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Approximation of a Continuous Core-periphery Model by Core-periphery Models with a Large Number of Small Regions

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Abstract

For a continuous core-periphery model, we construct a core-periphery model with n regions for each $n \geq 2$. Assuming that the known functions of the core-periphery model with n regions and the diameters of n regions converge to the known functions of the continuous core-periphery model and 0 respectively as the number n tends to infinity, this paper proves approximate relations between the continuous core-periphery model and core-periphery models with a large number of small regions when the models are in short-run equilibrium.

Keywords Short-run equilibrium · Core-periphery model · Wage equation · Equilibrium approximation · Error estimate

JEL Classification R12

1 Introduction

The most fundamental model in the New Economic Geography (NEG) is *Krugman's core-periphery model* (*KCP model*) (Krugman 1991). This model is a two-region model. However, an economy should sometimes be considered as consisting of n regions (*economy with n regions*), where $n > 2$ denotes an integer (Bosker et al. 2010). Hence, *core-periphery models* (*CP models*) *with n regions* are constructed as a multiregional version of KCP model (Fujita et al. 2001, pp. 61–77). Moreover, an economy should sometimes be considered as spread across a continuous space (*continuous*

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economy). Hence, to model a continuous economy, *continuous core-periphery models* (*CCP models*) are constructed as a continuous-space version of KCP model (Fujita et al. 2001, pp. 49, 79–95, 141). These models are ones of the most fundamental models in the NEG.

Many researchers have studied equilibria, agglomerations and bifurcations in the CP models with n regions (see, e.g., Ago et al. 2006; Akamatsu and Takayama 2009; Akamatsu et al. 2012; Barbero and Zofío 2016; Beckmann and Puu 1985; Behrens and Thisse 2007; Bosker et al. 2010; Castro et al. 2012; Forslid and Ottaviano 2003; Fujita et al. 2001, 2004; Fujita and Thisse 1996; Gallego and Zofío 2018; Gaspar et al. 2018; Ikeda et al. 2012, 2019; Ikeda and Murota 2014; Ikeda 2017; Krugman and Elizondo 1996; Krugman 1996, 1993; Mori and Nishikimi 2002; Mossay 2006; Mossay and Picard 2020; Pavlidis et al. 2007; Pflüger 2004; Tabata et al. 2014, 2015a, b; Tabata and Eshima 2018a, 2019). However, there are only a few studies on the CCP models (see, e.g., Fujita et al. 2001, pp. 82–95, 309–327; Ohtake and Yagi 2021; Tabata et al. 2013; Tabata and Eshima 2015a). Hence, in order to study the CCP models, it is a promising attempt to apply the predictions and implications derived from the CP models with n regions to the CCP models. For this attempt, this paper studies approximate relationships between the CP models with n regions and the CCP models.

It is reasonable to consider a continuous space as a set consisting of an infinite number of infinitesimally small regions. Hence, we can consider every CCP model as a CP model with an infinite number of infinitesimally small regions. Therefore, it is reasonable to conjecture that every CCP model is considered as the limit of a CP model with n regions as the number n and the diameter of each region tend to infinity and 0, respectively, that is, that every CCP model is approximated by CP models with a large number of small regions.

This approximation is intuitively plausible, but has not been hitherto proved. If we attempt to prove the approximation, then we encounter thefollowing difficulties:

1. To prove the approximate relation, we need to analyze how the CP models with n regions behave as the number n and the diameter of each region tend to infinity and 0, respectively. However, the CP models with n regions are increasingly complicated as the number n increases.
2. In the CP models with n regions, production, consumption and transportation are conducted discretely in space, and the distribution of wages is described by the *wage equation with n regions*, which is a discrete equation (Fujita et al. 2001, (4.35), (5.5)). In contrast, in the CCP models, production, consumption and transportation are conducted continuously in space, and the distribution of wages is described by the *continuous wage equation*, which is an integral equation (Fujita et al. 2001, (9.14)). There is an essential difference between the CP models with n regions and the CCP models. It is difficult to compare them.
3. The mathematical structure of the wage equation with n regions is clarified (Tabata and Eshima 2018b). However, the continuous wage equation has not been fully studied. The continuous wage equation is a nonlinear integral equa-

tion with a *double singular structure*, that is, it contains a nonlinear integral operator whose kernel is described as a fraction whose denominator is expressed by another nonlinear singular integral operator. It is difficult to approximate the continuous wage equation with high precision using linear integral operators (Tabata et al. 2013, Remark 2.3(ii)).

Considering these difficulties, as a first step, we study the approximate relation when the models are in *short-run equilibrium*, where each short-run equilibrium of the CCP model (the CP model with n regions) is defined as a solution of the continuous wage equation (the wage equation with n regions) when the distribution of workers, the distribution of farmers and the transport-cost function are given.

The main results of this paper manifest in the form of Theorems 9–12. First, we construct a CCP model and a CP model with n regions for each $n \geq 2$. Assuming that the known functions of the CP model with n regions and the diameters of n regions converge to the known functions of the CCP model and 0 respectively as the number n tends to infinity, we analyze how an arbitrary short-run equilibrium of the CP model with n regions behaves as the number n tends to infinity (Theorem 9). Second, we prove approximate relations between the set of all short-run equilibria of all CP models with a large number of small regions and a nonempty compact set of short-run equilibria of the CCP model (Theorem 10). Third, Theorem 11 provides a sufficient condition for the CCP model and every CP model with a large number of small regions to have unique short-run equilibria that are approximately equal to each other. Finally, Theorem 12 provides an error estimate for the approximation proved in Theorem 11.

The remainder of this paper consists of 6 sections and Appendices A, B, C, and D. Section 2 provides the methods. Section 3 presents the preliminaries. Section 4 reviews the wage equation with n regions and the continuous wage equation. Section 5 provides sufficient conditions and necessary conditions for the CCP model and the CP model with n regions to have short-run equilibria (Propositions 4 and 5). Section 6 states and discusses Theorems 9–12. Moreover, to help us to understand the main results, Section 6 provides numerical simulations. Section 7 is the conclusion section. Appendices A, B, and C prove Theorems 9–12. Appendix D analyzes the numerical simulations. Advanced knowledge of neither integral equations nor functional analysis is used.

2 Methods

This paper uses the following methods:

1. We construct a CCP model (Fujita et al. 2001, pp. 140–143; Tabata et al. 2013). Let the CCP model be fixed, and denote it by M. For each $n \geq 2$, we construct a CP model with n regions. We denote by $\{\mathbf{M}_n\}$ the sequence of models thus constructed. To overcome the difficulty mentioned in (2) of Section 1, we equivalently transform the wage equation with n regions to an integral equation whose

unknown function is a step function that is constant in each region. This integral equation is also called the wage equation with n regions. No confusion should arise. We assume that the maximum of diameters of the n regions and the known functions of $\mathbf{M}_n$ converge to 0 and the known functions of M, respectively, as n tends to infinity (Conditions 6 and 7 in Section 6).

2. Proposition 4 provides sufficient conditions and necessary conditions for $\mathbf{M}_n$ to have a short-run equilibrium for each $n \geq 2$. In particular, Proposition 4(i)(iii) proves that $\mathbf{W}_n$ is nonempty and compact for each $n \geq 2$, where $\mathbf{W}_n$ denotes the set of all short-run equilibria of $\mathbf{M}_n$ (the set of all solutions of the wage equation with n regions) for each $n \geq 2$. To overcome the difficulty mentioned in (3) of Section 1, Proposition 5 provides sufficient conditions and necessary conditions for M to have a short-run equilibrium. In particular, Proposition 5(i)(iii) proves that W is nonempty and compact, where W denotes the set of all short-run equilibria of M (the set of all solutions of the continuous wage equation).
3. Using Conditions 6 and 7, we define *a CP model with a large number of small regions* as $\mathbf{M}_n$ when n is sufficiently large. Moreover, we define *the set of all short-run equilibria of a CP model with a large number of small regions* as $\mathbf{W}_n$ when n is sufficiently large. Hence, if n is sufficiently large, then we refer to $\cup_{m\geq n}\mathbf{W}_m$ as *the set of all short-run equilibria of all CP models with a large number of small regions*.
4. For each n, we attempt to substitute an arbitrary $\mathbf{w}_n \in \mathbf{W}_n$ in the continuous wage equation. Even if n is sufficiently large, then the equality does not hold in general. However, the difference between the left-hand side and the right-hand side is small when n is sufficiently large (Proposition 8). In this sense, every short-run equilibrium of all CP models with a large number of small regions approximately satisfies the continuous wage equation. We use this result to prove Theorems 9–12.
5. To overcome the difficulty mentioned in (1) of Section 1, we analyze an arbitrary sequence $\{\mathbf{w}_n\}$ such that $\mathbf{w}_n \in \mathbf{W}_n$ for all $n \geq 2$. In general, $\{\mathbf{w}_n\}$ itself is not convergent. However, Theorem 9(i) proves that $\{\mathbf{w}_n\}$ is relatively compact, and Theorem 9(ii) proves that every accumulation point of $\{\mathbf{w}_n\}$ is a short-run equilibrium of M (see, e.g., Yoshida 1965, pp. 4, 5). Theorem 9(iii) clarifies the structure of $\{\mathbf{w}_n\}$.
6. Theorem 10(i) proves that $\cup_{m\geq n}\mathbf{W}_m$ is relatively compact for each $n \geq 2$. We prove that there exists a nonempty compact set $W_\infty \subseteq W$ such that if N is sufficiently large, then $\cup_{m\geq n}\mathbf{W}_m$ approximately coincides with W_∞ for all $n \geq N$, that is, $\cup_{m\geq n}\mathbf{W}_m$ and W_∞ can be approximated by each other for every $n \geq N$. Hence, to obtain approximate relations between W and $\cup_{m\geq n}\mathbf{W}_m$, in the following (7) and (8) we analyze how $W_\infty \subseteq W$ and $\cup_{m\geq n}\mathbf{W}_m$ can be approximated by each other.
7. To describe how $\cup_{m\geq n}\mathbf{W}_m$ can be approximated by W_∞, we prove that there exist an integer $N \geq 2$ and a nonempty finite subset $W_0 \subseteq W_\infty$ such that $\cup_{m\geq n}\mathbf{W}_m$ is included in the small neighborhood of W_0 for every $n \geq N$ (Theorem 10(ii)). Hence, if n is sufficiently large, then $\cup_{m\geq n}\mathbf{W}_m$ can be approximated by W_0.
8. To describe how W_∞ can be approximated by $\cup_{m\geq n}\mathbf{W}_m$, we prove that for each $n \geq 2$ there exists a finite subset $\mathbf{W}_0 \subseteq \cup_{m\geq n}\mathbf{W}_m$ such that W_∞ is included in the

small neighborhood of $\mathbf{W}_0$ (Theorem 10(iii)). Hence, W_∞ can be approximated by $\mathbf{W}_0$.

9. To analyze the set of all short-run equilibria of all CP models with a large number of small regions, we describe how $\cup_{m\geq n}\mathbf{W}_m$ behaves as n increases. We prove that if n is sufficiently large, then there exists a finite subset $\mathbf{W}_0 \subseteq \cup_{m\geq n}\mathbf{W}_m$ such that $\cup_{m\geq n}\mathbf{W}_m$ is included in the small neighborhood of $\mathbf{W}_0$, that is, $\cup_{m\geq n}\mathbf{W}_m$ can be approximated by $\mathbf{W}_0$ (Theorem 10(iv)).
10. Theorem 9 cannot imply that $\{\mathbf{w}_n\}$ itself is convergent. Theorem 10 cannot imply that $W_\infty = W$. Moreover, Theorem 10 cannot provide an error estimate for the approximations proved in Theorem 10. To prove these results, we need to fully observe how $\mathbf{w}_n$ behaves as the number n increases. Recalling Conditions 6 and 7, we find it necessary to analyze how the wage distribution changes along with the worker distribution, the farmer distribution, and the transport-cost function. However, if the number n of regions is large, then the wage distribution changes significantly and complexly. Hence, to facilitate this analysis, we seek a sufficient condition for the wage distribution to change gradually even when the number n is large. As such a condition, this paper assumes that the elasticity of substitution and the maximum of transport costs are sufficiently small in comparison with the manufacturing expenditure (this condition is discussed fully in Section 5). Using this condition, we can prove that if the changes in the worker distribution, the farmer distribution, and the transport-cost function are small, then also the change in the wage distribution is small even when the number n is large (Lemma 18). Using this lemma, Theorem 11 proves that $W = W_\infty = \{w\}$, and $\mathbf{w}_n \to w$ as $n \to +\infty$. Theorem 12 provides an error estimate for the approximation obtained in Theorem 11.

3 Preliminaries

We assume that economic activities are conducted in D, where D denotes a closed bounded domain of an Euclidean space. For simplicity we assume that the boundary of D is piece-wise smooth. We seek solutions of the continuous wage equation in Banach spaces of continuous functions of $x \in D$, and solutions of the wage equation with n regions in spaces of step functions of $x \in D$. We compare these solutions in Banach spaces of Lebesgue-measurable functions of $x \in D$. This section defines these function spaces.

First, we define the Banach spaces of Lebesgue-measurable functions of $x \in D$. We denote, by $L^1(D)$ ($L^\infty(D)$), the Banach space of all Lebesgue-summable (essentially bounded) functions of $x \in D$. These spaces are equipped with the following respective norms:

$$||g||_{L^1(D)} := \int_{x\in D} |g(x)|dx, \ \ ||g||_{L^\infty(D)} := \operatorname*{ess\,sup}_{x\in D}|g(x)|. \tag{1}$$

We define the following subsets of $L^i(D), i = 1, \infty$:

$$L^i_{0+}(D) := \{g(x) \in L^i(D);\ \operatorname*{ess\,inf}_{x\in D} g(x) \geq 0\},\ i = 1, \infty, \tag{2}$$

$$L^i_{+}(D) := \{g(x) \in L^i(D);\ \operatorname*{ess\,inf}_{x\in D} g(x) > 0\},\ i = 1, \infty. \tag{3}$$

For every $h(x) \in S$ we define

$$S(h) := \{g(x) \in L^\infty_{0+}(D);\ g(x)h(x) \in S\}, \tag{4}$$

where

$$S := \{h(x) \in L^1_{0+}(D);\ \int_{y\in D} h(y)dy = 1\}. \tag{5}$$

Second, we define the Banach spaces of continuous functions of $x \in D$. We denote by $C(D)$ the Banach space of all continuous functions of $x \in D$. This space is equipped with the following norm:

$$||g||_{C(D)} := \max_{x\in D} |g(x)|. \tag{6}$$

We define the following subsets of $C(D)$:

$$C_{0+}(D) := C(D) \cap L^\infty_{0+}(D), \tag{7}$$

$$C_{+}(D) := C(D) \cap L^\infty_{+}(D). \tag{8}$$

Third, we define the spaces of step functions of $x \in D$. For each integer $n \geq 2$, we divide D as follows:

$$D = \bigcup_{i=1}^{n} d^i_n, \tag{9}$$

where d^i_n, $i = 1, ..., n$, are disjoint non-null subsets of D. For simplicity, we assume that these subsets are domains whose boundaries are piece-wise smooth. Each subset represents a *region*. For each $n \geq 2$, we denote by $\mathbf{L}^n(D)$ the set of all step functions of $x \in D$ that are constant for all $x \in d^i_n$, $i = 1, ..., n$, where we add superscript n to $\mathbf{L}^n(D)$ to emphasize that this space depends on division (9). Note that $\mathbf{L}^n(D)$ is an n-dimensional subspace of $L^\infty(D)$ for each $n \geq 2$. We define the following subsets of $\mathbf{L}^n(D)$:

$$\mathbf{L}^n_{0+}(D) := \mathbf{L}^n(D) \cap L^\infty_{0+}(D), \tag{10}$$

$$\mathbf{L}^n_{+}(D) := \mathbf{L}^n(D) \cap L^\infty_{+}(D), \tag{11}$$

$$\mathbf{S}^n := \mathbf{L}^n(D) \cap S. \tag{12}$$

To consider transportation costs, we define spaces of functions of $(x, y) \in D \times D$. We denote, by $L^\infty(D \times D)$ $(C(D \times D))$, the Banach space of all essentially bounded (continuous) functions of $(x, y) \in D \times D$. By $\mathbf{L}^n(D \times D)$ we denote the space of all step functions of $(x, y) \in D \times D$ that are constant for all $(x, y) \in d_n^i \times d_n^j, i, j = 1, ..., n$. Replacing D with $D \times D$ in (1), (3), (6), (8) and (11), we can define $|| \cdot ||_{L^\infty(D\times D)}$, $L_+^\infty(D \times D)$, $|| \cdot ||_{C(D\times D)}$, $C_+(D \times D)$ and $\mathbf{L}_+^n(D \times D)$.

4 Wage Equations

First, assuming that workers and farmers are distributed continuously in D, we introduce the continuous wage equation. The continuous wage equation is the following nonlinear integral equation (Fujita et al. 2001, (9.14); Tabata et al. 2013, (2.4)–(2.7)):

$$w(x)^\sigma = f(w;\lambda, \phi, \tau)(x), \tag{13}$$

where

$$f(u;\lambda, \phi, \tau)(x) := \int_{y\in D} G(u;\lambda, \tau)(y)^{\sigma-1} Y(u;\lambda, \phi)(y)\tau(x, y)^{-(\sigma-1)} dy, \tag{14}$$

$$G(u;\lambda, \tau)(y) := \{\frac{1}{\int_{z\in D} \lambda(z)(1/u(z))^{\sigma-1}\tau(y, z)^{-(\sigma-1)} dz}\}^{1/(\sigma-1)}, \tag{15}$$

$$Y(u;\lambda, \phi)(y) := \mu\lambda(y)u(y) + (1 - \mu)\phi(y), \tag{16}$$

and $w(x)$ is an unknown function that represents the nominal wage at point $x \in D$. Integral operator (15) represents the *price index* at point $y \in D$, and (16) is *income* at point $y \in D$. The elasticity of substitution σ and the manufacturing expenditure μ are constants such that

$$\sigma > 1, \tag{17}$$

$$0 < \mu < 1. \tag{18}$$

Moreover, $\lambda(x)$, $\phi(x)$ and $\tau(x, y)$ are known functions that satisfy the following condition:

Condition 1

$$\lambda(x) \in S, \tag{19}$$

$$\phi(x) \in S, \tag{20}$$

$$\tau(x, y) \in L_+^\infty(D \times D),\ \ \tau(x, y) \geq 1 \textit{ for a.e. } (x, y) \in D \times D, \tag{21}$$

$$\tau(x,y) = \tau(y,x) \textit{ for a.e. } (x,y) \in D \times D. \tag{22}$$

Let us explain this condition. We denote, by $\lambda(x)$ and $\phi(x)$, the shares of point $x \in D$ in the worker supply and the farmer supply, respectively (Fujita et al. 2001, p. 62). Hence, we assume (19) and (20). Recalling that μ is the manufacturing expenditure, we see that $\mu\lambda(x)$ and $(1-\mu)\phi(x)$ are equal to the densities of workers and farmers at $x \in D$. Moreover, we denote, by $\tau(x,y)$, the transportation cost from point y to point x (Fujita et al. 2001, p. 49). It follows from (22) that the cost of transportation between any two points is independent of the direction of transportation.

Second, we introduce the wage equation with n regions for each $n \geq 2$. Dividing D as done in (9), we choose a point contained in d_n^i and denote it by p_n^i for each $i = 1, ..., n$. We assume that workers and farmers are distributed in the following finite set of points:

$$D_n := \{p_n^1, ..., p_n^n\}.$$

The wage equation with n regions is the following discrete equation (Fujita et al. 2001, (5.3)–(5.5)):

$$\mathbf{v}_n(i)^\sigma = \sum_{j=1}^{n} \mathbf{G}(\mathbf{v}_n;\mathbf{\Lambda}_n,\mathbf{T}_n)(j)^{\sigma-1}\mathbf{Y}(\mathbf{v}_n;\mathbf{\Lambda}_n,\mathbf{\Phi}_n)(j)\mathbf{T}_n(i,j)^{-(\sigma-1)}, \tag{23}$$

where $\mathbf{v}_n(i)$ is an unknown that represents the wage at p_n^i for each $i = 1, ..., n$, and

$$\mathbf{G}(\mathbf{v}_n;\mathbf{\Lambda}_n,\mathbf{T}_n)(j) := \left\{\frac{1}{\sum_{k=1}^{n}\mathbf{\Lambda}_n(k)(1/\mathbf{v}_n(k))^{\sigma-1}\mathbf{T}_n(j,k)^{-(\sigma-1)}}\right\}^{1/(\sigma-1)}, \tag{24}$$

$$\mathbf{Y}(\mathbf{v}_n;\mathbf{\Lambda}_n,\mathbf{\Phi}_n)(j) := \mu\mathbf{\Lambda}_n(j)\mathbf{v}_n(j) + (1-\mu)\mathbf{\Phi}_n(j). \tag{25}$$

For each $i = 1, ..., n$, we denote, by $\mathbf{\Lambda}_n(i)$ and $\mathbf{\Phi}_n(i)$, nonnegative known values, and for each $i,j = 1, ..., n$, we denote, by $\mathbf{T}_n(i,j)$, a positive known value. For each $j = 1, ..., n$, (24) represents the price index at p_n^j, and (25) is income at p_n^j. We assume the following condition (Fujita et al. 2001, pp. 61–64):

Condition 2

$$\sum_{j=1}^{n}\mathbf{\Lambda}_n(j) = 1,\ \mathbf{\Lambda}_n(i) \geq 0 \textit{ for each } i = 1, ..., n, \tag{26}$$

$$\sum_{j=1}^{n}\mathbf{\Phi}_n(j) = 1,\ \mathbf{\Phi}_n(i) \geq 0 \textit{ for each } i = 1, ..., n, \tag{27}$$

$$\mathbf{T}_n(i,j) \geq 1,\ \textit{for each } i,j = 1, ..., n, \tag{28}$$

$$\mathbf{T}_n(i,j) = \mathbf{T}_n(j,i),\ \textit{for each } i,j = 1, ..., n. \tag{29}$$

Let us explain this condition. For each $i = 1, ..., n$, in the same way as Fujita et al. (2001, p. 62), $\mathbf{\Lambda}_n(i)$ and $\mathbf{\Phi}_n(i)$ denote the shares of point p_n^i in the worker supply and the farmer supply, respectively. Hence, we assume conditions (26) and (27). Recalling that μ is the manufacturing expenditure, we see that $\mu\mathbf{\Lambda}_n(i)$ and $(1-\mu)\mathbf{\Phi}_n(i)$ are equal to the numbers of workers and farmers at p_n^i for each $i = 1, ..., n$. Workers and farmers are distributed in D_n, and there are no workers or farmers in the complement of D_n. Moreover, $\mathbf{T}_n(i,j)$ represents the transportation cost from p_n^j to p_n^i for each $i,j = 1, ..., n$, (Fujita et al. 2001, p. 49). It follows from (29) that the transportation cost between every two points of D_n is independent of the direction of transportation.

