by Assumption 3.1 2(b)

$$\eta_p \leq K. \tag{4.14}$$

Hence, we have

$$2\lambda_n \eta_p \langle c'(y_n) - c'(w_n), y_n - w_n \rangle$$
$$\leq 2\lambda_n \eta_p \|c'(y_n) - c'(w_n)\| \|y_n - w_n\| \tag{4.15}$$
$$\leq 2\lambda_n K \|c'(y_n) - c'(w_n)\| \|y_n - w_n\|.$$

Applying (4.2), (4.13) and (4.15) in (4.8), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2$$
$$+ 2\lambda_n K \|c'(y_n) - c'(w_n)\|\|y_n - w_n\|$$
$$\le \|w_n - p\|^2 - \left(1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2$$
$$+ 2\delta K \frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 \qquad (4.16)$$
$$= \|w_n - p\|^2 - \left[1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K \frac{\lambda_n}{\lambda_{n+1}}\right]$$
$$\|w_n - y_n\|^2,$$

which is the required inequality (4.1). $\qquad\square$

Since the limit of $\{\lambda_n\}$ exists, $\lim\limits_{n\to\infty} \lambda_n = \lim\limits_{n\to\infty} \lambda_{n+1}$. Hence, by the conditions on the control parameters we have

$$\lim_{n\to\infty}\left[1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K \frac{\lambda_n}{\lambda_{n+1}}\right] = \left[1 - \delta^2 - 2\delta K\right] > 0. \qquad (4.17)$$

Therefore, there exists $n_0 \ge 1$ such that for all $n \ge n_0$ we have

$$\lim_{n\to\infty}\left[1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K \frac{\lambda_n}{\lambda_{n+1}}\right] > 0.$$

Thus, from (4.1) we have that for all $n \ge n_0$,

$$\|z_n - p\| \le \|w_n - p\|. \qquad (4.18)$$

Lemma 4.3 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then, $\{x_n\}$ is bounded.*

Proof Let $p \in VI(C, A)$. Then, by the definition of w_n we have

$$\|w_n - p\| = \|x_n + \theta_n(x_n - x_{n-1}) - p\|$$
$$\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \qquad (4.19)$$
$$= \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|.$$

By Remark 3.4 (ii.), there exists $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1, \quad \forall\, n \ge 1.$$

Thus, it follows from (4.19) that

$$\|w_n - p\| \le \|x_n - p\| + \alpha_n M_1, \quad \forall n \ge 1. \tag{4.20}$$

By the definition of x_{n+1}, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p) - \alpha_n p\| \\
&\le \|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\| + \alpha_n \|p\|.
\end{aligned} \tag{4.21}$$

Applying Lemma 2.1(ii) and using (4.18) we have

$$\begin{aligned}
\|(1 &- \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\|^2 \\
&= (1 - \alpha_n - \beta_n)^2 \|w_n - p\|^2 \\
&\quad + 2(1 - \alpha_n - \beta_n)\beta_n \langle w_n - p, z_n - p \rangle \\
&\quad + \beta_n^2 \|z_n - p\|^2 \\
&\le (1 - \alpha_n - \beta_n)^2 \|w_n - p\|^2 \\
&\quad + 2(1 - \alpha_n - \beta_n)\beta_n \|w_n - p\| \|z_n - p\| \\
&\quad + \beta_n^2 \|z_n - p\|^2 \\
&\le (1 - \alpha_n - \beta_n)^2 \|w_n - p\|^2 \\
&\quad + (1 - \alpha_n - \beta_n)\beta_n \big[\|w_n - p\|^2 \\
&\quad + \|z_n - p\|^2 \big] + \beta_n^2 \|z_n - p\|^2 \\
&= (1 - \alpha_n - \beta_n)(1 - \alpha_n) \|w_n - p\|^2 \\
&\quad + \beta_n(1 - \alpha_n) \|z_n - p\|^2 \\
&\le (1 - \alpha_n - \beta_n)(1 - \alpha_n) \|w_n - p\|^2 \\
&\quad + \beta_n(1 - \alpha_n) \|w_n - p\|^2 \\
&= (1 - \alpha_n)^2 \|w_n - p\|^2,
\end{aligned}$$

which implies that

$$\|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\| \le (1 - \alpha_n) \|w_n - p\|. \tag{4.22}$$

Now, applying (4.20) and (4.22) in (4.21), we have for all $n \ge n_0$

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1 - \alpha_n) \|w_n - p\| + \alpha_n \|p\| \\
&\le (1 - \alpha_n) \big[\|x_n - p\| + \alpha_n M_1 \big] + \alpha_n \|p\| \\
&\le (1 - \alpha_n) \|x_n - p\| + \alpha_n \big[\|p\| + M_1 \big] \\
&\le \max \big\{ \|x_n - p\|, \|p\| + M_1 \big\} \\
&\quad \vdots \\
&\le \max \big\{ \|x_{n_0} - p\|, \|p\| + M_1 \big\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded. $\qquad \square$

Lemma 4.4 *Suppose* $\{x_n\}$ *is a sequence generated by Algorithm* 3.2 *such that Assumption* 3.1 *holds. Then, the following inequality holds for all* $p \in VI(C, A)$:

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n)^2 \|x_n - p\|^2 + 3M_2\alpha_n(1 - \alpha_n)^2 \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|$$

$$- \beta_n(1 - \alpha_n)\left[1 - \delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K \frac{\lambda_n}{\lambda_{n+1}}\right]\|w_n - y_n\|^2 + 2\alpha_n\langle p, p - x_{n+1}\rangle.$$

Proof Let $p \in VI(C, A)$. Then, by applying Lemma 2.1(ii) together with the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\le \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|(\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|) \quad (4.23) \\
&\le \|x_n - p\|^2 + 3M_2\theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|,
\end{aligned}
$$

where $M_2 := \sup_{n \in \mathbb{N}}\{\|x_n - p\|, \theta_n\|x_n - x_{n-1}\|\} > 0$.

