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Control of linear extended nD systems with minimized sensitivity to parameter uncertainties

Ewaryst Rafajłowicz · **Wojciech Rafajłowicz**

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Abstract A new view on uncertain system parameters is presented considering them in the same way as other independent variables, e.g., time or space variables. After re-interpreting the well known equations for the sensitivities of a system to parameter changes, we consider the problem of optimal control that takes into account not only the quality of control itself, but also a reduction in the influence of parameter changes. Firstly, we re-derive and elucidate known results for systems described by linear ordinary differential equations in the state-space form. Then, it is shown how to extend the well known theory of designing optimal controllers with quadratic criterion so as to cover the reduction of uncertainties in systems described by a class of linear partial differential equations. As a result, we obtain a controller that has a new modal structure in space. Furthermore, the controller incorporates additional sensitivity signals for each mode.

Keywords nD systems · Control systems · Control systems with reduced sensitivity

1 Introduction

Bode, in his seminal paper (Bode 1945) was the first to warn of considerable risks if there are small changes in closed-loop transfer functions caused by small variations of parameters in the plant. Since then, many approaches in attaining a robustness of control systems against parameter changes have been discussed. The most important contributions have been provided by the H_∞ theory. Note, however, that the H_∞ theory and related approaches impose constraints on the admissible influence of parameter uncertainties (see Bhattacharyya et al. 1995; Boyd and Barratt 1991; Boyd 1986; Gu and Petkov 2005; Sainchez-Pena and Sznaier 1998; Vidyasagar and Kimura 1986; Zhou and Doyle 1998). In a sense, it is *a passive approach*, because there is no feedback from sensitivities. Conversely, an approach discussed in this

E. Rafajłowicz (✉) · W. Rafajłowicz
Faculty of Electronics, Institute of Computer Engineering, Control and Robotics,
Wrocław University of Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland
e-mail: ewaryst.rafajlowicz@pwr.wroc.pl

paper is active in the sense that additional feedback input signals are supplied that are obtained from a sensitivity models to controllers. Firstly, we recall known results for systems described by linear ordinary differential equations (ODE's) that can be traced back to Kreindler (1968) (see also Cruz 1973 Chap. 28 for a reprint of Kreindler 1968), where a synthesis procedure ensuring good control system performance by adding to the performance index term involving sensitivity coefficients has been proposed. Later, this direction of research is discussed in Frank (1978) (pp. 318–323), Zames (1981), Weinmann (1991) (Chap. 10 pp. 139–142), Rosenwasser and Yusupov (2000) (pp. 403–405), Eslami (1994) (pp. 322–323). In Sect. 2 we elucidate and formulate them in a form convenient for generalization to systems described by a class of linear partial differential equations (PDE's) with constant coefficients. This is necessary, because a full analysis of the control system behaviour relies on a PDE if sensitivity signals are present in the feedback. This PDE depends at least on the time variable and the uncertain parameter(s), that can also be treated as independent variables (see Sect. 3.1, where it was shown that a hyperbolic PDE arises even in the simplest case of first order systems). Only a full analysis provides a clear base for pointing out which simplifications we are introducing when we also allow for feedback from sensitivity signals.

After re-stating the results for desensitizing control of systems described by ODE (Sect. 3), we provide their generalization to systems with spatio-temporal dynamics described by a class of PDE's with constant coefficients that allow for a modal decomposition in space variables. We state the problem of optimal control of such systems, including an additional penalty for a system state sensitivity into a quadratic quality criterion. In Sect. 4 it is shown that this problem can be reduced to the one considered in Sect. 3 when only zone control is allowed or to a sequence of such problems, when we are able to apply a control action more freely in space. Furthermore, the structure of a new class of controllers arises in a natural way, as a result of solving an extended optimal control problem with the quadratic criterion that aims to minimize not only the output error, but also excessive sensitivity to parameter uncertainty. These additional sensitivity signals are generated from equations for the modal sensitivities of a system state to parameter changes.

By construction, our approach is local in the sense that we design a control system in the vicinity of a nominal parameter a_0. In the monograph by Cruz (1973) and in Boyd and Barratt (1991), Cruz and Perkins (1964), Eslami (1994) one can find an account on calculating local (differential) sensitivities and their applications.

These facts motivate *extension of an nD systems concept*, see, e.g., (Gałkowski 2001; Kaczorek 1985; Rogers et al. 2007). By treating uncertain parameters of a system as additional independent variables (in addition to time and space variables).

Quite recently, an interesting and in-depth approach to defining disturbance attenuation has been proposed in Dąbkowski et al. (2012) for the class of 2D systems, the so-called discrete linear repetitive processes. This paper uses strong practical stability and measures of disturbance attenuation defined in terms of the evolution over one independent variable with the other fixed and conversely. This approach forces us to rethink notions of stability not only for nD systems, but also for systems described by PDE's, nevertheless, this is outside the scope of this paper. We refer the reader also to Przedwojski et al. (2011), where a new approach to possible structural uncertainty is considered as a generalization of parameter uncertainty.

Remark 1 Formal links between parametric uncertainty modeling using linear fractional transformations (LFT) and nD systems have already been exploited in Lambrechts (1993), where it was shown that the problem of finding an LFT representation of a linear system with uncertain parameters can be reduced to an nD realization problem. In a similar vein in Cockburn and Morton (1997) LFT realizations of polynomial parameter uncertainties were obtained. Note however that our approach is based on the time domain (or space-time domain) system representation that allows for a direct generation of sensitivity signals and can be generalized to nonlinear systems, but it is outside the scope of this paper.

In Beck et al. (1996) (a paragraph just before Remark 1, p. 1467) the possibility of considering deviations from nominal parameters as additional independent variables is briefly discussed, but this topic was not further developed in Beck et al. (1996) as it is less useful for model reduction.