Third, as mentioned in (1) of Section 2, to approximate integral equation (13) by discrete equation (23), we equivalently transform discrete equation (23) to an integral equation whose unknown function is a step function. Using $\mathbf{\Lambda}_n(i)$, $\mathbf{\Phi}_n(i)$, and $\mathbf{T}_n(i,j)$, we define the following step functions contained in $\mathbf{L}^n(D)$ and $\mathbf{L}^n(D\times D)$:

$$\lambda_n(x) := \mathbf{\Lambda}_n(i)/|d_n^i|, \text{ for all } x \in d_n^i, \ i = 1, ..., n, \tag{30}$$

$$\boldsymbol{\phi}_n(x) := \mathbf{\Phi}_n(i)/|d_n^i|, \text{ for all } x \in d_n^i, \ i = 1, ..., n, \tag{31}$$

$$\boldsymbol{\tau}_n(x,y) := \mathbf{T}_n(i,j) \text{ for all } (x,y) \in d_n^i \times d_n^j, \ i,j = 1, ..., n, \tag{32}$$

where $|d_n^i|$ is the square measure of d_n^i, $i = 1, ..., n$. Moreover, using $\mathbf{v}_n(i)$, we define the following new unknown step function $\mathbf{w}_n(x) \in \mathbf{L}_{0+}^n(D)$:

$$\mathbf{w}_n(x) := \mathbf{v}_n(i) \text{ for all } x \in d_n^i, \ i = 1, ..., n. \tag{33}$$

Substituting (30)–(33) in (23)–(25), and using symbols (14)–(16), we can rewrite (23) as follows:

$$\mathbf{w}_n(x)^\sigma = f(\mathbf{w}_n;\lambda_n, \boldsymbol{\phi}_n, \boldsymbol{\tau}_n)(x), \tag{34}$$

where

$$f(\mathbf{w}_n;\lambda_n, \boldsymbol{\phi}_n, \boldsymbol{\tau}_n)(x)$$

$$= \int_{y\in D} G(\mathbf{w}_n;\lambda_n, \boldsymbol{\tau}_n)(y)^{\sigma-1} Y(\mathbf{w}_n;\lambda_n, \boldsymbol{\phi}_n)(y)\boldsymbol{\tau}_n(x,y)^{-(\sigma-1)}dy, \tag{35}$$

$$G(\mathbf{w}_n;\lambda_n, \boldsymbol{\tau}_n)(y) = \{\frac{1}{\int_{z\in D} \lambda_n(z)(1/\mathbf{w}_n(z))^{\sigma-1}\boldsymbol{\tau}_n(y,z)^{-(\sigma-1)}dz}\}^{1/(\sigma-1)}, \tag{36}$$

$$Y(\mathbf{w}_n;\lambda_n, \boldsymbol{\phi}_n)(y) = \mu\lambda_n(y)\mathbf{w}_n(y) + (1-\mu)\boldsymbol{\phi}_n(y). \tag{37}$$

Integral equation (34) is equivalent to discrete equation (23). Moreover, substituting (30)–(32) in (26)–(29), we see that Condition 2 is equivalent to the following condition:

Condition 3

$$\lambda_n(x) \in \mathbf{S}^n, \tag{38}$$

$$\boldsymbol{\phi}_n(x) \in \mathbf{S}^n, \tag{39}$$

$$\boldsymbol{\tau}_n(x,y) \in \mathbf{L}_+^n(D \times D),\ \ \boldsymbol{\tau}_n(x,y) \geq 1 \textit{ for all } (x,y) \in D \times D, \tag{40}$$

$$\boldsymbol{\tau}_n(x,y) = \boldsymbol{\tau}_n(y,x) \textit{ for all } (x,y) \in D \times D. \tag{41}$$

Considering (10)–(12), we see that if $\boldsymbol{\lambda}_n(x)$, $\boldsymbol{\phi}_n(x)$ and $\boldsymbol{\tau}_n(x,y)$ satisfy Condition 3, then they satisfy Condition 1 also. Hence, comparing (13) and (34), we can consider (34) with Condition 3 as a special case of (13) with Condition 1. Hence, in the same way as (13) with Condition 1, from (38) and (39), we see that $\mu\boldsymbol{\lambda}_n(x)$ and $(1-\mu)\boldsymbol{\phi}_n(x)$ are equal to the densities of workers and farmers at point $x \in D$. Moreover, it follows from (41) that the cost of transportation between any two points is independent of the direction of transportation.

Indeed (34) with Condition 3 is equivalent to (23) with Condition 2, but these equations are quite different in form. In fact, the former is an integral equation, but the latter is a discrete equation. Noting that $\boldsymbol{\lambda}_n(x)$ and $\boldsymbol{\phi}_n(x)$ are step functions that are constant in small domain d_n^i for each $i = 1, ..., n$, we see that workers and farmers are distributed uniformly in each small domain. We see that $\mathbf{w}_n(x)$ is an unknown step function that is equal to a constant representing the wage in each small domain. From (41) we see that $\boldsymbol{\tau}_n(x,y)$ is a step function that is constant for all $(x,y) \in d_n^i \times d_n^j$ for each $i,j = 1, ..., n$. Hence, this step function can be considered as the transportation cost between any two small domains.

From this explanation of Condition 3, we see that economic activities described by (34) are conducted uniformly in small domain d_n^i for each $i = 1, ..., n$. In contrast, from the explanation of Condition 2, we see that economic activities described by (23) are conducted at point p_n^i for each $i = 1, ..., n$.

Throughout the paper, we use not (23) with Condition 2 but (34) with Condition 3. As mentioned in (1) of Section 2, integral equation (34) is called the wage equation with n regions.

5 Short-run Equilibria

Each short-run equilibrium of the CP model with n regions is defined as a solution $\mathbf{w}_n(x) \in \mathbf{L}_+^n(D)$ of (34). The following proposition provides sufficient conditions and necessary conditions for the CP model with n regions to have a short-run equilibrium for each $n \geq 2$.

Proposition 4 *If Condition 3 holds for each $n \geq 2$, then the following statements (i)–(iv) hold for all $n \geq 2$:*

(i) *Equation* (34) *has a solution* $\mathbf{w}_n(x) \in \mathbf{L}^n_+(D)$.
(ii) *Every solution* $\mathbf{w}_n(x) \in \mathbf{L}^n_+(D)$ *of* (34) *satisfies that*

$$\mathbf{w}_n(x) \in S(\lambda_n), \tag{42}$$

$$1/\,\overline{\mathbf{w}}_n \le \mathbf{w}_n(x) \le \overline{\mathbf{w}}_n \text{ for all } x \in D, \tag{43}$$

$$1/\,\overline{\mathbf{w}}_n^{\sigma-1} \le G(\mathbf{w}_n;\lambda_n, \boldsymbol{\tau}_n)(x)^{\sigma-1} \le \overline{\mathbf{w}}_n^{\sigma} \text{ for all } x \in D, \tag{44}$$

where $\overline{\mathbf{w}}_n$ is a positive constant defined in terms of $\boldsymbol{\tau}_n(x, y)$ as follows:

$$\overline{\mathbf{w}}_n := \max_{x,y\in D} \boldsymbol{\tau}_n(x, y)^{\sigma-1}. \tag{45}$$

(iii) *The set of all solutions to* (34) *is a compact set of* $\mathbf{L}^n_+(D)$.
(iv) *If*

$$a_0(\mu, \sigma, \overline{\mathbf{w}}_n) < 1, \tag{46}$$

then (34) has a unique solution in $\mathbf{L}^n_+(D)$, where

$$a_0(\mu, \sigma, r) := \frac{\mu r^{2\sigma-1} + (\sigma - 1)r^{4\sigma-1}}{\sigma}. \tag{47}$$

Proof of Proposition 4(i)(ii) Recalling that Condition 2 and (23) are equivalent to Condition 3 and (34), respectively, we see that Tabata and Eshima (2018a, Theorem 1; 2018b, Appendix) immediately imply (i) and (ii). Appendices B and C prove Proposition 4(iii)(iv).

Let us discuss condition (46). Noting the superscript attached to (45), we consider the following inequality:

$$a_0(\mu, \sigma, t^{\sigma-1}) < 1, \tag{48}$$

where $t > 1$ is a variable. We can easily solve this inequality with respect to μ as follows:

$$\mu < m(\sigma, t), \tag{49}$$

where

$$m(\sigma, t) := \sigma t^{-(2\sigma-1)(\sigma-1)} - (\sigma - 1)t^{2\sigma(\sigma-1)}.$$

Figure 1 shows the graph of $\mu = m(\sigma, t)$ in (σ, t, μ) space. Observing this figure, we see that if $\sigma > 1$ and $t > 1$ are sufficiently small in comparison with $\mu \in (0, 1)$, then (49) holds, that is, (48) holds. Hence, recalling (45), we see that if the elasticity of substitution and the maximum of transportation costs are sufficiently small in comparison with the manufacturing expenditure, then (46) holds. Figure 2 shows the graph of $\mu = (\sigma - 1)/\sigma$ in (σ, t, μ) space. Comparing Figs. 1 and 2, we see that

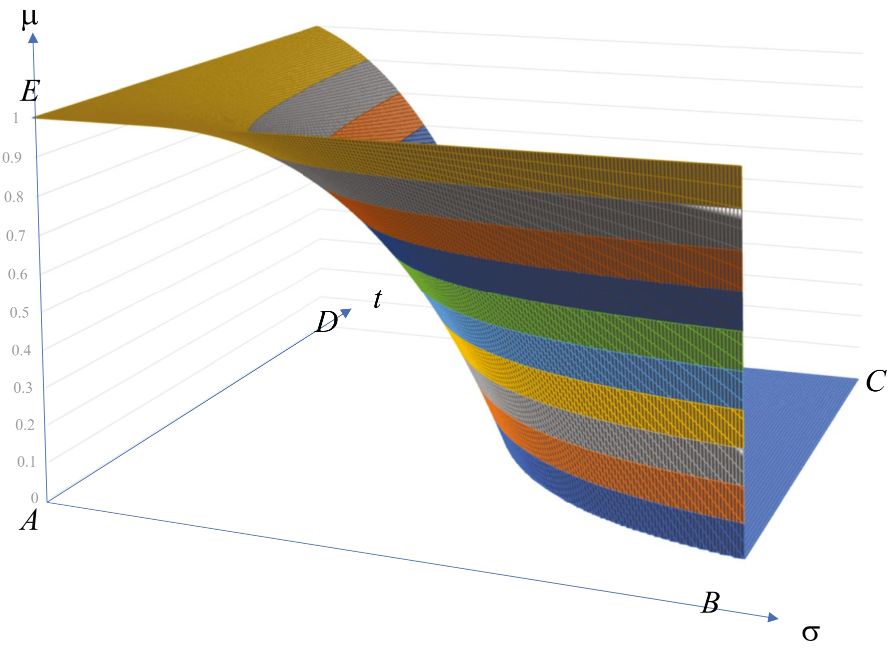

Fig. 1 The graph of $\mu = m(\sigma, t)$. $A = (1,1,0)$, $B = (2,1,0)$, $C = (2,2,0)$, $D = (1,2,0)$, $E = (1,1,1)$. If point $(\sigma, t, \mu) \in (1, +\infty) \times [1, +\infty) \times (0, 1)$ is under this surface, then condition (49) hold

domain (49) intersects not only the no-black-hole domain $\mu < (\sigma - 1)/\sigma$ but also the black-hole domain $\mu \geq (\sigma - 1)/\sigma$. Hence, condition (46) is compatible with the no-black-hole condition and the black-hole condition (Fujita et al. (2001), pp. 58, 59).

Lemma 18 proves that if the elasticity of substitution and the maximum of transport costs are sufficiently small in comparison with the manufacturing expenditure, then the distribution of wages changes *Lipschitz continuously* along with the worker distribution, the farmer distribution, and the transport-cost function. Lipschitz continuity is a strong form of uniform continuity (see, e.g., Sohrab 2003). If the wage distribution changes Lipschitz continuously along with the worker distribution, the farmer distribution, and the transport-cost function, then the change in the wage distribution is not greater than a linear combination of changes in the worker distribution, the farmer distribution, and the transport-cost function. In Lemma 18, the coefficients of the linear combination are positive constants that are dependent on the elasticity of substitution, the manufacturing expenditure and the maximum of transport costs but are independent of the number n of regions. Hence, as mentioned in (10) of Section 2, if the changes in the worker distribution, the farmer distribution, and the transport-cost function are small, then also the change in the wage distribution is small even when the number n is large. Using this result, we can easily observe how $\mathbf{w}_n$ behaves as the number n increases.

Proposition 5(iv), Theorems 11 and 12 assume the same conditions as (46) for the same reason that we assume (46) in Proposition 4(iv). Applying these

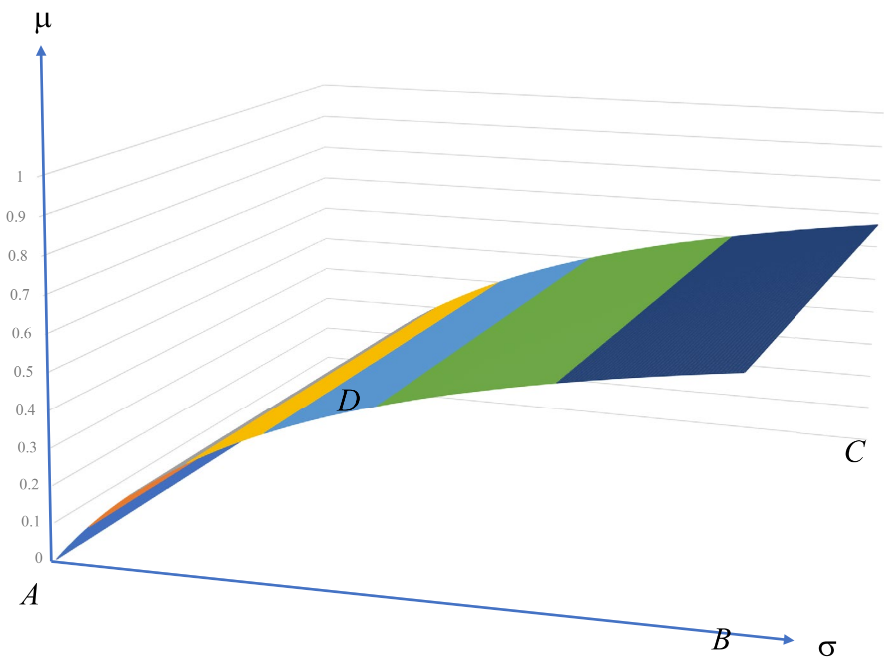

Fig. 2 The graph of $\mu = (\sigma - 1)/\sigma$. If point $(\sigma, t, \mu) \in (1, +\infty) \times [1, +\infty) \times (0, 1)$ is under this surface, then the no-black-holes condition holds. Points A, B, C, and D are the same as in Fig. 1

assumptions to Lemma 18, Appendix C proves Propositions 4(iv) and 5(iv) and Theorems 11 and 12.

Each short-run equilibrium of the CCP model is defined as a solution $w(x) \in L^{\infty}_{+}(D)$ of (13). The following proposition provides sufficient conditions and necessary conditions for the CCP model to have a short-run equilibrium.

Proposition 5 *If Condition 1 holds, then the following statements (i)–(iv) hold:*

(i) *If*

$$\lambda(x), \phi(x) \in C_{0+}(D), \tag{50}$$

$$\tau(x, y) \in C_{+}(D \times D), \tag{51}$$

then (13) has a solution $w(x) \in C_{+}(D)$.

(ii) *If (13) has a solution $w(x) \in L^{\infty}_{+}(D)$, then*

$$w(x) \in S(\lambda), \tag{52}$$

$$1/\overline{w} \leq w(x) \leq \overline{w} \text{ for a.e. } x \in D, \tag{53}$$

$$1/\overline{w}^{\sigma-1} \leq G(w;\lambda,\tau)(x)^{\sigma-1} \leq \overline{w}^{\sigma} \text{ for a.e. } x \in D, \tag{54}$$

where $\overline{w}$ is a positive constant defined in terms of $\tau(x,y)$ as follows:

$$\overline{w} := \operatorname*{ess\,sup}_{x,y \in D} \tau(x,y)^{\sigma-1}. \tag{55}$$

(iii) *If* (51) *holds, then the set of all solutions to* (13) *is a compact set of* $C(D)$.
(iv) *If* (13) *has solutions* $w_i(x) \in L^{\infty}_{+}(D), i = 1,2,$ *and*

$$a_0(\mu,\sigma,\overline{w}) < 1, \tag{56}$$

then $w_1(x) = w_2(x)$ for a.e. $x \in D$.

Appendices B and C prove Proposition 5. Let us compare Proposition 5 and our previous research (Tabata et al. 2013). Proposition 5 is a continuous version of Proposition 4. It was proved in Tabata et al. (2013, Theorem 3.1, (3.3)) that if the following inequality holds in addition to Condition 1 and the conditions of Proposition 5(i):

$$\mu < 1/\overline{w},$$

then (13) has a solution $w(x) \in C_{+}(D)$. Moreover, (53), (54) and (56) are more accurate than Tabata et al. (2013, (3.7), (3.8), (3.11)). Moreover, (52) was not obtained in Tabata et al. (2013, Theorem 3.1). For these reasons, Proposition 5 is an extension of Tabata et al. (2013, Theorem 3.1).

Applying (51) to (55), we see that

$$\overline{w} = \max_{x,y \in D} \tau(x,y)^{\sigma-1}. \tag{57}$$

Using Propositions 5(i)(iv), we see that if the elasticity of substitution and the maximum of transportation costs are sufficiently small in comparison with the manufacturing expenditure, then the CCP model has a unique short-run equilibrium. Applying (4), (5) and (12) to (42) and (52), we see that

$$\int_{y \in D} w(y)\lambda(y)dy = 1, \quad \int_{y \in D} \mathbf{w}_n(y)\boldsymbol{\lambda}_n(y)dy = 1 \text{ for all } n \geq 2.$$

Hence, the CCP model and the CP model with n regions satisfy that the total of wages is equal to 1.

6 Results and Discussion

This section states and discusses the main results (Theorems 9–12). Assume that $(\lambda(x), \phi(x), \tau(x,y))$ satisfies Condition 1 and the conditions of Proposition 5(i). Let $(\lambda(x), \phi(x), \tau(x,y))$ be fixed in what follows. Substituting it in (13), we define the

CCP model. As mentioned in (1) of Section 2, we denote the CCP model by M. We assume that division (9) satisfies the following condition:

Condition 6

$$\max_{i=1,\dots,n} \sup_{x,y\in d_n^i} |x-y| \to 0, \ \textit{as } n \to +\infty.$$

We assume that $\{(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))\}$ satisfies the following condition in addition to Condition 3:

Condition 7

$$\lim_{n\to+\infty} ||\boldsymbol{\lambda}_n - \lambda||_{L^1(D)} = 0, \tag{58}$$

$$\lim_{n\to+\infty} ||\boldsymbol{\phi}_n - \phi||_{L^1(D)} = 0, \tag{59}$$

$$\lim_{n\to+\infty} ||\boldsymbol{\tau}_n - \tau||_{L^\infty(D\times D)} = 0. \tag{60}$$

Substituting $(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))$ in (34), we define the CP model with n regions for each $n \geq 2$. As mentioned in (1) of Section 2, we denote by $\{\mathbf{M}_n\}$ the sequence of models thus defined.

It follows from Condition 6 that the maximum of diameters of n regions converges to 0 as n tends to infinity. Moreover, from Condition 7 we see that the known functions of $\mathbf{M}_n$ converges to those of M as n tends to infinity.

Note that $\{\mathbf{W}_n\}$ depends on $\{(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))\}$ and division (9) (recall (2) of Section 2). Conditions 6 and 7 have no influence on Proposition 4. Hence, $\mathbf{W}_n$ is nonempty and compact for each $n \geq 2$.

As mentioned in (4) of Section 2, even if n is sufficiently large, then $\mathbf{w}_n \in \mathbf{W}_n$ does not satisfy (13) in general, that is,

$$\mathbf{w}_n(x)^\sigma \neq f(\mathbf{w}_n; \lambda, \phi, \tau)(x).$$

However, the following proposition provides an estimate for the difference between the left-hand side and the right-hand side,

$$\theta_n(\mathbf{w}_n; \lambda, \phi, \tau)(x) := \mathbf{w}_n(x)^\sigma - f(\mathbf{w}_n; \lambda, \phi, \tau)(x). \tag{61}$$

Proposition 8 *If* $(\lambda(x), \phi(x), \tau(x,y))$, $(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))$ *and division* (9) *satisfy Conditions* 1, 3, 6, 7, *and the conditions of Proposition* 5(*i*), *then there exists an integer* $N_0 \geq 2$ *such that difference* (61) *satisfies the following inequality for each* $n \geq N_0$:

$$\sup_{\mathbf{w}_n\in\mathbf{W}_n} ||\theta_n(\mathbf{w}_n; \lambda, \phi, \tau)||_{L^\infty(D)}$$

$$\leq \overline{b}_1 ||\boldsymbol{\lambda}_n - \lambda||_{L^1(D)} + \overline{b}_2 ||\boldsymbol{\phi}_n - \phi||_{L^1(D)} + \overline{b}_3 ||\boldsymbol{\tau}_n - \tau||_{L^\infty(D\times D)}, \tag{62}$$

where $\overline{b}_i, i = 1, 2, 3,$ *are positive constants defined as follows*:

$$\overline{b}_i := b_i(\mu, \sigma, \overline{w}),\ i = 1, 2, \tag{63}$$

$$\overline{b}_3 := b_3(\sigma, \overline{w}). \tag{64}$$

Here $\overline{w}$ *is defined in* (57), *and*

$$b_1(\mu, \sigma, r) := 2(\mu r^{\sigma+1} + r^{3\sigma-1}), \tag{65}$$

$$b_2(\mu, \sigma, r) := 2(1 - \mu) r^{\sigma}, \tag{66}$$

$$b_3(\sigma, r) := 2(\sigma - 1)(r^{\sigma} + r^{3\sigma-1}). \tag{67}$$

Let us discuss Proposition 8. Inequality (62) shows that the norm of difference (61) is smaller than the linear combination of

$$||\boldsymbol{\lambda}_n - \lambda||_{L^1(D)},\ ||\boldsymbol{\phi}_n - \phi||_{L^1(D)},\ ||\boldsymbol{\tau}_n - \tau||_{L^\infty(D\times D)}. \tag{68}$$

These norms represent the differences between the known functions of $\mathbf{M}_n$ and those of M. Coefficients (63) and (64) are positive constants that are dependent on the manufacturing expenditure μ, the elasticity of substitution σ, and the maximum of transport costs $\overline{w}$ (see (57)) but independent of the number n of regions.

Let us consider how and why coefficients (63) and (64) depend on σ, μ, and $\overline{w}$. The coefficient $\overline{b}_1$ represents how much the upper bound of the norm of difference (61) depends on $||\boldsymbol{\lambda}_n - \lambda||_{L^1(D)}$. To analyze $\overline{b}_1$, we examine how the wage equation includes the manufacturing-worker distribution and the transportation-cost function. The price index includes the product of the manufacturing-worker distribution and the $-(\sigma - 1)$ power of the transportation-cost function. The income contains the manufacturing-worker distribution. Moreover, in the wage equation, the product of the income and the $(\sigma - 1)$ power of the price index is multiplied by the $-(\sigma - 1)$ power of the transportation-cost function. In the wage equation, the manufacturing-worker distribution is multiplied by the $-(\sigma - 1)$ power of the transportation-cost function in this way. Examining Proofs of Proposition 8 and Lemma 14 (see Appendices A and B), we see that it is a reflection of these multiplications that $\overline{b}_1$ includes $\overline{w}^{\sigma+1}$ and $\overline{w}^{3\sigma-1}$.

The first term $2\mu\overline{w}^{\sigma+1}$ of $\overline{b}_1$ represents how much the upper bound of the norm of difference (61) depends on the manufacturing-worker distribution included in the income. The first term includes μ. This inclusion reflects the fact that μ is the coefficient of the manufacturing-worker distribution included in the income. The second term $2\overline{w}^{3\sigma-1}$ of $\overline{b}_1$ represents how much the upper bound of the norm of difference (61) depends on the manufacturing-worker distribution included in the price index. The second term does not include μ. This reflects the fact that the price index does not contain μ.

The coefficient $\overline{b}_2$ represents how much the upper bound of the norm of difference (61) depends on $||\boldsymbol{\phi}_n - \phi||_{L^1(D)}$. The wage equation contains the product of the income and the $-(\sigma - 1)$ power of the transportation-cost function. The income contains the agricultural-worker distribution. Hence, the agricultural-worker distribution is multiplied by the $-(\sigma-1)$ power of the transportation-cost function. Examining Proofs of Proposition 8 and Lemma 14, we see that it is a reflection of this multiplication that $\bar{b}_2$ includes $\overline{w}^{\sigma}$. The coefficient $\bar{b}_2$ includes $(1-\mu)$. This inclusion reflects the fact that $(1-\mu)$ is the coefficient of the agricultural-worker distribution included in the income.