Next, by applying (4.2) together with the nonexpansiveness of P_{C_n} we have

$$
\begin{aligned}
\|z_n - w_n\| &\le \|z_n - y_n\| + \|w_n - y_n\| \\
&= \|P_{C_n}(w_n - \lambda_n Ay_n) - P_{C_n}(w_n - \lambda_n Aw_n)\| + \|w_n - y_n\| \\
&\le \lambda_n\|Aw_n - Ay_n\| + \|w_n - y_n\| \\
&\le \delta\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\| + \|w_n - y_n\| \\
&= \left(1 + \delta\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|.
\end{aligned}
\quad (4.24)
$$

Using (4.23) and applying Lemmas 2.1 and 4.2, we have

$$\|x_{n+1} - p\|^2 = \|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p) - \alpha_n p\|^2$$
$$\leq \|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\|^2$$
$$\quad - 2\alpha_n\langle p, x_{n+1} - p\rangle$$
$$= (1 - \alpha_n - \beta_n)^2\|w_n - p\|^2 + \beta_n^2\|z_n - p\|^2$$
$$\quad + 2\beta_n(1 - \alpha_n - \beta_n)\langle w_n - p, z_n - p\rangle$$
$$\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle$$
$$\leq (1 - \alpha_n - \beta_n)^2\|w_n - p\|^2 + \beta_n^2\|z_n - p\|^2$$
$$\quad + 2\beta_n(1 - \alpha_n - \beta_n)\|w_n - p\|\|z_n - p\|$$
$$\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle$$
$$\leq (1 - \alpha_n - \beta_n)^2\|w_n - p\|^2 + \beta_n^2\|z_n - p\|^2$$
$$\quad + \beta_n(1 - \alpha_n - \beta_n)\big[\|w_n - p\|^2 + \|z_n - p\|^2\big]$$
$$\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle$$
$$\leq (1 - \alpha_n - \beta_n)(1 - \alpha_n)\|w_n - p\|^2$$
$$\quad + \beta_n(1 - \alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle p, p - x_{n+1}\rangle$$
$$\leq (1 - \alpha_n - \beta_n)(1 - \alpha_n)\|w_n - p\|^2$$
$$\quad + \beta_n(1 - \alpha_n)\|w_n - p\|^2$$
$$\quad - \beta_n(1 - \alpha_n)\Big[1 - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K\frac{\lambda_n}{\lambda_{n+1}}\Big]$$
$$\|w_n - y_n\|^2 + 2\alpha_n\langle p, p - x_{n+1}\rangle$$
$$= (1 - \alpha_n)^2\|w_n - p\|^2$$
$$\quad - \beta_n(1 - \alpha_n)\Big[1 - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K\frac{\lambda_n}{\lambda_{n+1}}\Big]$$
$$\|w_n - y_n\|^2 + 2\alpha_n\langle p, p - x_{n+1}\rangle$$
$$\leq (1 - \alpha_n)^2\|x_n - p\|^2$$
$$\quad + 3M_2\alpha_n(1 - \alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|$$
$$\quad - \beta_n(1 - \alpha_n)\Big[1 - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2\delta K\frac{\lambda_n}{\lambda_{n+1}}\Big]\|w_n - y_n\|^2$$
$$\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle,$$

which is the required inequality. $\qquad\square$

Lemma 4.5 *Let $\{w_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3.2 under Assumption 3.1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$, which converges weakly to $x^* \in H$ and $\lim\limits_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, then $x^* \in VI(C, A)$.*

Proof Suppose $\{w_n\}$ and $\{y_n\}$ are two sequences generated by Algorithm 3.2 with subsequences $\{w_{n_k}\}$ and $\{y_{n_k}\}$, respectively such that $w_{n_k} \rightharpoonup x^*$. Then by the hypothesis of the lemma we have $y_{n_k} \rightharpoonup x^*$. Also, since $y_{n_k} \in C_{n_k}$, then by the definition of C_n we have

$$c(w_{n_k}) + \langle c'(w_{n_k}), y_{n_k} - w_{n_k} \rangle \le 0.$$

Applying the Cauchy-Schwartz inequality, we obtain from the last inequality

$$c(w_{n_k}) \le \|c'(w_{n_k})\| \|w_{n_k} - y_{n_k}\|. \tag{4.25}$$

Since $c'(\cdot)$ is continuous and $\{w_{n_k}\}$ is bounded, then $\{c'(w_{n_k})\}$ is bounded, that is, there exists a constant $M > 0$ such that $\|c'(w_{n_k})\| \le M$ for all $k \ge 0$. Then, from (4.25) we obtain

$$c(w_{n_k}) \le M \|y_{n_k} - w_{n_k}\|. \tag{4.26}$$

Since $c(\cdot)$ is continuous, then it is lower semi-continuous. Also, since $c(\cdot)$ is convex, by Lemma 2.6 $c(\cdot)$ is weakly lower semi-continuous. Hence, it follows from (4.26) and the definition of weakly lower semi-continuity that

$$c(x^*) \le \liminf_{k \to \infty} c(w_{n_k}) \le \lim_{k \to \infty} M \|y_{n_k} - w_{n_k}\| = 0, \tag{4.27}$$

which implies that $x^* \in C$. By property (2.1) of P_{C_n}, we obtain

$$\langle y_{n_k} - w_{n_k} + \lambda_{n_k} A w_{n_k}, z - y_{n_k} \rangle \ge 0, \quad \forall z \in C \subseteq C_{n_k}.$$

Since A is monotone, we have

$$
\begin{aligned}
0 &\le \langle y_{n_k} - w_{n_k}, z - y_{n_k} \rangle + \lambda_{n_k} \langle A w_{n_k}, z - y_{n_k} \rangle \\
&= \langle y_{n_k} - w_{n_k}, z - y_{n_k} \rangle + \lambda_{n_k} \langle A w_{n_k}, z - w_{n_k} \rangle + \lambda_{n_k} \langle A w_{n_k}, w_{n_k} - y_{n_k} \rangle \\
&\le \langle y_{n_k} - w_{n_k}, z - y_{n_k} \rangle + \lambda_{n_k} \langle Az, z - w_{n_k} \rangle + \lambda_{n_k} \langle A w_{n_k}, w_{n_k} - y_{n_k} \rangle.
\end{aligned}
$$

Letting $k \to \infty$ in the last inequality, and applying $\lim_{k \to \infty} \|y_{n_k} - w_{n_k}\| = 0$ and $\lim_{k \to \infty} \lambda_{n_k} = \lambda > 0$, we have

$$\langle Az, z - x^* \rangle \ge 0, \quad \forall z \in C.$$

Applying Lemma 2.8, we obtain $x^* \in VI(C, A)$. $\qquad\square$

At this point, we state and prove the strong theorem for our proposed algorithm.

