



On H_∞ Finite-Time Boundedness and Finite-Time Stability for Discrete-Time Neural Networks with Leakage Time-Varying Delay

Le A. Tuan¹

Accepted: 4 December 2023 / Published online: 6 February 2024
© The Author(s) 2024

Abstract

This paper investigates the problems of H_∞ finite-time boundedness (FTB) and finite-time stability (FTS) for discrete-time neural networks (NNs) with both leakage delay and discrete delay, as well as different generalized activation functions. To this end, we construct suitable Lyapunov–Krasovskii (L–K) functionals and apply the extended reciprocally convex approach to derive delay-dependent criteria for the addressed problems. The obtained conditions are expressed as linear matrix inequalities (LMIs), which can be efficiently solved by the LMI Toolbox in Matlab. The paper also admits of two numerical examples to point up the advantages and reliability of the proposed method.

Keywords Discrete-time neural networks · Finite-time stability · H_∞ finite-time boundedness · Leakage delay · Time-varying delay

1 Introduction

The idea of the finite-time stability of systems originated in Soviet scientific literature in the mid-20th century [19] and has since been applied to various real-world systems, such as biochemical reactions and communication networks, etc. [4, 12, 27]. The main goal of this analysis is to determine whether a system can maintain its desired behavior within a given time interval, which may be very short. By using the L–K functional approach and LMI techniques, a series of research works on stability, boundedness, stabilization, and H_∞ control over a finite time interval for both linear and nonlinear discrete-time systems were announced of late, e.g., see [3, 32, 38, 41, 44, 45].

NNs with delays are a type of artificial NNs that can model the temporal dynamics of complex systems. They can capture the effects of past inputs and outputs on the current state of the network and thus can handle non-Markovian processes. NNs with delays can learn the patterns and trends of time series data, such as stock prices, weather, or traffic, and forecast

✉ Le A. Tuan
leanhtuan@hueuni.edu.vn

¹ Department of Mathematics, University of Sciences, Hue University, 77 Nguyen Hue, Hue 49100, Thua Thien Hue, Vietnam

future values based on historical data; NNs with delays can be also used to design adaptive controllers for nonlinear systems, such as robots, vehicles, or power plants; and at last, NNs with delays can be applied to various signal-processing tasks, such as speech recognition, image processing, or biomedical signal analysis [5, 10, 17, 25, 26, 39, 40]. Especially, for the class of discrete-time NNs, there are a number of fascinating articles [1, 27, 37, 40] that deal with one of the topics of stability, passivity, or boundedness in a finite time interval. The paper [14] highlighted the importance of considering both discrete and leakage delays in the analysis of NNs. In particular, leakage delay in the negative feedback term can lead to instability and complex behaviors in NNs. Therefore, it is essential to investigate the effects of leakage delay on the stability and performance of NNs. Based on that fact, stability and passivity for several different classes of continuous-time NNs with leakage delay were studied in [5, 11, 21, 31], while the stability and dissipativity for some discrete-time analogs were considered in [6–8, 16, 22, 29, 34–36]. Delay-dependent stability criteria for complex-valued NNs with leakage delay were also established, e.g., see [9, 33]. The works mentioned seem to be enough to demonstrate the considerable attractiveness of leakage delay for researchers who have been working in various fields over the years. However, despite that, for the sake of convenience of analysis, leakage delay was unfortunately ignored while studying the finite-time stability, passivity, and boundedness of the systems mentioned in the papers [27, 40]. The same situation also holds for H_∞ FTB of the systems discussed in [1, 37]. On the other hand, from [15, 20], we know that nonlinear functions satisfying the sector-bounded condition are more general than the usual class of Lipschitz functions. However, up to this point, very few authors have investigated general NNs with activation functions satisfying the sector-bounded condition, and [10, 30] are some typical works among them.

To the best of our knowledge, in the literature, there are no results on the problem of H_∞ FTB for discrete-time NNs with leakage delay and discrete delay as well as different generalized activation functions. That is the main reason why we want to fill that gap in the existing literature. More specifically, in this paper, by a logical combination of appropriately constructed L–K functionals and an extended reciprocally convex matrix inequality, we first suggest conditions that ensure not only FTB of the involved NNs but also a finite-time H_∞ performance. With the same technique, the FTS of the corresponding nominal system is also obtained. Some numerical examples are offered to demonstrate the validity of the achieved conditions. The core contributions of this paper consist of:

- It is the first time to propose delay-dependent criteria for H_∞ FTB and FTS of discrete-time NNs with a pair of time-varying delays (leakage delay and discrete delay). These criteria contain information about the upper and lower bounds of both types of delay.
- The neuron activation functions are different and are supposed to satisfy the sector-bounded conditions, which are known to be more general than the usual Lipschitz conditions.
- The extended reciprocally convex technique is entirely exploited, so the number of decision variables is limited as much as possible.

The arrangement of the rest of this paper is as follows. In Sect. 2, some relevant definitions and technical propositions are presented in detail, including the problem we aim to handle. Delay-dependent criteria in the form of matrix inequalities for H_∞ FTB and FTS, respectively, together with two illustrative examples, are delivered in Sect. 3. Sections on conclusion and references close the paper.

Notation \mathbb{Z}_+ represents the set of all non-negative integers; \mathbb{R}^n and $\mathbb{R}^{m \times n}$ symbol for the n -dimensional Euclidean space and the set of real matrices of size $m \times n$ respectively; A^{-1} and A^\top are the inverse and transpose of a matrix A respectively; $A > 0$ means that A is a

positive-definite matrix and $A > B$ implies $A - B > 0$. The symbol $*$ stands for entries of a matrix implied by symmetry (in a symmetric matrix).

2 Preliminaries

Let us take the following discrete-time NNs with a pair of time-varying delays and any external disturbance into consideration

$$\begin{aligned} x(k + 1) &= Ax(k - \sigma(k)) + Bf(x(k)) + B_1g(x(k - h(k))) + C\omega(k), \quad k \in \mathbb{Z}_+, \\ z(k) &= A_1x(k) + Dx(k - h(k)) + D_1x(k - \sigma(k)) + C_1\omega(k), \\ x(k) &= \phi(k), \quad k \in \{-\rho, -\rho + 1, \dots, 0\}, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; n is the number of neurons; $z(k) \in \mathbb{R}^r$ is the observation output; the diagonal matrix $A \in \mathbb{R}^{n \times n}$ characterizes the self-feedback terms; $B, B_1 \in \mathbb{R}^{n \times n}$ are connection weight matrices; $C \in \mathbb{R}^{n \times s}, C_1 \in \mathbb{R}^{r \times s}$ are known matrices; $A_1, D, D_1 \in \mathbb{R}^{r \times n}$ are the observation matrices. The discrete delay function $h(k)$ and leakage delay function $\sigma(k)$ satisfy the condition

$$0 < h_1 \leq h(k) \leq h_2, \quad 0 < \sigma_1 \leq \sigma(k) \leq \sigma_2 \quad \forall k \in \mathbb{Z}_+, \tag{2}$$

where h_1, h_2, σ_1 and σ_2 are given natural numbers; $\rho := \max\{\sigma_2, h_2\}$ and $\phi(k)$ is the initial function. $\omega(k) \in \mathbb{R}^s$ is the external disturbance that is assumed to satisfy

$$\sum_{k=0}^N \omega^\top(k)\omega(k) < d, \tag{3}$$

where d is a given positive scalar. Besides, in the framework of this article, we adopt the following assumption for neuron activation functions $f(\cdot)$ and $g(\cdot)$.

Assumption 1 [10, 30] The diagonal neuron activation functions

$$\begin{aligned} f(x(k)) &= [f_1(x_1(k)) \ f_2(x_2(k)) \ \dots \ f_n(x_n(k))]^\top, \\ g(x(k - h(k))) &= [g_1(x_1(k - h(k))) \ g_2(x_2(k - h(k))) \ \dots \ g_n(x_n(k - h(k)))]^\top \end{aligned}$$

are assumed to be continuous and satisfy $f_i(0) = 0, g_i(0) = 0$ for $i = 1, \dots, n$ as well as the sector-bounded conditions

$$\begin{aligned} [f(x) - f(y) - F_1(x - y)]^\top [f(x) - f(y) - F_2(x - y)] &\leq 0, \\ [g(x) - g(y) - G_1(x - y)]^\top [g(x) - g(y) - G_2(x - y)] &\leq 0, \end{aligned} \tag{4}$$

where F_1, F_2, G_1 and G_2 are real matrices of appropriate dimensions.

Remark 1 As the authors explained in detail in the articles [10, 30], the sector-bounded condition covers the standard Lipschitz condition as a special case, so the NNs model (1) is more general than those were depicted in [17, 25, 26, 36, 37, 39].

Definition 1 (FTB [2, 40]) Suppose that c_1, c_2, N are given scalars with $0 < c_1 < c_2, N \in \mathbb{Z}_+$, and that $R > 0$ is a symmetric matrix. The discrete-time delay NNs with exogenous disturbances $\omega(k)$ satisfying (3)

$$x(k + 1) = Ax(k - \sigma(k)) + Bf(x(k)) + B_1g(x(k - h(k))) + C\omega(k), \quad k \in \mathbb{Z}_+,$$

$$x(k) = \phi(k), \quad k \in \{-\rho, -\rho + 1, \dots, 0\}, \tag{5}$$

is FTB w.r.t. (c_1, c_2, R, N) if

$$\max_{k \in \{-\rho, -\rho+1, \dots, 0\}} \phi^\top(k)R\phi(k) \leq c_1 \implies x^\top(k)Rx(k) < c_2 \quad \forall k \in \{1, 2, \dots, N\}.$$

Remark 2 FTS w.r.t. (c_1, c_2, R, N) is a special case of FTB w.r.t. (c_1, c_2, R, N) which happens when $\omega(k) = 0$.