The aims of this paper include:

(1) to discuss the extension of an nD systems concept, and to re-derive known equations for the differential sensitivity of the system state to parameter changes in a form suitable for our purposes, i.e., including a feedback from sensitivity signals, (see Sect. 2),
(2) to discuss the problem of *optimal control* of linear systems *extended by sensitivity equations* with the quadratic criterion that additionally contains a penalty for an excessive sensitivity to imprecise knowledge or changes in parameters (see Sect. 3) in a way similar to Kreindler (1968) and to re-derive its solution in a way convenient for extensions,
(3) to show how to extend this approach to systems with spatio-temporal dynamics that are described by a class of linear PDE's (see Sect. 4).

The solution of the problem mentioned in (2) is relatively easy, using the well known results for LQ optimal problems. Its extension in (3) leads to the optimal, controller which is a modal (in space). *Each modal controller is fed additionally by sensitivity signals*. Thus, we obtain a non-classical controller structure, which consists of

- the classical part,
- the model for sensitivity equations (for each mode separately),
- a new part of the modal controllers, which processes sensitivity signals, producing additional control signals.

Numerical examples indicate that the behavior of a control system with reduced sensitivity is different than that of classical systems, This is the price that we pay for parameter uncertainty and its reduction.

We have sketched the preliminary results on extending *nD* systems and their applications in control system design in Rafajłowicz and Rafajłowicz (2011). In this paper we generalize these ideas from systems described by ODE's to systems described in the state space form and to a class of systems described by linear PDE's. Additionally, we have derived a PDE for a system state when the feedback from the differential sensitivity is present. Simultaneously and independently of (Rafajłowicz and Rafajłowicz 2011) the idea of taking PDE's sensitivities into account appeared in Rauh et al. (2011, 2012). Note however that in these papers there is no explicit feedback from parameter sensitivity signals. The sensitivities with respect to control signal values and uncertain parameters are used for predicting system states and for improving control system design. This indicates that calculating sensitivities is a useful tool in this area.

2 Systems extended by sensitivity equations as nD systems

Differential equations that describe our system do not describe directly how their solutions depend on uncertain or unknown parameters. Thus, we have to extend them by so-called sensitivity equations. The way of deriving them is well known (see Cruz 1972, 1973; Frank 1978). Possible difficulties in an interpretation may arise when a feedback from sensitivity equations to a system is allowed. This case is crucial for the rest of this paper and we discuss it in detail in Sect. 2.4.

2.1 Extended nD systems: $(n + r)D$ way of thinking

To motivate the proposed extension of nD systems we start from two simple examples.

Example 1 The following simple system is usually considered as a 1D system (with t as the independent variable)

$$\frac{d\,y(t)}{d\,t} = a\,y(t), \quad y(0) = y_0. \tag{1}$$

Its solution is $y(t) = y_0 \exp(a\,t)$. However, if a in this system is unknown (or uncertain, i.e., not exactly known), then its response depends on two variables and we have $y(t; a) = y_0 \exp(a\,t)$. Note that a enters into (1) exactly in the same way as t, which is reflected by displaying a in the notation.

The initial condition y_0 can also be included into independent variables if it is unknown or uncertain. However, it seems reasonable not to include it, because, usually y_0 (or $y(0+)$) can be directly observed.

Example 2 Consider the well known example of the transport equation

$$a_1\,\frac{\partial\,y(\chi,\,t)}{\partial\,t} + \frac{\partial\,y(\chi,\,t)}{\partial\,\chi} + a_2\,y(\chi,\,t) = 0, \quad \chi > 0,\; t > 0, \tag{2}$$

with the boundary condition $y(0, t) = \phi(t)$, where $y(\chi,\,t)$ is the system state at spatial point χ and time t, while $\phi(t)$, is a given function, which is continuously differentiable. It is easy to verify, using the method of characteristics (see, e.g., Tveito and Wither 2005), that the solution of (2) has the form:

$$y(\chi,\,t) = \phi(t - a_1\,\chi)\,e^{-a_2\,\chi}. \tag{3}$$

Again, (2) is usually considered as a 2D system with χ and t as independent variables. If $\bar{a} = [a_1,\,a_2]$ is a vector of uncertain parameters, then it is reasonable to consider y as $y(\chi,\,t;\,\bar{a})$, i.e., depending on four variables. Note that in this case a_1 and a_2 enter into (3) exactly the same way as χ.

In more complicated cases, parameters of (PDE's) appear in their solutions as dimensionless complexes known as e.g., the Reynolds number, the Fourier number etc.

Extending original independent variables by adding unknown parameters can be useful in:

- system identification both for constructing gradient based identification methods, for sensors' allocation (Uciński 2005) and selecting the most sensitizing input signals,
- control system synthesis with reduced sensitivity of outputs to imprecise knowledge of parameter values.

The notation $y(t; a)$ (or $y(t; \bar{a})$) in the above examples emphasizes that *a is also treated as the independent variable(s), but it has a slightly different status than t or space variables.* One can consider the system in Example 1 as a 2D one. We prefer to denote this system as a (1+1)D system. Analogously, the system in Example 2 is formally 4D, but we use the notation $(2+2)D$ for it. In a similar vein, a 3D system with four uncertain parameters will be denoted as a $(3+4)D$ system. Thus, $(n+r)D$ means that we have n independent variables such as time or space variables and r uncertain parameters, also considered as independent variables.

2.2 A system description

As a vehicle for presenting ideas we consider a linear, time-invariant (LTI) system described by the ODE of 1-st order:

$$\dot{x}(t;\ a) = A(a)\,x(t;\ a) + b\,u(t), \quad x(0) = x_0 \tag{4}$$

where $x(t;\ a) \in R^k$ is the system state that depends on an uncertain parameter a, $\dot{x}(t;\ a) \stackrel{def}{=} \frac{d\,x(t;\,a)}{d\,t}$ and $u(t)$ is the input at time t. Additionally, the initial conditions for (4) are given and independent of a. For simplicity of the exposition we consider only one unknown or uncertain parameter a, i.e., we consider $(k+1)D$ systems. $b \in R^k$ is a given vector and $A(a)$ is $k \times k$ matrix. One of its elements is a. We skip an easy extension arising when $A(a)$ is a differentiable function of a.