Theorem 4.6 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then $\{x_n\}$ converges strongly to $\hat{x} \in VI(C, A)$, where $\hat{x} = \min\{\|p\| : p \in VI(C, A)\}$.*

Proof Since $\hat{x} = \min\{\|p\| : p \in VI(C, A)\}$, we have $\hat{x} = P_{VI(C,A)}(0)$. From Lemma 4.4, we have

$$\|x_{n+1} - \hat{x}\|^2 \leq (1 - \alpha_n)\|x_n - \hat{x}\|^2 + \alpha_n \left[3M_2(1 - \alpha_n)^2 \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| + 2\langle \hat{x}, \hat{x} - x_{n+1} \rangle \right]$$

$$= (1 - \alpha_n)\|x_n - \hat{x}\|^2 + \alpha_n b_n,$$

(4.28)

where $b_n = 3M_2(1 - \alpha_n)^2 \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| + 2\langle \hat{x}, \hat{x} - x_{n+1} \rangle$. Now, we claim that the sequence $\{\|x_n - \hat{x}\|\}$ converges to zero. To establish this, by Lemma 2.10 it suffices to show that $\limsup\limits_{k \to \infty} b_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - \hat{x}\|\}$ of $\{\|x_n - \hat{x}\|\}$ satisfying

$$\liminf_{k \to \infty} \left(\|x_{n_k+1} - \hat{x}\| - \|x_{n_k} - \hat{x}\| \right) \geq 0.$$

(4.29)

Suppose that $\{\|x_{n_k} - \hat{x}\|\}$ is a subsequence of $\{\|x_n - \hat{x}\|\}$ such that (4.29) holds. Again, from Lemma 4.4 we obtain

$$\beta_{n_k}(1 - \alpha_{n_k}) \left[1 - \delta^2 \frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2} - 2\delta K \frac{\lambda_{n_k}}{\lambda_{n_k+1}} \right] \|w_{n_k} - y_{n_k}\|^2$$

$$\leq (1 - \alpha_{n_k})^2 \|x_{n_k} - \hat{x}\|^2 - \|x_{n_k+1} - \hat{x}\|^2 + 3M_2 \alpha_{n_k}(1 - \alpha_{n_k})^2 \frac{\theta_{n_k}}{\alpha_{n_k}} \|x_{n_k} - x_{n_k-1}\|$$

$$+ 2\alpha_{n_k} \langle \hat{x}, \hat{x} - x_{n_k+1} \rangle.$$

Applying (4.29), Remark 3.4 and the fact that $\lim\limits_{k \to \infty} \alpha_{n_k} = 0$, we get

$$\beta_{n_k}(1 - \alpha_{n_k}) \left[1 - \delta^2 \frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2} - 2\delta K \frac{\lambda_{n_k}}{\lambda_{n_k+1}} \right] \|w_{n_k} - y_{n_k}\|^2 \to 0, \quad k \to \infty.$$

By the conditions on $\alpha_{n_k}, \beta_{n_k}$ and (4.17), we have

$$\|w_{n_k} - y_{n_k}\| \to 0, \quad k \to \infty.$$

(4.30)

Consequently, from (4.24) we get

$$\|z_{n_k} - w_{n_k}\| \to 0, \quad k \to \infty.$$

(4.31)

By Remark 3.4 (ii.), we have

$$\|x_{n_k} - w_{n_k}\| = \theta_{n_k} \|x_{n_k} - x_{n_k-1}\| \to 0, \quad k \to \infty.$$

(4.32)

Next, applying (4.31) and (4.32) we have

$$\|x_{n_k} - z_{n_k}\| \leq \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - z_{n_k}\| \to 0, \quad k \to \infty.$$

(4.33)

Now, using (4.32), (4.33) and the fact that $\lim\limits_{k \to \infty} \alpha_{n_k} = 0$ we obtain

$$\|x_{n_k+1} - x_{n_k}\| = \|(1 - \alpha_{n_k} - \beta_{n_k})(w_{n_k} - x_{n_k}) + \beta_{n_k}(z_{n_k} - x_{n_k}) - \alpha_n x_{n_k}\|$$

$$\leq (1 - \alpha_{n_k} - \beta_{n_k})\|w_{n_k} - x_{n_k}\| + \beta_{n_k}\|z_{n_k} - x_{n_k}\| + \alpha_{n_k}\|x_{n_k}\| \to 0, \quad k \to \infty.$$

(4.34)

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. It follows from (4.32) that $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Moreover, by Lemma 4.5 and (4.30) we have $x^* \in VI(C, A)$. Consequently, we have $w_\omega(x_n) \subset VI(C, A)$.

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup q$ and

$$\limsup_{k \to \infty} \langle \hat{x}, \hat{x} - x_{n_k} \rangle = \lim_{j \to \infty} \langle \hat{x}, \hat{x} - x_{n_{k_j}} \rangle.$$

Since $\hat{x} = P_{VI(C,A)}(0)$, we have

$$\limsup_{k \to \infty} \langle \hat{x}, \hat{x} - x_{n_k} \rangle = \lim_{j \to \infty} \langle \hat{x}, \hat{x} - x_{n_{k_j}} \rangle = \langle \hat{x}, \hat{x} - q \rangle \leq 0,$$

Hence, it follows from the last inequality and (4.34) that

$$\limsup_{k \to \infty} \langle \hat{x}, \hat{x} - x_{n_{k+1}} \rangle \leq 0. \tag{4.35}$$

Next, by Remark 3.4 (ii.), (4.31) and (4.35) we have $\limsup_{k \to \infty} b_{n_k} \leq 0$. Consequently, by invoking Lemma 2.10 it follows from (4.28) that $\{\|x_n - \hat{x}\|\}$ converges to zero as required. $\qquad\square$

5 Numerical Examples

In this section, we present some numerical experiments to illustrate the performance of our method, Algorithm 3.2 in comparison with Algorithms 7.1, 7.2, 7.3, and 7.4. All numerical computations were carried out using Matlab version R2019(b).

In our computations, we choose $\alpha_n = \frac{2}{3n+2}$, $\beta_n = \frac{1-\alpha_n}{2}$, $\xi_n = (\frac{2}{3n+2})^2$, $\phi_n = \frac{20}{(2n+5)^2}$, $\theta = 0.87$, $\lambda_1 = 0.93$ in our Algorithm 3.2, we choose $\tau = 0.0018$, $\rho_n = \frac{n}{4n+1}$ in Algorithm 7.1, $\lambda_{-1} = 0.0018$, $\varphi = 0.6$, $\mu = 0.8$ in Algorithm 7.2, $l = 0.018$ in Algorithm 7.3, and $f(x) = \frac{1}{3}x$ in Algorithms 7.3 and 7.4.