Definition 2 (H_∞ FTB [32, 37]) Suppose that c_1, c_2, N are given scalars with $0 < c_1 < c_2, N \in \mathbb{Z}_+,$ and that $R > 0$ is a symmetric matrix. System (1) is H_∞ FTB w.r.t. (c_1, c_2, R, N) if the following two conditions hold:

1. System (5) is FTB w.r.t. (c_1, c_2, R, N) .
2. Under zero initial condition, for any nonzero $\omega(k)$ satisfying (3), the output $z(k)$ satisfies the condition

$$\sum_{k=0}^N z^\top(k)z(k) \leq \gamma \sum_{k=0}^N \omega^\top(k)\omega(k) \tag{6}$$

with a prescribed scalar $\gamma > 0$.

Proposition 1 (Discrete Jensen inequality, [18]) Suppose that $M > 0$ is a symmetric matrix of order n, k_1, k_2 are positive integers satisfying $k_1 \leq k_2,$ and $\chi : \{k_1, k_1 + 1, \dots, k_2\} \rightarrow \mathbb{R}^n$ is a vector function. Then

$$\left(\sum_{i=k_1}^{k_2} \chi(i) \right)^\top M \left(\sum_{i=k_1}^{k_2} \chi(i) \right) \leq (k_2 - k_1 + 1) \sum_{i=k_1}^{k_2} \chi^\top(i)M\chi(i).$$

Proposition 2 (Extended reciprocally convex matrix inequality, [42]) Suppose that $R > 0$ is a symmetric matrix of order $n.$ Then the following matrix inequality

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geq \begin{bmatrix} R + (1-\alpha)T_1 & S \\ S^\top & R + \alpha T_2 \end{bmatrix},$$

holds for some matrix S (of order n) and for all $\alpha \in (0, 1),$ where $T_1 = R - SR^{-1}S^\top, T_2 = R - S^\top R^{-1}S.$

3 Main Results

Let $y(k) = x(k + 1) - x(k)$ and assume that $\max_{k \in \{-\rho, -\rho+1, \dots, -1\}} y^\top(k)y(k) < \tau$ with τ is given positive real constant. For $p, q \in \{1, 2\},$ the following notations are used to facilitate the statement of the main results.

$$\begin{aligned} \sigma_{12} &= \sigma_2 - \sigma_1, \quad h_{12} = h_2 - h_1, \quad \bar{F}_1 = \frac{1}{2} \left(F_1^\top F_2 + F_2^\top F_1 \right), \\ \bar{F}_2 &= -\frac{1}{2} \left(F_1^\top + F_2^\top \right), \quad \bar{G}_1 = \frac{1}{2} \left(G_1^\top G_2 + G_2^\top G_1 \right), \quad \bar{G}_2 = -\frac{1}{2} \left(G_1^\top + G_2^\top \right), \\ \Sigma_{h_p, \sigma_q}^{11} &= -\delta(P + R_1 + S_1) + (\sigma_{12} + 1)Q - \bar{F}_1, \quad \Sigma_{h_p, \sigma_q}^{12} = \delta R_1, \quad \Sigma_{h_p, \sigma_q}^{15} = \delta S_1, \\ \Sigma_{h_p, \sigma_q}^{18} &= -\bar{F}_2, \quad \Sigma_{h_p, \sigma_q}^{1,12} = -h_1^2 R_1, \quad \Sigma_{h_p, \sigma_q}^{1,13} = -h_{12}^2 R_2, \quad \Sigma_{h_p, \sigma_q}^{1,14} = -\sigma_1^2 S_1, \quad \Sigma_{h_p, \sigma_q}^{1,16} = A_1^\top, \end{aligned}$$

$$\begin{aligned}
 \Sigma_{h_p, \sigma_q}^{1,15} &= -\sigma_{12}^2 S_2, \quad \Sigma_{h_1, \sigma_q}^{22} = -\delta R_1 - 2\delta^{h_1+1} R_2, \quad \Sigma_{h_2, \sigma_q}^{22} = -\delta R_1 - \delta^{h_1+1} R_2, \\
 \Sigma_{h_1, \sigma_q}^{23} &= \Sigma_{h_2, \sigma_q}^{34} = \delta^{h_1+1} [2R_2 - Y_1], \quad \Sigma_{h_2, \sigma_q}^{23} = \Sigma_{h_1, \sigma_q}^{34} = \delta^{h_1+1} [R_2 - Y_1], \\
 \Sigma_{h_p, \sigma_q}^{24} &= \Sigma_{h_1, \sigma_q}^{3,17} = \delta^{h_1+1} Y_1, \quad \Sigma_{h_1, \sigma_q}^{2,17} = -\delta^{h_1+1} Y_1, \quad \Sigma_{h_2, \sigma_q}^{3,17} = -\delta^{h_1+1} Y_1^T, \\
 \Sigma_{h_p, \sigma_q}^{33} &= -\delta^{h_1+1} [3R_2 - Y_1 - Y_1^T] - \overline{G}_1, \quad \Sigma_{h_p, \sigma_q}^{39} = -\overline{G}_2, \quad \Sigma_{h_p, \sigma_q}^{3,16} = D^T, \\
 \Sigma_{h_1, \sigma_q}^{44} &= -\delta^{h_1+1} R_2, \quad \Sigma_{h_2, \sigma_q}^{44} = -2\delta^{h_1+1} R_2, \quad \Sigma_{h_2, \sigma_q}^{4,17} = \delta^{h_1+1} Y_1^T, \\
 \Sigma_{h_p, \sigma_1}^{55} &= -\delta S_1 - 2\delta^{\sigma_1+1} S_2, \quad \Sigma_{h_p, \sigma_2}^{55} = -\delta S_1 - \delta^{\sigma_1+1} S_2, \\
 \Sigma_{h_p, \sigma_1}^{56} &= \Sigma_{h_p, \sigma_2}^{67} = \delta^{\sigma_1+1} [2S_2 - Y_2], \quad \Sigma_{h_p, \sigma_2}^{56} = \Sigma_{h_p, \sigma_1}^{67} = \delta^{\sigma_1+1} [S_2 - Y_2], \\
 \Sigma_{h_p, \sigma_q}^{57} &= \Sigma_{h_p, \sigma_1}^{6,18} = \delta^{\sigma_1+1} Y_2, \quad \Sigma_{h_p, \sigma_1}^{5,18} = -\delta^{\sigma_1+1} Y_2, \quad \Sigma_{h_p, \sigma_q}^{6,11} = AP, \quad \Sigma_{h_p, \sigma_q}^{6,12} = h_1^2 AR_1, \\
 \Sigma_{h_p, \sigma_q}^{66} &= -\delta^{\sigma_1} Q - \delta^{\sigma_1+1} [3S_2 - Y_2 - Y_2^T], \quad \Sigma_{h_p, \sigma_q}^{6,13} = h_{12}^2 AR_2, \quad \Sigma_{h_p, \sigma_q}^{6,14} = \sigma_1^2 AS_1, \\
 \Sigma_{h_p, \sigma_q}^{6,15} &= \sigma_{12}^2 AS_2, \quad \Sigma_{h_p, \sigma_q}^{6,16} = D_1^T, \quad \Sigma_{h_p, \sigma_2}^{6,18} = -\delta^{\sigma_1+1} Y_2^T, \quad \Sigma_{h_p, \sigma_1}^{77} = -\delta^{\sigma_1+1} S_2, \\
 \Sigma_{h_p, \sigma_2}^{77} &= -2\delta^{\sigma_1+1} S_2, \quad \Sigma_{h_p, \sigma_2}^{7,18} = \delta^{\sigma_1+1} Y_2^T, \quad \Sigma_{h_p, \sigma_q}^{88} = \Sigma_{h_p, \sigma_q}^{99} = \Sigma_{h_p, \sigma_q}^{16,16} = -I, \\
 \Sigma_{h_p, \sigma_q}^{8,11} &= B^T P, \quad \Sigma_{h_p, \sigma_q}^{8,12} = h_1^2 B^T R_1, \quad \Sigma_{h_p, \sigma_q}^{8,13} = h_{12}^2 B^T R_2, \quad \Sigma_{h_p, \sigma_q}^{8,14} = \sigma_1^2 B^T S_1, \\
 \Sigma_{h_p, \sigma_q}^{8,15} &= \sigma_{12}^2 B^T S_2, \quad \Sigma_{h_p, \sigma_q}^{9,11} = B_1^T P, \quad \Sigma_{h_p, \sigma_q}^{9,12} = h_1^2 B_1^T R_1, \quad \Sigma_{h_p, \sigma_q}^{9,13} = h_{12}^2 B_1^T R_2, \\
 \Sigma_{h_p, \sigma_q}^{9,14} &= \sigma_1^2 B_1^T S_1, \quad \Sigma_{h_p, \sigma_q}^{9,15} = \sigma_{12}^2 B_1^T S_2, \quad \Sigma_{h_p, \sigma_q}^{10,11} = C^T P, \quad \Sigma_{h_p, \sigma_q}^{10,12} = h_1^2 C^T R_1, \\
 \Sigma_{h_p, \sigma_q}^{10,13} &= h_{12}^2 C^T R_2, \quad \Sigma_{h_p, \sigma_q}^{10,14} = \sigma_1^2 C^T S_1, \quad \Sigma_{h_p, \sigma_q}^{10,15} = \sigma_{12}^2 C^T S_2, \quad \Sigma_{h_p, \sigma_q}^{10,16} = C_1^T, \\
 \Sigma_{h_p, \sigma_q}^{10,10} &= -\frac{\gamma}{\delta N} I, \quad \Sigma_{h_p, \sigma_q}^{11,11} = -P, \quad \Sigma_{h_p, \sigma_q}^{12,12} = -h_1^2 R_1, \quad \Sigma_{h_p, \sigma_q}^{13,13} = -h_{12}^2 R_2, \\
 \Sigma_{h_p, \sigma_q}^{14,14} &= -\sigma_1^2 S_1, \quad \Sigma_{h_p, \sigma_q}^{15,15} = -\sigma_{12}^2 S_2, \quad \Sigma_{h_p, \sigma_q}^{17,17} = -\delta^{h_1+1} R_2, \\
 \Sigma_{h_p, \sigma_q}^{18,18} &= -\delta^{\sigma_1+1} S_2, \quad \Sigma_{h_p, \sigma_q}^{ij} = 0 \text{ for any other } i, j : j > i, \\
 \Sigma_{h_p, \sigma_q}^{ij} &= \left(\Sigma_{h_p, \sigma_q}^{ji} \right)^T \text{ for } i > j, \quad \Pi^{11} = \gamma d - c_2 \delta \lambda_1, \quad \Pi^{12} = c_1 \delta^{N+1} \lambda_2, \\
 \Pi^{13} &= \frac{1}{2} c_1 (\sigma_1 + \sigma_2) (\sigma_{12} + 1) \delta^{N+\sigma_2} \lambda_3, \quad \Pi^{14} = \frac{1}{2} \tau h_1^2 (h_1 + 1) \delta^{N+h_1} \lambda_4, \\
 \Pi^{15} &= \frac{1}{2} \tau h_{12}^2 (h_1 + h_2 + 1) \delta^{N+h_2} \lambda_5, \quad \Pi^{16} = \frac{1}{2} \tau \sigma_1^2 (\sigma_1 + 1) \delta^{N+\sigma_1} \lambda_6, \\
 \Pi^{17} &= \frac{1}{2} \tau \sigma_{12}^2 (\sigma_1 + \sigma_2 + 1) \delta^{N+\sigma_2} \lambda_7, \quad \Pi^{22} = -\Pi^{12}, \quad \Pi^{33} = -\Pi^{13}, \\
 \Pi^{44} &= -\Pi^{14}, \quad \Pi^{55} = -\Pi^{15}, \quad \Pi^{66} = -\Pi^{16}, \quad \Pi^{77} = -\Pi^{17}, \\
 \Pi^{ij} &= 0 \text{ for any other } i, j : j > i, \quad \Pi^{ij} = \left(\Pi^{ji} \right)^T \text{ for } i > j.
 \end{aligned}$$