2.3 Sensitivity equations: no feedback

It is known that in our case $x(t; a)$ is an analytic function of a.

Remark 2 This follows immediately from the expansion of $\exp(A(a)\,t)$ into the power series. (see also Taylor 2011 p. 33 and Exercise 2, p. 34). Thus, $x(t; a)$ can be differentiated w.r.t. a as many times as required, since the analyticity of $x(t; a)$ implies that it has infinitely many derivatives.

The well known method of deriving ODE's for sensitivities (see, e.g., Cruz 1972, 1973; Frank 1978) is as follows.

1. Differentiate both sides of (4) w.r.t. a.
2. Change the order of differentiation w.r.t. to a and t.

As a result we obtain for $t > 0$

$$\frac{\partial^2 x(t; a)}{\partial t\,\partial a} = A(a)\,\frac{\partial\,x(t; a)}{\partial\,a} + \frac{\partial\,A(a)}{\partial\,a}\,x(t; a), \tag{5}$$

with the initial condition $\frac{\partial\,x(0;a)}{\partial\,a}$. We shall introduce the following notation

$$w(t;\ a) = \frac{\partial\,x(t;\ a)}{\partial\,a}, \tag{6}$$

i.e., $w(t;\ a)$ is $k \times 1$ vector of unknown functions, for which we obtain [from (5)] the following equation

$$w'(t;\ a) = A(a)\,w(t;\ a) + \frac{\partial\,A(a)}{\partial\,a}\,x(t;\ a), \quad t > 0 \tag{7}$$

with the initial condition $w(0, a) = 0$, since x_0 does not depend on a (by assumption).

The proper way of calculating w in a vicinity of a nominal value of a, denoted further by a_0, is[1] as follows:

[1] When the sensitivity equations are used for system identification we use a current estimate of a instead of a_0.

1. solve (4) for given $u(\cdot)$ and for $a = a_0$,
2. substitute $x(t, a_0)$ on the right hand side of (7),
3. solve (7) for a_0 and zero initial conditions.

Note that the structure of (7) is the same as (4). Thus, it can be expedient to find (or approximate) the impulse response (Green's function). In general, we have to solve $2\,k$ equations. When we have $r > 1$ unknown (or uncertain) parameters, then we have to extend (4) by $r\,k$ equations, i.e., our $(1 + r)D$ system is described by $(1 + r)\,k$ ODE's.

We provide several remarks on possible generalizations.

1. When a feedback from x to u is present, then include the term $\partial\, u(t)/\partial\, a$ (see below for details).
2. The above works also for nonlinear equations, provided that their solutions are differentiable sufficiently many times.
3. The alternative way of calculating sensitivities is based on the so called adjoint equations (see, e.g., Kalaba and Spingarn 1982; Schittkowski 2002). However, this way requires solving ODE's backward in time. Hence, if our system is stable, the adjoint equations are unstable, which leads to numerical difficulties. In some cases the method of Perkins and Kokotovic can also be used (see Frank 1978).

2.4 Sensitivity equations when a feedback is present

When feedback from x to u is present, then u also depends on a and we obtain

$$w'(t;\ a) = A(a)\, w(t;\ a) + \frac{\partial\, A(a)}{\partial\, a}\, x(t;\ a) + b\, \frac{\partial\, u(t;\ a)}{\partial\, a}, \quad t > 0, \tag{8}$$

$w(0;\ a) = \bar{0}$. It is expedient to consider two, as we shall see, essentially different cases. Namely, when a feedback is from the system state only and when we admit also a feedback from the sensitivity equations, i.e., from w.

2.4.1 Feedback from the system state only

When a feedback from the system state is present, i.e., $u(t;\ a) = q^T\, x(t;\ a)$, where $q \in R^k$ does not depend on a (by assumption), and T denotes the transposition, then the system is described by

$$x'(t; a) = A(a)\, x(t; a) + b\, q^T\, x(t;\ a), \quad x(0) = x_0 \tag{9}$$

and, after the differentiation of (9), the sensitivity equations have the form:

$$\frac{\partial^2\, x(t; a)}{\partial\, a\, \partial\, t} = \left[A(a) + b\, q^T\right] \frac{\partial\, x(t; a)}{\partial\, a} + \frac{\partial\, A(a)}{\partial\, a}\, x(t;\ a), \tag{10}$$

with the initial condition $\frac{\partial\, x(0;a)}{\partial\, a} \equiv 0$. Notice that (9) and (10) form a system of mixed ODE's and PDE's that is—in general—not easy to solve. In this case, however, it suffices to use (6) in order to reduce (10) to the following ODE

$$w'(t;\ a) = \left[A(a) + b\, q^T\right] w(t;\ a) + \frac{\partial\, A(a)}{\partial\, a}\, x(t;\ a), \quad t > 0. \tag{11}$$

Combining (9) and (11), we obtain the closed system of ODE's that can be solved by solving firstly (9) and then substituting the result into (11).