Example 5.1 Let the feasible set $C = \{x \in \mathbb{R}^2 : c(x) := x_1^2 + x_2 - 2 \leq 0\}$ and define the operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ by $A(x) = (6h(x_1), 4x_1 + 2x_2)$, where $x = (x_1, x_2) \in \mathbb{R}^2$ and

$$h(s) := \begin{cases} e(s-1) + e, & \text{if } s > 1, \\ e^s, & \text{if } -1 \leq s \leq 1, \\ e^{-1}(s+1) + e^{-1}, & \text{if } s < -1. \end{cases}$$

Then, it can easily be verified that A is monotone and $2\sqrt{9e^2 + 5}$-Lipschitz continuous. Also, c is a continuously differentiable convex function and c' is 2-Lipschitz

Table 1 Numerical Results for Example 5.1

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Algorithm 7.1	28	0.0142	28	0.0133	28	0.0152	28	0.0146
Algorithm 7.2	38	0.0065	38	0.0059	34	0.0066	37	0.0063
Algorithm 7.3	45	0.0065	45	0.0062	45	0.0069	45	0.0071
Algorithm 7.4	45	0.0100	45	0.0109	45	0.0103	46	0.0104
Algorithm 3.2	22	0.0094	22	0.0097	22	0.0091	22	0.0094

continuous. Moreover, we have that $K = 6\sqrt{e^2 + 1}$ (see He et al. 2018). Hence, we choose $\delta = 0.025$.

We test the algorithms for four different initial points as follows:

Case 1: $x_0 = (0.5, 1), x_1 = (1, 0.7)$;
Case 2: $x_0 = (1.3, 0.2), x_1 = (0.3, 1.5)$;
Case 3: $x_0 = (0.7, 0.9), x_1 = (0.4, 0.8)$;
Case 4: $x_0 = (1.2, 0.3), x_1 = (0.9, 1.1)$.

The stopping criterion used for this example is $|x_{n+1} - x_n| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figs. 1, 2, 3, and 4 and Table 1.

Example 5.2 Let $H = (\ell_2(\mathbb{R}), \| \cdot \|_2)$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < +\infty\}$, $\|x\|_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ and $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ for all $x \in \ell_2(\mathbb{R})$. Let $C = \{x \in H : c(x) := \|x\|^2 - 1 \leq 0\}$, and we define the operator $A : H \to H$

Fig. 1 Example 5.1 Case 1
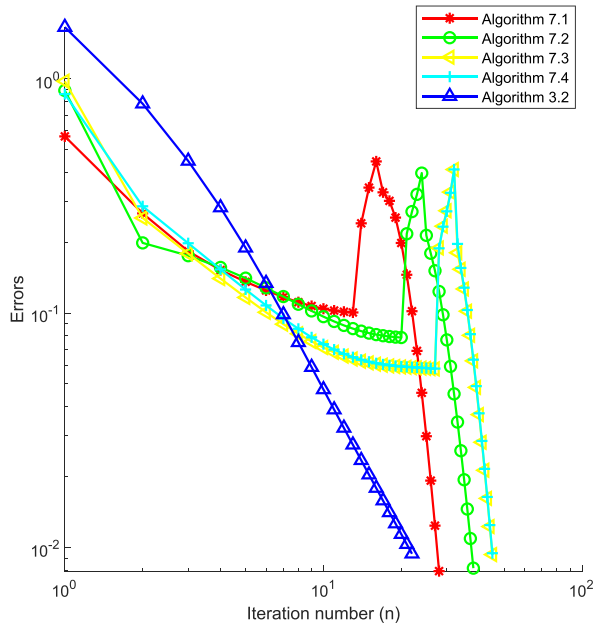
Fig. 2 Example 5.1 Case 2
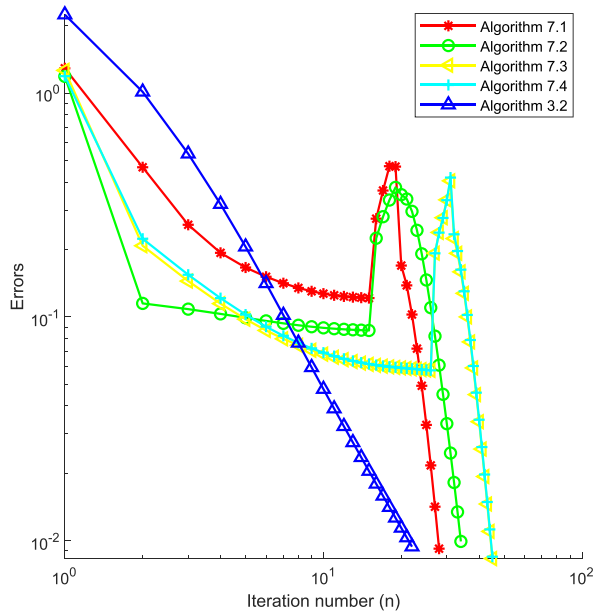
by $A(x) = 2x$, $\forall x \in H$. Then A is monotone and 2-Lipschitz continuous. Moreover, $K = 1$ and we choose $\delta = 0.4$.

We choose different initial values as follows:

Case 1: $x_0 = (2, 1, \frac{1}{2}, \cdots)$, $x_1 = (-3, 1, -\frac{1}{3}, \cdots)$;