Theorem 1 Suppose that c_1, c_2, γ, N are given scalars such that $0 < c_1 < c_2, \gamma > 0, N \in \mathbb{Z}_+,$ and that $R > 0$ is a symmetric matrix. If there exist some positive-definite symmetric matrices $P, Q, R_1, R_2, S_1, S_2 \in \mathbb{R}^{n \times n},$ two any matrices $Y_1, Y_2 \in \mathbb{R}^{n \times n}$ and some positive scalars $\lambda_i, i = \overline{1, 7}, \delta \geq 1,$ such that the following matrix inequalities are satisfied

$$\lambda_1 R < P < \lambda_2 R, \quad Q < \lambda_3 R, \quad R_1 < \lambda_4 I, \quad R_2 < \lambda_5 I, \quad S_1 < \lambda_6 I, \quad S_2 < \lambda_7 I, \quad (7)$$

$$\Sigma_{h_p, \sigma_q} = \left[\Sigma_{h_p, \sigma_q}^{ij} \right]_{18 \times 18} < 0 \text{ for } p, q \in \{1, 2\}, \quad (8)$$

$$\Pi = [\Pi^{ij}]_{7 \times 7} < 0, \tag{9}$$

then system (1) is H_∞ FTB w.r.t. (c_1, c_2, R, N) .

Proof Consider the L–K functional candidate $V(k) = \sum_{i=1}^4 V_i(k)$, where

$$\begin{aligned} V_1(k) &= x^\top(k) P x(k), \\ V_2(k) &= \sum_{s=-\sigma_2+1}^{-\sigma_1+1} \sum_{t=k-1+s}^{k-1} \delta^{k-1-t} x^\top(t) Q x(t), \\ V_3(k) &= \sum_{s=-h_1+1}^0 \sum_{t=k-1+s}^{k-1} h_1 \delta^{k-1-t} y^\top(t) R_1 y(t) \\ &\quad + \sum_{s=-h_2+1}^{-h_1} \sum_{t=k-1+s}^{k-1} h_{12} \delta^{k-1-t} y^\top(t) R_2 y(t), \\ V_4(k) &= \sum_{s=-\sigma_1+1}^0 \sum_{t=k-1+s}^{k-1} \sigma_1 \delta^{k-1-t} y^\top(t) S_1 y(t) \\ &\quad + \sum_{s=-\sigma_2+1}^{-\sigma_1} \sum_{t=k-1+s}^{k-1} \sigma_{12} \delta^{k-1-t} y^\top(t) S_2 y(t). \end{aligned}$$

Denote

$$\begin{aligned} \xi(k) &= [x^\top(k) \ x^\top(k-h_1) \ x^\top(k-h(k)) \ x^\top(k-h_2) \ x^\top(k-\sigma_1) \\ &\quad x^\top(k-\sigma(k)) \ x^\top(k-\sigma_2) \ f^\top(x(k)) \ g^\top(x(k-h(k))) \ \omega^\top(k)]^\top, \\ \Upsilon &:= [0 \ 0 \ 0 \ 0 \ 0 \ A \ 0 \ B \ B_1 \ C], \end{aligned}$$

then it is not hard to get the following evaluation

$$V_1(k+1) - \delta V_1(k) = \xi^\top(k) \Upsilon^\top P \Upsilon \xi(k) - \delta x^\top(k) P x(k), \tag{10}$$

$$V_2(k+1) - \delta V_2(k) \leq (\sigma_{12} + 1) x^\top(k) Q x(k) - \delta^{\sigma_1} x^\top(k - \sigma(k)) Q x(k - \sigma(k)), \tag{11}$$

$$\begin{aligned} V_3(k+1) - \delta V_3(k) &\leq y^\top(k) [h_1^2 R_1 + h_{12}^2 R_2] y(k) - h_1 \delta \sum_{s=k-h_1}^{k-1} y^\top(s) R_1 y(s) \\ &\quad - h_{12} \delta^{h_1+1} \sum_{s=k-h_2}^{k-1-h_1} y^\top(s) R_2 y(s), \end{aligned} \tag{12}$$

$$\begin{aligned} V_4(k+1) - \delta V_4(k) &\leq y^\top(k) [\sigma_1^2 S_1 + \sigma_{12}^2 S_2] y(k) - \sigma_1 \delta \sum_{s=k-\sigma_1}^{k-1} y^\top(s) S_1 y(s) \\ &\quad - \sigma_{12} \delta^{\sigma_1+1} \sum_{s=k-\sigma_2}^{k-1-\sigma_1} y^\top(s) S_2 y(s). \end{aligned} \tag{13}$$

By discrete Jensen inequality (Proposition 1),

$$-h_{11}\delta \sum_{s=k-h_1}^{k-1} y^\top(s)R_1y(s) \leq -\delta[x(k) - x(k - h_1)]^\top R_1[x(k) - x(k - h_1)], \quad (14)$$

$$-\sigma_1\delta \sum_{s=k-\sigma_1}^{k-1} y^\top(s)S_1y(s) \leq -\delta[x(k) - x(k - \sigma_1)]^\top S_1[x(k) - x(k - \sigma_1)],$$

$$-h_{12}\delta^{h_1+1} \sum_{s=k-h_2}^{k-1-h_1} y^\top(s)R_2y(s) \leq -\delta^{h_1+1} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\alpha_1}R_2 & 0 \\ 0 & \frac{1}{1-\alpha_1}R_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad (15)$$

where $\zeta_1 = x(k - h_1) - x(k - h(k))$, $\zeta_2 = x(k - h(k)) - x(k - h_2)$ and $\alpha_1 = \frac{h(k)-h_1}{h_{12}}$. Using Proposition 2 to further evaluate the right hand side of the last inequality, we gain

$$\begin{aligned} & -h_{12}\delta^{h_1+1} \sum_{s=k-h_2}^{k-1-h_1} y^\top(s)R_2y(s) \\ & \leq -\delta^{h_1+1} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^\top \begin{bmatrix} R_2 + (1 - \alpha_1)M_1 & Y_1 \\ * & R_2 + \alpha_1M_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \\ & = -\delta^{h_1+1} [\zeta_1^\top (R_2 + (1 - \alpha_1)M_1)\zeta_1 + \zeta_1^\top Y_1\zeta_2 + \zeta_2^\top Y_1^\top \zeta_1 + \zeta_2^\top (R_2 + \alpha_1M_2)\zeta_2], \quad (16) \end{aligned}$$

where $M_1 = R_2 - Y_1R_2^{-1}Y_1^\top$ and $M_2 = R_2 - Y_1^\top R_2^{-1}Y_1$.