2.4.2 Additional feedback from sensitivity equations

Difficulties arise when our feedback contains the sensitivity signal $\frac{\partial x(t;a)}{\partial a}$. To illustrate these difficulties, let us consider the following feedback $u(t;\ a) = q^T\ x(t;\ a) + h^T\ w(t;\ a)$, where $h \in R^k$, by assumption, does not depend on a. Then, instead of (10) we obtain

$$\frac{\partial^2 x(t;a)}{\partial a\, \partial t} = \left[A(a) + b\, q^T\right] \frac{\partial x(t;a)}{\partial a} + \frac{\partial A(a)}{\partial a} x(t;\ a) + b\, h^T \frac{\partial^2 x(t;a)}{\partial a^2} \tag{12}$$

with the initial condition $\frac{\partial x(0;a)}{\partial a} \equiv 0$. Substituting $w(t;a) = \frac{\partial x(t;a)}{\partial a}$ we obtain

$$w'(t;\ a) = A(a)\, w(t;\ a) + \frac{\partial A(a)}{\partial a} x(t;\ a) + b\, q^T\ w(t;\ a) + b\, h^T \frac{\partial w(t;\ a)}{\partial a}, \tag{13}$$

but the difficulty is that function $\frac{\partial w(t;\,a)}{\partial a}$ is unknown. Later on we impose an additional, *simplifying assumption*: $\frac{\partial^2 x(t;\,a)}{\partial a^2} = \frac{\partial w(t;\,a)}{\partial a} \approx 0$ and this term is omitted. This is equivalent to assuming that we truncate the Taylor series for $x(t;a)$ after the first derivative w.r.t. a. This, in turn, implies that the system of ODE's for x and w is closed and it again has the form (9) and (10) [or (11)].

Remark 3 It was already noticed Kreindler (1968) (see also Eslami 1994; Frank 1978) that the simplifying assumption may lead to inexact approximations. The remedy is to obtain equations for $\frac{\partial^2 x(t;\,a)}{\partial a^2}$ at the expense of solving an additional system of equations that can be obtained by the differentiation of (10) w.r.t a again (by Remark 2 we know that the second derivative exists). Notice, however, that we have to assume $\frac{\partial^3 x(t;\,a)}{\partial a^3} \approx 0$ in order to obtain a closed system of equations. This process of constructing higher order sensitivities can be continued until a desired accuracy is attained. The computational power of available computer systems is sufficient for solving such problems.

3 Optimal control systems with reduced sensitivity to parameter changes

In Sect. 3.3 we state the problem of optimal control of this system in such a way that not only the system output is close to a target value, but also the sensitivity of the optimal output with respect to uncertain parameter(s) is taken into account. The results presented in Sect. 3.3 are close to Kreindler (1968). They are stated in a form that is suitable for generalization to PDE's in Sect. 4. Considerations in Sect. 3.1 seem to be new. They elucidate why one can not expect that a control system can be completely insensitive to parameter uncertainties.

3.1 A formal approach to desensitizing a control system to parameter changes

In this subsection we discuss the possibility of desensitizing a control system to parameter changes. As we shall see, our considerations lead to the conclusion that even in the simplest case, exact desensitizing is not possible, since it may lead to an unstable control system. On the other hand, they contain hints on a proper problem statement.

Consider the following simple, univariate system: $x'(t;a) = a\, x(t;a) + b\, u(t),\ b > 0$, with the feedback of the form: $u(t) = q\, x(t;a) + h\, \frac{\partial x(t;a)}{\partial a}$. After closing the loop, we obtain the following equation:

$$\frac{\partial x(t;a)}{\partial t} - \beta\, \frac{\partial x(t;a)}{\partial a} - \zeta\, x(t;a) = 0, \quad x(0,a) = x_0, \tag{14}$$

where $\zeta \overset{def}{=} a + b\,q,\ \beta \overset{def}{=} b\,h$. In (14) one can easily recognize the first order hyperbolic PDE. Their general solution is of the form: $x(t;a) = \Phi(a + \beta\,t)\,exp(\zeta\,t)$, where Φ is an arbitrary, continuously differentiable function, while its particular solution is $x(t;a) = x_0\,exp(\zeta\,t)$.

Selecting $\zeta = 0$ we can formally force $x(t;a)$ to be independent of a. Note, however, that $\zeta = 0$ implies $x(t;a) = x_0$, which means that our control system is not exponentially. Furthermore, condition $\zeta = 0$ means that we select $q = -a/b$, but a is an uncertain parameter. Replacing it by a nominal value a_0 and selecting $q_0 = -a_0/b$ may lead to $\zeta = a + b\,q_0 = a - a_0$ that can be positive and then, $x(t;a) = x_0\,exp(\zeta\,t)$ approaches to infinity.

From these formal speculations it follows that we can only attempt to reduce the influence of uncertainties in a, keeping in mind the requirement of the stability of the closed-loop system. The above considerations can be repeated for (4). Instead of (14) we obtain a system of hyperbolic PDE's, but the overall pattern is qualitatively the same.

The second formal ingredient that has to be mentioned is the fact that the signal $\frac{\partial\,x(t;a)}{\partial\,a}$ in $u(t) = q\,x(t;a) + h\,\frac{\partial\,x(t;a)}{\partial\,a}$ is not available. Taking into account that our feedback contains the sensitivity, we have to impose our simplifying assumption that was discussed in Sect. 2.4.2. Otherwise, we have to derive a differential equation for $\frac{\partial^2\,x(t;a)}{\partial\,a^2}$. This equation will contain the unknown term $\frac{\partial^3\,x(t;a)}{\partial\,a^3}$, for which we need one more differential equation etc. In fact, we can obtain any desired accuracy, but this (potentially infinite) sequence of sensitivity equations must be somewhere truncated.

Notice that we again obtain PDE (10), but in a manageable form (11).

3.2 Optimal control with reduced sensitivity to parameter uncertainty

In this subsection we shall state the problem of optimal control with reduced sensitivity to changes of one parameter only.

As it will be clear from the derivations, the problem and its solution can readily be extended to the general case when all parameters are uncertain. However, this generalization complicates formulas. For the same reasons we present the ideas for the so called quadratic criterion in its simplest form and for the infinite control horizon.