The coefficient $\bar{b}_3$ represents how much the upper bound of the norm of difference (61) depends on $||\boldsymbol{\tau}_n - \tau||_{L^\infty(D\times D)}$. The wage equation contains the $-(\sigma-1)$ power of the transportation-cost function in two places. The exponential function $\overline{w}^{\sigma} + \overline{w}^{3\sigma-1}$ reflects this structure of the wage equation. Moreover, the coefficient $(\sigma-1)$ is included in $\bar{b}_3$. Examining Proofs of Proposition 8 and Lemma 14, we see that this inclusion reflects the fact that the power $-(\sigma-1)$ is attached to the transportation-cost function in the wage equation.

Let us compare (65)–(67) where $r>1$ is a constant. We see easily that

$$b_1(\mu,\sigma,r) \geq b_2(\mu,\sigma,r) \text{ for all } \mu \in (0,1) \text{ and } \sigma > 1.$$

Moreover, if σ is large, then

$$b_3(\sigma,r) \geq b_1(\mu,\sigma,r) \text{ for all } \mu \in (0,1).$$

To help us to understand (63) and (64), Figs. 3, 4, and 5 show the graphs of (65)–(67) with $r=2.0$.

Applying Condition 7 to (62), we see that if $N \geq N_0$ is sufficiently large, then difference (61) is small for every $n \geq N$. In this sense, every short-run equilibrium of all CP models with a large number of small regions satisfies the continuous wage equation approximately. The integer N_0 will be defined in the last of this section. We use N_0 also in Theorems 11 and 12.

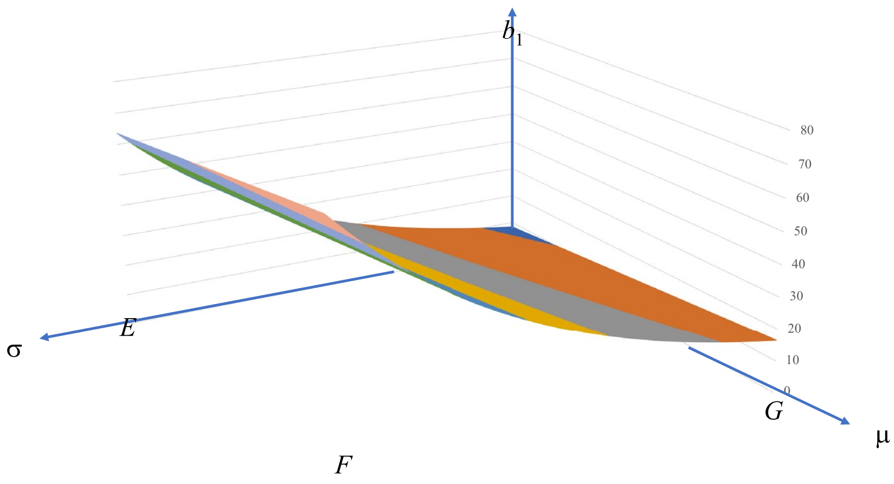

Fig. 3 The graph of $b_1(\mu,\sigma,2.0)$. $E=(0,2,0)$, $F=(1,2,0)$, $G=(1,1,0)$. The maximum scale of the vertical axis is equal to 80.0

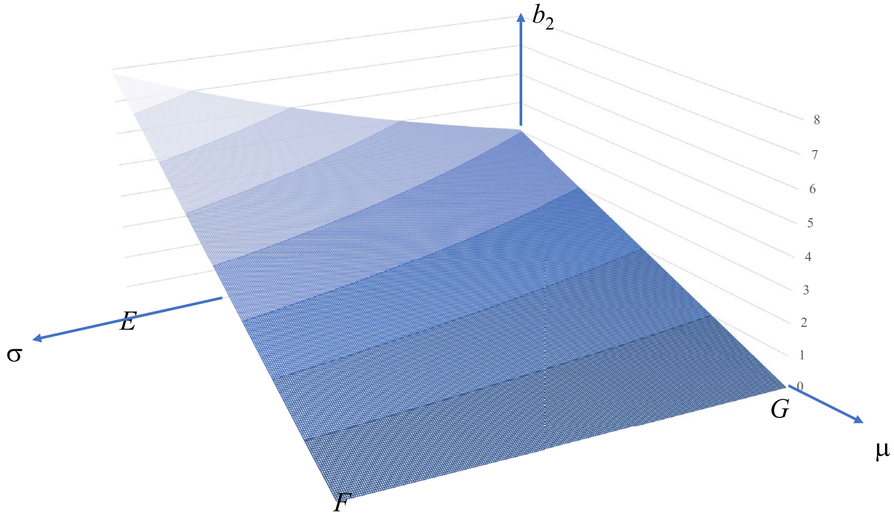

Fig. 4 The graph of $b_2(\mu, \sigma, 2.0)$. Points E, F, and G are the same as in Fig. 3. The maximum scale of the vertical axis is equal to 8.0

Let us examine the consistency of the conditions of Proposition 8 (Conditions 1, 3, 6, 7, and the conditions of Proposition 5(i)). For example, we define $(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x, y))$ as follows using $(\lambda(x), \phi(x), \tau(x, y))$ for each $n \geq 2$:

$$\boldsymbol{\lambda}_n(x) := \int_{u \in d_n^i} \lambda(u) du / |d_n^i|, \text{ for all } x \in d_n^i, \ i = 1, ..., n, \tag{69}$$

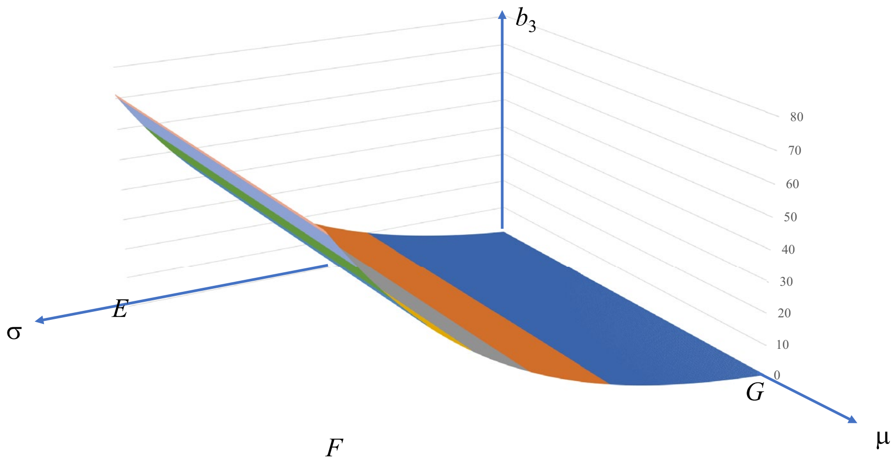

Fig. 5 The graph of $b_3(\sigma, 2.0)$. Points E, F, and G are the same as in Fig. 3. The scale of the vertical axis is the same as in Fig. 3

$$\boldsymbol{\phi}_n(x) := \int_{u \in d_n^i} \phi(u)du/|d_n^i|, \text{ for all } x \in d_n^i,\ i = 1, ..., n, \tag{70}$$

$$\boldsymbol{\tau}_n(x, y) := \frac{\int_{v \in d_n^j} \int_{u \in d_n^i} \tau(u, v)dudv}{|d_n^i||d_n^j|}, \text{ for all } x \in d_n^i, y \in d_n^j, i, j = 1, ..., n, \tag{71}$$

where $|d_n^i|$ denotes the square measure of d_n^i, $i = 1, ..., n$ (see (9)). These step functions are equal to the mean values of $\lambda(x)$, $\phi(x)$, and $\tau(x, y)$ when $x \in d_n^i$ and $y \in d_n^j$, $i, j = 1, ..., n$. Applying Conditions 1, 6, and the conditions of Proposition 5(i) to (69)–(71), we see that these step functions satisfy Conditions 3 and 7. Hence, this example shows that the conditions of Proposition 8 are consistent with each other. Moreover, this example helps us to understand the conditions of Proposition 8.

Using Proposition 4(i), we construct a sequence $\{\mathbf{w}_n\}$ such that

$$\mathbf{w}_n \in \mathbf{W}_n, \text{ for all } n \geq 2. \tag{72}$$

As mentioned in (5) of Section 2, this sequence is not convergent in general. However, the following theorem can describe how $\{\mathbf{w}_n\}$ behaves:

Theorem 9 *Assume the conditions of Proposition 8, and let $\{\mathbf{w}_n\}$ be an arbitrary sequence that satisfies (72). The following statements (i)–(iii) hold:*

(i) *$\{\mathbf{w}_n\}$ is a relatively compact sequence of $L^\infty_+(D)$.*
(ii) *Every accumulation point of $\{\mathbf{w}_n\}$ is a short-run equilibrium of M.*
(iii) *$\{\mathbf{w}_n\}$ itself converges to a short-run equilibrium of M in $L^\infty(D)$, or if not, $\{\mathbf{w}_n\}$ consists of a (finite or infinite) countable number of disjoint subsequences that converge to different short-run equilibria of M in $L^\infty(D)$, respectively.*

Appendix B proves this theorem. From Theorem 9(i)(ii) we see that every subsequence of $\{\mathbf{w}_n\}$ has a convergent subsequence, and every convergent subsequence of $\{\mathbf{w}_n\}$ converges to a short-run equilibrium of M (see, e.g., Yoshida 1965, pp. 4, 5). It follows from Theorem 9(iii) that $\{\mathbf{w}_n\}$ itself is an approximate sequence of a short-run equilibrium of M, or if not, $\{\mathbf{w}_n\}$ is the union of a (finite or infinite) countable number of disjoint approximate sequences of different short-run equilibria of M.

As mentioned in (6)–(9) of Section 2, we obtain the approximate relations between a nonempty compact subset $W_\infty \subseteq W$ and $\cup_{m \geq n} \mathbf{W}_m$ when n is sufficiently large (recall (2) of Section 2).

Theorem 10 *If the conditions of Proposition 8 hold, then the following statements (i)–(iv) hold:*

(i) *$\cup_{m \geq n} \mathbf{W}_m$ is a relatively compact set of $L^\infty_+(D)$ for every $n \geq 2$.*

(ii) *There exists a nonempty compact subset* $W_\infty \subseteq W$ *such that for every* $\varepsilon > 0$ *there exist an integer* $N \geq 2$ *and a finite set* W_0 *that satisfy the following inclusion relations*:

$$W_0 \subseteq W_\infty, \tag{73}$$

$$\bigcup_{m \geq n} \mathbf{W}_m \subseteq V(\varepsilon, W_0) \textit{ for every } n \geq N, \tag{74}$$

where $V(\varepsilon, U)$ denotes a small neighborhood of $U \subseteq L^\infty(D)$ defined as follows:

$$V(\varepsilon, U) := \{v \in L^\infty(D); ||v - u||_{L^\infty(D)} < \varepsilon \textit{ for some } u \in U\}. \tag{75}$$

(iii) *For every* $\varepsilon > 0$ *and every* $n \geq 2$ *there exists a finite set* $\mathbf{W}_0$ *that satisfies the following inclusion relations*:

$$\mathbf{W}_0 \subseteq \bigcup_{m \geq n} \mathbf{W}_m, \tag{76}$$

$$W_\infty \subseteq V(\varepsilon, \mathbf{W}_0), \tag{77}$$

where W_∞ is the nonempty compact subset obtained in (ii).

(iv) *For every* $\varepsilon > 0$ *there exists an integer* $N \geq 2$ *such that for every* $n \geq N$ *there exists a finite set* $\mathbf{W}_0$ *that satisfies* (76) *and the following inclusion relation*:

$$\bigcup_{m \geq n} \mathbf{W}_m \subseteq V(\varepsilon, \mathbf{W}_0). \tag{78}$$

Appendix B proves Theorem 10. Let us discuss this theorem. First, we discuss W_∞. Note that W_∞ is independent of ε, n and N. From (73) and (74) we see that for every $\varepsilon > 0$ there exists an integer $N \geq 2$ such that

$$\bigcup_{m \geq n} \mathbf{W}_m \subseteq V(\varepsilon, W_\infty), \text{ for every } n \geq N. \tag{79}$$

Moreover, from (76) and (77) we see that for every $\varepsilon > 0$,

$$W_\infty \subseteq V(\varepsilon, \bigcup_{m \geq n} \mathbf{W}_m), \text{ for every } n \geq 2. \tag{80}$$

Combining (79) and (80), we see that if N is sufficiently large, then W_∞ approximately coincides with $\cup_{m \geq n} \mathbf{W}_m$ for all $n \geq N$, that is, W_∞ and $\cup_{m \geq n} \mathbf{W}_m$ are approximated by each other for every $n \geq N$. There is no approximate relation between the complement $W \setminus W_\infty$ and $\cup_{m \geq n} \mathbf{W}_m$ when n is sufficiently large. Hence, to obtain approximate relations between W and $\cup_{m\geq n}\mathbf{W}_m$,we have only to analyze how $W_\infty \subseteq W$ and $\cup_{m\geq n}\mathbf{W}_m$ can be approximated by each other.

Let us discuss (73) and (74). Note that W_0 is a finite set that is dependent on ε. It follows from (74) that if n is sufficiently large, then $\cup_{m\geq n}\mathbf{W}_m$ is included in the ε -neighborhood of W_0. Hence, considering (75), we see that if n is sufficiently large, then for every $\mathbf{w} \in \cup_{m\geq n}\mathbf{W}_m$ there exists $w \in W_0$ that is close to $\mathbf{w}$. Hence, W_0 can be used as an approximation of $\cup_{m\geq n}\mathbf{W}_m$. This result shows that the set of all short-run equilibria of all CP models with a large number of small regions can be approximated by a finite number of short-run equilibria contained in W_∞. Using the finite number of short-run equilibria, we can approximately describe the set of all short-run equilibria of all CP models with a large number of small regions.

Let us discuss (76) and (77). Note that $\mathbf{W}_0$ is a finite set that is dependent on ε and n. It follows from (77) that W_∞ is included in the ε -neighborhood of $\mathbf{W}_0$. Hence, $\mathbf{W}_0$ can be used as an approximation of W_∞. This result shows that $W_\infty$is approximated by a finite number of short-run equilibria of CP models with a large number of small regions. Using the finite number of short-run equilibria, we can approximately describe W_∞. We have no need to use an infinite number of short-run equilibria of CP models with a large number of small regions.

Let us discuss (76) and (78). Note that $\mathbf{W}_0$ is a finite set that is dependent on ε and n. It follows from (76) and (78) that if n is sufficiently large, then $\cup_{m\geq n}\mathbf{W}_m$ is included in the ε-neighborhood of $\mathbf{W}_0 \subseteq \cup_{m\geq n}\mathbf{W}_m$. Hence, $\cup_{m\geq n}\mathbf{W}_m$ can be approximated by $\mathbf{W}_0$ when n is sufficiently large. This result shows that the set of all short-run equilibria of all CP models with a large number of small regions can be approximated by a finite number of short-run equilibria contained in the set. Using the finite number of short-run equilibria, we can approximately describe the set of all short-run equilibria of all CP models with a large number of small regions.

In general, $\{\mathbf{w}_n\}$ itself is not convergent, W_∞ is a proper subset of W, and W_∞ depends on $\{\mathbf{W}_n\}$, which is dependent on $\{(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))\}$. As mentioned in (10) of Section 2, we obtain a sufficient condition for W_∞ to coincide with $W$and for $\{\mathbf{w}_n\}$ to converge to a short-run equilibrium of M.

Theorem 11 *Assume the conditions of Proposition 8. If the elasticity of substitution and the maximum of transport costs are so small that (56) can hold, then M has a unique short-run equilibrium $w \in L^\infty_+(D)$,*

$$W_\infty = W = \{w\}, \tag{81}$$

$\mathbf{M}_n$ has a unique short-run equilibrium $\mathbf{w}_n$for each $n \geq N_0$, and

$$\lim_{n\to+\infty} ||\mathbf{w}_n - w||_{L^\infty(D)} = 0, \tag{82}$$

where N_0 is the integer defined in Proposition 8.

Appendix C proves this theorem. Theorem 11 complements Theorems 9 and 10. From Theorem 11, we see that the CCP model and every CP model with a large number of small regions have unique short-run equilibria, respectively, and the short-run equilibria are approximately equal to each other. Hence, the CCP model and every CP model with a large number of small regions are approximately the same when the models are in short-run equilibrium.

The following theorem provides an error estimate for the approximation given in Theorem 11:

Theorem 12 *If the conditions of Theorem 11 hold, then for each* $n \geq N_0$,

$$||\mathbf{w}_n - w||_{L^\infty(D)}$$

$$\leq \frac{\overline{a}_1||\lambda_n - \lambda||_{L^1(D)} + \overline{a}_2||\boldsymbol{\phi}_n - \phi||_{L^1(D)} + \overline{a}_3||\boldsymbol{\tau}_n - \tau||_{L^\infty(D\times D)}}{1-\overline{a}_0}, \tag{83}$$

where N_0 *is the integer defined in Proposition 8,* w *and* $\mathbf{w}_n$ *denote the unique short-run equilibria of* M *and* $\mathbf{M}_n$ *proved in Theorem 11, and*

$$\overline{a}_i := a_i(\mu, \sigma, \overline{w}), \;\; i = 0, 1, 2, \tag{84}$$

$$\overline{a}_3 := a_3(\sigma, \overline{w}). \tag{85}$$

Here $\overline{w}$ *is defined in (57), and* $a_0(\mu, \sigma, r)$ *is defined in (47), and* $a_i(\mu, \sigma, r), i = 1, 2,$ *and* $a_3(\sigma, r)$ *are a constant multiple of (65)–(67) defined as follows:*

$$a_i(\mu, \sigma, r) := 2k(\sigma, r)b_i(\mu, \sigma, r), \;\; i = 1, 2, \tag{86}$$

$$a_3(\sigma, r) := 2k(\sigma, r)b_3(\sigma, r), \tag{87}$$

where

$$k(\sigma, r) := r^{\sigma-1}/\sigma, \;\; r > 0. \tag{88}$$

Appendix C proves Theorem 12. Inequality (83) shows that the norm $||\mathbf{w}_n - w||_{L^\infty(D)}$ is smaller than the linear combination of norms (68). Inequality (83) is an error estimate for the approximation given by Theorem 11.

Noting that (84) with $i = 0$ is equal to the left-hand side of (56), we see that the denominator of the right-hand side of (83) is positive. Comparing (63), (64) and (84)–(87), we see that the right-hand side of (83) is obtained by multiplying the right-hand side of (62) by $2k(\sigma, \overline{w})/(1 - \overline{a}_0)$.

To help us to understand how $\mathbf{w}_n(x)$ behaves as the number n of regions increases, we obtain $\mathbf{w}_n(x)$ numerically. We define

$$D := \{x \in \mathbb{R}; 0 \leq x \leq 1\}, \tag{89}$$

$$(\sigma, \mu) := (4.0, 0.5), \tag{90}$$

$$\tau(x, y) := \exp(|x - y|), \;\; \text{for all } x, y \in D, \tag{91}$$

$$\phi(x) := 1,\ \lambda(x) := c\exp(-|x-1|^2/0.02),\ \text{for all } x \in D, \tag{92}$$

where c denotes a positive constant such that $||\lambda||_{L^1(D)} = 1$. We divide (89) as (9), where d_n^i denotes the following small segment for each $i = 1, ..., n$:

$$d_n^i := [(i-1)/n, i/n]. \tag{93}$$

Substituting (91)–(93) in (69)–(71), we define $(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))$. Substituting $(\boldsymbol{\lambda}_n(x), \boldsymbol{\phi}_n(x), \boldsymbol{\tau}_n(x,y))$ in (34)–(37), we can define a CP model with n regions for each $n \geq 2$.

Manufacturing workers are distributed uniformly in each small segment. The manufacturing-worker distribution is highest in the rightmost small segment and decreases sharply toward the left. Figure 6 shows the graph of $\lambda_{80}(x)$. Agricultural workers are distributed uniformly throughout the line segment [0, 1]. Transportation-cost function is a step function that increases exponentially with distance. In this line economy, economic activities are conducted uniformly in each small segment.

Using iterative scheme (Tabata et al. 2015a, (8.1), (8.2)), we can obtain $\mathbf{w}_n(x)$ numerically for each $n \geq 2$. Figures 7, 8, 9, 10, and 11 show the graphs of $\mathbf{w}_n(x)$, $n = 5, 10,\ 20,\ 40,\ 80$. Let us explain these graphs.

Figure 7 shows the graph of $\mathbf{w}_5(x)$. When the number of small segments is small ($n = 5$), $\mathbf{w}_5(x)$ are highest in the left-most small segment [0, 1/5], and lowest in the right-most small segment [4/5, 5/5]. Wages decrease in a stepwise fashion to the right.

Figure 8 shows the graph of wage $\mathbf{w}_{10}(x)$. Comparing Figs. 7 and 8, we can observe how the wage distribution changes when the number of small segments becomes

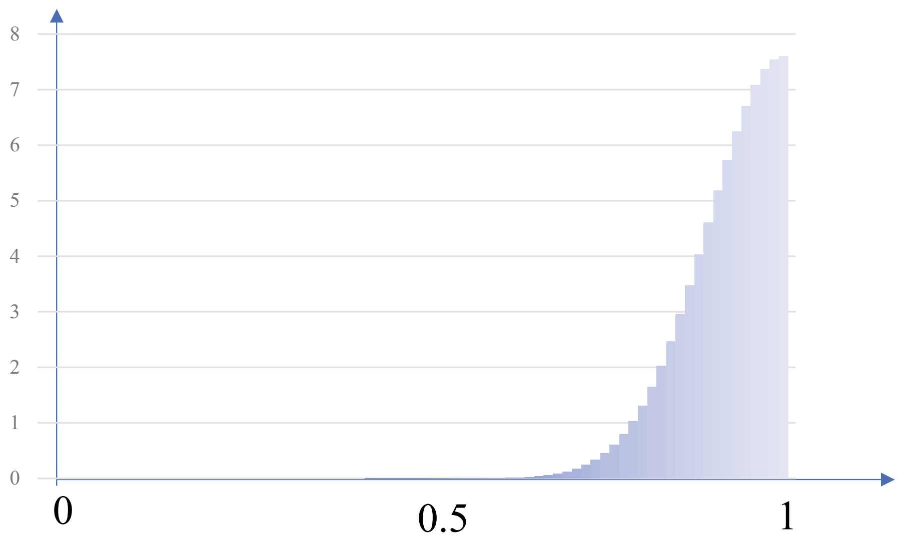

Fig. 6 The graph of $\lambda_{80}(x)$. The vertical axis represents the density of workers in each small segment. The line segment [0, 1] is contained in the horizontal axis. The left half section [0, 0.5] has few manufacturing workers

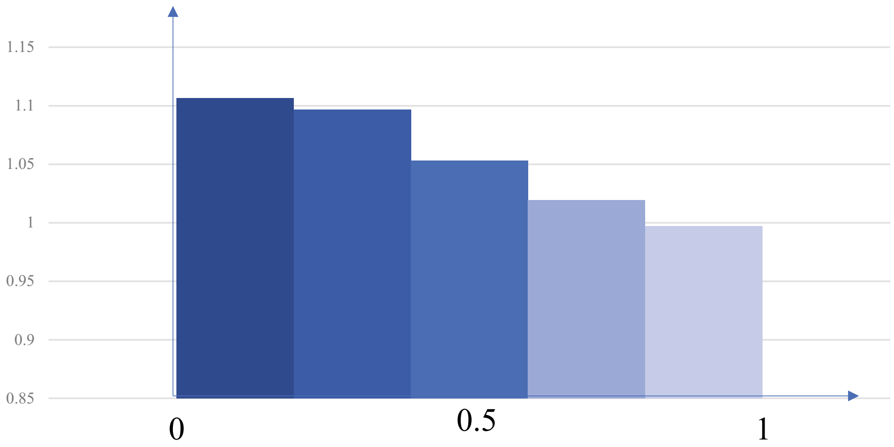

Fig. 7 The wage distribution $\mathbf{w}_5(\mathrm{x})$ when $(\sigma, \mu) = (4.0, 0.5)$. The length of each small segment is equal to 1/5. The vertical axis represents the wages in each small segment. The wage distribution attains the maximum value in [0/5, 1/5]. All the vertical scales in Figs. 7, 8, 9, 10, and 11 are the same

larger ($n = 10$). Wages are highest not in the left-most small segment [0, 1/10], but in the second small segment from the left end [1/10, 2/10]. This phenomenon is not observed when $n = 5$.