In an entirely similar way, we obtain

$$\begin{aligned} & -\sigma_{12}\delta^{\sigma_1+1} \sum_{s=k-\sigma_2}^{k-1-\sigma_1} y^\top(s)S_2y(s) \\ & \leq -\delta^{\sigma_1+1} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}^\top \begin{bmatrix} S_2 + (1 - \alpha_2)N_1 & Y_2 \\ * & S_2 + \alpha_2N_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \\ & = -\delta^{\sigma_1+1} [\eta_1^\top (S_2 + (1 - \alpha_2)N_1)\eta_1 + \eta_1^\top Y_2\eta_2 + \eta_2^\top Y_2^\top \eta_1 + \eta_2^\top (S_2 + \alpha_2N_2)\eta_2], \quad (17) \end{aligned}$$

where $\eta_1 = x(k - \sigma_1) - x(k - \sigma(k))$, $\eta_2 = x(k - \sigma(k)) - x(k - \sigma_2)$, $\alpha_2 = \frac{\sigma(k)-\sigma_1}{\sigma_{12}}$, $N_1 = S_2 - Y_2S_2^{-1}Y_2^\top$ and $N_2 = S_2 - Y_2^\top S_2^{-1}Y_2$.

Insert (14), (16) into (12) and (15), (17) into (13) then put (10)–(13) together, we have

$$\begin{aligned} & V(k + 1) - \delta V(k) \\ & \leq \xi^\top(k)\Upsilon^\top P\Upsilon\xi(k) + x^\top(k)[-\delta(P + R_1 + S_1) + (\sigma_{12} + 1)Q + A_1^\top A_1]x(k) \\ & \quad + x^\top(k)[2\delta R_1]x(k - h_1) + x^\top(k)[2A_1^\top D]x(k - h(k)) \\ & \quad + x^\top(k)[2\delta S_1]x(k - \sigma_1) + x^\top(k)[2A_1^\top D_1]x(k - \sigma(k)) + x^\top(k)[2A_1^\top C_1]\omega(k) \\ & \quad + x^\top(k - h_1)[-\delta R_1 - \delta^{h_1+1}(R_2 + (1 - \alpha_1)M_1)]x(k - h_1) \\ & \quad + x^\top(k - h_1)[2\delta^{h_1+1}(R_2 + (1 - \alpha_1)M_1 - Y_1)]x(k - h(k)) \\ & \quad + x^\top(k - h_1)[2\delta^{h_1+1}Y_1]x(k - h_2) \\ & \quad + x^\top(k - h(k))[-\delta^{h_1+1}(2R_2 + (1 - \alpha_1)M_1 + \alpha_1M_2 - Y_1 - Y_1^\top + D^\top D)]x(k - h(k)) \\ & \quad + x^\top(k - h(k))[2\delta^{h_1+1}(R_2 + \alpha_1M_2 - Y_1)]x(k - h_2) \\ & \quad + x^\top(k - h(k))[2D^\top D_1]x(k - \sigma(k)) + x^\top(k - h(k))[2D^\top C_1]\omega(k) \end{aligned}$$

$$\begin{aligned}
 &+ x^\top(k - h_2)[- \delta^{h_1+1}(R_2 + \alpha_1 M_2)]x(k - h_2) \\
 &+ x^\top(k - \sigma_1)[- \delta S_1 - \delta^{\sigma_1+1}(S_2 + (1 - \alpha_2)N_1)]x(k - \sigma_1) \\
 &+ x^\top(k - \sigma_1)[2\delta^{\sigma_1+1}(S_2 + (1 - \alpha_2)N_1 - Y_2)]x(k - \sigma(k)) \\
 &+ x^\top(k - \sigma_1)[2\delta^{\sigma_1+1}Y_2]x(k - \sigma_2) \\
 &+ x^\top(k - \sigma(k))[- \delta^{\sigma_1}Q - \delta^{\sigma_1+1}(2S_2 + (1 - \alpha_2)N_1 + \alpha_2 N_2 - Y_2 - Y_2^\top) \\
 &+ D_1^\top D_1]x(k - \sigma(k)) \\
 &+ x^\top(k - \sigma(k))[2\delta^{\sigma_1+1}(S_2 + \alpha_2 N_2 - Y_2)]x(k - \sigma_2) + x^\top(k - \sigma(k))[2D_1^\top C_1]\omega(k) \\
 &+ x^\top(k - \sigma_2)[- \delta^{\sigma_1+1}(S_2 + \alpha_2 N_2)]x(k - \sigma_2) + \omega^\top(k)[- \frac{\gamma}{\delta^N}I + C_1^\top C_1]\omega(k) \\
 &+ y^\top(k)[h_1^2 R_1 + h_{12}^2 R_2 + \sigma_1^2 S_1 + \sigma_{12}^2 S_2]y(k) + \frac{\gamma}{\delta^N}\omega^\top(k)\omega(k) - z^\top(k)z(k). \tag{18}
 \end{aligned}$$

In addition, we receive the following evaluations from constraint (4)

$$\begin{aligned}
 &0 \leq f^\top(x(k))[-I]f(x(k)) + x^\top(k)[-2\bar{F}_2]f(x(k)) + x^\top(k)[- \bar{F}_1]x(k), \\
 &0 \leq g^\top(x(k - h(k)))[-I]g(x(k - h(k))) + x^\top(k - h(k))[-2\bar{G}_2]g(x(k - h(k))) \\
 &\quad + x^\top(k - h(k))[- \bar{G}_1]x(k - h(k)). \tag{19}
 \end{aligned}$$

Thenceforth, by the combination of (18) and (19), we get

$$\begin{aligned}
 V(k + 1) - \delta V(k) \leq \xi^\top(k) &\left(\Xi_{h(k),\sigma(k)} + \Lambda^\top \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ 0 & h_1^2 R_1 & 0 & 0 & 0 \\ 0 & 0 & h_{12}^2 R_2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1^2 S_1 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{12}^2 S_2 \end{bmatrix}^{-1} \right) \Lambda \xi(k) \\
 &+ \frac{\gamma}{\delta^N}\omega^\top(k)\omega(k) - z^\top(k)z(k), \tag{20}
 \end{aligned}$$

where

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & PA & 0 & PB & PB_1 & PC \\ -h_1^2 R_1 & 0 & 0 & 0 & h_1^2 R_1 A & 0 & h_1^2 R_1 B & h_1^2 R_1 B_1 & h_1^2 R_1 C \\ -h_{12}^2 R_2 & 0 & 0 & 0 & h_{12}^2 R_2 A & 0 & h_{12}^2 R_2 B & h_{12}^2 R_2 B_1 & h_{12}^2 R_2 C \\ -\sigma_1^2 S_1 & 0 & 0 & 0 & \sigma_1^2 S_1 A & 0 & \sigma_1^2 S_1 B & \sigma_1^2 S_1 B_1 & \sigma_1^2 S_1 C \\ -\sigma_{12}^2 S_2 & 0 & 0 & 0 & \sigma_{12}^2 S_2 A & 0 & \sigma_{12}^2 S_2 B & \sigma_{12}^2 S_2 B_1 & \sigma_{12}^2 S_2 C \end{bmatrix},$$

and

$$\Xi_{h(k),\sigma(k)} = \left[\Xi_{h(k),\sigma(k)}^{ij} \right]_{10 \times 10}$$

with the entries of the last matrix are defined as

$$\begin{aligned}
 \Xi_{h(k),\sigma(k)}^{11} &= \Sigma_{h_1,\sigma_1}^{11} + A_1^\top A_1, \quad \Xi_{h(k),\sigma(k)}^{12} = \Sigma_{h_1,\sigma_1}^{12}, \quad \Xi_{h(k),\sigma(k)}^{13} = A_1^\top D, \\
 \Xi_{h(k),\sigma(k)}^{15} &= \Sigma_{h_1,\sigma_1}^{15}, \quad \Xi_{h(k),\sigma(k)}^{16} = A_1^\top D_1, \quad \Xi_{h(k),\sigma(k)}^{18} = \Sigma_{h_1,\sigma_1}^{18}, \\
 \Xi_{h(k),\sigma(k)}^{1,10} &= A_1^\top C_1, \quad \Xi_{h(k),\sigma(k)}^{22} = -\delta R_1 - \delta^{h_1+1}[R_2 + (1 - \alpha_1)M_1], \\
 \Xi_{h(k),\sigma(k)}^{23} &= \delta^{h_1+1}[R_2 + (1 - \alpha_1)M_1 - Y_1], \quad \Xi_{h(k),\sigma(k)}^{24} = \Sigma_{h_1,\sigma_1}^{24}, \\
 \Xi_{h(k),\sigma(k)}^{33} &= -\delta^{h_1+1}[2R_2 + (1 - \alpha_1)M_1 + \alpha_1 M_2 - Y_1 - Y_1^\top] - \bar{G}_1 + D^\top D, \\
 \Xi_{h(k),\sigma(k)}^{34} &= \delta^{h_1+1}[R_2 + \alpha_1 M_2 - Y_1], \quad \Xi_{h(k),\sigma(k)}^{36} = D^\top D_1, \quad \Xi_{h(k),\sigma(k)}^{39} = \Sigma_{h_1,\sigma_1}^{39},
 \end{aligned}$$