Fig. 3 Example 5.1 Case 3
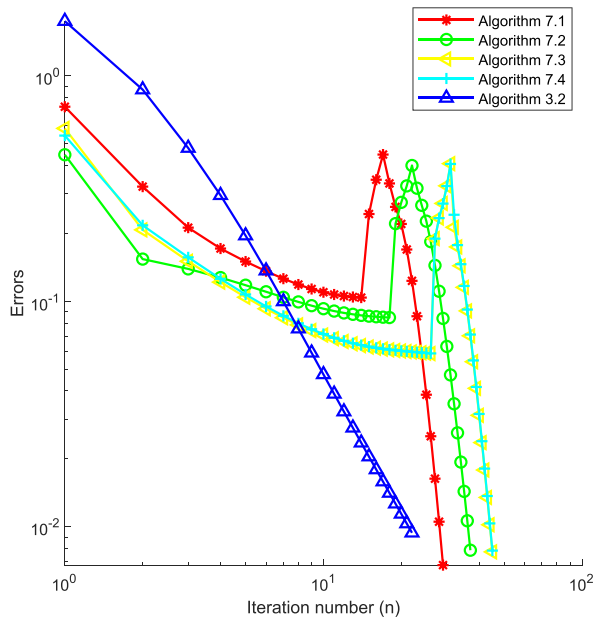
Fig. 4 Example 5.1 Case 4
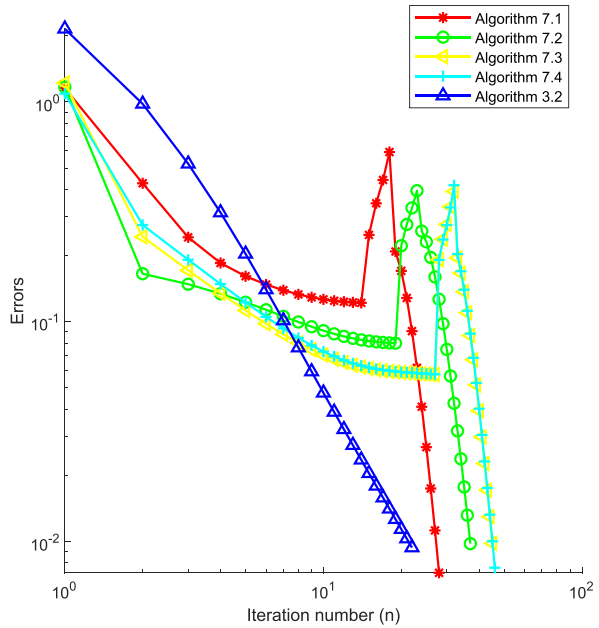
Case 2: $x_0 = (-2, 1, -\frac{1}{2}, \cdots), x_1 = (-4, 1, -\frac{1}{4}, \cdots)$;
Case 3: $x_0 = (2, 1, \frac{1}{2}, \cdots), x_1 = (-5, 1, -\frac{1}{5}, \cdots)$;
Case 4: $x_0 = (-2, 1, -\frac{1}{2}, \cdots), x_1 = (-3, 1, -\frac{1}{3}, \cdots)$.

The stopping criterion used for this example is $\|x_{n+1} - x_n\| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figs. 5, 6, 7, and 8 and Table 2.

Example 5.3 (**Application to Image Restoration Problem**)

In this last example, we apply our result to image restoration problem. We compare the efficiency of our Algorithm 3.2 with Algorithms 7.1, 7.3, and 7.4.

We recall that the image restoration problem can be formulated as the following linear inverse problem:

Table 2 Numerical Results for Example 5.2

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Algorithm 7.1	43	0.0240	48	0.0130	44	0.0125	50	0.0122
Algorithm 7.2	69	0.0085	77	0.0038	83	0.0033	69	0.0034
Algorithm 7.3	34	0.0097	39	0.0042	43	0.0043	34	0.0039
Algorithm 7.4	34	0.0064	39	0.0075	43	0.0077	34	0.0058
Algorithm 3.2	18	0.0067	18	0.0078	18	0.0082	18	0.0088

Fig. 5 Example 5.2 Case 1
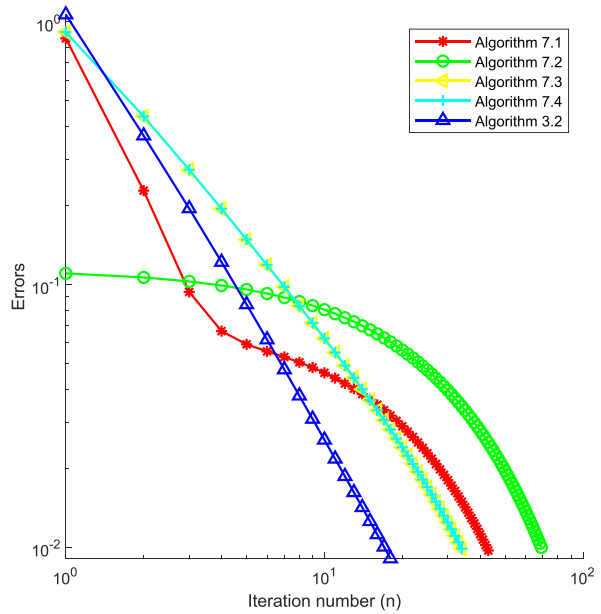
$$v = Dx + e \tag{5.1}$$

where $x \in \mathbb{R}^N$ is the original image, $D \in \mathbb{R}^{M \times N}$ is the blurring matrix, $v \in \mathbb{R}^M$ is the observed blurred image while e is the Gaussian noise. It is known that solving Problem (5.1) is equivalent to solving the convex minimization problem

Fig. 6 Example 5.2 Case 2
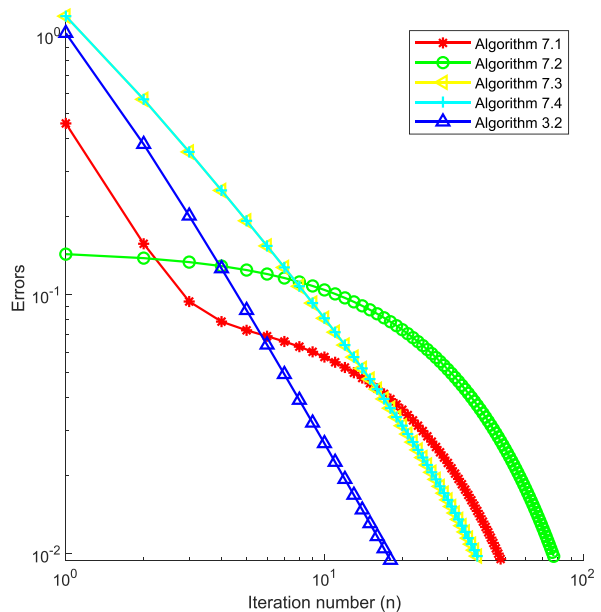
Fig. 7 Example 5.2 Case 3
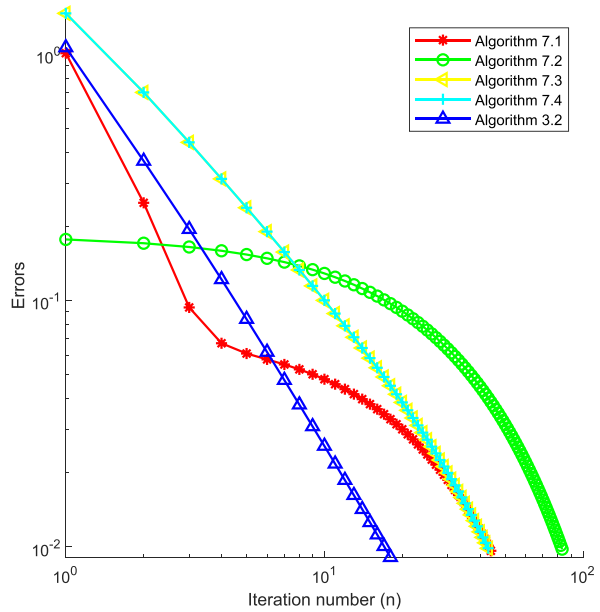
$$\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} ||Dx - v||_2^2 + \lambda ||x||_1 \right\}, \tag{5.2}$$

where $\lambda > 0$ is the regularization parameter, $\| \cdot \|_2$ denotes the Euclidean norm and $\| \cdot \|_1$ is the ℓ_1-norm. Our task here is to restore the original image x given the data