Figures 9 and 10 show the graphs of $\mathbf{w}_{20}(x)$ and $\mathbf{w}_{40}(x)$. Comparing Figs. 9 and 10, we can observe how the wage distribution changes as the number of small

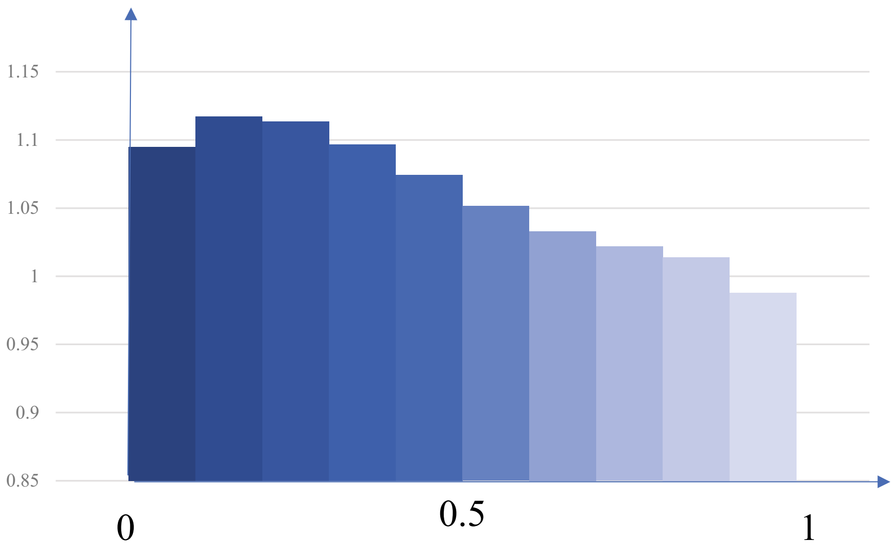

Fig. 8 The wage distribution $\mathbf{w}_{10}(\mathrm{x})$ when $(\sigma, \mu) = (4.0, 0.5)$. The wage distribution attains the maximum value in [1/10, 2/10]

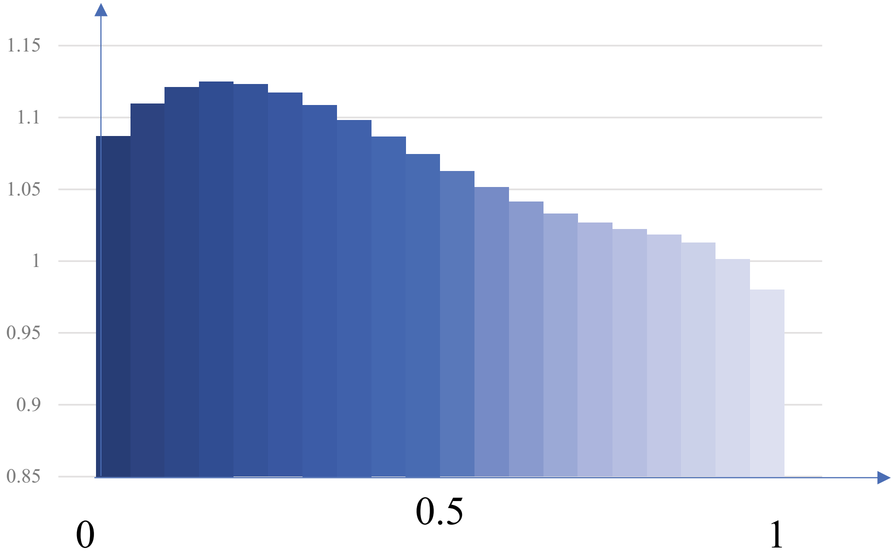

Fig. 9 The wage distribution $\mathbf{w}_{20}(\mathrm{x})$ when $(\sigma, \mu) = (4.0, 0.5)$. The wage distribution attains the maximum value in [3/20, 4/20]

segments increases ($n = 20$, 40). We see that $\mathbf{w}_{20}(x)$ increases as point x moves from the left end to the right and attains the maximum value in the fourth small

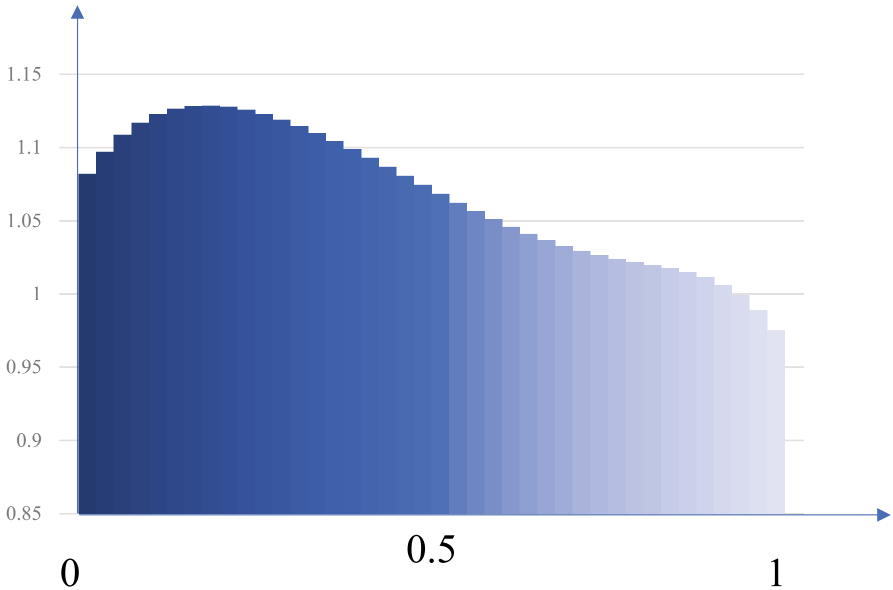

Fig. 10 The wage distribution $\mathbf{w}_{40}(\mathrm{x})$ when $(\sigma, \mu) = (4.0, 0.5)$. The wage distribution attains the maximum value in [7/40, 8/40]

segment [3/20, 4/20]. Moreover, $\mathbf{w}_{20}(x)$ decreases monotonically as x moves from [3/20, 4/20] to the right and attains the minimum value at the right end.

We see that $\mathbf{w}_{20}(x)$ falls at the right end and the left end. As the number of small segments increases ($n = 40$), this wage drop becomes steeper (see Fig. 10). However, if the number of small segments is not larger ($n = 10$), then this drop is not clearly observed (see Fig. 8).

Figure 11 show the graph of $\mathbf{w}_{80}(x)$. Comparing Figs. 10 and 11, we see that the wage distribution curve converges to a smooth curve as the number of small segments increases.

Let us analyze this simulation result. All the wage distributions are pulled to the left. Let us explain the reason for $n = 80$. First, we observe the income. From Fig. 11, we see that

$$0.971 < \mathbf{w}_{80}(x) < 1.131,$$

that is, that the wage distribution is close to 1.0. Hence, the manufacturing-income distribution is close to the manufacturing-worker distribution. Agricultural workers are distributed uniformly in [0, 1], but manufacturing workers are concentrated in the neighborhood of $x = 1$ (see Fig. 6 and (92)). Hence, the income distribution is pulled to the right (see Fig. 12).

Second, we observe the price index. The denominator of the price index is expressed as the sum of the products of the manufacturing-worker distribution, $(1/\mathbf{w}_{80}(x))^{\sigma-1}$, and the $-(\sigma - 1)$ power of the transportation-cost function. The

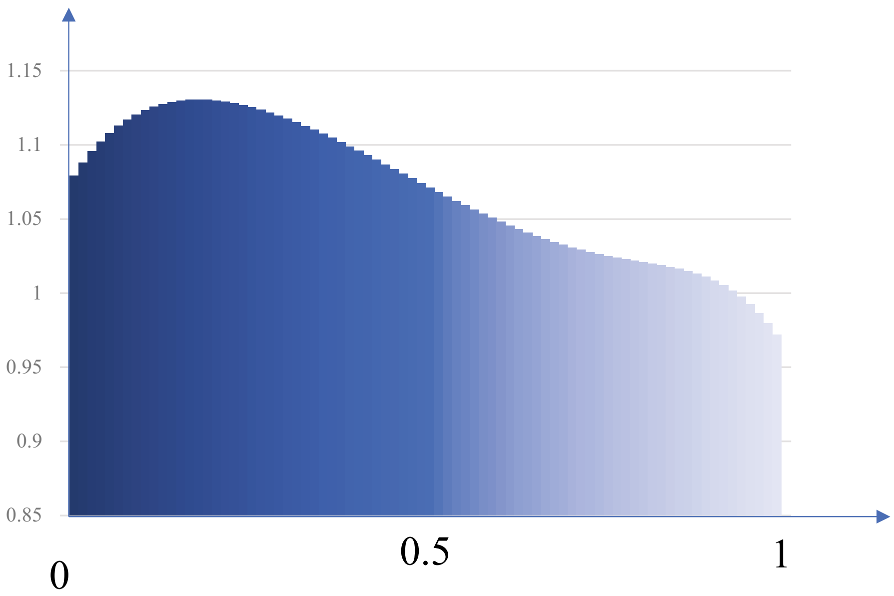

Fig. 11 The wage distribution $\mathbf{w}_{80}(\mathrm{x})$ when $(\sigma, \mu) = (4.0, 0.5)$. The wage distribution attains the maximum in [14/80, 15/80]. The maximum value is equal to 1.130479. The minimum value is equal to 0.971967

wage distribution is close to 1.0, and the transportation-cost function is an exponential function of distance (see (91)). However, the left half section [0, 0.5] has few manufacturing workers (see Fig. 6). Applying these observations to (36), we see that the denominator of the price index at point x decreases exponentially as x approaches the left end $x = 0$. We see that the $(\sigma - 1)$ power of the price index at x increases exponentially as x approaches the left end (see Fig. 13).

Comparing Figs. 12 and 13, we see that the $(\sigma - 1)$ power of the price index is larger than the income. Hence, the product of the $(\sigma - 1)$ power of the price index and the income is pulled to the left by the the price index (see Fig. 14). The right-hand side of the wage equation is expressed as the sum of the products of the $(\sigma - 1)$ power of the price index, the income, and the $-(\sigma - 1)$ power of the transportation-cost function. Hence, the right-hand side of the wage equation is pulled to the left by the price index. Therefore, the wage distribution is pulled to the left as shown in Fig. 11. Because of the same reason for $n = 80$, the wage distributions shown in Figs. 7, 8, 9, and 10 are pulled to the left by the price index.

Let us observe Figs. 7, 8, 9, 10, and 11 in the neighborhood of both ends $x = 0, 1$. Economic activities in each small segment depend on economic activities in the neighborhood of the small segment. However, there is no economic activity in $(-\infty, 0) \cup (1, +\infty)$. Hence, small segments in the neighborhood of $x = 0, 1$ are economically disadvantaged. This economic disadvantage causes the steep wage decline in the neighborhoods of both ends (see Figs. 9, 10, and 11).

Since the transportation-cost function is an exponential function of distance, the economic disadvantage is smaller at points away from $x = 0, 1$. As explained above, the wage distributions are pulled to the left. Hence, the wage distributions attain

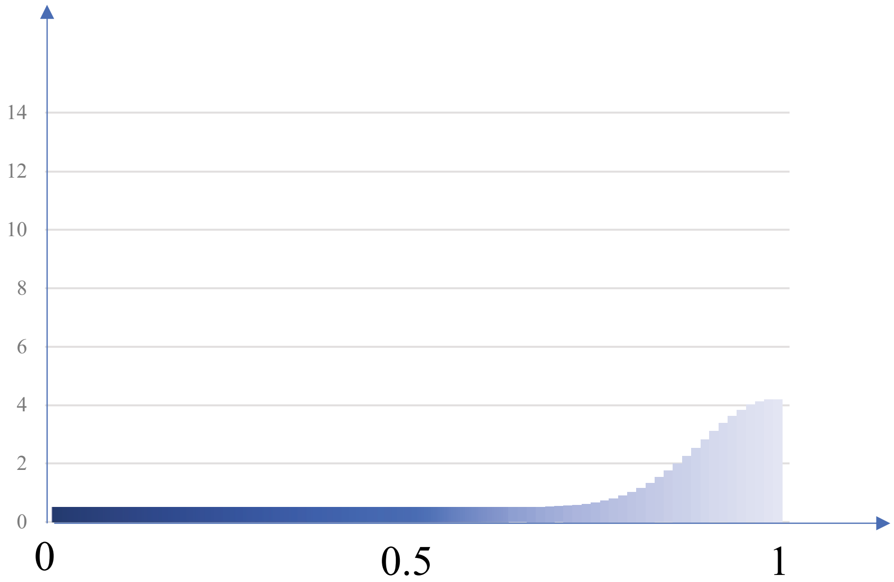

Fig. 12 The graph of the income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.5)$. The scale of the vertical axis of this graph is not equal to that in Fig. 6. The maximum value is equal to 4.193407, and the minimum value is equal to 0.5

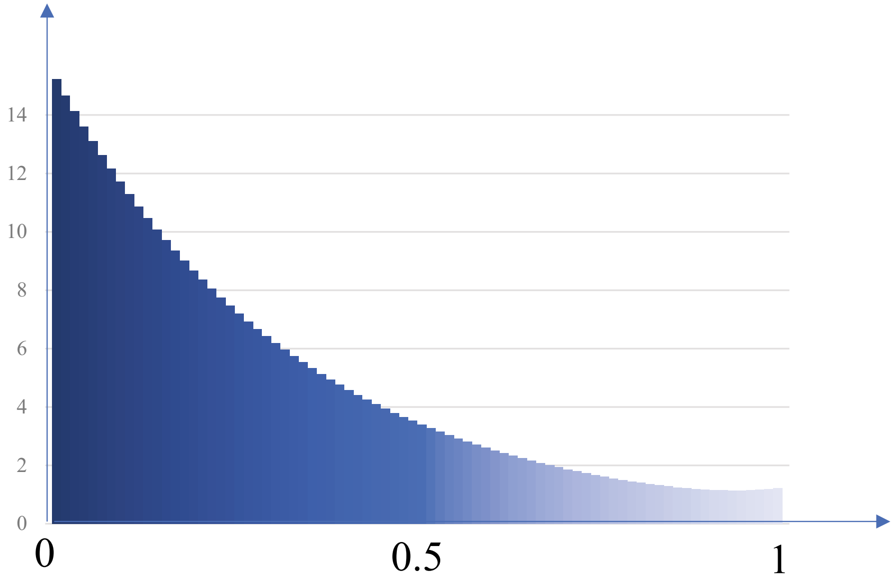

Fig. 13 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(4.0,0.5)$. The scale of the vertical axis is the same as in Fig. 12. The maximum value is equal to 15.233695, and the minimum value is equal to 1.143667

the maximum values in the small segments slightly to the right of the left end (see Figs. 8, 9, 10, and 11).

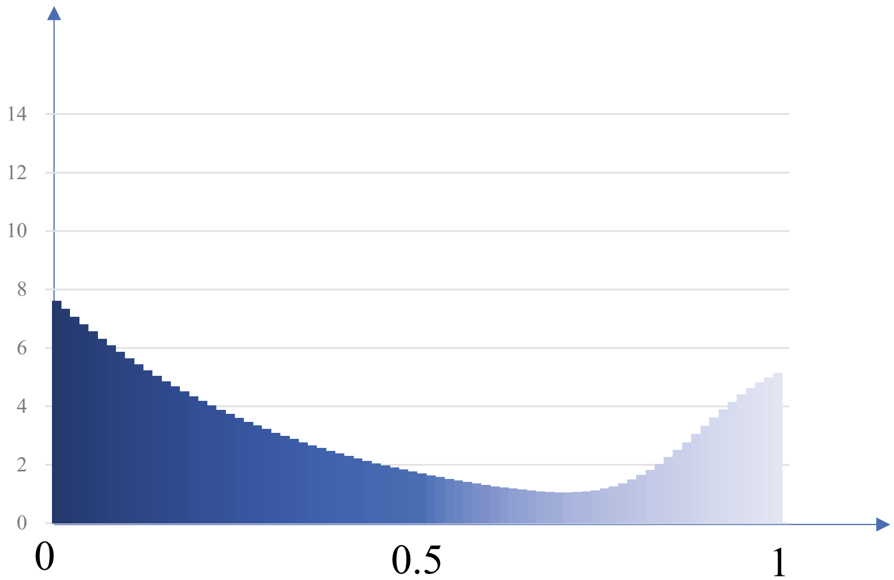

Fig. 14 The graph of the product $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}\,Y(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\phi}_{80})(x)$ when $(\sigma,\mu)=(4.0,0.5)$. The scale of the vertical axis is the same as in Figs. 12 and 13. This product is pulled by the income and the price index. The latter pulls the product more strongly than the former

In contrast, when the number of small segments is small ($n = 5$), the width of each segment is large. In the segment $[0, 1/5]$, wages are averaged over the leftmost segment $[0, 1/10]$ and the second segment from the leftmost $[1/10, 2/10]$ (see Figs. 7 and 8). Because of this averaging effect, for $n = 5$, the wage distribution attains the maximum value in the leftmost segment $[0, 1/5]$ (see Fig. 7). For the same reason, for $n = 5, 10,$ there is no steep wage decline in the neighborhood of both ends (see Figs. 7 and 8).

If n is sufficiently large, then we can use $\mathbf{w}_n(x)$ as an approximation of the continuous wage distribution $w(x)$. To help us to understand this approximation, we examine how $\mathbf{w}_{80}(x)$ changes with μ and σ.

Figures 15, 16, 17, 18, and 19 show the graphs of $\mathbf{w}_{80}(x)$ for $\sigma = 4.0$ and $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$. When $\mu = 0.1$, $\mathbf{w}_{80}(x)$ is weakly pulled to the left (see Fig. 15). As μ increases ($\mu = 0.3, 0.5, 0.7, 0.9$), $\mathbf{w}_{80}(x)$ is gradually pulled to the right (see Figs. 16, 17, 18, and 19).

Figures 20, 21, 22, 23, and 24 show the graphs of $\mathbf{w}_{80}(x)$ for $\mu = 0.5$ and $\sigma = 2.0, 4.0, 6.0, 8.0, 10.0$. When $\sigma = 2.0$, $\mathbf{w}_{80}(x)$ is weakly pulled to the right. As σ increases ($\sigma = 4.0, 6.0, 8.0, 10.0$), $\mathbf{w}_{80}(x)$ is gradually pulled to the left (Figs. 21, 22, 23, and 24).

From these simulation results, we see that also the continuous wage distribution $w(x)$ is pulled to the right (the left) as μ increases (σ increases). Appendix D analyzes the simulation results shown in Figs. 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24.

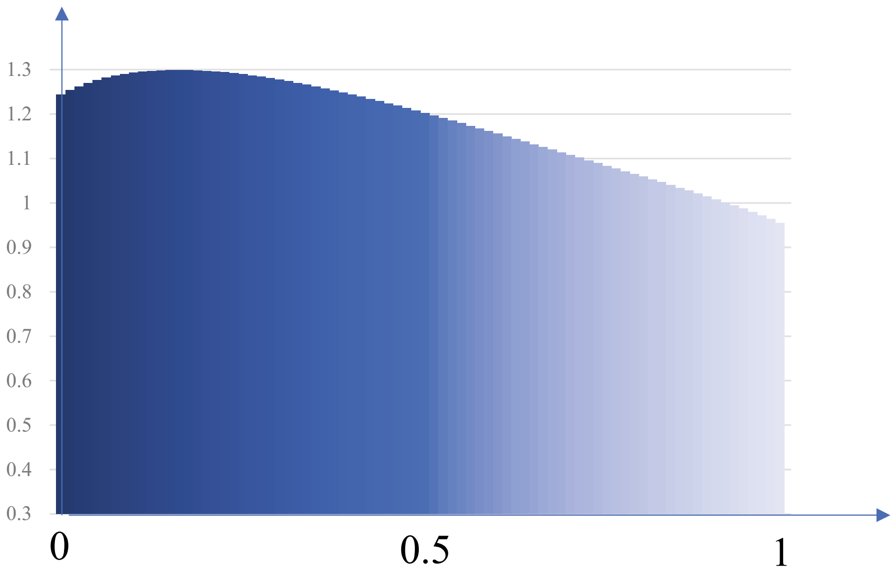

Fig. 15 The distribution of wages when $(n, \sigma, \mu) = (80, 4.0, 0.1)$. All the vertical scales in Figs. 15, 16, 17, 18, and 19 are the same. The maximum value is equal to 1.299017. The minimum value is equal to 0.955199

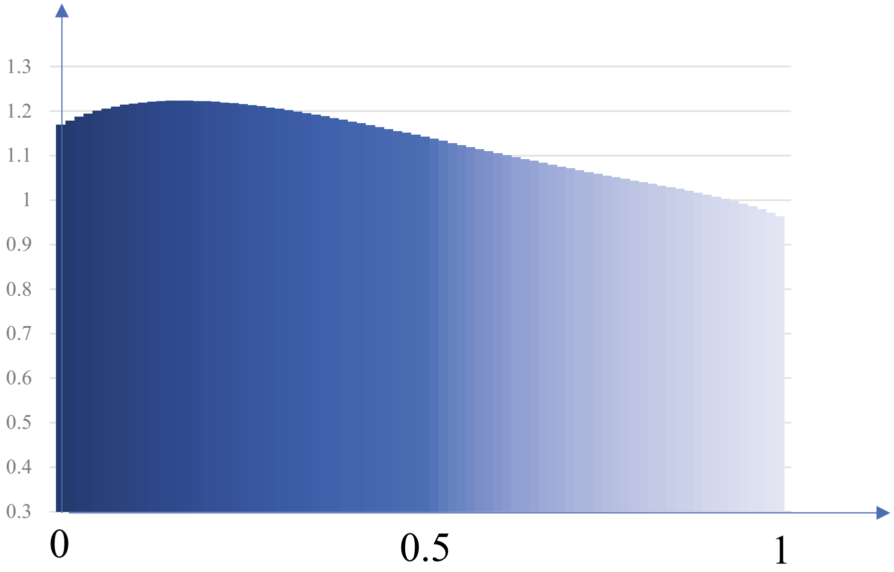

Fig. 16 The distribution of wages when $(n, \sigma, \mu) = (80, 4.0, 0.3)$

Let us define the integer N_0 that is used in Proposition 8 and Theorems 11 and 12. For this purpose, we introduce some symbols. Applying Condition 6, (17), (21), (40), (51) and (60) to (45), (57) and

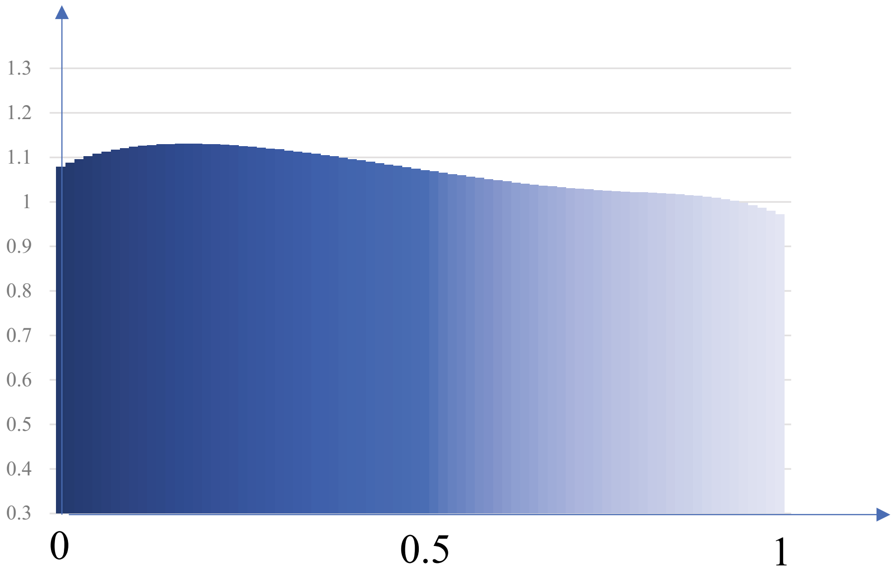

Fig. 17 The distribution of wages when $(n, \sigma, \mu) = (80, 4.0, 0.5)$. This graph is the same as in Fig. 11 except for the scale of the vertical axis