$$\begin{aligned} \Xi_{h(k),\sigma(k)}^{3,10} &= D^\top C_1, \quad \Xi_{h(k),\sigma(k)}^{44} = -\delta^{h_1+1}[R_2 + \alpha_1 M_2], \\ \Xi_{h(k),\sigma(k)}^{55} &= -\delta S_1 - \delta^{\sigma_1+1}[S_2 + (1 - \alpha_2)N_1], \\ \Xi_{h(k),\sigma(k)}^{56} &= \delta^{\sigma_1+1}[S_2 + (1 - \alpha_2)N_1 - Y_2], \\ \Xi_{h(k),\sigma(k)}^{66} &= -\delta^{\sigma_1} Q - \delta^{\sigma_1+1}[2S_2 + (1 - \alpha_2)N_1 + \alpha_2 N_2 - Y_2 - Y_2^\top] + D_1^\top D_1, \\ \Xi_{h(k),\sigma(k)}^{57} &= \Sigma_{h_1,\sigma_1}^{57}, \quad \Xi_{h(k),\sigma(k)}^{67} = \delta^{\sigma_1+1}[S_2 + \alpha_2 N_2 - Y_2], \quad \Xi_{h(k),\sigma(k)}^{6,10} = D_1^\top C_1, \\ \Xi_{h(k),\sigma(k)}^{77} &= -\delta^{\sigma_1+1}[S_2 + \alpha_2 N_2], \quad \Xi_{h(k),\sigma(k)}^{88} = \Xi_{h(k),\sigma(k)}^{99} = -I, \\ \Xi_{h(k),\sigma(k)}^{10,10} &= -\frac{\gamma}{\delta N} I + C_1^\top C_1, \quad \Xi_{h(k),\sigma(k)}^{ij} = 0 \text{ for any other } i, j : j > i, \\ \Xi_{h(k),\sigma(k)}^{ij} &= \left(\Xi_{h(k),\sigma(k)}^{ji} \right)^\top \text{ for } i > j. \end{aligned}$$

Next, well-known Schur Complement Lemma gives us

$$\Xi_{h(k),\sigma(k)} + \Lambda^\top \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ 0 & h_1^2 R_1 & 0 & 0 & 0 \\ 0 & 0 & h_{12}^2 R_2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1^2 S_1 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{12}^2 S_2 \end{bmatrix}^{-1} \Lambda < 0 \iff \Theta_{h(k),\sigma(k)} < 0,$$

where

$$\Theta_{h(k),\sigma(k)} := \left[\Theta_{h(k),\sigma(k)}^{ij} \right]_{16 \times 16}$$

possesses its entries which are defined as follows

$$\begin{aligned} \Theta_{h(k),\sigma(k)}^{11} &= \Sigma_{h_1,\sigma_1}^{11}, \quad \Theta_{h(k),\sigma(k)}^{12} = \Sigma_{h_1,\sigma_1}^{12}, \quad \Theta_{h(k),\sigma(k)}^{15} = \Sigma_{h_1,\sigma_1}^{15}, \quad \Theta_{h(k),\sigma(k)}^{18} = \Sigma_{h_1,\sigma_1}^{18}, \\ \Theta_{h(k),\sigma(k)}^{1,12} &= \Sigma_{h_1,\sigma_1}^{1,12}, \quad \Theta_{h(k),\sigma(k)}^{1,13} = \Sigma_{h_1,\sigma_1}^{1,13}, \quad \Theta_{h(k),\sigma(k)}^{1,14} = \Sigma_{h_1,\sigma_1}^{1,14}, \quad \Theta_{h(k),\sigma(k)}^{1,15} = \Sigma_{h_1,\sigma_1}^{1,15}, \\ \Theta_{h(k),\sigma(k)}^{1,16} &= \Sigma_{h_1,\sigma_1}^{1,16}, \quad \Theta_{h(k),\sigma(k)}^{22} = \Xi_{h(k),\sigma(k)}^{22}, \quad \Theta_{h(k),\sigma(k)}^{23} = \Xi_{h(k),\sigma(k)}^{23}, \\ \Theta_{h(k),\sigma(k)}^{33} &= -\delta^{h_1+1}[2R_2 + (1 - \alpha_1)M_1 + \alpha_1 M_2 - Y_1 - Y_1^\top] - \bar{G}_1, \\ \Theta_{h(k),\sigma(k)}^{24} &= \Sigma_{h_1,\sigma_1}^{24}, \quad \Theta_{h(k),\sigma(k)}^{34} = \Xi_{h(k),\sigma(k)}^{34}, \quad \Theta_{h(k),\sigma(k)}^{39} = \Sigma_{h_1,\sigma_1}^{39}, \\ \Theta_{h(k),\sigma(k)}^{3,16} &= \Sigma_{h_1,\sigma_1}^{3,16}, \quad \Theta_{h(k),\sigma(k)}^{44} = \Xi_{h(k),\sigma(k)}^{44}, \quad \Theta_{h(k),\sigma(k)}^{55} = \Xi_{h(k),\sigma(k)}^{55}, \\ \Theta_{h(k),\sigma(k)}^{56} &= \Xi_{h(k),\sigma(k)}^{56}, \quad \Theta_{h(k),\sigma(k)}^{57} = \Sigma_{h_1,\sigma_1}^{57}, \quad \Theta_{h(k),\sigma(k)}^{67} = \Xi_{h(k),\sigma(k)}^{67}, \\ \Theta_{h(k),\sigma(k)}^{66} &= -\delta^{\sigma_1} Q - \delta^{\sigma_1+1}[2S_2 + (1 - \alpha_2)N_1 + \alpha_2 N_2 - Y_2 - Y_2^\top], \\ \Theta_{h(k),\sigma(k)}^{6,11} &= \Sigma_{h_1,\sigma_1}^{6,11}, \quad \Theta_{h(k),\sigma(k)}^{6,12} = \Sigma_{h_1,\sigma_1}^{6,12}, \quad \Theta_{h(k),\sigma(k)}^{6,13} = \Sigma_{h_1,\sigma_1}^{6,13}, \quad \Theta_{h(k),\sigma(k)}^{6,14} = \Sigma_{h_1,\sigma_1}^{6,14}, \\ \Theta_{h(k),\sigma(k)}^{6,15} &= \Sigma_{h_1,\sigma_1}^{6,15}, \quad \Theta_{h(k),\sigma(k)}^{6,16} = \Sigma_{h_1,\sigma_1}^{6,16}, \quad \Theta_{h(k),\sigma(k)}^{77} = \Xi_{h(k),\sigma(k)}^{77}, \\ \Theta_{h(k),\sigma(k)}^{88} &= \Theta_{h(k),\sigma(k)}^{99} = \Theta_{h(k),\sigma(k)}^{16,16} = -I, \quad \Theta_{h(k),\sigma(k)}^{8,11} = \Sigma_{h_1,\sigma_1}^{8,11}, \\ \Theta_{h(k),\sigma(k)}^{8,12} &= \Sigma_{h_1,\sigma_1}^{8,12}, \quad \Theta_{h(k),\sigma(k)}^{8,13} = \Sigma_{h_1,\sigma_1}^{8,13}, \quad \Theta_{h(k),\sigma(k)}^{8,14} = \Sigma_{h_1,\sigma_1}^{8,14}, \quad \Theta_{h(k),\sigma(k)}^{8,15} = \Sigma_{h_1,\sigma_1}^{8,15}, \\ \Theta_{h(k),\sigma(k)}^{9,11} &= \Sigma_{h_1,\sigma_1}^{9,11}, \quad \Theta_{h(k),\sigma(k)}^{9,12} = \Sigma_{h_1,\sigma_1}^{9,12}, \quad \Theta_{h(k),\sigma(k)}^{9,13} = \Sigma_{h_1,\sigma_1}^{9,13}, \quad \Theta_{h(k),\sigma(k)}^{9,14} = \Sigma_{h_1,\sigma_1}^{9,14}, \\ \Theta_{h(k),\sigma(k)}^{9,15} &= \Sigma_{h_1,\sigma_1}^{9,15}, \quad \Theta_{h(k),\sigma(k)}^{10,10} = \Sigma_{h_1,\sigma_1}^{10,10}, \quad \Theta_{h(k),\sigma(k)}^{10,11} = \Sigma_{h_1,\sigma_1}^{10,11}, \quad \Theta_{h(k),\sigma(k)}^{10,12} = \Sigma_{h_1,\sigma_1}^{10,12}, \\ \Theta_{h(k),\sigma(k)}^{10,13} &= \Sigma_{h_1,\sigma_1}^{10,13}, \quad \Theta_{h(k),\sigma(k)}^{10,14} = \Sigma_{h_1,\sigma_1}^{10,14}, \quad \Theta_{h(k),\sigma(k)}^{10,15} = \Sigma_{h_1,\sigma_1}^{10,15}, \quad \Theta_{h(k),\sigma(k)}^{10,16} = \Sigma_{h_1,\sigma_1}^{10,16}, \\ \Theta_{h(k),\sigma(k)}^{11,11} &= \Sigma_{h_1,\sigma_1}^{11,11}, \quad \Theta_{h(k),\sigma(k)}^{12,12} = \Sigma_{h_1,\sigma_1}^{12,12}, \quad \Theta_{h(k),\sigma(k)}^{13,13} = \Sigma_{h_1,\sigma_1}^{13,13}, \quad \Theta_{h(k),\sigma(k)}^{14,14} = \Sigma_{h_1,\sigma_1}^{14,14} \end{aligned}$$

$$\Theta_{h(k),\sigma(k)}^{15,15} = \Sigma_{h_1,\sigma_1}^{15,15}, \Theta_{h(k),\sigma(k)}^{ij} = 0 \text{ for any other } i, j : j > i,$$

$$\Theta_{h(k),\sigma(k)}^{ij} = \left(\Theta_{h(k),\sigma(k)}^{ji}\right)^T \text{ for } i > j.$$