Fig. 8 Example 5.2 Case 4
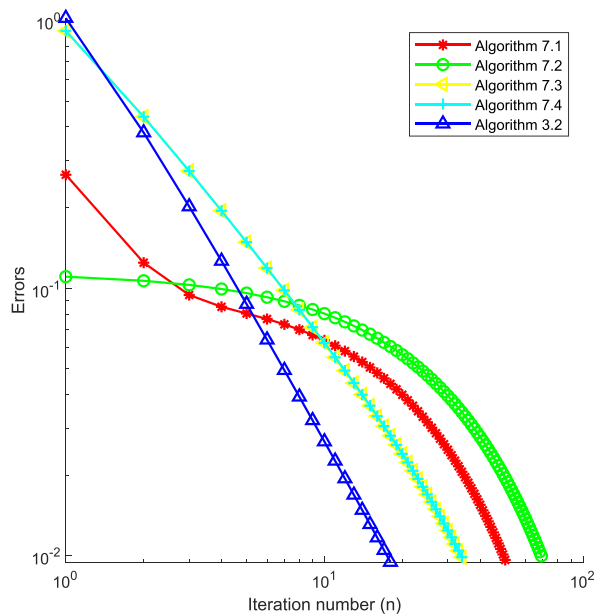
Table 3 Numerical Results for Example 5.3

	Pout Iter.	Cell Iter.	Shadow Iter.	Cameraman Iter.
Algorithm 7.1	3.1789	3.1389	3.1402	3.1231
Algorithm 7.3	3.6632	3.5917	3.6374	3.6827
Algorithm 7.4	4.8854	3.1389	4.1088	3.6827
Algorithm 3.2	5.3130	5.2272	5.2886	5.2501

of the blurred image v. The minimization problem (5.2) can be expressed as a variational inequality problem by setting $A := D^T(Dx - v)$. It is known in this case that the operator A is monotone and $\|D^T D\|$-Lipschitz continuous. We consider the 291×240 Pout, 159×191 Cell, 223×298 Shadow, and 256×256 Cameraman images from MATLAB Image Processing Toolbox. Moreover, we use the Gaussian blur of size 7×7 and standard deviation $\sigma = 4$ to create the blurred and noisy image (observed image) and use the algorithms to recover the original image from the blurred image. Also, we measure the quality of the restored image using the signal-to-noise ratio defined by

$$\text{SNR} = 20 \times \log_{10} \left(\frac{\|x\|_2}{\|x - x^*\|_2} \right),$$

where x is the original image and x^* is the restored image. Note that, the larger the SNR, the better the quality of the restored image. We choose the initial values as $x_0 = \mathbf{0} \in \mathbb{R}^N$ and $x_1 = \mathbf{1} \in \mathbb{R}^N$. The results are reported in Table 3, which shows the SNR values for each algorithm, and Figs. 9, 10, 11, 12, 13, 14, 15, and 16 shows the

Fig. 9 Example 5.3 Pout Figure
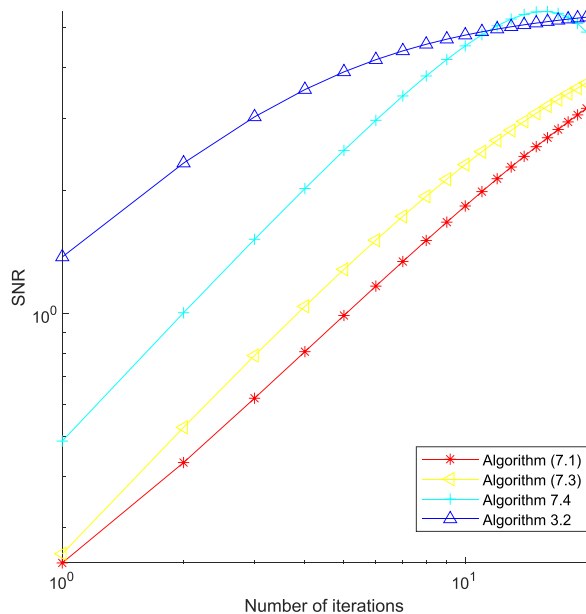
Fig. 10 Example 5.3 Pout Image
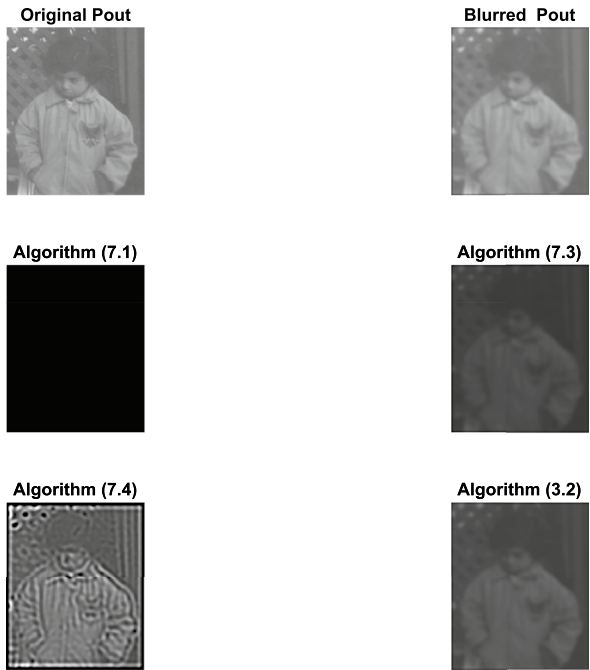
original, blurred and restored images. The major advantages of our proposed Algorithm 3.2 over the other algorithms compared with are the higher SNR values for generating the recovered images.