For system (4), nominal parameters a_0 are given, but we are not certain whether a_0 is a proper value for a. This can happen, e.g., if a_0 results from a system identification procedure. We have also (7), considered also for $a = a_0$. Our aim is to find a square integrable signal $u^*(t)$ that minimizes:

$$J(u) = \int_0^\infty \left[x^T(t)\,Q_1\,x(t) + w^T(t)\,Q_2\,w(t) + \gamma\,u^2(t)\right] dt \tag{15}$$

with respect to $u(\cdot)$, where Q_1 and Q_2 are given $k \times k$ nonnegative definite matrices, while $\gamma > 0$ is a given constant. The constraints for our problem are the following:

$$x'(t) = A(a_0)\,x(t) + b\,u(t), \quad x(0) = x_0 \tag{16}$$

$$w'(t) = A(a_0)\,w(t) + \frac{\partial\,A(a)}{\partial\,a}\Big|_{a=a_0} x(t), \quad w(0) = 0. \tag{17}$$

The quadratic criterion (15) contains a new summand $w^T(t)\,Q_2\,w(t)$ that is expected to select $u^*(\cdot)$ with reduced sensitivity to changes in a. Additionally, a penalty for excessive use of energy is present and the resulting state is close to the target value $x^* = 0$, i.e., we set the reference signal to zero (see Fig. 1).

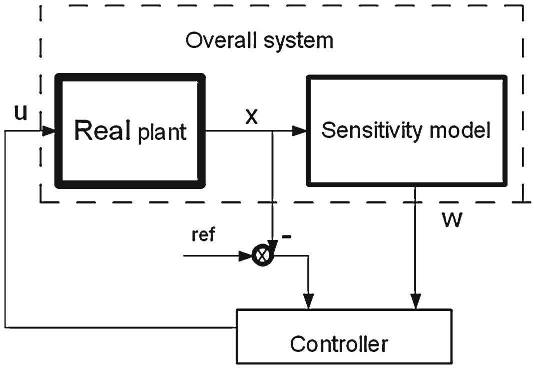

Fig. 1 Extended system and its controller

Remark 4 $J(u)$ defined by (15) looks familiar, but it contains the summand $w^T(t)\,Q_2\,w(t)$ together with (17) that was explicitly introduced in Kreindler (1968) for designing controllers with reduced sensitivity (see also Cruz 1973; Frank 1978 Chap. 28). In opposite to Kreindler (1968), we do not allow for cross-terms $x^{(i)}(t)\,w^{(k)}(t)$ in (15), because there are difficulties in their interpretation. Furthermore, their absence leads to a simpler and interpretable controller structure (see (18) below).

Note that in H_∞ optimal controller design quite different criterions (based on a transfer matrix from disturbances to outputs) are used (see Bhattacharyya et al. 1995; Sainchez-Pena and Sznaier 1998; Vidyasagar and Kimura 1986; Zhou and Doyle 1998).

The system to be controlled consists of a real plant with input $u(\cdot)$ and state $x(\cdot)$, which is fed—as the input—to the sensitivity model. Its state, in turn, is also treated as a part of the system state. Thus, our system contains the real plant plus virtual sub-system, as illustrated in Fig. 1.

3.3 The structure of optimal controller with reduced sensitivity

A nice feature of the problem stated in the previous subsection is that the minimization of $J(u)$ can be embedded into the theory of well known LQ problems. An additional advantage is that we do not need a new software for designing optimal control systems with reduced sensitivity to uncertain parameters. One can directly use the existing software by feeding an extended system as its input data.

Define $\bar{x}^T(t) = [x^T(t),\ w^T(t)]^T$ and $\bar{b}^T = [b^T,\ \bar{0}^T]$ as well as $2k \times 2k$ matrices

$$Q = \begin{bmatrix} Q_1 & \mathbf{O} \\ \mathbf{O} & Q_2 \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} A(a_0) & \mathbf{O} \\ D & A(a_0) \end{bmatrix}, \tag{18}$$

where $\mathbf{O}$ is a $k \times k$ matrix of zeros, while $D \stackrel{def}{=} \left.\frac{\partial A(a)}{\partial a}\right|_{a=a_0}$ is a $k \times k$ matrix.

In the above notations the problem (15), (16), and (17) reads as follows: minimize w.r.t. $u(\cdot) \in L_2(0, \infty)$

$$J(u) = \int_0^\infty \bar{x}^T(t)\,Q\,\bar{x}(t) + \gamma\,u^2(t)\,dt, \tag{19}$$

for the extended system:

$$\bar{x}'(t) = \bar{A}\,\bar{x}(t) + \bar{b}\,u(t) \quad \bar{x}(0) = \bar{x}_0 \tag{20}$$

Thus, from the view point of the optimization theory, we have reformulated our problem in such a way that its solution can be obtained by re-interpreting the well known results (see, e.g., Curtain 1992; Sontag 1998) concerning optimal control of linear, time invariant systems with quadratic criterions. From this theory we immediately obtain the following.

Theorem 1 *If for $\bar{A}$, $\bar{b}$, defined by* (18)*, the extended system* (20) *is stabilizable,*[2] *then the optimal solution of* (19), (20) *is of the form:*

$$u^*(t) = \frac{1}{2}\,\gamma^{-1}\,\bar{b}^T\,K\,\bar{x}(t) \tag{21}$$

where K is a $2\,k \times 2\,k$ matrix that solves

$$\frac{1}{2}\,\gamma^{-1}\,K\,\bar{b}\,\bar{b}^T\,K + \bar{A}^T\,K + K\,\bar{A} - 2\,Q = 0. \tag{22}$$

Later on we shall denote by $\mathcal{K}_{opt}$ the vector $\frac{1}{2}\,\gamma^{-1}\,\bar{b}^T\,K$ of optimal controller gains. The structure of the optimal control system is shown in Fig. 1. From (21) it follows that the optimal controller receives signals from the plant and from its sensitivity model and additively combines them into the optimal control signal.