The convex combination technique allows us to assert that $\Theta_{h(k),\sigma(k)} < 0$ if the following four inequalities

$$\Theta_{h_1,\sigma_1} < 0, \quad \Theta_{h_1,\sigma_2} < 0, \quad \Theta_{h_2,\sigma_1} < 0 \quad \text{and} \quad \Theta_{h_2,\sigma_2} < 0$$

hold. In turn, by Schur Complement Lemma again, these inequalities hold if the inequalities stated in (8) hold. Combine these remarks with (20), we have

$$V(k + 1) - \delta V(k) \leq \frac{\gamma}{\delta^N} \omega^T(k) \omega(k) \quad \forall k \in \mathbb{Z}_+.$$

This implies that

$$\begin{aligned} V(k) &\leq \delta V(k - 1) + \frac{\gamma}{\delta^N} \omega^T(k - 1) \omega(k - 1) \\ &\leq \delta^2 V(k - 2) + \frac{\gamma}{\delta^{N-1}} \omega^T(k - 2) \omega(k - 2) + \frac{\gamma}{\delta^N} \omega^T(k - 1) \omega(k - 1) \\ &\leq \dots \\ &\leq \delta^k V(0) + \frac{\gamma}{\delta^N} \sum_{s=0}^{k-1} \delta^{k-1-s} \omega^T(s) \omega(s) \quad \forall k \in \mathbb{N}. \end{aligned}$$

By condition (3), it is obvious that

$$V(k) < \delta^N V(0) + \frac{\gamma}{\delta} d \quad \forall k \in \{1, \dots, N\}. \tag{21}$$

Moreover, according to condition (7), we can estimate

$$\begin{aligned} V_1(0) &= x^T(0) P x(0) < \lambda_2 c_1, \\ V_2(0) &= \sum_{s=-\sigma_2+1}^{-\sigma_1+1} \sum_{t=-1+s}^{-1} \delta^{-1-t} x^T(t) Q x(t) < \lambda_3 \delta^{\sigma_2-1} \frac{\sigma_2(\sigma_2 + 1) - \sigma_1(\sigma_1 - 1)}{2} c_1, \\ V_3(0) &= \sum_{s=-h_1+1}^0 \sum_{t=-1+s}^{-1} h_1 \delta^{-1-t} y^T(t) R_1 y(t) + \sum_{s=-h_2+1}^{-h_1} \sum_{t=-1+s}^{-1} h_{12} \delta^{-1-t} y^T(t) R_2 y(t) \\ &< \left[\lambda_4 \delta^{h_1-1} h_1 \frac{h_1(h_1 + 1)}{2} + \lambda_5 \delta^{h_2-1} h_{12} \frac{h_2(h_2 + 1) - h_1(h_1 + 1)}{2} \right] \tau, \\ V_4(0) &= \sum_{s=-\sigma_1+1}^0 \sum_{t=-1+s}^{-1} \sigma_1 \delta^{-1-t} y^T(t) S_1 y(t) + \sum_{s=-\sigma_2+1}^{-\sigma_1} \sum_{t=-1+s}^{-1} \sigma_{12} \delta^{-1-t} y^T(t) S_2 y(t) \\ &< \left[\lambda_6 \delta^{\sigma_1-1} \sigma_1 \frac{\sigma_1(\sigma_1 + 1)}{2} + \lambda_7 \delta^{\sigma_2-1} \sigma_{12} \frac{\sigma_2(\sigma_2 + 1) - \sigma_1(\sigma_1 + 1)}{2} \right] \tau. \end{aligned}$$

Substitute these upper bounds of $V_i(0)$, $i = 1, \dots, 4$, into (21), it can be derived that

$$V(k) < \delta^N \kappa + \frac{\gamma}{\delta} d \quad \forall k \in \{1, \dots, N\}, \tag{22}$$

where

$$\kappa := c_1 \lambda_2 + \frac{1}{2} c_1 (\sigma_1 + \sigma_2) (\sigma_{12} + 1) \delta^{\sigma_2-1} \lambda_3 + \frac{1}{2} \tau h_1^2 (h_1 + 1) \delta^{h_1-1} \lambda_4$$

$$\begin{aligned}
 &+ \frac{1}{2} \tau h_{12}^2 (h_1 + h_2 + 1) \delta^{h_2-1} \lambda_5 + \frac{1}{2} \tau \sigma_1^2 (\sigma_1 + 1) \delta^{\sigma_1-1} \lambda_6 \\
 &+ \frac{1}{2} \tau \sigma_{12}^2 (\sigma_1 + \sigma_2 + 1) \delta^{\sigma_2-1} \lambda_7.
 \end{aligned}$$

Furthermore, by (7), we see that (for $k \in \mathbb{Z}_+$)

$$V(k) \geq x^\top(k) P x(k) \geq \lambda_1 x^\top(k) R x(k). \tag{23}$$

Using Schur Complement Lemma to perform equivalent transformations on (9), we find

$$\begin{aligned}
 \gamma d - c_2 \delta \lambda_1 + c_1 \delta^{N+1} \lambda_2 + \frac{1}{2} c_1 (\sigma_1 + \sigma_2) (\sigma_{12} + 1) \delta^{N+\sigma_2} \lambda_3 \\
 + \frac{1}{2} \tau h_1^2 (h_1 + 1) \delta^{N+h_1} \lambda_4 + \frac{1}{2} \tau h_{12}^2 (h_1 + h_2 + 1) \delta^{N+h_2} \lambda_5 \\
 + \frac{1}{2} \tau \sigma_1^2 (\sigma_1 + 1) \delta^{N+\sigma_1} \lambda_6 + \frac{1}{2} \tau \sigma_{12}^2 (\sigma_1 + \sigma_2 + 1) \delta^{N+\sigma_2} \lambda_7 < 0,
 \end{aligned}$$

or

$$\gamma d - c_2 \delta \lambda_1 + \delta^{N+1} \kappa < 0. \tag{24}$$

As a result of (22)–(24), it can be concluded that

$$x^\top(k) R x(k) < \frac{1}{\delta \lambda_1} [\delta^{N+1} \kappa + \gamma d] < c_2 \quad \forall k = 1, 2, \dots, N.$$

According to Definition 1, we can confirm that system (5) is FTB w.r.t. (c_1, c_2, R, N) . The rest of the proof is to show the system (1) has a finite-time l_2 -gain γ , i.e., the condition (6) is satisfied. However, this step can be done in the same way as in [37]. The theorem has been completely proved. \square

Corollary 2 *Suppose that c_1, c_2, N are given scalars such that $0 < c_1 < c_2, N \in \mathbb{Z}_+$, and that $R > 0$ is a symmetric matrix. Then, the nominal system, which is the system in Eq. (5) without any disturbance ($\omega(k) = 0$), is FTS w.r.t. (c_1, c_2, R, N) if there exist some positive-definite symmetric matrices $P, Q, R_1, R_2, S_1, S_2 \in \mathbb{R}^{n \times n}$, two any matrices $Y_1, Y_2 \in \mathbb{R}^{n \times n}$ and some positive scalars $\lambda_i, i = \overline{1, 7}, \delta \geq 1$, that satisfy the LMIs in Eq. (7) and also the following matrix inequalities*

$$\overline{\Sigma}_{h_p, \sigma_q} = \left[\overline{\Sigma}_{h_p, \sigma_q}^{ij} \right]_{16 \times 16} < 0 \quad \text{for } p, q \in \{1, 2\}, \tag{25}$$

$$\overline{\Pi} = \left[\overline{\Pi}^{ij} \right]_{7 \times 7} < 0, \tag{26}$$

where $\overline{\Sigma}_{h_p, \sigma_q}$ as a submatrix of Σ_{h_p, σ_q} that is obtained by removing the 10th and 16th rows and columns and $\overline{\Pi}^{11} = -c_2 \delta \lambda_1, \overline{\Pi}^{ij} = \Pi^{ij}$ for any other i, j .

Proof We can omit the proof of this result, since it follows the same steps as the one given for Theorem 1. \square

Remark 3 Theorem 1 and Corollary 2 provide theoretical analysis for H_∞ FTB and FTS for a class of discrete-time NNs with leakage delay, discrete delay, and sector-bounded neuron activation functions. Leakage delay is the time-delay in leakage term of the systems and a significant factor that influences the network dynamics; in other words, the influence of leakage delay is not trivial. However, since the time-delay in the leakage term is often not easy to deal with, up to this point, such delay has yet to receive sufficient attention in the

qualitative study of NNs. Moreover, in contrast to most existing literature, where the neuron activation functions are assumed to satisfy standard Lipschitz conditions [17, 25, 26, 36, 37, 39], in this paper, we consider more general sector-bounded nonlinearities that include the Lipschitz case as a special one. In short, the criteria stated in Theorem 1 and Corollary 2 are fresh and meaningful contributions.