Fig. 11 Example 5.3 Cell Figure
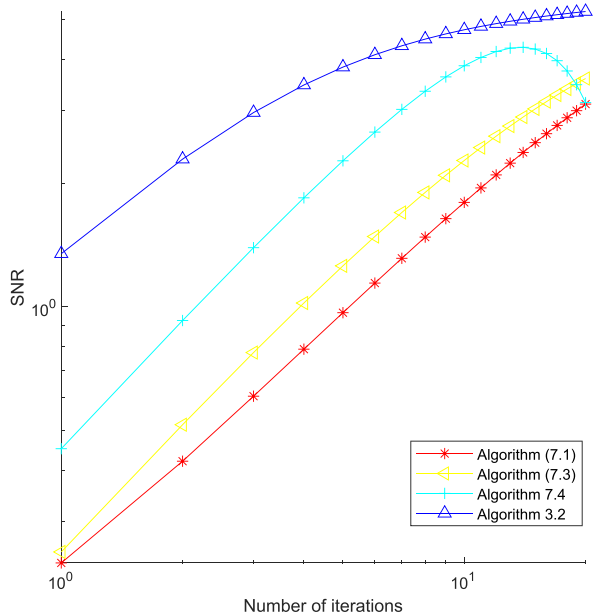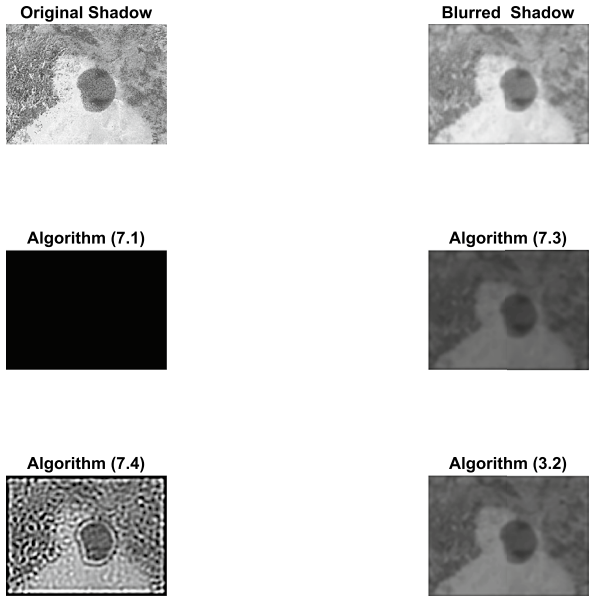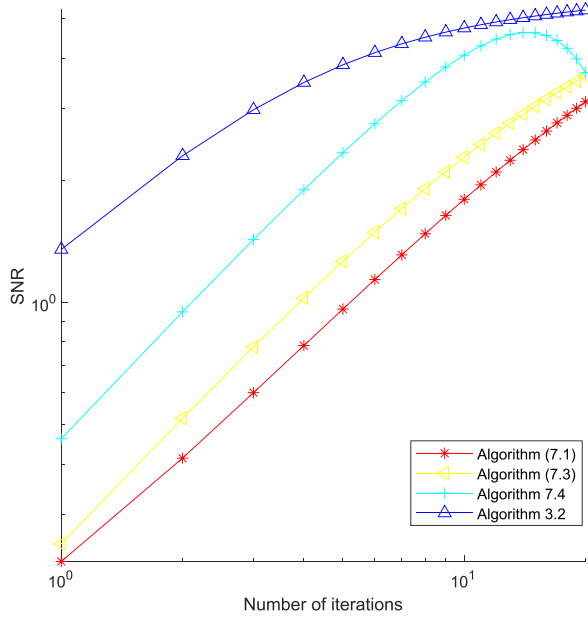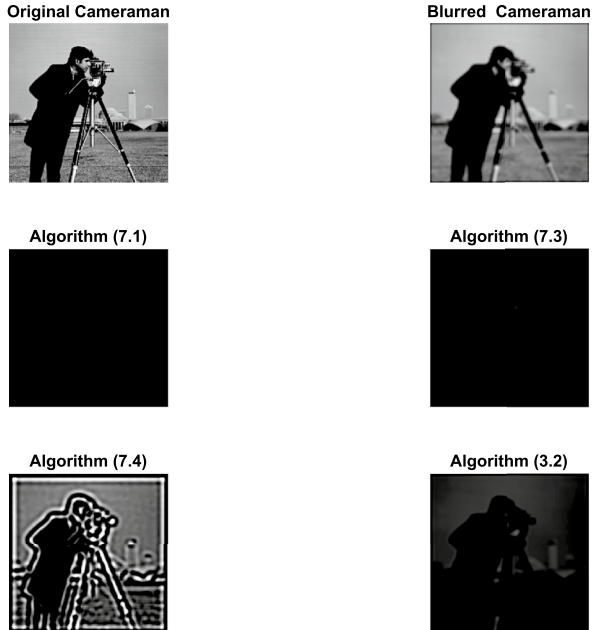
Fig. 12 Example 5.3 Cell Image
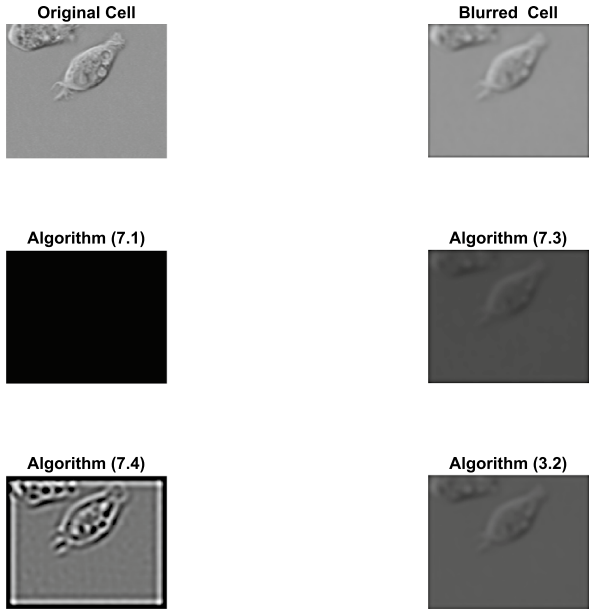
Fig. 13 Example 5.3 Shadow Figure
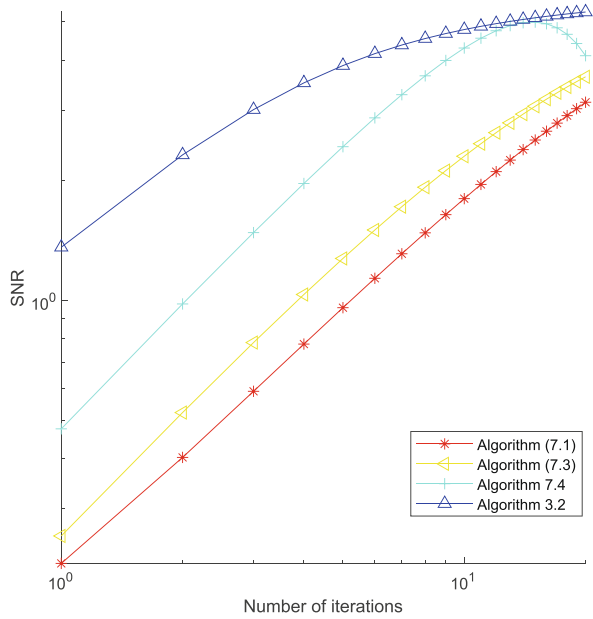
Fig. 14 Example 5.3 Shadow Image