If uncertain parameter a is directly one of the elements of $A(a)$, then sub-matrix D contains zeros with the exception of one element that equals 1. This, in turn, yields that matrix $\bar{A}$ has a special structure that can be useful in solving the Riccati equations (22). A more extensive discussion of this fact is outside the scope of our paper.

Remark 5 In Fig. 1 the sensitivity model is interpreted as a part of the system to be controlled. This interpretation is crucial for finding an easy way for solving the optimal control problem. Note, however, that after finding the optimal controller, we have to reconfigure our thinking in the sense that both the sensitivity model and the optimal controller should be implemented as one (usually digital) control unit.

Remark 6 Let us note that the controller defined in Theorem 1 is very similar but not identical to the one obtained in Kreindler (1968), which results in the possibility of using available packages (e.g., from Matlab) directly. The reason is in a different way of approximating sensitivity signals. In our case, we assume from the beginning that a controller operates in a closed loop (see Sect. 2.4.2). This point deserves further studies, but they are outside the scope of this paper.

3.4 Example: DC motor

To illustrate the above results consider the LTI system, which is a simplified model of a DC motor with the integrator added in order to ensure zero error in the steady state (see Chiasson 2005 and the MATLAB documentation). The matrices $A,\ b$ are given by

$$A(a) = \begin{bmatrix} -5 & -2 & 0 \\ 2 & 0 & a \\ 0 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

while the matrix D is the lower left block of $\bar{A}$. As the blocks Q_1 and Q_2 of Q in (18) we take $Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and Q_2 being the 3×3 unit matrix. As the nominal value of a, that is

[2] For the definition of stabilizability see (Sontag 1998) p. 214.

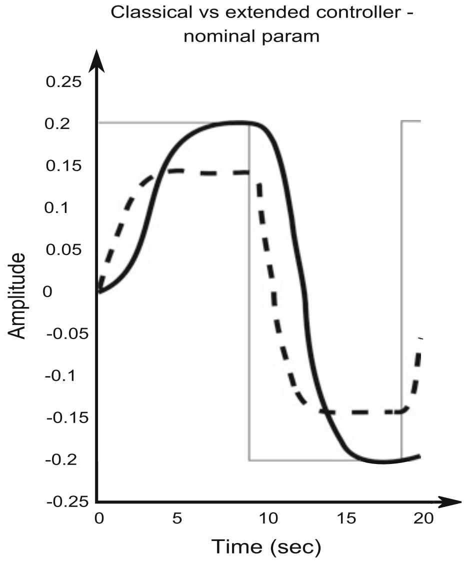

Fig. 2 Close loop response—nominal parameter $a_0 = 0.1$. *Solid line* the response of the optimal extended system, *dashed line* the response of the classical LQ system

used for the control system design, we take $a_0 = 0.1$. The response of the open loop system for a small input signal $0.2\, sin(t/3)$ indicates that the system can not work properly without a closed loop controller.

According to Theorem 1, the optimal gains (obtained by `lqry MATLAB` function, applied to the extended system) of the controller are the following:

$$\mathcal{K}_{opt} = -[0.8260,\ 2.4061,\ 3.4516,\ 0.5935,\ 1.5164,\ 1.0]. \tag{24}$$

Although it is somewhat risky to compare systems that are optimal under different optimality criterion, we also have designed the classical LQ controller for the same system. The optimality criterion differs only by the absence of $\int_0^\infty w^T(t)\, Q_2\, w(t)\, dt$ term. The resulting controller that is optimal in the classical LQ sense has the following gains (obtained by `lqry MATLAB` function, applied to the original system):

$$\mathcal{K}_{clas} = -[0.5000,\ 1.3750,\ 1.5697]. \tag{25}$$

Then, the performance of both closed loop control systems was simulated with the same reference signal: $0.2\, signum(sin(t/3))$. The results of the simulations are shown in Fig. 2. We plot only the third component of the system state that is considered as the system output. As one can notice, the response of the control system with $\mathcal{K}_{opt}$ (solid curve) much better follows the reference signal than the controller with gains $\mathcal{K}_{clas}$ (dashed line). The aim of the second experiment was to verify the performance of both controllers when instead of $a_0 = 0.1$ that was used for their design, the 'true' value of a is zero. To this end, the simulations were repeated with $A(0)$, but keeping the same controller gains (24), (25) and the same reference signal. The results are shown in Fig. 3. Comparing the corresponding curves in Figs. 2 and 3, the following conclusions can be drawn:

- The classical LQ controller behaves similarly in both cases, but it is not able to attain the reference signal level 0.2. Furthermore, it attains a level of about 0.15 when the same $a_0 = 0.1$ is used for the control system design and in simulations, but it is able to attain only the level 0.125 when the "true" value is $a = 0$.

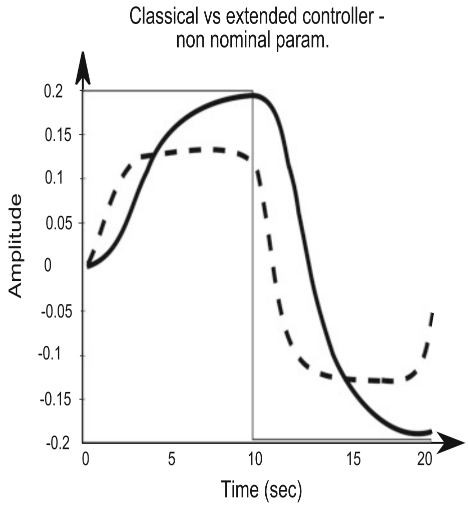

Fig. 3 Close loop response—non-nominal parameter $a = 0$. *Solid line* the response of the optimal extended system, *dashed line* the response of the classical LQ system

- The extended controller attains the level 0.2 in both cases. The only difference is that in the first case it appears after about 7 s, while in the second case this level is attained about 3 s later.