Remark 4 To verify the practicability of the matrix inequalities in (8), (9), (25), and (26), we can fix the value of δ and transform them into LMIs. Then, we can use the LMI Toolbox in MATLAB [13] to solve them efficiently.

Remark 5 The reciprocally convex combination technique, proposed in [24], is a method that minimizes the number of decision variables in matrix inequalities. In the proof of Theorem 1, we have applied a modified version of this technique to deal with double summation terms, which has the potential to achieve less conservative solutions than the widely used original method [42]. Namely, the matrix inequalities (8) and (25) contain only two free-weighting matrices each. Another advantage of Theorem 1 is that the results are derived without using any model transformations or other free-weighting matrix methods. We will demonstrate that the criteria given in Theorem 1 and Corollary 2 are succinct, valid, and practical through some examples.

Example 1 Considering the NNs (1) with its parameters are as follows

$$\begin{aligned}
 A &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.175 \end{bmatrix}, \quad B = \begin{bmatrix} -0.02 & 0.03 \\ 0.03 & 0.05 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.05 & 0.045 \\ -0.05 & 0.05 \end{bmatrix}, \quad C = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, \\
 A_1 &= [0.3 \ -0.4], \quad D = [0.2 \ -0.15], \quad D_1 = [0.12 \ -0.055], \quad C_1 = [0.1], \\
 \sigma(k) &= 1 + 3 \cos^2 \frac{k\pi}{2}, \quad h(k) = 1 + 6 \sin^2 \frac{k\pi}{2}, \quad k \in \mathbb{Z}_+.
 \end{aligned}$$

Moreover, the nonlinear activation functions $f(x(k))$ and $g(x(k - h(k)))$ are chosen as

$$\begin{aligned}
 f(x(k)) &= \begin{bmatrix} 0.2x_1(k) + \tanh(0.4x_1(k)) \\ 0.3x_2(k) + \tanh(0.2x_2(k)) \end{bmatrix}, \\
 g(x(k - h(k))) &= \begin{bmatrix} 0.1x_1(k - h(k)) + \tanh(0.2x_1(k - h(k))) \\ 0.1x_2(k - h(k)) + \tanh(0.2x_2(k - h(k))) \end{bmatrix}.
 \end{aligned}$$

Then we can confirm that (4) is satisfied with

$$F_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad G_1 = 0.1I, \quad G_2 = 0.3I.$$

Since $\bar{F}_1 = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.15 \end{bmatrix}$, $\bar{F}_2 = -0.4I \neq 0$, $\bar{G}_1 = 0.03I$, $\bar{G}_2 = -0.2I \neq 0$, the neuron activation functions $f(\cdot)$, $g(\cdot)$ in this case are sector-bounded.

For given $h_1 = 1$, $h_2 = 7$, $\sigma_1 = 1$, $\sigma_2 = 4$, $d = 1$, $\tau = 1$, $\gamma = 1$, $c_1 = 1$, $c_2 = 9$, $N = 90$ and $R = \begin{bmatrix} 2.25 & 0.35 \\ 0.35 & 1.75 \end{bmatrix}$, the LMIs (7)–(9) are feasible with $\delta = 1.0001$ and the following values of the decision variables

$$\begin{aligned}
 P &= \begin{bmatrix} 39.0159 & -5.0480 \\ -5.0480 & 26.4993 \end{bmatrix}, \quad Q = \begin{bmatrix} 5.4035 & 0.0171 \\ 0.0171 & 3.6365 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1.4784 & -0.6092 \\ -0.6092 & 1.0255 \end{bmatrix}, \\
 R_2 &= \begin{bmatrix} 0.1473 & -0.0496 \\ -0.0496 & 0.1064 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.6555 & -0.1779 \\ -0.1779 & 0.5015 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.1324 & -0.0359 \\ -0.0359 & 0.1028 \end{bmatrix},
 \end{aligned}$$

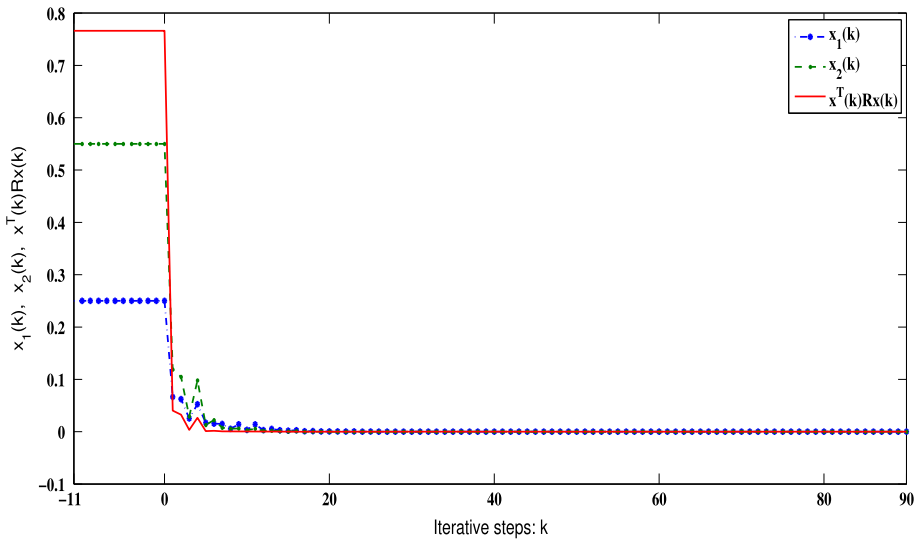


Fig. 1 The trajectories of the system are considered in Example 2

$$Y_1 = \begin{bmatrix} 0.002 & 0.0002 \\ 0 & 0.002 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.0008 & -0.0003 \\ -0.0001 & 0.001 \end{bmatrix}, \quad \lambda_1 = 11.4154, \quad \lambda_2 = 24.1403, \\ \lambda_3 = 2.8687, \quad \lambda_4 = 4.9184, \quad \lambda_5 = 0.1908, \quad \lambda_6 = 4.4446, \quad \lambda_7 = 0.2454.$$

As a result of this, the system under consideration is H_∞ FTB w.r.t. $(1, 9, R, 90)$ according to Theorem 1.

Example 2 Considering the nominal system with the same matrices $A, B, B_1, F_1, F_2, G_1, G_2, R$ as in Example 1. Also, we choose parameters $h_1, \sigma_1, \sigma_2, c_1, c_2, \tau$ and N as in Example 1 except for $h_2 = 11$. By solving the LMIs (7), (25) and (26) with $\delta = 1.0001$, we can gain the solution matrices and scalars as follows:

$$P = \begin{bmatrix} 42.5262 & 1.5871 \\ 1.5871 & 33.8294 \end{bmatrix}, \quad Q = \begin{bmatrix} 6.1590 & 0.4737 \\ 0.4737 & 4.4940 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.9622 & -0.0022 \\ -0.0022 & 0.9291 \end{bmatrix}, \\ R_2 = \begin{bmatrix} 0.0579 & 0.0005 \\ 0.0005 & 0.0586 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.6917 & -0.0161 \\ -0.0161 & 0.6249 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.1473 & -0.0042 \\ -0.0042 & 0.1340 \end{bmatrix}, \\ Y_1 = \begin{bmatrix} 0.0028 & 0 \\ -0.0002 & 0.0029 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.0015 & 0.0001 \\ 0.0001 & 0.0019 \end{bmatrix}, \quad \lambda_1 = 16.4062, \quad \lambda_2 = 25.3133, \\ \lambda_3 = 3.5150, \quad \lambda_4 = 9.6686, \quad \lambda_5 = 0.0732, \quad \lambda_6 = 9.5124, \quad \lambda_7 = 0.5622.$$

Therefore, by Corollary 2, we can affirm that the nominal system is FTS w.r.t. $(1, 9, R, 90)$. With the initial condition is chosen as

$$\phi(k) = \begin{bmatrix} 0.25 \\ 0.55 \end{bmatrix} \quad \forall k \in \{-11, -10, \dots, 0\},$$

the trajectories of the system are described in Fig. 1.

Remark 6 In the proof of Theorem 1, we have applied the Jensen inequality without applying the Wirtinger-based summation inequality or any refined Jensen summation inequalities to

obtain better conditions, although they are useful tools for this purpose, see [23, 28, 43] and references therein. The reason is that system (1) is a nonlinear system with leakage delay and discrete delay, which makes the analysis more complicated than the linear systems or NNs with lonely discrete delay in [23, 27, 28, 42, 43]. Therefore, we have used the Jensen inequality and the extended reciprocally convex technique in this paper which are more suitable for our problems. However, we think that trying the use of Wirtinger-based summation inequality or some refined Jensen summation inequalities in qualitative analysis for system (1) are some potential directions for future research to reduce conservatism. Moreover, our results can be also extended to switched NNs or Markovian jump NNs such as [30, 40].