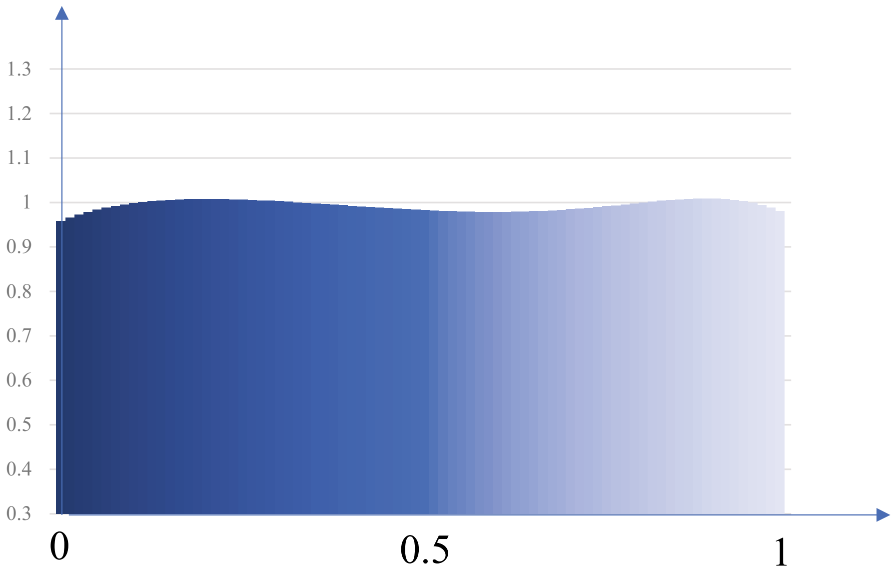

Fig. 18 The distribution of wages when $(n, \sigma, \mu) = (80, 4.0, 0.7)$

$$\overline{\overline{\mathbf{w}}}_n := \sup_{m \geq n} \overline{\mathbf{w}}_m, \text{ for each } n \geq 2, \tag{94}$$

we see that

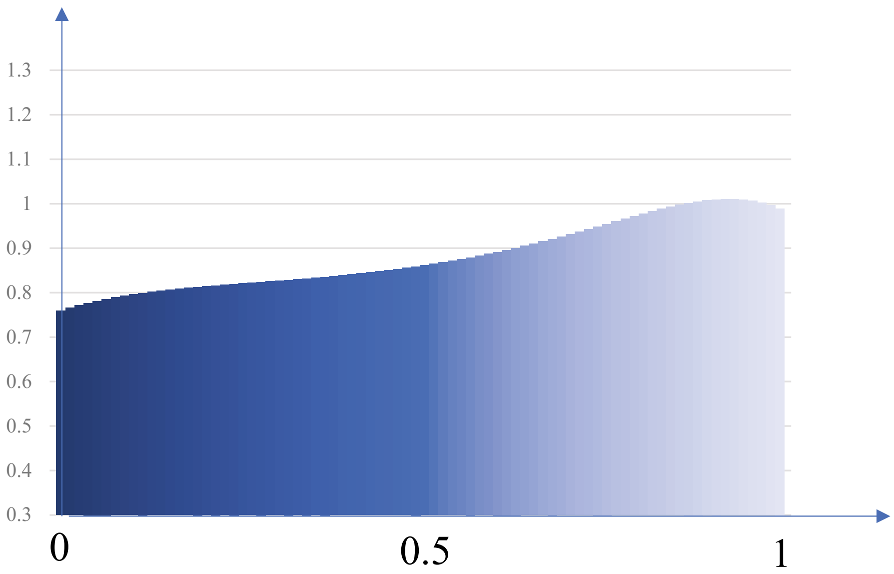

Fig. 19 The distribution of wages when $(n, \sigma, \mu) = (80, 4.0, 0.9)$. The wage distribution is pulled to the right. The maximum value is equal to 1.01053. The minimum value is equal to 0.759448

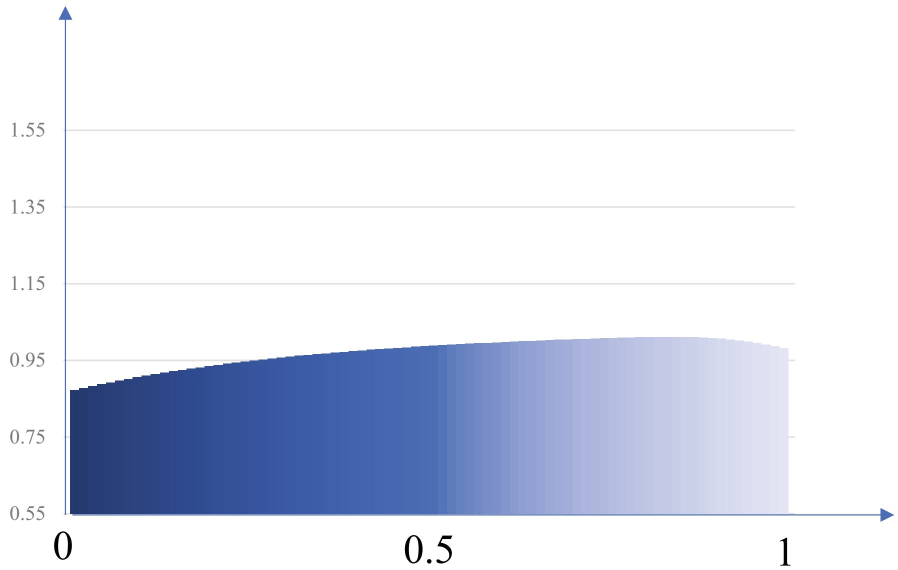

Fig. 20 The distribution of wages when $(n, \sigma, \mu) = (80, 2.0, 0.5)$. All the vertical scales in Figs. 20, 21, 22, 23, and 24 are the same. The maximum value is equal to 1.011046. The minimum value is equal to 0.872614

$$\overline{w} = \lim_{n \to +\infty} \overline{\mathbf{w}}_n = \lim_{n \to +\infty} \overline{\overline{\mathbf{w}}}_n, \tag{95}$$

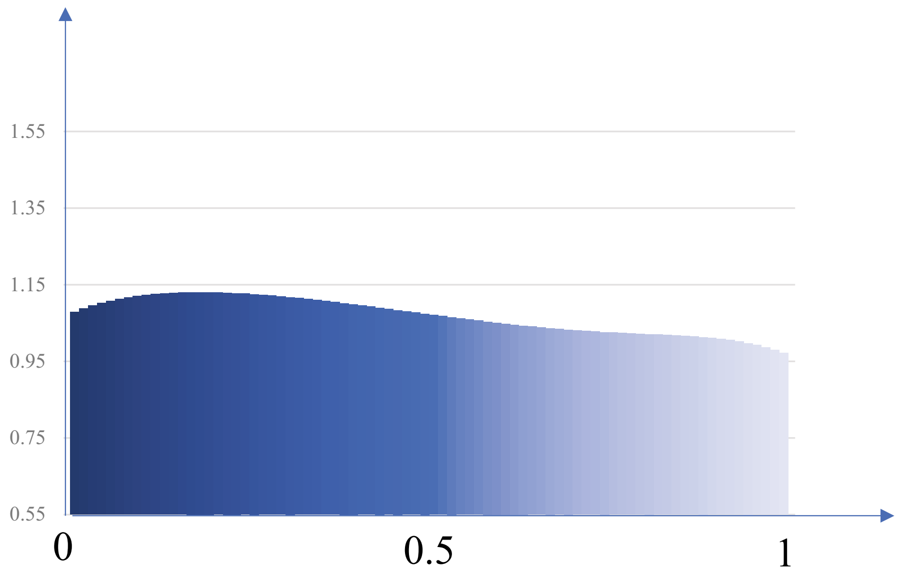

Fig. 21 The distribution of wages when $(n, \sigma, \mu) = (80, 4.0, 0.5)$. This graph is the same as in Fig. 11 except for the scale of the vertical axis

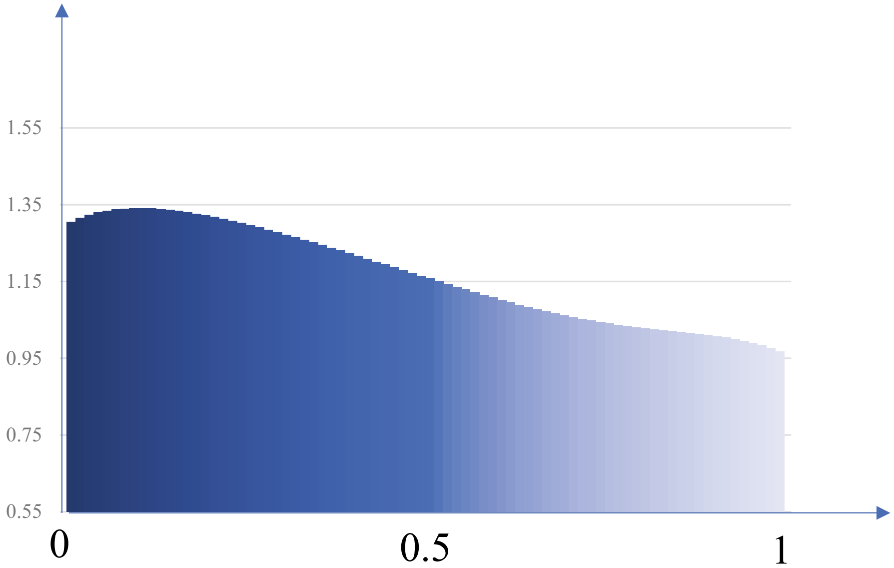

Fig. 22 The distribution of wages when $(n, \sigma, \mu) = (80, 6.0, 0.5)$

$$1 \leq \overline{\mathbf{w}}_n \leq \overline{\overline{\mathbf{w}}}_n,\ 1 \leq \overline{w} \leq \overline{\overline{\mathbf{w}}}_n,\ \text{for each } n \geq 2. \tag{96}$$

Applying (47), (63), (64), (84), (85) and (95) to the following constants for each $n \geq 2$:

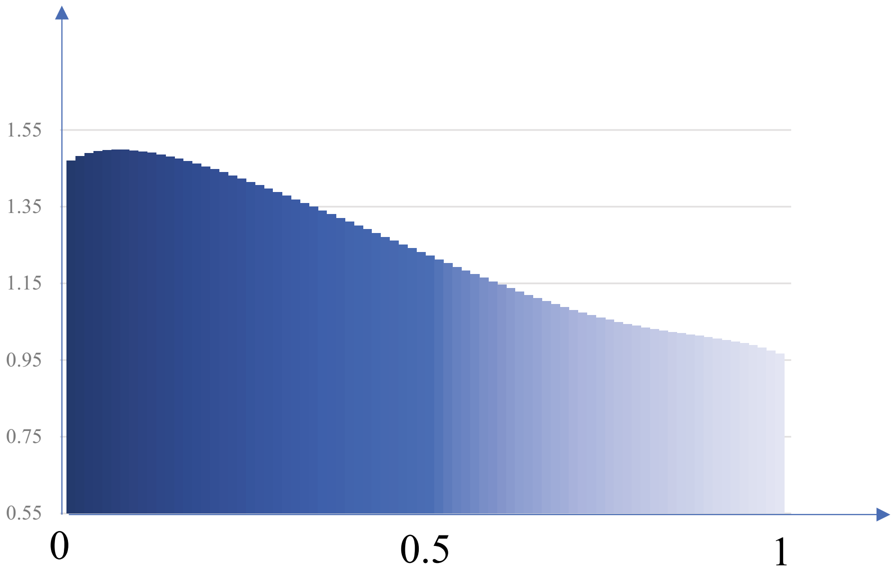

Fig. 23 The distribution of wages when $(n, \sigma, \mu) = (80, 8.0, 0.5)$

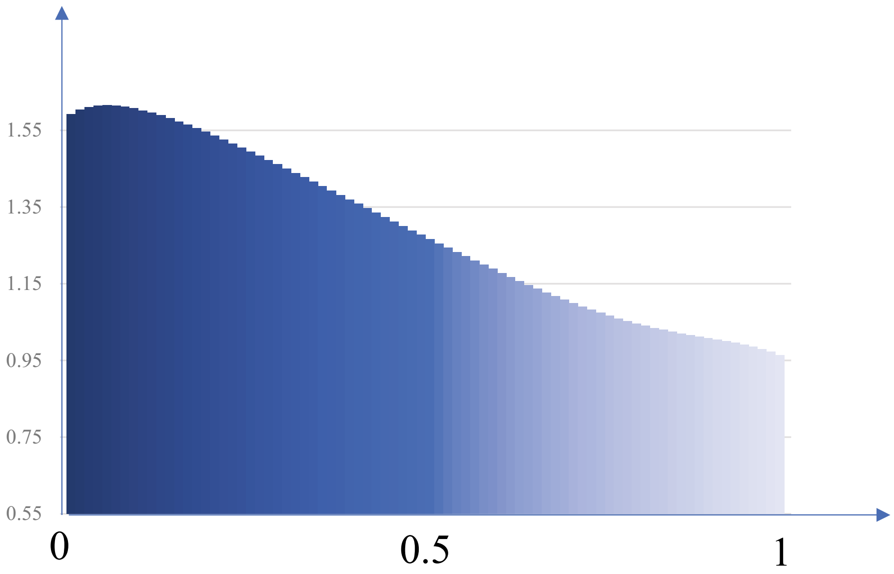

Fig. 24 The distribution of wages when $(n, \sigma, \mu) = (80, 10.0, 0.5)$. The wage distribution is pulled to the left. The maximum value is equal to 1.61568. The minimum value is equal to 0.964153

$$\bar{\bar{\mathbf{a}}}_{i,n} := a_i(\mu, \sigma, \bar{\bar{\mathbf{w}}}_n),\ i = 0, 1, 2,\ \bar{\bar{\mathbf{a}}}_{3,n} := a_3(\sigma, \bar{\bar{\mathbf{w}}}_n), \tag{97}$$

$$\bar{\bar{\mathbf{b}}}_{i,n} := b_i(\mu, \sigma, \bar{\bar{\mathbf{w}}}_n),\ i = 1, 2,\ \bar{\bar{\mathbf{b}}}_{3,n} := b_3(\sigma, \bar{\bar{\mathbf{w}}}_n), \tag{98}$$

we see that

$$\lim_{n\to+\infty} \bar{\bar{\mathbf{a}}}_{i,n} = \bar{a}_i,\ i = 0, ..., 3,\ \lim_{n\to+\infty} \bar{\bar{\mathbf{b}}}_{i,n} = \bar{b}_i,\ i = 1, 2, 3. \tag{99}$$

Note that constants (99) are positive, and that $\bar{a}_0$ is equal to the left-hand side of (56). Using (99), we see that if (56) holds, then there exists an integer $N \geq 2$ such that if $n \geq N$, then

$$|\bar{\bar{\mathbf{a}}}_{0,n} - \bar{a}_0| \leq (1 - \bar{a}_0)/2, \tag{100}$$

$$\bar{\bar{\mathbf{a}}}_{i,n} \leq 2\bar{a}_i,\ i = 1, 2, 3, \tag{101}$$

$$\bar{\bar{\mathbf{b}}}_{i,n} \leq 2\bar{b}_i,\ i = 1, 2, 3. \tag{102}$$

We define N_0 as the smallest integer $N \geq 2$ that satisfies (100)–(102).

7 Conclusions

We construct a continuous core-periphery model (CCP model). For each $n \geq 2$ we construct a core-periphery model (CP model) with n regions. We assume that the known functions of the CP model with n regions and the maximum of diameters of n regions converge to the known functions of the CCP model and 0, respectively, as the number n tends to infinity (see (1) of Section 2). Using the sequence of models thus constructed, we define the set of all short-run equilibria of a CP model with a large number of small regions and the set of all short-run equilibria of all CP models with a large number of small regions (see (3) of Section 2). We obtained the following results (1)–(6):

1. Consider an arbitrary sequence whose n-th term is a short-run equilibrium of the CP model with n regions for each $n \geq 2$. The sequence is a relatively compact sequence whose all accumulation points are short-run equilibria of the CCP model (Theorem 9(i)(ii)). The sequence converges to a short-run equilibrium of the CCP model, or if not, the sequence consists of a (finite or infinite) countable number of disjoint subsequences that converge to different short-run equilibria of the CCP model (Theorem 9(iii)).
2. The set of all short-run equilibria of the CCP model has a nonempty compact subset W_∞ that approximately coincides with the set of all short-run equilibria of all CP models with a large number of small regions. Hence, W_∞ and the set of all short-run equilibria of all CP models with a large number of small regions are approximated by each other, and there is no approximate relation between the complement of W_∞ and the set of all short-run equilibria of all CP models with a large number of small regions.
3. The set of all short-run equilibria of all CP models with a large number of small regions is a relatively compact set that can be approximated by a finite number of short-run equilibria contained in W_∞ (Theorem 10(i)(ii)). Using the finite number of short-run equilibria of the CCP model, we can approximately describe the set of all short-run equilibria of all CP models with a large number of small regions.
4. The nonempty compact set W_∞ can be approximated by a finite number of short-run equilibria of CP models with a large number of small regions (Theorem 10(iii)). Using the finite number of short-run equilibria, we can approximately describe W_∞.
5. The set of all short-run equilibria of all CP models with a large number of small regions can be approximated by a finite number of short-run equilibria contained in the set (Theorem 10(iv)). Using the finite number of short-run equilibria, we can approximately describe the set of all short-run equilibria of all CP models with a large number of small regions.
6. If the elasticity of substitution and the maximum of transportation costs are sufficiently small in comparison with the manufacturing expenditure, then the CCP model and every CP model with a large number of small regions have unique short-run equilibria, respectively, and these short-run equilibria are approximately equal to each other (Theorem 11). In this sense, the CCP model and every CP model with a large number of small regions are approximately the same when

the models are in short-run equilibrium. Theorem 12 provides the error estimate for the approximation proved in Theorem 11.

This paper proved the approximate relations between a CCP model and CP models with a large number of small regions when these models are in short-run equilibrium. These approximate relations allow us to approximately study the CCP model using the CP models with a large number of small regions. Depending on what we try to model, it is sometimes convenient to use CP models with n regions, sometimes to use CCP models (Fujita et al. 2001, p. 49). To choose these models appropriately, we need to compare them. The approximate relations are useful for this comparison. However, there is significant room for further investigation. For example, the following issues remain unresolved:

1. It is important to clarify the characteristics of short-run equilibria of the CCP models using the approximations proved in this paper. However, the behavior of the short-run equilibria is complex. In order to clarify the characteristics, we need to conduct many numerical experiments on short-run equilibria of CP models with n regions when the number n is sufficiently large.
2. In this paper the distributions of workers, the distributions of farmers and the transport-cost functions were fixed in the CCP model and the CP model with n regions for each $n \geq 2$. If workers can move to regions with higher real wages, then we need to discuss whether continuous evolutionary game (Tabata and Eshima 2015b, (3), (11)) can be approximated by evolutionary games with a large number of small regions (Fujita et al. 2001, (5.1)–(5.6); Tabata and Eshima 2016, (2), (19)). Moreover, it is important to discuss whether a long-run equilibrium of the CCP model can be approximated by long-run equilibria of CP models with a large number of small regions. This paper could not study these approximations.

Appendix A

This appendix proves estimates for integral operator (14). We use (4), (13)–(16) and (55) in the following lemma:

Lemma 13 *Assume Condition 1. If $u \in L^\infty(D)$ satisfies that*

$$U_1 \leq u(x) \leq U_2 \text{ for a.e. } x \in D, \tag{103}$$

where $U_i, i = 1, 2,$ are constants such that

$$0 < U_1 \leq U_2, \tag{104}$$

then the following statements (i)–(iii) hold:

(i) $$V_1/\overline{w} \leq f(u;\lambda,\phi,\tau)(x) \leq V_2\overline{w}, \text{ for a.e. } x \in D, \tag{105}$$

where

$$V_i := (\mu U_i + (1-\mu))U_i^{\sigma-1},\ i = 1, 2. \tag{106}$$

(ii) *If* $u(x) \in S(\lambda)$, *then*

$$U_1^{\sigma-1}/\overline{w} \leq f(u;\lambda,\phi,\tau)(x) \leq U_2^{\sigma-1}\overline{w} \textit{ for a.e. } x \in D. \tag{107}$$

(iii) *If* (51) *holds*, *then*

$$|f(u;\lambda,\phi,\tau)(x) - f(u;\lambda,\phi,\tau)(y)| \leq V_2\overline{w}T(x,y),\ \textit{for all } x, y \in D, \tag{108}$$

where

$$T(x,y) := (\sigma-1)\max_{z\in D}|\tau(x,z) - \tau(y,z)|.$$

Proof Let us obtain estimates for

$$\mathbf{Y}(u) := \int_{y\in D} Y(u;\lambda,\phi)(y)dy. \tag{109}$$

Applying (5), (16) and (20) to (109), we see that

$$\mathbf{Y}(u) = \mu\int_{x\in D}\lambda(x)u(x)dx + (1-\mu). \tag{110}$$

Applying (19) and (4) with $h = \lambda$ to (110), we see that

$$\mathbf{Y}(u) = 1 \text{ for all } u \in S(\lambda). \tag{111}$$

Applying (5), (18), (19) and (103) to (110), we see that

$$\mu U_1 + (1-\mu) \leq \mathbf{Y}(u) \leq \mu U_2 + (1-\mu). \tag{112}$$

It follows from (17), (21) and (55) that

$$1/\overline{w} \leq \tau(x,y)^{-(\sigma-1)} \leq 1 \text{ for a.e. } x, y \in D. \tag{113}$$

Applying this inequality, (17) and (103) to (15), we obtain

$$\frac{U_1^{\sigma-1}}{\int_{z\in D}\lambda(z)dz} \leq G(u;\lambda,\tau)(y)^{\sigma-1} \leq \frac{U_2^{\sigma-1}\overline{w}}{\int_{z\in D}\lambda(z)dz} \text{ for a.e. } y \in D.$$

Applying (5) and (19) to this inequality, we see that

$$U_1^{\sigma-1} \leq G(u;\lambda,\tau)(y)^{\sigma-1} \leq U_2^{\sigma-1}\overline{w},\ \text{for a.e. } y \in D. \tag{114}$$

It follows from (5) and (16)–(20) that each term of the integrand of (109) is non-negative. Hence, applying (109), (113) and (114) to (14), we obtain

$$(U_1^{\sigma-1}/\overline{w})\mathbf{Y}(u) \leq f(u;\lambda,\phi,\tau)(x) \leq U_2^{\sigma-1}\overline{w}\mathbf{Y}(u), \text{ for a.e. } x \in D.$$

Applying (111) and (112) to this inequality, we obtain (107) and (105), respectively.
Applying (14), (17), (112), (114), and the following trivial inequality with $(X_1, X_2) = (\tau(x,z), \tau(y,z))$:

$$|X_1^{-(\sigma-1)} - X_2^{-(\sigma-1)}| \leq (\sigma-1)|X_1 - X_2| \text{ for all } X_1, X_2 \geq 1, \tag{115}$$

to the left-hand side of (108), we obtain (108) in the same way as (105).