Fig. 15 Example 5.3 Cameraman Figure

Fig. 16 Example 5.3 Camera-
man Image

Original Cameraman	Blurred Cameraman

Algorithm (7.1)	Algorithm (7.3)

Algorithm (7.4)	Algorithm (3.2)

6 Conclusion

In this paper we study the monotone VIP. We introduce a new inertial two-sub-gradient extragradient method for approximating the solution of the problem in Hilbert spaces. Unlike several of the existing results in the literature, our method does not require any linesearch technique which could be time-consuming to implement. Rather, we employ a more efficient self-adaptive step size technique which generates a non-monotonic sequence of step sizes. Moreover, under mild conditions we prove that the sequence generated by our proposed algorithm converges strongly to a minimum-norm solution of the VIP. Finally, we presented several numerical experiments and applied our result to image restoration problem. Our result complements the existing results in the literature in this direction.

Appendix

Algorithm 7.1 (Algorithm 8 in Cao and Guo 2020)

Step 0. Let $x_0, x_1 \in H$ be two arbitrary initial points and set $n = 1$.
Step 1. Compute

$$w_n = x_n + \rho_n(x_n - x_{n-1}),$$

Step 2. Construct the half-space

$$C_n = \{x \in H : c(w_n) + \langle c'(w_n), x - w_n \rangle \leq 0\},$$

and compute

$$y_n = P_{C_n}(w_n - \tau A w_n)$$

$$x_{n+1} = P_{C_n}(w_n - \tau A y_n)$$

where

$$0 < \tau \leq \frac{-\eta_p L_1 + \sqrt{\eta_p^2 L_1^2 + L_2^2 v^2}}{L_2^2},$$

$$0 < v \leq \frac{1 - 3\rho - \gamma}{1 - \rho + 2\rho^2 + \gamma}, \quad 0 < \gamma < 1 - 3\rho,$$

$$0 \leq \rho_n \leq \rho < \frac{1}{3}.$$

Set $n := n + 1$ and return to **Step 1**,

where $A : \mathcal{H} \to \mathcal{H}$ is monotone and L_2-Lipschitz continuous, $c'(\cdot)$ is L_1-Lipschitz continuous and η_p is the parameter in Lemma 2.7 (2).

Algorithm 7.2 (Algorithm 2 in Ma and Wang 2022)

Step 0. Let $x_{-1}, x_0, y_{-1} \in H; \varphi, \mu \in [a, b] \subset (0, 1); \lambda_{-1} \in (0, \frac{1-\varphi^2}{2\eta_p L_1}]$, set $n = 0$.

Step 1. Given λ_{n-1}, y_{n-1} and x_{n-1}. Let $p_{n-1} = x_{n-1} - y_{n-1}$.

$$\lambda_n = \begin{cases} \lambda_{n-1}, & \lambda_{n-1} \|A(x_{n-1}) - A(y_{n-1})\| \leq \varphi \|p_{n-1}\|, \\ \lambda_{n-1}\mu, & \text{Otherwise.} \end{cases}$$

Step 2. Compute

$$y_n = P_{C_n}(x_n - \lambda_n A x_n).$$

Step 3. Compute

$$x_{n+1} = P_{C_n}(y_n - \lambda_n(A y_n - A x_n)),$$

where

$$C_n = \{x \in H : c(w_n) + \langle c'(w_n), x - x_n \rangle \leq 0\}.$$

Step 4. Set $n := n + 1$ and return to **Step 1**,

where $A : \mathcal{H} \to \mathcal{H}$ is monotone and L_2-Lipschitz continuous, $c'(\cdot)$ is L_1-Lipschitz continuous and η_p is the parameter in Lemma 2.7 (2).

Algorithm 7.3 (Algorithm 3.1 in Shehu and Iyiola 2017)

Step 0. Given $l \in (0, 1)$, $\mu \in (0, 1)$. Let $x_1 \in \mathcal{H}$ be arbitrary.

Step 1. Compute

$$y_n = P_\mathcal{C}(x_n - \lambda_n Ax_n),$$

where $\lambda_n = l^{m_n}$, where m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|.$$

Step 2. Compute

$$z_n = P_{T_n}(x_n - \lambda_n Ay_n),$$

where

$$T_n = \{x \in \mathcal{H} : \langle x_n - \lambda_n Ax_n - y_n, x - y_n \rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n,$$

Set $n := n + 1$ and return to **Step 1**,

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where $A : \mathcal{H} \to \mathcal{H}$ is monotone and Lipschitz continuous, and $f : \mathcal{H} \to \mathcal{H}$ is a contraction.

Algorithm 7.4 (Algorithm 1 in Yang and Liu 2019)

Initialization: Given $\lambda_1 > 0$, $\mu \in (0, 1)$. Let $x_1 \in \mathcal{H}$ be arbitrary.

Step1. Given the current iterate x_n, compute

$$y_n = P_\mathcal{C}(x_n - \lambda_n Ax_n),$$

If $x_n = y_n$, then stop: x_n is a solution of the VIP. Otherwise,

Step 2: Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n,$$

and

$$\lambda_{n+1} = \begin{cases} \min \{\frac{\mu \|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\}, & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n, & \text{otherwise} \end{cases}$$

where

$$z_n = y_n + \lambda_n(Ax_n - Ay_n),$$

Set $n := n + 1$ and return to **Step 1**,

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where $A : \mathcal{H} \to \mathcal{H}$ is monotone and Lipschitz continuous, and $f : \mathcal{H} \to \mathcal{H}$ is a contraction.

Declarations

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