4 Conclusion

This paper addresses the FTB and finite-time H_∞ performance for a general class of discrete-time NNs subjected to leakage time-varying delay, discrete time-varying delay, and different sector-bounded neuron activation functions. We have developed some novel delay-dependent criteria based on designing suitable L–K functionals and applying an enhanced reciprocally convex technique. These criteria can be easily implemented and solved by MATLAB software in a fast and efficient way.

Acknowledgements The paper benefited greatly from the constructive feedback and insights of the editor and the anonymous reviewers. We are grateful for their time and expertise in helping us enhance the quality of our work.

Funding The author acknowledges the financial support from Hue University under the project DHH2021-01-188.

Data Availability This article does not involve any data sharing because it did not produce or examine any data sets in the course of this study.

Declarations

Conflict of interest The work reported in this paper was not influenced by any competing financial interests or personal relationships that the author is aware of.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Ali MS, Meenakshi K, Gunasekaran N (2017) Finite-time H_∞ boundedness of discrete-time neural networks norm-bounded disturbances with time-varying delay. *Int J Control Autom Syst* 15:2681–2689
2. Amato F, Ariola M, Dorato P (2001) Finite-time control of linear systems subject to parametric uncertainties and disturbances. *Automatica* 37:1459–1463
3. Amato F, Ariola M, Cosentino C (2010) Finite-time control of discrete-time linear systems: analysis and design conditions. *Automatica* 46:919–924

4. Amato F, Ambrosino R, Ariola M, Cosentino C, Tommasi GD (2014) *Finite-time stability and control*. Springer, London
5. Balasubramaniam P, Vembarasan V, Rakkiyappan R (2012) Global robust asymptotic stability analysis of uncertain switched Hopfield neural networks with time delay in the leakage term. *Neural Comput Appl* 21:1593–1616
6. Balasundaram K, Raja R, Zhu Q, Chandrasekaran S, Zhou H (2016) New global asymptotic stability of discrete-time recurrent neural networks with multiple time-varying delays in the leakage term and impulsive effects. *Neurocomputing* 214:420–429
7. Banu LJ, Balasubramaniam P (2016) Robust stability analysis for discrete-time neural networks with time-varying leakage delays and random parameter uncertainties. *Neurocomputing* 179:126–134
8. Banu LJ, Balasubramaniam P, Ratnavelu K (2015) Robust stability analysis for discrete-time uncertain neural networks with leakage time-varying delay. *Neurocomputing* 151:808–816
9. Chen X, Song Q (2013) Global stability of complex-valued neural networks with both leakage time delay and discrete time delay on time scales. *Neurocomputing* 121:254–264
10. Chen Y, Zheng WX (2012) Stochastic state estimation for neural networks with distributed delays and Markovian jump. *Neural Netw* 25:14–20
11. Chen Y, Fu Z, Liu Y, Alsaadid FE (2017) Further results on passivity analysis of delayed neural networks with leakage delay. *Neurocomputing* 224:135–141
12. Dorato P (2006) An overview of finite-time stability. In: Menini L, Zaccarian L, Abdallah CT (eds) *Current trends in nonlinear systems and control, in Honor of Kokotovic P and Nicosia Turi*. Birkhauser, Boston, pp 185–194
13. Gahinet P, Nemirovskii A, Laub AJ, Chilali M (1995) *LMI control toolbox for use with MATLAB*. The MathWorks Inc, Massachusetts
14. Gopalsamy K (2007) Leakage delays in BAM. *J Math Anal Appl* 325:1117–1132
15. Han QL (2005) Absolute stability of time-delay systems with sector-bounded nonlinearity. *Automatica* 41:2171–2176
16. Hu M, Cao J, Hu A (2014) Exponential stability of discrete-time recurrent neural networks with time-varying delays in the leakage terms and linear fractional uncertainties. *IMA J Math Control Inf* 31:345–362
17. Jia T, Chen X, Zhao F, Cao J, Qiu J (2023) Adaptive fixed-time synchronization of stochastic memristor-based neural networks with discontinuous activations and mixed delays. *J Frankl Inst* 360:3364–3388
18. Jiang X, Han QL, Yu X (2005) Stability criteria for linear discrete-time systems with interval-like time-varying delay. *Proc Am Control Conf* 2005:2817–2822
19. Kamenkov G (1953) On stability of motion over a finite interval of time [in Russian]. *J Appl Math Mech* 17
20. Khalil HK (2002) *Nonlinear systems*, 3rd edn. Prentice Hall, New York
21. Li XD, Rakkiyappan R (2013) Stability results for Takagi–Sugeno fuzzy uncertain BAM neural networks with time delays in the leakage term. *Neural Comput Appl* 22:203–219
22. Maa Z, Sun G, Liu D, Xing X (2016) Dissipativity analysis for discrete-time fuzzy neural networks with leakage and time-varying delays. *Neurocomputing* 175:579–584
23. Nam PT, Trinh H, Pathirana PN (2015) Discrete inequalities based on multiple auxiliary functions and their applications to stability analysis of time-delay systems. *J Frankl Inst* 352:5810–5831
24. Park PG, Ko JW, Jeong C (2011) Reciprocally convex approach to stability of systems with time-varying delays. *Automatica* 47:235–238
25. Rajchakit G, Sriraman R (2021) Robust passivity and stability analysis of uncertain complex-valued impulsive neural networks with time-varying delays. *Neural Process Lett* 53:581–606
26. Rajchakit G, Sriraman R, Boonsatit N, Hammachukiattikul P, Lim CP, Agarwal P (2021) Global exponential stability of Clifford-valued neural networks with time-varying delays and impulsive effects. *Adv Differ Equ* 208:1–21
27. Saravanakumar R, Stojanovic SB, Radosavljevic DD, Ahn CK, Karimi HR (2019) Finite-time passivity-based stability criteria for delayed discrete-time neural networks via new weighted summation inequalities. *IEEE Trans Neural Netw Learn Syst* 30:58–71
28. Seuret A, Gouaisbaut F, Fridman E (2015) Stability of discrete-time systems with time-varying delays via a novel summation inequality. *IEEE Trans Autom Control* 60:2740–2745
29. Shan Y, Zhong S, Cui J, Hou L, Li Y (2017) Improved criteria of delay-dependent stability for discrete-time neural networks with leakage delay. *Neurocomputing* 266:409–419
30. Shi P, Zhang Y, Agarwal RK (2015) Stochastic finite-time state estimation for discrete time-delay neural networks with Markovian jumps. *Neurocomputing* 151:168–174
31. Song Q, Cao J (2012) Passivity of uncertain neural networks with both leakage delay and time-varying delay. *Nonlinear Dyn* 67:1695–1707

32. Song H, Yu L, Zhang D, Zhang WA (2012) Finite-time H_∞ control for a class of discrete-time switched time-delay systems with quantized feedback. *Commun Nonlinear Sci Numer Simul* 17:4802–4814
33. Song Q, Shu H, Zhao Z, Liu Y, Alsaadi FE (2017) Lagrange stability analysis for complex-valued neural networks with leakage delay and mixed time-varying delays. *Neurocomputing* 244:33–41
34. Sowmiya C, Raja R, Cao J, Li X, Rajchakit G (2018) Discrete-time stochastic impulsive BAM neural networks with leakage and mixed time delays: an exponential stability problem. *J Franklin Inst* 355:404–4435
35. Sowmiya C, Raja R, Cao J, Rajchakit G (2018) Impulsive discrete-time BAM neural networks with random parameter uncertainties and time-varying leakage delays: an asymptotic stability analysis. *Nonlinear Dyn* 91:2571–2592
36. Suntornsinsoungvon E, Udpin S (2020) Exponential stability of discrete-time uncertain neural networks with multiple time-varying leakage delays. *Math Comput Simul* 171:233–245
37. Tuan LA (2020) H_∞ finite-time boundedness for discrete-time delay neural networks via reciprocally convex approach. *VNU J Sci Math Phys* 36:10–23
38. Tuan LA, Phat VN (2016) Finite-time stability and H_∞ control of linear discrete-time delay systems with norm-bounded disturbances. *Acta Math Vietnam* 41:481–493
39. Yi Z, Tan KK (2004) *Convergence analysis of recurrent neural networks*. Springer, Berlin
40. Zhang Y, Shi P, Nguang SK, Zhang J, Karimi HR (2014) Finite-time boundedness for uncertain discrete neural networks with time-delays and Markovian jumps. *Neurocomputing* 140:1–7
41. Zhang Z, Zhang Z, Zhang H, Zheng B, Kamiri HR (2014) Finite-time stability analysis and stabilization for linear discrete-time system with time-varying delay. *J Frankl Inst* 351:3457–3476
42. Zhang CK, He Y, Jiang L, Wang QG, Wu M (2017) Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality. *IEEE Trans Cybern* 47:3040–3049
43. Zhang CK, He Y, Jiang L, Wu M, Zeng HB (2017) Summation inequalities to bounded real lemmas of discrete-time systems with time-varying delay. *IEEE Trans Autom Control* 62:2582–2588
44. Zong G, Wang R, Zheng W, Hou L (2015) Finite-time H_∞ control for discrete-time switched nonlinear systems with time delay. *Int J Robust Nonlinear Control* 25:914–936
45. Zuo Z, Li H, Wang Y (2013) New criterion for finite-time stability of linear discrete-time systems with time-varying delay. *J Frankl Inst* 350:2745–2756

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.