We use (47), (65)–(67) and (88) in the following lemma:

Lemma 14 *Assume that*

$$(\lambda, \phi, \tau) = (\lambda_i, \phi_i, \tau_i), \ i = 1, 2, \tag{116}$$

satisfy Condition 1. If $u_i(x) \in L^\infty(D), i = 1, 2,$ *satisfy the following conditions*:

$$u_2 \in S(\lambda_2), \tag{117}$$

$$1/c_i \leq u_i(x) \leq c_i \textit{ for a.e. } x \in D, \ i = 1, 2, \tag{118}$$

where $c_i, i = 1, 2,$ *are positive constants such that* $c_i \geq 1, \ i = 1, 2,$ *then*

$$||f(u_1;\lambda_1,\phi_1,\tau_1) - f(u_2;\lambda_2,\phi_2,\tau_2)||_{L^\infty(D)} \leq \overline{B}_0\varepsilon_u + \overline{B}_1\varepsilon_\lambda + \overline{B}_2\varepsilon_\phi + \overline{B}_3\varepsilon_\tau, \tag{119}$$

where

$$\varepsilon_u := ||u_1 - u_2||_{L^\infty(D)}, \ \varepsilon_\lambda := ||\lambda_1 - \lambda_2||_{L^1(D)}, \tag{120}$$

$$\varepsilon_\phi := ||\phi_1 - \phi_2||_{L^1(D)}, \ \varepsilon_\tau := ||\tau_1 - \tau_2||_{L^\infty(D\times D)}, \tag{121}$$

$$\overline{B}_0 := a_0(\mu, \sigma, \overline{W})/k(\sigma, \overline{W}), \tag{122}$$

$$\overline{B}_i := b_i(\mu, \sigma, \overline{W})/2, \ i = 1, 2, 3, \tag{123}$$

$$\overline{W} := \max\{\overline{w}_1, \overline{w}_2, c_1, c_2\}, \tag{124}$$

$$\overline{w}_i := \operatorname*{ess\,sup}_{x,y\in D} \tau_i(x,y)^{\sigma-1}, \ i = 1, 2. \tag{125}$$

Proof We denote $f(u_i;\lambda_i,\phi_i,\tau_i)(x)$, $G(u_i;\lambda_i,\tau_i)(y)$ and $Y(u_i;\lambda_i,\phi_i)(y)$, $i=1,2$, by $f_i(x)$, $G_i(y)$ and $Y_i(y)$, $i=1,2$, respectively, for simplicity (see (14)–(16)). Using (14), we see that

$$f_1(x)-f_2(x)=I_1+I_2+I_3, \tag{126}$$

where

$$I_1 := \int_{y\in D} G_1(y)^{\sigma-1}\{Y_1(y)-Y_2(y)\}\tau_1(x,y)^{-(\sigma-1)}dy, \tag{127}$$

$$I_2 := \int_{y\in D} \{G_1(y)^{\sigma-1}-G_2(y)^{\sigma-1}\}Y_2(y)\tau_1(x,y)^{-(\sigma-1)}dy, \tag{128}$$

$$I_3 := \int_{y\in D} G_2(y)^{\sigma-1}Y_2(y)\{\tau_1(x,y)^{-(\sigma-1)}-\tau_2(x,y)^{-(\sigma-1)}\}dy. \tag{129}$$

Let us obtain an estimate for $G_1(y)^{\sigma-1}-G_2(y)^{\sigma-1}$, which is contained in (128). Using (15), we see that

$$G_1(y)^{\sigma-1}-G_2(y)^{\sigma-1}=G_1(y)^{\sigma-1}G_2(y)^{\sigma-1}(I_4+I_5+I_6), \tag{130}$$

where

$$I_4 := \int_{z\in D} \{\lambda_2(z)-\lambda_1(z)\}u_2(z)^{-(\sigma-1)}\tau_2(y,z)^{-(\sigma-1)}dz, \tag{131}$$

$$I_5 := \int_{z\in D} \lambda_1(z)\{u_2(z)^{-(\sigma-1)}-u_1(z)^{-(\sigma-1)}\}\tau_2(y,z)^{-(\sigma-1)}dz, \tag{132}$$

$$I_6 := \int_{z\in D} \lambda_1(z)u_1(z)^{-(\sigma-1)}\{\tau_2(y,z)^{-(\sigma-1)}-\tau_1(y,z)^{-(\sigma-1)}\}dz. \tag{133}$$

Using (118), we can substitute

$$(u,U_1,U_2)=(u_i,1/c_i,c_i), i=1,2, \tag{134}$$

in (103). Hence, using (116) and (125) in the same way as in proving (113) and (114), we can prove the following inequalities:

$$1/\overline{w}_i \le \tau_i(x,y)^{-(\sigma-1)} \le 1 \text{ for a.e. } x,y\in D,\ i=1,2, \tag{135}$$

$$1/c_i^{\sigma-1} \le G_i(y)^{\sigma-1} \le c_i^{\sigma-1}\overline{w}_i, \text{ for a.e. } y\in D,\ i=1,2. \tag{136}$$

Applying (124) to (136), we see that

$$||G_i^{\sigma-1}||_{L^\infty(D)} \le \overline{W}^{\sigma}, \ i = 1, 2. \tag{137}$$

Applying (118), (124) and (135) to (131), we see that

$$||I_4||_{L^\infty(D)} \le \overline{W}^{\sigma-1} \varepsilon_\lambda. \tag{138}$$

Applying Cauchy's mean-value theorem to the functions X^β and X^γ, where $\beta, \gamma \ne 0$, we see easily that if $X_i \in (U_1, U_2), i = 1, 2$, where $U_i, i = 1, 2$, satisfy (104), then

$$|X_1^\beta - X_2^\beta| \le |\frac{\beta}{\gamma}| \max_{U_1 \le \xi \le U_2} \xi^{\beta-\gamma} |X_1^\gamma - X_2^\gamma|. \tag{139}$$

Using (118) and (124), we can substitute $(X_i, U_1, U_2) = (u_i, 1/\overline{W}, \overline{W}), i = 1, 2$, in (139). Hence,

$$||u_1^\beta - u_2^\beta||_{L^\infty(D)} \le |\frac{\beta}{\gamma}| \overline{W}^{|\beta-\gamma|} ||u_1^\gamma - u_2^\gamma||_{L^\infty(D)}. \tag{140}$$

Applying (5), (17), (19), (118), (124), (135) and (140) with $(\beta, \gamma) = (-(\sigma-1), 1)$ to (132), we see that

$$||I_5||_{L^\infty(D)} \le (\sigma - 1)\overline{W}^{\sigma} \varepsilon_u. \tag{141}$$

Applying (21) to (115) with $X_i = \tau_i(x, y), i = 1, 2$, we see that

$$||\tau_1^{-(\sigma-1)} - \tau_2^{-(\sigma-1)}||_{L^\infty(D\times D)} \le (\sigma - 1)\varepsilon_\tau. \tag{142}$$

Applying (5), (17), (19), (118), (124) and (142) to (133), we see that

$$||I_6||_{L^\infty(D)} \le \overline{W}^{\sigma-1} (\sigma - 1)\varepsilon_\tau. \tag{143}$$

Applying (137), (138), (141) and (143) to (130), we obtain

$$||G_1^{\sigma-1} - G_2^{\sigma-1}||_{L^\infty(D)} \le \overline{W}^{2\sigma} (\overline{W}^{\sigma-1} \varepsilon_\lambda + (\sigma - 1)\overline{W}^{\sigma} \varepsilon_u + \overline{W}^{\sigma-1} (\sigma - 1)\varepsilon_\tau). \tag{144}$$

Recalling that each term of the integrand of (109) is nonnegative, and applying (111) with $(u, \lambda) = (u_2, \lambda_2)$, (117) and (135) to (128), we see that

$$||I_2||_{L^\infty(D)} \le ||G_1^{\sigma-1} - G_2^{\sigma-1}||_{L^\infty(D)}. \tag{145}$$

Applying (5), (18), (19), (118) and (124) to (16) with (116), we obtain

$$||Y_1 - Y_2||_{L^1(D)} \le \mu\varepsilon_u + \mu\overline{W}\varepsilon_\lambda + (1 - \mu)\varepsilon_\phi. \tag{146}$$

Applying (135) and (137) to (127), we see that

$$||I_1||_{L^\infty(D)} \le \overline{W}^{\sigma} ||Y_1 - Y_2||_{L^1(D)}. \tag{147}$$

Applying (111) with $(u, \lambda) = (u_2, \lambda_2)$, (117), (137) and (142) to (129), we see that

$$||I_3||_{L^\infty(D)} \leq \overline{W}^\sigma(\sigma - 1)\varepsilon_\tau. \tag{148}$$

Apply (144)–(148) to (126). Applying (47), (65)–(67), (88), (122) and (123) to the inequality thus obtained, we obtain (119).

Appendix B

For simplicity, we denote $f(u;\lambda, \phi, \tau)(x)$, $G(u;\lambda, \tau)(x)$ and $Y(u;\lambda, \phi)(x)$ by $f(u)(x)$, $G(u)(x)$ and $Y(u)(x)$, respectively. If $w(x) \in L^\infty_+(D)$, then (13) is equivalent to

$$w(x) = F(w)(x), \tag{149}$$

where

$$F(u)(x) := f(u)(x)/u(x)^{\sigma-1}. \tag{150}$$

Noting that (15) and (150) contain the singular term $(1/u(x))^{\sigma-1}$, we see that if Condition 1 holds, then the operators $f(u)$and $F(u)$ can act on all $u \in L^\infty_+(D)$. The following lemma is the key lemma of Appendices A, B, and C (see (109)):

Lemma 15 *If Condition 1 holds, then*

$$\int_{x\in D} \lambda(x)F(u)(x)dx = \mathbf{Y}(u) \textit{ for all } u \in L^\infty_+(D). \tag{151}$$

Proof Multiply both sides of (150) by $\lambda(x)$, and integrate both sides over $x \in D$. Using (14), we obtain

$$\int_{x\in D} \lambda(x)F(u)(x)dx$$

$$= \int_{x\in D} \{\int_{y\in D} \lambda(x)(1/u(x))^{\sigma-1}\tau(x, y)^{-(\sigma-1)}G(u)(y)^{\sigma-1}Y(u)(y)dy\}dx.$$

Using the Fubini–Tonelli theorem, we can exchange $\int_{x\in D}$ and $\int_{y\in D}$ in the right-hand side (see, e.g., Yoshida (1965), p.18). Moreover, we apply (22) to the right-hand side. Recalling definition (15), we see that the integrand of the right-hand side of the equality thus obtained contains $1/G(u)(y)^{\sigma-1}$ and $G(u)(y)^{\sigma-1}$, which cancel out each other. Recalling (109), we obtain (151).

This lemma is a continuous version of Tabata and Eshima (2018b, (40)). Considering (14)–(16), we see that the left-hand side of (151) is expressed as the triple integral, but that the right-hand side is expressed as the single integral of affine function (16). As mentioned in (3) of Section 1, Eq. (13) is complicated. Using this lemma, we can clarify the double singular structure of the right-hand side of (149), which is equivalent to (13).

Remark 16 We stated Propositions 4 and 5 before Proposition 8 and Theorem 9, and proved Proposition 4(i)(ii). However, first, we will prove Propositions 5(ii) and 8 and Theorem 9 without using Propositions 4(iii)(iv) and 5(i)(iii)(iv). Second, using Propositions 5(ii) and 8 and Theorem 9, we will prove Propositions 4(iii) and 5(i)(iii) and Theorem 10. Third, we will prove Propositions 4(iv) and 5(iv) and Theorems 11 and 12 in Appendix C.

Proof of Proposition 5(ii) Let $w(x) \in L_+^\infty(D)$ be a solution of (13). Note that (50) and (51) are not assumed in Proposition 5(ii). Recall that (13) is equivalent to (149). Substituting (149) in (151) with $u = w$, we see that

$$\int_{x\in D} \lambda(x)w(x)dx = \mathbf{Y}(w).$$

Applying (18) and (110) to this equality, and recalling (4) with $h = \lambda$, we obtain (52). Using Lemma 13(ii), (52) and (103) with

$$U_1 := \operatorname*{ess\,inf}_{x\in D} w(x),\ U_2 := \operatorname*{ess\,sup}_{x\in D} w(x),\ u(x) := w(x), \tag{152}$$

we obtain (107) with (152). Substituting (13) in (107) with (152), we see that

$$U_1^{\sigma-1}/\overline{w} \le w(x)^\sigma \le U_2^{\sigma-1}\overline{w} \text{ for a.e. } x \in D.$$

Applying (152) to $w(x)$ contained in this inequality, we see that

$$U_1^{\sigma-1}/\overline{w} \le U_1^\sigma,\ U_2^\sigma \le U_2^{\sigma-1}\overline{w}. \tag{153}$$

Solving these inequalities for U_i, $i = 1, 2$, and using (114) and (152), we obtain (53) and (54).

Proof of Proposition 8 Let $\mathbf{w}_n \in \mathbf{W}_n$. Substituting (34) in (61), we see that

$$\theta_n(\mathbf{w}_n;\lambda,\phi,\tau)(x) = -f(\mathbf{w}_n;\lambda,\phi,\tau)(x) + f(\mathbf{w}_n;\boldsymbol{\lambda}_n,\boldsymbol{\phi}_n,\boldsymbol{\tau}_n)(x). \tag{154}$$

Define

$$(\lambda_1,\phi_1,\tau_1) := (\lambda,\phi,\tau),\ (\lambda_2,\phi_2,\tau_2) := (\boldsymbol{\lambda}_n,\boldsymbol{\phi}_n,\boldsymbol{\tau}_n), \tag{155}$$

$$u_i := \mathbf{w}_n,\ i = 1, 2, \tag{156}$$

$$c_i := \overline{\mathbf{w}}_n,\ i = 1, 2, \tag{157}$$

where $\overline{\mathbf{w}}_n$ is defined in (45). Note that (50) and (51) are assumed in Proposition 8, but not assumed in Lemma 14. Hence, using (42) and (43), we can substitute

(155)–(157) in (117)–(125). Substituting (155) in (125), and using (45) and (55), we see that

$$\overline{w}_1 = \overline{w},\ \overline{w}_2 = \overline{\mathbf{w}}_n. \tag{158}$$

Substitute (157) and (158) in (124). Applying (96) to $\overline{W}$ thus obtained, we see that

$$\overline{W} \le \overline{\overline{\mathbf{w}}}_n. \tag{159}$$

It follows from (17) and (18) that (65)–(67) are monotone increasing functions of $r \ge 0$. Apply (98) and (159) to (123). Applying the inequality thus obtained to (119) with (155) and (156), we obtain

$$||\theta_n(\mathbf{w}_n;\lambda,\phi,\tau)||_{L^\infty(D)}$$

$$\le \frac{1}{2}(\overline{\overline{\mathbf{b}}}_{1,n}||\lambda_n - \lambda||_{L^1(D)} + \overline{\overline{\mathbf{b}}}_{2,n}||\boldsymbol{\phi}_n - \phi||_{L^1(D)} + \overline{\overline{\mathbf{b}}}_{3,n}||\boldsymbol{\tau}_n - \tau||_{L^\infty(D\times D)}).$$

Applying (102) to this inequality, we obtain (62) for each $n \ge N_0$.

Proof of Theorem 9 Let $\{\mathbf{w}_n\}$ be an arbitrary sequence that satisfies (72). Applying (58)–(60) to (62), we obtain

$$||\theta_n(\mathbf{w}_n;\lambda,\phi,\tau)||_{L^\infty(D)} \to 0 \text{ as } n \to +\infty. \tag{160}$$

From (94) and (96), we see that

$$1 \le \overline{\mathbf{w}}_n \le \overline{\overline{\mathbf{w}}}_2 \text{ for all } n \ge 2. \tag{161}$$

Using this result, (43) and (51), we can substitute

$$(u, U_1, U_2) = (\mathbf{w}_n, 1/\overline{\overline{\mathbf{w}}}_2, \overline{\overline{\mathbf{w}}}_2), \tag{162}$$

in (103), (105), (106) and (108). Applying the inequalities thus obtained to the following sequence of functions:

$$\{f(\mathbf{w}_n;\lambda,\phi,\tau)(x)\}, \tag{163}$$

we see that this sequence of functions is uniformly bounded and equicontinuous. Using the Ascoli-Arzelà theorem, we see that sequence of functions (163) is relatively compact in $C(D) \subset L^\infty(D)$ (see, e.g., Yoshida 1965, pp. 3–5, 85). Applying this result and (160) to the right-hand side of (61), we obtain (i).

It follows from (i) that every subsequence of $\{\mathbf{w}_n\}$ has a convergent subsequence. We denote the convergent subsequence by the same symbol $\{\mathbf{w}_n\}$, and define

$$w := \lim_{n\to+\infty} \mathbf{w}_n \text{ in } L^\infty(D). \tag{164}$$

We define

$$(\lambda_i, \phi_i, \tau_i) := (\lambda, \phi, \tau),\ i = 1, 2, \tag{165}$$

$$u_2 := w. \tag{166}$$

Applying (4), (58), and (164)–(166) to (42), we obtain (117). Moreover, from (43), (96), (164) and (103) with (162), we can substitute (166) and

$$u_1 := \mathbf{w}_n,\ c_i := \overline{\overline{\mathbf{w}}}_2,\ i = 1, 2, \tag{167}$$

in (118). Hence, we can substitute (165)–(167) in (119). Using (164) and (119) with (165)–(167), we see that

$$||f(\mathbf{w}_n;\lambda, \phi, \tau) - f(w;\lambda, \phi, \tau)||_{L^\infty(D)} \to 0,\ \text{as } n \to +\infty. \tag{168}$$

Applying this result, (160) and (164) to (61), we see that (164) is a solution of (13), that is, we obtain (ii). From (i) and (ii), we see that every subsequence of $\{\mathbf{w}_n\}$ has a convergent subsequence, and every convergent subsequence converges to a short-run equilibrium of M. Applying this result to $\{\mathbf{w}_n\}$ itself iteratively, we obtain (iii).

Remark 17 Condition (117) is assumed in Lemma 14. Hence, we need to prove that (117) holds in Proof of Theorem 9. However, it is not assumed in Lemma 14 that $u_1 \in S(\lambda_1)$. Hence, in Proof of Theorem 9, we have no need to prove that $u_1 \in S(\lambda_1)$.

Proof of Propositions 4(iii) and 5(i)(iii) Using Proposition 4(i), we can construct a sequence $\{\mathbf{w}_n\}$ that satisfies (72). Applying Theorem 9(i)(ii) to this sequence, we obtain Proposition 5(i).

Let us prove Proposition 5(iii). Proposition 5(i) implies that W is nonempty. If W is finite, then Proposition 5(iii) is trivial. Hence, we assume that W is infinite. Using (51)–(55) in the same way as in proving that (163) is relatively compact, we can prove that

$$\{f(w;\lambda, \phi, \tau)(x)^{1/\sigma}; w \in W\}, \tag{169}$$

is relatively compact in $C(D) \subset L^\infty(D)$. Applying (13) and the definition of W to (169), we see easily that (169) coincides with W. Hence, W is relatively compact. Hence, every sequence $\{w_n\} \subseteq W$ has a convergent subsequence. Denote the convergent subsequence by the same symbol $\{w_n\}$, and define

$$w_\infty := \lim_{n\to+\infty} w_n.$$

Hence, using (52), (53) and (55), we can substitute (165) and

$$u_1 := w_\infty,\ u_2 := w_n,\ c_i := \overline{w},\ i = 1, 2,$$

in (117)–(125). Applying (13) with $w = w_n$ to the inequality thus obtained, we see that $w_\infty \in W$. Hence, we obtain Proposition 5(iii).

Recalling that $\mathbf{L}^n(D)$ is a finite dimensional subspace of $L^\infty(D)$ for each $n \geq 2$, and performing calculations similar to, but easier than, those proving Proposition 5(iii), we can obtain Proposition 4(iii).

Proof of Theorem 10 Assume that $\{u_m\}$ is an arbitrary sequence included in $\underset{m\geq n}{\cup}\mathbf{W}_m$, where n is an arbitrary integer. If this sequence is included in $\underset{N\geq m\geq n}{\cup}\mathbf{W}_m$ for some $N \geq n$, then Proposition 4(iii) implies that $\{u_m\}$ has a convergent subsequence. If not, then $\{u_m\}$ has a subsequence which can be considered as a subsequence of a sequence $\{\mathbf{w}_n\}$ that satisfies (72). Hence, applying Theorem 9(i) to this subsequence, we see that $\{u_m\}$ has a convergent subsequence. Hence, we obtain (i).

Using Theorem 9(i)(ii), we define W_∞ as the set of all accumulation points of *all* $\{\mathbf{w}_n\}$ that satisfy (72). It follows from Proposition 4(i) and Theorem 9(i)(ii) that W_∞ is a nonempty subset of W. Let us prove that W_∞ is compact. If W_∞ is finite, then this statement is trivial. Hence, we assume that W_∞ is infinite. Using Proposition 5(iii), we see that every sequence $\{w_n\} \subseteq W_\infty$ has a subsequence that converges to a short-run equilibrium $w \in W$. We denote this convergent subsequence by the same symbol $\{w_n\}$, that is, we define

$$w := \lim_{n\to\infty} w_n \in W. \tag{170}$$

We prove that $w \in W_\infty$. Noting that $\{w_n\} \subseteq W_\infty$, and recalling the definition of W_∞, we see that for each $n \geq 2$ there exists a sequence $\{\mathbf{w}_m(n)\}$such that $\mathbf{w}_m(n) \in \mathbf{W}_m$ for each $m \geq 2$, and $w_n$is an accumulation point of $\{\mathbf{w}_m(n)\}$.Hence, there exists a monotone-increasing integer-valued sequence $\{m_n\}$ such that

$$||\mathbf{w}_{m_n}(n) - w_n||_{L^\infty(D)} \to 0 \text{ as } n \to \infty. \tag{171}$$

Noting that $\mathbf{w}_{m_n}(n) \in \mathbf{W}_{m_n}$ for each $n \geq 2$, combining (170) and (171), and recalling the definition of W_∞, we see that $w \in W_\infty$. Hence, we see that W_∞ is nonempty and compact.

Let us prove that for every $\varepsilon > 0$ there exists $N \geq 2$ such that

$$\bigcup_{m\geq n}\mathbf{W}_m \subseteq V(\varepsilon/2, W_\infty), \text{ for all } n \geq N. \tag{172}$$

Assume that (172) does not hold. Then, there exist a constant $\varepsilon > 0$ and a sequence $\{\mathbf{w}_n\}$ that satisfies (72) and has a subsequence included in the complement of $V(\varepsilon/2, W_\infty)$. Applying Theorem 9(i)(ii) to this subsequence of $\{\mathbf{w}_n\}$, and recalling the definition of W_∞, we see that the subsequence of $\{\mathbf{w}_n\}$ has a subsequence converging to a short-run equilibrium $v \in W_\infty$. However, v is contained in the closure of the complement of $V(\varepsilon/2, W_\infty)$. This is a contradiction. Hence we obtain (172). Recalling that W_∞ is a nonempty compact subset of W, we see that for every $\varepsilon > 0$ there exists a finite set W_0 that satisfies (73) and

$$W_\infty \subseteq V(\varepsilon/2, W_0).$$

Combining this inclusion relation and (172), and recalling (75), we obtain (74). Using Theorem 9(i)(ii), and recalling the definition of W_∞, we obtain (80). Noting that W_∞ is compact, we obtain (77) from (80). Replace ε with $\varepsilon/2$ in (77) and (79). Combining the inclusion relations thus obtained, we obtain (78).

Appendix C

We use (47), (65)–(67), (86)–(88) and (125) in the following lemma:

Lemma 18 *Assume that* (116) *satisfy Condition* 1. *Assume that* (13) *with* (116) *has a solution* $w_i(x) \in L_+^\infty(D), i = 1, 2,$ *respectively. If*

$$A_0 < 1, \tag{173}$$

then,

$$||w_1 - w_2||_{L^\infty(D)}$$

$$\leq \frac{A_1||\lambda_1 - \lambda_2||_{L^1(D)} + A_2||\phi_1 - \phi_2||_{L^1(D)} + A_3||\tau_1 - \tau_2||_{L^\infty(D\times D)}}{4(1 - A_0)}, \tag{174}$$

where

$$A_i := a_i(\mu, \sigma, \overline{\overline{W}}),\ i = 0, ..., 3, \tag{175}$$

$$\overline{\overline{W}} := \max\{\overline{w}_1, \overline{w}_2\}. \tag{176}$$

Proof Using (52) and (53), we can substitute

$$u_i := w_i,\ c_i := \overline{w}_i,\ i = 1, 2, \tag{177}$$

in (117) and (118). Substituting (177) in (124), we see that

$$\overline{W} = \overline{\overline{W}}. \tag{178}$$

Hence, we can substitute (177) and (178) in (119), (122) and (123). Applying (17), (88) and (178) to (140) with (177) when $(\beta, \gamma) = (1, \sigma)$, we see that

$$||w_1 - w_2||_{L^\infty(D)} \leq k(\sigma, \overline{\overline{W}})||w_1^\sigma - w_2^\sigma||_{L^\infty(D)}. \tag{179}$$

Subtract (13) with (116), $i = 2$, from (13) with (116), $i = 1$. Apply (119), (122) and (123) to the equality thus obtained. Apply (86)–(88), (175), (178) and (179) to the inequality thus obtained. Using (173), we divide both sides by $1 - A_0$. Hence, we obtain (174).