It seems that the conclusion about the better behavior of the extended controller is convincing. It is confirmed by other simulations performed by the authors that are not reported here. Although controllers extended by sensitivity signals have been studied for a long time (see Kreindler 1968), it seems that an additional effort in this direction is needed, before suggesting extended controllers as an alternative to well understood LQ controllers.

4 Generalization to systems described by PDE

Our aim in this section is to show how the results from Sect. 3 can be used for nD systems that are described by a class of PDE's with uncertain parameters.

System description Denote by $q(\chi, t)$ a system state at spatial point $\chi \in \Omega \subset R^d$ and time t, where Ω is a bounded and open domain. We shall also write $q(\chi, t; \bar{a})$ to indicate its dependence on uncertain parameters $\bar{a}$. Consider the following class of PDE's:

$$\frac{\partial q(\chi, t)}{\partial t} = A_\chi(\bar{a})\, q(\chi, t) + U(\chi, t), \quad \chi \in \Omega, \tag{26}$$

where $A_\chi(\bar{a})$—an operator of the elliptic type that depends on a vector of uncertain constant parameters. Boundary conditions for (26) are included in the definition of a class of functions for which operator $A_\chi(\bar{a})$ is defined. This class is denoted as $\mathcal{D}(A_\chi) \subset L^2(\Omega)$, where $L^2(\Omega)$ is the Hilbert of squared integrable functions with the inner product $< f, g >= \int_\Omega f(\chi)\, g(\chi)\, d\chi$. In addition, (26) is equipped with the initial condition $q(\chi, 0) = \varphi(\chi),\ \chi \in \Omega$. $U(\chi, t)$ denotes an input signal that is at our disposal.

Remark 7 Equation (26) is of the parabolic type and it can describe a heat transfer or mass diffusion. Note however that hyperbolic PDE's can be embedded into the same formalism by replacing $q(\chi, t)$ by a vector function and properly defining operator $A_\chi(\bar{a})$ (see, e.g., Curtain 1992 for details).

Assumptions We admit operators A_χ of the form:

$$A_\chi(\bar{a})\, q(\chi, t) = \sum_{i=1}^{p} a_i\, P_\chi^{(i)}\, q(\chi, t), \tag{27}$$

where $P_\chi^{(i)}$, $i = 1, \ldots, p$ are differential operators w.r.t. spatial variables and such that:

(A1) A_χ is symmetric, i.e., $\forall f\, g \in \mathcal{D}(A_\chi)$: $< A_\chi(\bar{a})\, f,\ g > = < f,\ A_\chi(\bar{a})\, g >$. and is positive definite, i.e., $\forall f \in \mathcal{D}(A_\chi),\ f \neq 0 \ \Rightarrow\ < f,\ A_\chi(\bar{a}) f >$ is positive.

(A2) Eigenfunctions $v_j \in \mathcal{D}(A_\chi),\ j = 1, 2, \ldots$ of operator $A_\chi(\bar{a})$ form complete and orthonormal basis of $L_2(\Omega)$.
We refer the reader to Yosida (1981) and Attouch et al. (2006) for the definitions of the above notions and the results on the existence of a complete and orthonormal set of eigenfunctions.

(A3) We additionally assume that eigenfunctions $v_1(\chi),\ v_2(\chi), \ldots$ of $A_\chi(\bar{a})$ are simultaneously the eigenfunction of all $P_\chi^{(i)}$, $i = 1, \ldots, p$.

Examples of $A_\chi(\bar{a})$ for which the above assumptions hold include:

$$A_\chi(\bar{a})\, q(\chi, t) = a_1 \frac{\partial^2 q(\chi, t)}{\partial \chi_1^2} + a_2 \frac{\partial^2 q(\chi, t)}{\partial \chi_2^2} + a_3\, q(\chi, t), \tag{28}$$

$$A_\chi(\bar{a})\, q(\chi, t) = a_1 \frac{\partial^4 q(\chi, t)}{\partial \chi_1^4} + a_2 \frac{\partial^4 q(\chi, t)}{\partial \chi_2^4} \tag{29}$$

with homogenous boundary conditions of the first and the second kind (see Taylor 2011 for the definitions).

It is easy to verify that under (A1)–(A3), eigenfunctions $v_1(\chi),\ v_2(\chi), \ldots$ do not depend on $\bar{a}$ and the eigenvalues λ_j of $A_\chi(\bar{a})$, i.e., constants for which $A_\chi(\bar{a})\, v_j(\chi) = -\lambda_j\, v_j(\chi)$, are linear functions of $\bar{a}$. Therefore, we shall denote them as $\lambda_j(\bar{a})$. By ^{tr} we denote the transposition of a vector. It is easy to verify that $\lambda_j(\bar{a}) = \bar{b}_j^{tr}\, \bar{a}$, for certain (known) vectors $\bar{b}_j,\ j = 1, 2, \ldots$. In fact, i-th element of $\bar{b}_j$ is the eigenvalue of $P_\chi^{(i)}$ operator that corresponds to its eigenfunction v_j.

Modal decomposition It is expedient to use the so called modal decomposition that allows us to express the solution of (26) as

$$q(\chi, t,\ \bar{a}) = \sum_{j=1}^{\infty} v_j(\chi)\, y_j(t,\ \bar{a}), \tag{30}$$

where $y_j(t)$'s are the solutions of

$$\dot{y}_j(t,\ \bar{a}) = -\lambda_j(\bar{a})\, y_j(t,\ \bar{a}) + \tilde{u}_j(t), \tag{31}$$

$\tilde{u}_j(t) \stackrel{def}{=} \int_\Omega U(\chi, t)\, v_j(\chi)\, d\chi,\ y_j(0) = \int_\Omega \varphi(\chi)\, v_j(\chi)\, d\chi$. Thus, eigenfunctions v_j's of A_χ play the role of spatial modes.