Remark 19 (i) To prove that M has a short-run equilibrium, we assume (50) and (51) in Proposition 5(i) and Lemma 13(iii). However, we have no need to assume these conditions in Lemmas 13(i)(ii), 14, 15 and 18.

(ii) It follows from (174) that the distribution of wages is Lipschitz continuous with respect to the worker distribution, the farmer distribution, and the transport-cost function. As mentioned in Section 4, (34) with Condition 3 is a special case of (13) with Condition 1. Hence, (174) can be applied to both the continuous wage equation and the wage equation with n regions. Hence, using Lemma 18, we can prove Propositions 4(iv) and 5(iv). In this proof, we use (46) and (56) to show that (173) holds.

(iii) The coefficients of the linear combination in the right-hand side of (174) are defined as positive functions of the elasticity of substitution, the manufacturing expenditure, and the maximum of transport costs (see (175)). These functions are independent of the number n of regions.

Proof of Propositions 4(iv) and 5(iv) Assume that (13) has two solutions $w_i(x)$, $i = 1, 2$. It follows from (55), (125) and (165) that

$$\overline{w}_i = \overline{w}, i = 1, 2.$$

Substitute these equalities, the solutions of (13), and (165) in (173)–(176). Using (56), we see that (173) holds. Hence, we obtain Proposition 5(iv). Performing calculations similar to, but easier than, those done in proving Proposition 5(iv), we obtain Proposition 4(iv).

Proof of Theorem 11 It follows from Theorem 10(ii) that W_∞ is a nonempty subset of W. Using Proposition 5(iv), we see that $W = \{w\}$. Combining these results, we obtain (81).

It follows from (17) and (18) that

$$\text{(47), (86) and (87) are monotone-increasing functions of } r \geq 0. \tag{180}$$

Applying this result and (96) to (97) with $i = 0$, and using (100), we see that

$$a_0(\mu, \sigma, \overline{\mathbf{w}}_n) \leq \overline{\overline{\mathbf{a}}}_{0,n} \leq (1 + \overline{a}_0)/2. \tag{181}$$

Applying (56) and (84) with $i = 0$ to this inequality, we obtain (46) for each $n \geq N_0$. Hence, from Proposition 4(iv) we see that $\mathbf{M}_n$has a unique short-run equilibrium $\mathbf{w}_n$ for each $n \geq N_0$. Applying this result and (81) to (79), we obtain (82).

Proof of Theorem 12 Substituting (45), (55) and (158) in (176), and using (96), we obtain

$$\overline{\overline{W}} \leq \overline{\overline{\mathbf{w}}}_n.$$

Applying this inequality and (180) to (175), and recalling (97), we see that

$$A_i \leq \bar{\bar{\mathbf{a}}}_{i,n}, \ \ i = 0, ..., 3.$$

Using these inequalities, (56), (84), (101) and (181), we see that (173) holds, and

$$\frac{1}{1 - A_0} \leq \frac{1}{1 - \bar{\bar{\mathbf{a}}}_{0,n}} \leq \frac{2}{1 - \bar{a}_0}, \text{ for all } n \geq N_0, \tag{182}$$

$$A_i \leq 2\bar{a}_i, i = 1, 2, 3. \tag{183}$$

Replacing $(w, \lambda, \phi, \tau)$with $(\mathbf{w}_n, \boldsymbol{\lambda}_n, \boldsymbol{\phi}_n, \boldsymbol{\tau}_n)$ in (13), we can obtain (34). Hence, defining

$$w_1 := w, \ w_2 := \mathbf{w}_n, \tag{184}$$

and recalling Remark 19(i), we can substitute (155), (158) and (184) in (174)–(176). Applying (182) and (183) to the inequality thus obtained, we obtain (83).

Appendix D

This appendix analyzes the simulation results shown in Figs. 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24. Let us analyze Figs. 15, 16, 17, 18, and 19. First, we observe how the income distribution changes as μ increases. It follows from Figs. 15, 16, 17, 18, and 19 that

$$0.759 < \mathbf{w}_{80}(x) < 1.3. \tag{185}$$

We see that even if μ increases, the wage distribution does not change significantly. Agricultural workers are distributed uniformly in [0, 1], but manufacturing workers are concentrated in the neighborhood of $x = 1$ (see Fig. 6). The ratio of manufacturing workers increases as μ increases. Hence, the income distribution is pulled to the right as μ increases (see Figs. 25, 26, 27, 28, and 29).

Second, we observe how the $(\sigma - 1)$ power of the price index changes as μ increases. The price index does not include μ, and μ only affects the price index indirectly through changes in the wage distribution. Using (185), we see that even if μ increases, then the $(\sigma - 1)$ power of the price index changes only slightly (see Figs. 30, 31, 32, 33, and 34).

It is difficult to determine whether the income or the price index has a stronger effect on the wage distribution by simply comparing Figs. 25, 26, 27, 28, and 29 and 30, 31, 32, 33, and 34. However, the income becomes equal to 0.1 in the left half segment [0, 0.5] as μ increases (see Figs. 25, 26, 27, 28, and 29). In contrast, the $(\sigma - 1)$ power of the price index is always larger than 1.13 in the right half segment [0.5, 1] (see Figs. 30, 31, 32, 33, and 34). Comparing Figs. 25, 26, 27, 28, and 29 and 30, 31, 32, 33, and 34 using these two results, we see that the product of the income and the $(\sigma - 1)$ power of the price index is pulled to the right as μ increases (see Fig. 35).

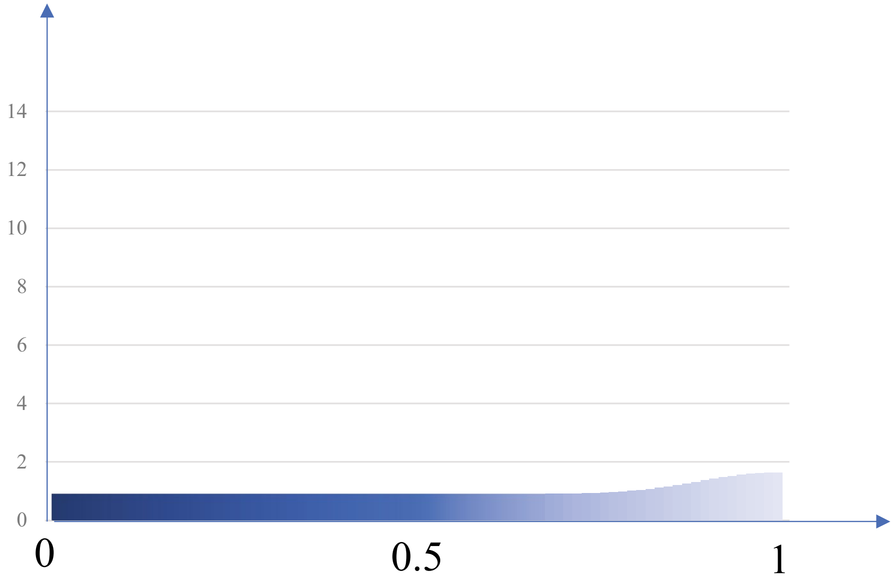

Fig. 25 The income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.1)$. The maximum value is equal to 1.625938, and the minimum value is equal to 0.9. All the vertical scales in Figs. 12, 13, 14 and 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, and 35 are the same

Using Fig. 14, Section 6 explains the reason why the wage distribution described in Fig. 11 is pulled to the left. In the same way, using Fig. 35, we see that the wage distribution is gradually pulled to the right as μ increases (see Figs. 15, 16, 17, 18, and 19).

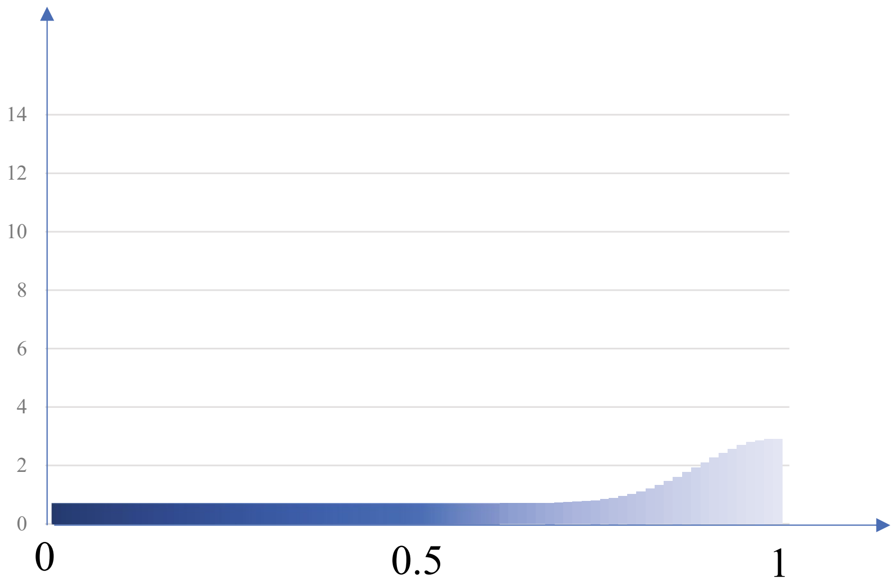

Fig. 26 The income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.3)$. The maximum value is equal to 2.896739, and the minimum value is equal to 0.7

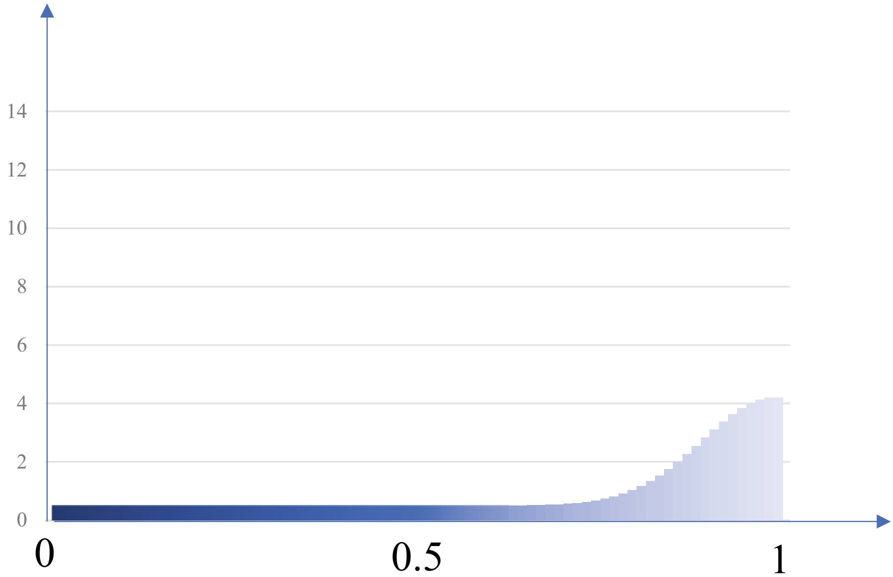

Fig. 27 The income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.5)$. The maximum value is equal to 4.193407, and the minimum value is equal to 0.5

Let us analyze Figs. 20, 21, 22, 23, and 24. First, we observe how the $(\sigma - 1)$ power of the price index changes as σ increases. The denominator of the price index is expressed as the sum of the products of the manufacturing-worker distribution, $(1/\mathbf{w}_{80}(x))^{\sigma-1}$, and the

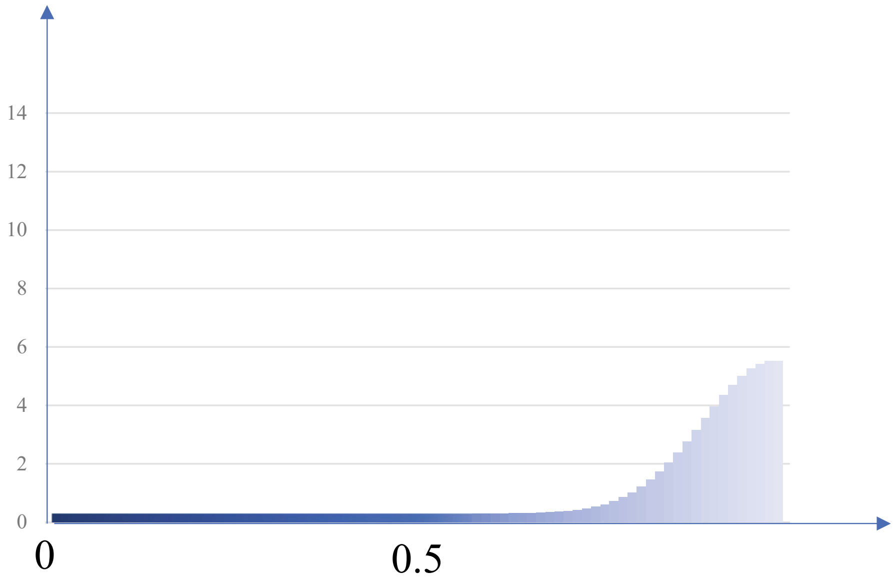

Fig. 28 The income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.7)$. The maximum value is equal to 5.517324, and the minimum value is equal to 0.3

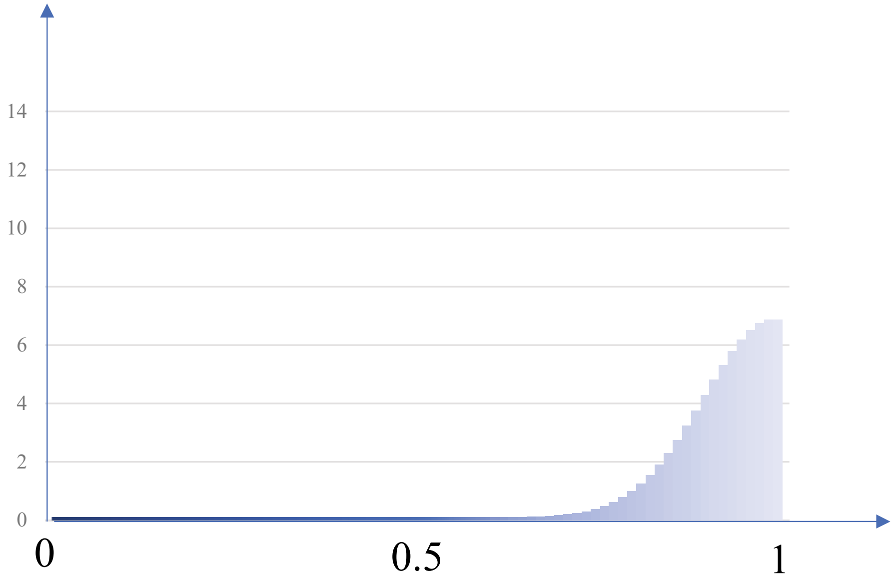

Fig. 29 The income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.9)$. The maximum value is equal to 6.868445, and the minimum value is equal to 0.1

$-(\sigma - 1)$ power of the transportation-cost function. The left half section [0, 0.5] has few manufacturing workers (see Fig. 6), and the transportation-cost function is an exponential function of distance (see (91)). Moreover, from Figs. 20, 21, 22, 23, and 24, we see that

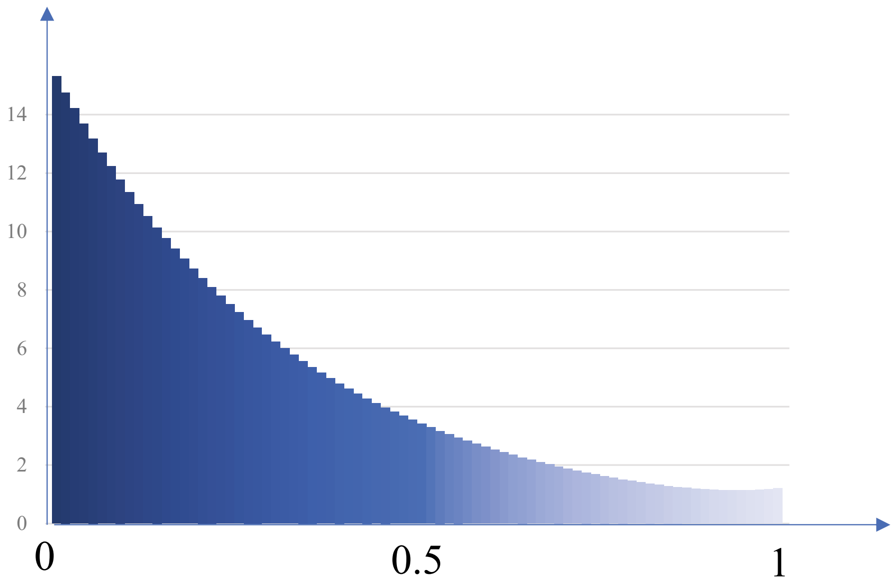

Fig. 30 The graph of $G(\mathbf{w}_{80}; \lambda_{80}, \boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma, \mu) = (4.0, 0.1)$. The maximum value is equal to 15.320182. The minimum value is equal to 1.132347

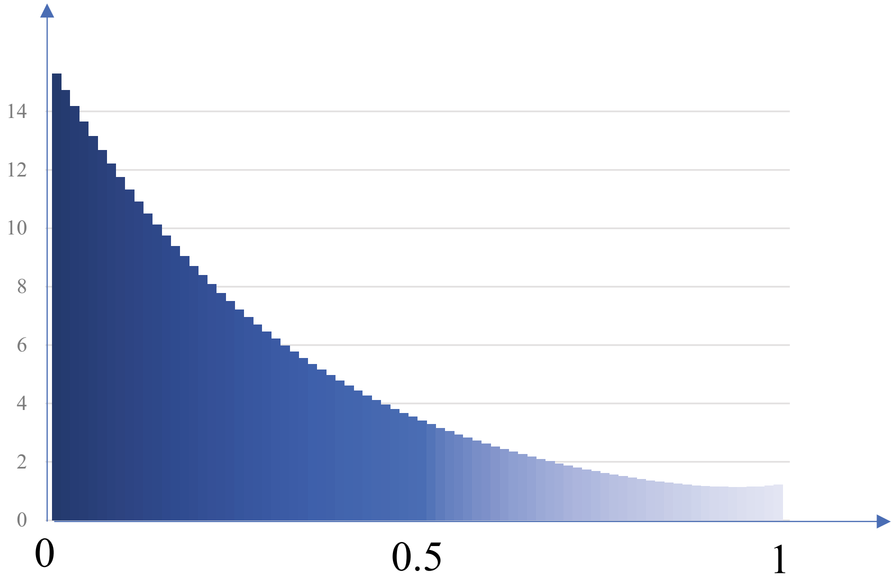

Fig. 31 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(4.0,0.3)$. The maximum value is equal to 15.289309. The minimum value is equal to 1.138684

$$0.872 < \mathbf{w}_{80}(x) < 1.616. \tag{186}$$

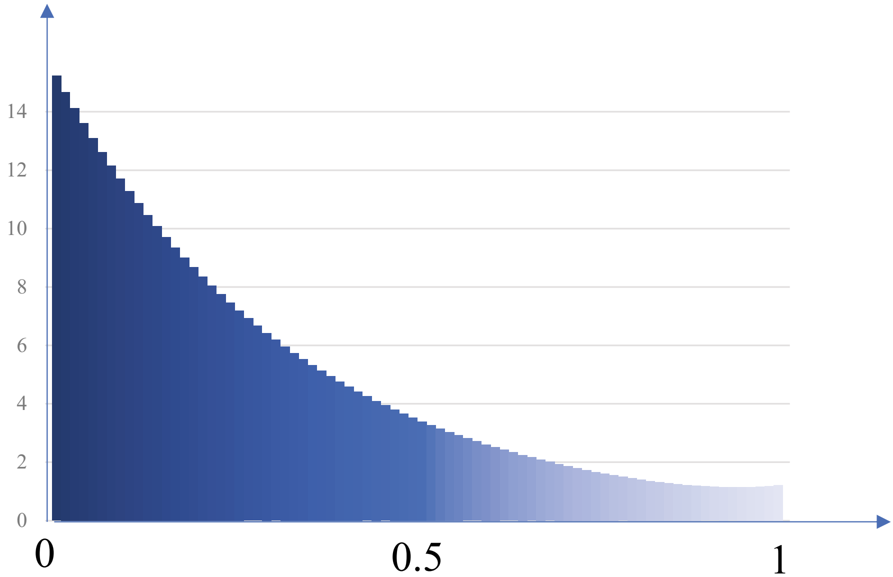

Fig. 32 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(4.0,0.5)$. The maximum value is equal to 15.233695. The minimum value is equal to 1.143667

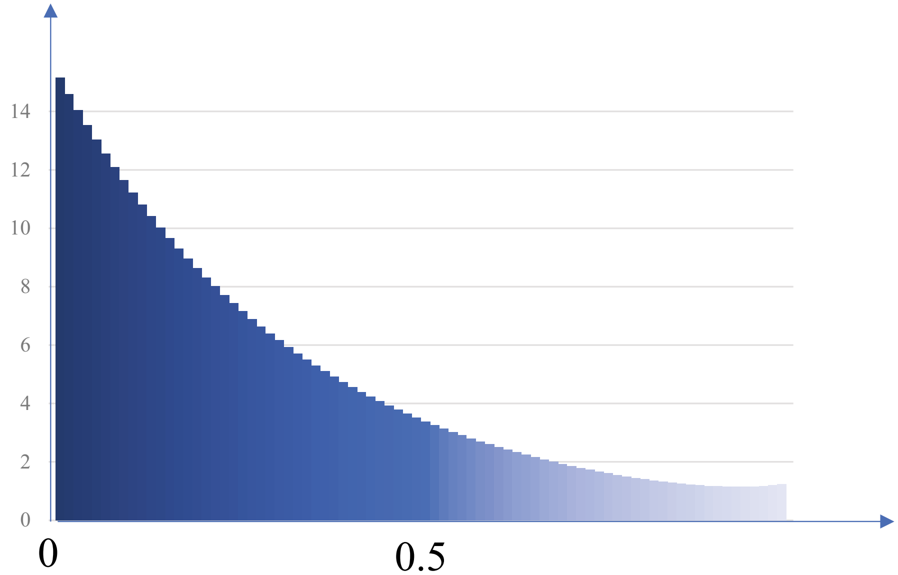

Fig. 33 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(4.0,0.7)$. The maximum value is equal to 15.152876. The minimum value is equal to 1.147924

Even if σ increases, then $\mathbf{w}_{80}(x)$ does not change greatly. Applying these results to (36), we see that the denominator of the $(\sigma-1)$ power of the price index at point x decreases exponentially as $x\in[0,0.5]$ approaches the left end and that the rate

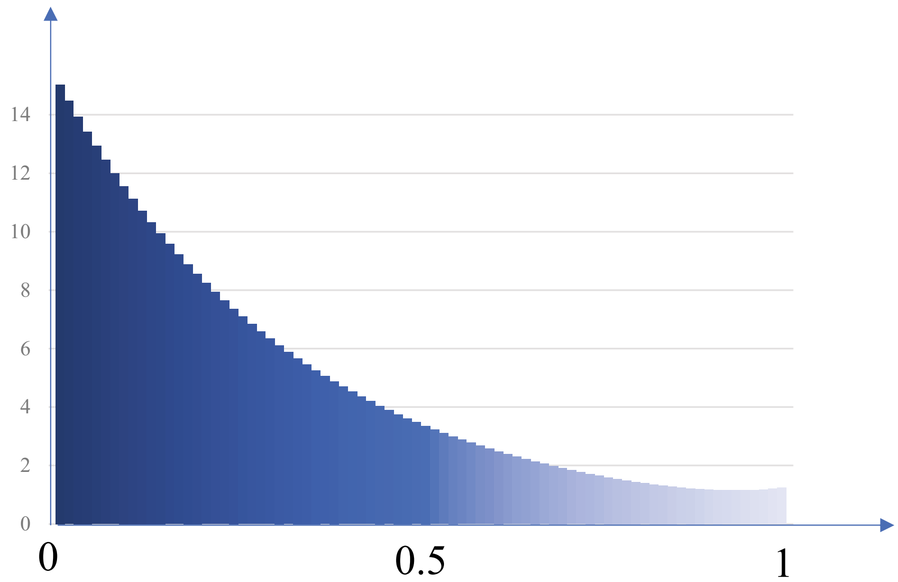

Fig. 34 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(4.0,0.9)$. The maximum value is equal to 15.026532. The minimum value is equal to 1.150564

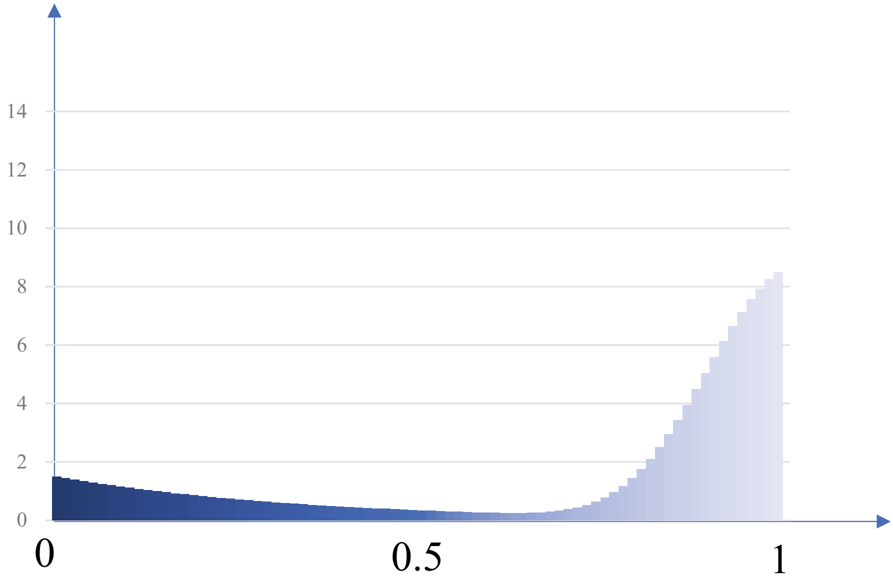

Fig. 35 The graph of the product $G(\mathbf{w}_{80};\boldsymbol{\lambda}_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}\,Y(\mathbf{w}_{80};\boldsymbol{\lambda}_{80},\boldsymbol{\phi}_{80})(x)$ when $(\sigma,\mu)=(4.0,0.9)$. This product is pulled by the income and the price index. The former pulls the product more strongly than the latter

of decrease increases with σ. Hence, the $(\sigma-1)$ power of the price index at point x increases exponentially as $x\in[0,0.5]$ approaches the left end, and the rate of increase increases with σ (see Figs. 36, 37, 38, 39, and 40).

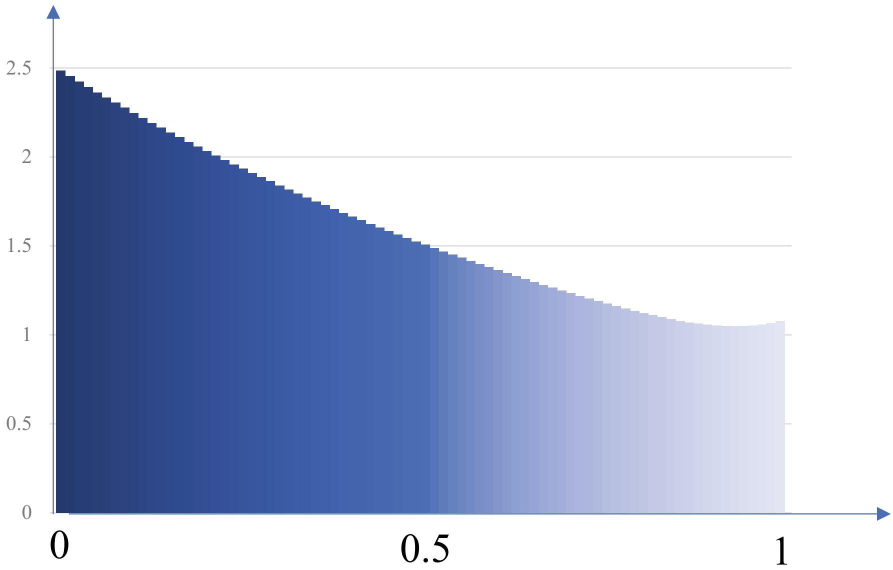

Fig. 36 The graph of $G(\mathbf{w}_{80};\boldsymbol{\lambda}_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(2.0,0.5)$. The maximum value is equal to 2.484539, and the minimum value is equal to 1.048297

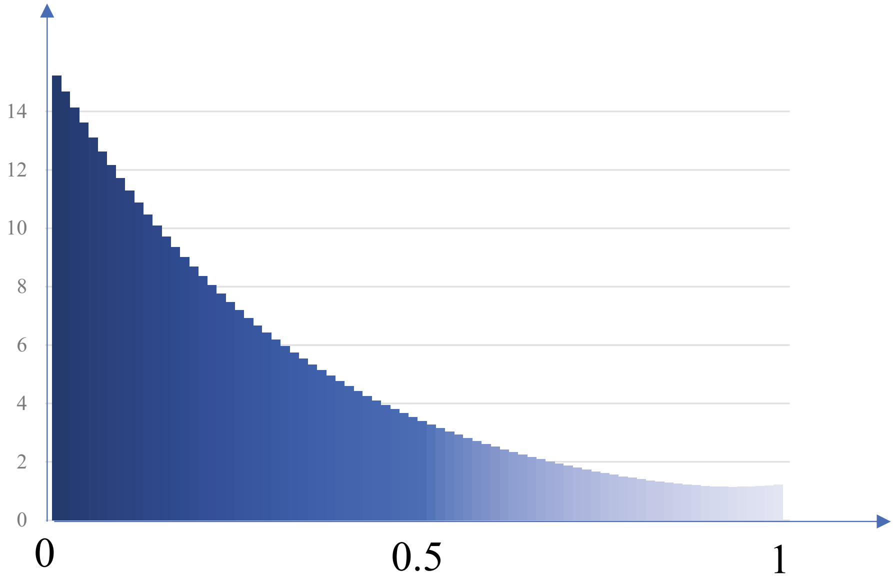

Fig. 37 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(4.0,0.5)$. The maximum value is equal to 15.233695, and the minimum value is equal to 1.143667

The income does not include σ, and σ only affects the income indirectly through changes in the wage distribution. Using (186), we see that the income distribution changes only slightly (see Figs. 41, 42, 43, 44, and 45).

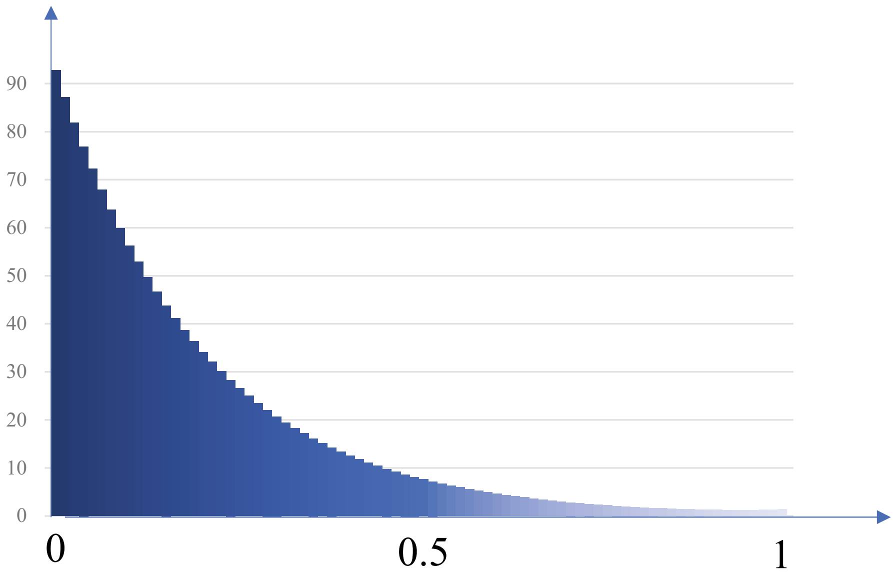

Fig. 38 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(6.0,0.5)$. The maximum value is equal to 92.784505. The minimum value is equal to 1.229771

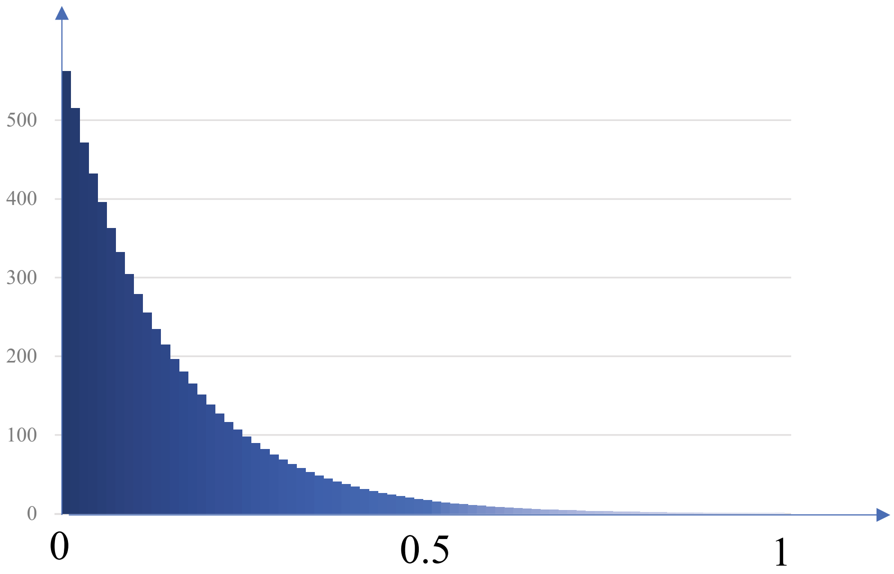

Fig. 39 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(8.0,0.5)$. The maximum value is equal to 562.317969. This graph appears to be zero in [0.5, 1]. However, the minimum value is 1.300977

Comparing Figs. 36, 37, 38, 39, and 40 and 41, 42, 43, 44, and 45, we see that the price index becomes very large in [0, 0.5] and that the income distribution is larger than 0.5 in [0, 0.5]. Hence, we see that the product of the income and the $(\sigma-1)$

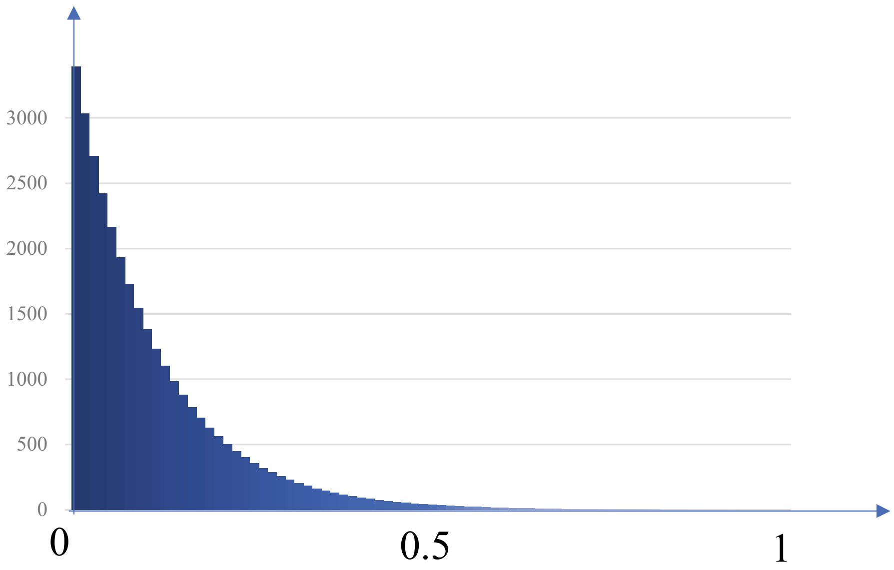

Fig. 40 The graph of $G(\mathbf{w}_{80};\lambda_{80},\boldsymbol{\tau}_{80})(x)^{\sigma-1}$ when $(\sigma,\mu)=(10.0,0.5)$. The maximum value is equal to 3393.442543. This graph appears to be zero in [0.5, 1]. However, the minimum value is equal to 1.354404

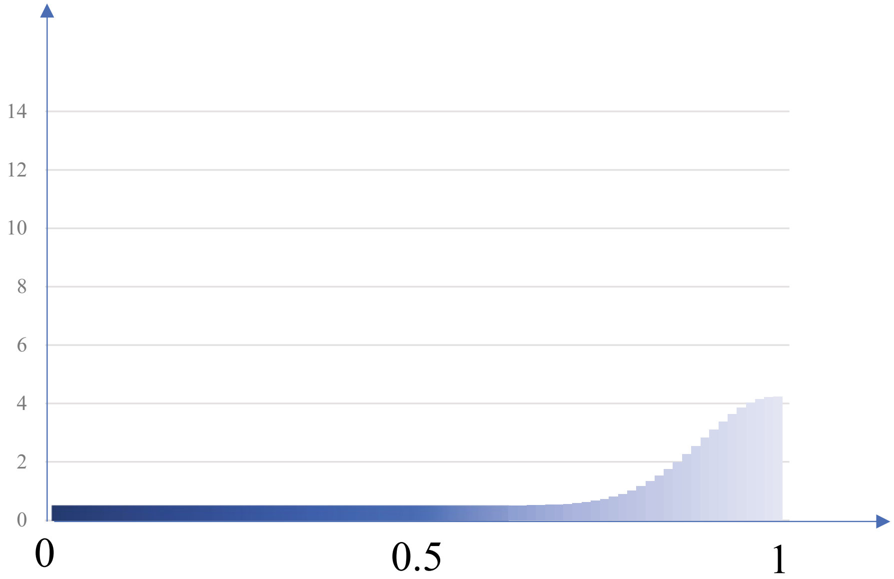

Fig. 41 The income distribution when $(n, \sigma, \mu) = (80, 2.0, 0.5)$. The maximum value is equal to 4.229456. The minimum value is equal to 0.5. All the vertical scales in Figs. 41, 42, 43, 44, and 45 are the same as in Figs. 12, 13, 14 and 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, and 35

power of the price index is pulled to the left by the price index (see Fig. 46). Hence, we see that the wage distribution is gradually pulled to the left as σ increases (see Figs. 20, 21, 22, 23, and 24).

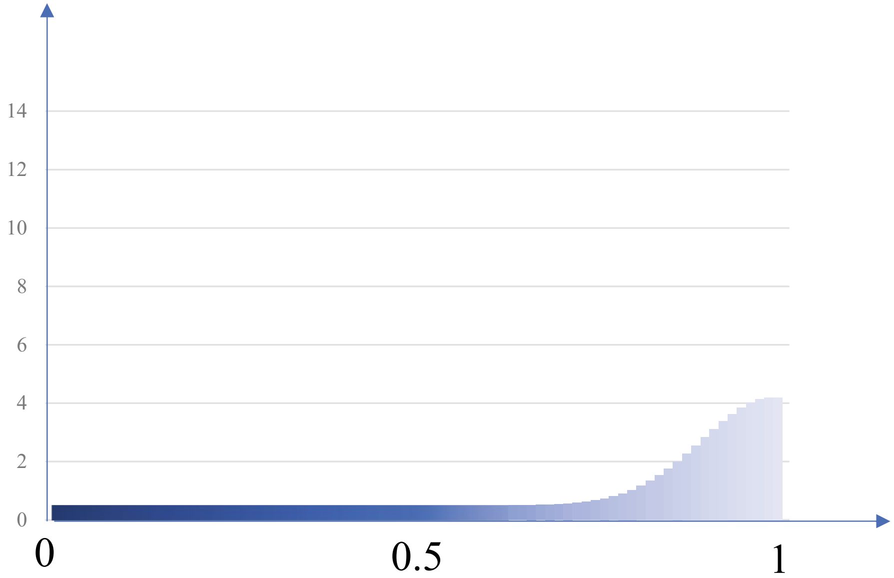

Fig. 42 The income distribution when $(n, \sigma, \mu) = (80, 4.0, 0.5)$. The maximum value is equal to 4.194367. The minimum value is equal to 0.5

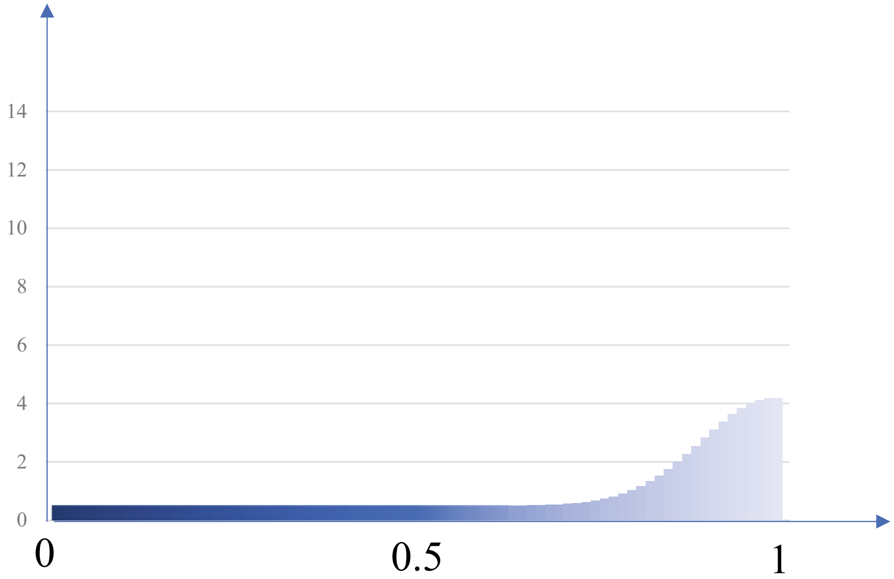

Fig. 43 The income distribution when $(n, \sigma, \mu) = (80, 6.0, 0.5)$. The maximum value is equal to 4.183171. The minimum value is equal to 0.5

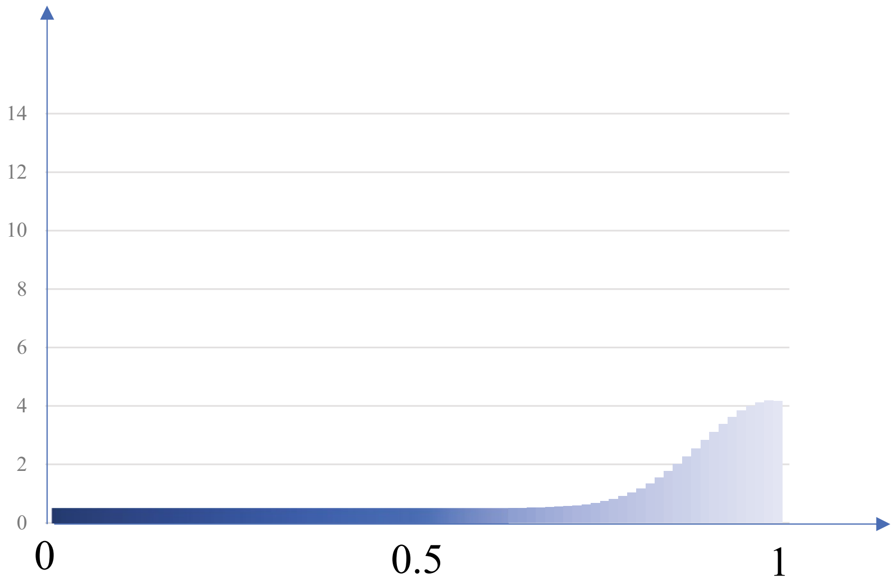

Fig. 44 The income distribution when $(n, \sigma, \mu) = (80, 8.0, 0.5)$. The maximum value is equal to 4.175064. The minimum value is equal to 0.5

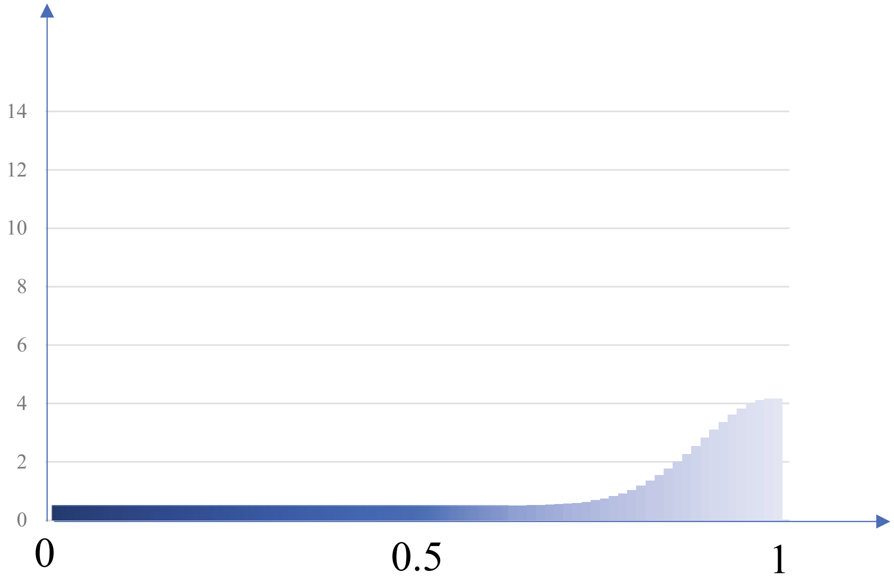

Fig. 45 The income distribution when $(n, \sigma, \mu) = (80, 10.0, 0.5)$. The maximum value is equal to 4.167675. The minimum value is equal to 0.5

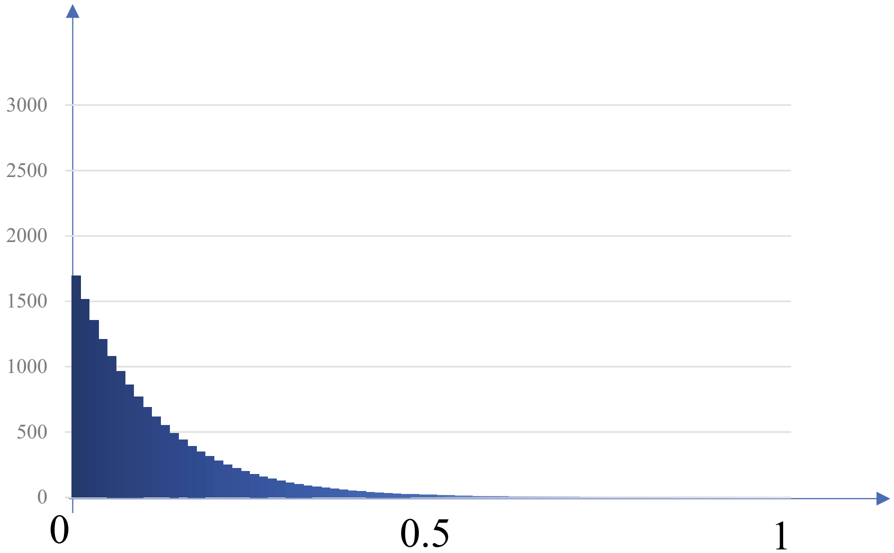

Fig. 46 The graph of the product $G(\mathbf{w}_{80};\boldsymbol{\lambda}_{80}, \boldsymbol{\tau}_{80})(x)^{\sigma-1}\,Y(\mathbf{w}_{80};\boldsymbol{\lambda}_{80}, \boldsymbol{\phi}_{80})(x)$ when $(\sigma, \mu) = (10.0, 0.5)$. The scale of the vertical axis is the same as in Fig. 40. The maximum value is equal to 1696.721272. The minimum value is equal to 2.981878. This product is strongly pulled to the left by the price index

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Code Availability The code that supports the findings of this study is available from the corresponding author upon request.

Declarations

Ethical Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication Not applicable.

Competing Interests The authors (Minoru Tabata and Nobuoki Eshima) have no competing interests to declare.

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