Sensitivity For the simplicity of the exposition, let us suppose that only one of the parameters in $\bar{a}$ is uncertain. We shall denote it by α. As in the previous sections, it is not necessary to assume the differentiability of $y_j(t, \bar{a})$ w.r.t α, since it follows from the analyticity of $y_j(t, \bar{a})$ w.r.t. $\bar{a}$. Denoting $w_j(t) = \frac{\partial y_j(t,\bar{a})}{\partial \alpha}$, imposing the same assumptions as in Sect. 2 and differentiating both sides of (31) we obtain

$$\dot{w}_j(t) = -\lambda_j(\bar{a})\, w_j(t) - b_j^{(\alpha)}\, y_j(t, \bar{a}), \quad w_j(0) = 0, \tag{32}$$

where $b_j^{(\alpha)}$ denotes this element of $\bar{b}_j$ that has the same position as α in $\bar{a}$. Notice that the series in (30) is uniformly convergent w.r.t. elements of $\bar{a}$. Thus, it can be differentiated term-by-term and for $w(\chi, t) \stackrel{def}{=} \frac{\partial q(\chi,t,\bar{a})}{\partial \alpha}$ we obtain

$$w(\chi, t) = \sum_{j=1}^{\infty} v_j(\chi)\, w_j(t). \tag{33}$$

Remark 8 One can obtain a PDE for $w(\chi, t)$ by differentiating (26) w.r.t. α, but this way of obtaining $w(\chi, t)$ is mathematically more subtle.

Optimal control with reduced sensitivity for PDE's The next ingredient that needs to be specified is a control signal $U(\chi, t)$. We shall consider two cases that differ, in that a control action is fully flexible in space or not.

Case I: When it is too expensive to influence a system at each spatial point $\chi \in \Omega$. In such cases it is frequently assumed that $U(\chi, t) = g(\chi)\, u(t)$, where $g(\chi)$ is a given function that describes a control action in space, while $u(t)$ is a control signal to be selected. By selecting $g(\chi)$ appropriately, one can cover the cases of zonal control, boundary control and approximate point-wise control. When more than one control action is admitted, a sum of such expressions is applied, but we skip this generalization here.

Case II: When it is possible to select $U(\chi, t)$, acting over the whole spatial domain Ω. Available new tools such as laser heating of surfaces and micro-electro-mechanical systems (MEMS) provide examples when such control action is (at least approximately) realizable.

Now, we are ready to state the problem of optimal control of the system (26) with reduced sensitivity to uncertainties in α.

Case I Select $u^*(t)$ which minimizes the following functional

$$J_I(u) = \int_0^{\infty} \int_{\Omega} \left[q^2(\chi, t, \bar{a}) + w^2(\chi, t) + g^2(\chi)\, u^2(t)\right] d\chi\, dt, \tag{34}$$

where $q(\chi, t, \bar{a})$ is the solution of (26) and $w(\chi, t)$ is defined by (33) and (32).

The completeness in $L_2(\Omega)$ and orthogonality of eigenfunctions v_j allows us to reformulate our problem equivalently. Namely, $J_I(u)$ can be expressed as

$$J_I(u) = \int_0^{\infty} \left[\sum_{j=1}^{\mathcal{J}} \left(y_j^2(t, \bar{a}) + w_j^2(t)\right) + \gamma\, u^2(t)\right] dt, \tag{35}$$

where $\gamma = \int_\Omega g^2(\chi)\, d\chi$, $y_j(t, \bar{a})$ and $w_j(t)$ are given by (31) and (32), respectively, while $\mathcal{J} = \infty$ for the strict equivalence of (35) and (34).

Corollary 1 *Define the infinite dimensional vectors* $x(t) = [y_1(t, \bar{a}), y_2(t, \bar{a}), \ldots]^T$ *and* $w(t) = [w_1(t), w_2(t), \ldots]^T$ *as well as matrix* $\bar{A}$ *of the form* (18)*, where matrix* $A(\bar{a})$ *is a* *diagonal matrix with entries* $-\lambda_j(\bar{a}),\ j = 1, 2, \ldots, \mathcal{J}, \ldots$*. Let matrix D be also diagonal with entries* $b_j^{(\alpha)},\ j = 1, 2, \ldots,$ *while* $\bar{b}^T = [< g, v_1 >, < g, v_2 >, \ldots, < g, v_{\mathcal{J}} >, \ldots, 0, 0, \ldots]$.

The problem of the optimal control for (34)*, with* $\mathcal{J} = \infty$ *in* (35)*, can be equivalently reduced to solving the problem* (19)*,* (20) *for a system with state vector of infinite length.*

We refer the reader to Bensoussan et al. (2007) Part IV Sect. 7 for the results concerning the existence of a solution of the Riccati equation and its properties in control problems for PDE's. Notice that the case considered here is the simplest one, since under (A1)–(A3) operator $A_\chi(\bar{a})$ generates an analytic semigroup w.r.t. time variable.

Remark 9 Fortunately, in practice one can select a finite $\mathcal{J}$. The reason is in that in typical cases $\lambda_j(\bar{a})$ grows as quickly as $c\, j^2$ for a certain constant $c > 0$. Thus, the components $y_j(t, \bar{a})$ of the solution of (26) decay as quickly as $\exp(-c\, j^2\, t)$ and it suffices to consider only the first few of them. Then, one can apply Theorem 1 with a finite $\mathcal{J}$. It remains to solve (19), (20) with truncated matrices defined in Corollary 1.

Case II Select $U^*(\chi, t)$ which minimizes the following functional

$$J_{II}(u) = \int_0^\infty \int_\Omega \left[q^2(\chi, t, \bar{a}) + \vartheta_1\, w^2(\chi, t) + \vartheta_2\, U^2(\chi, t)\right] d\chi\, dt, \tag{36}$$

where $\vartheta_1 > 0,\ \vartheta_2 > 0$ are selected weights, $q(\chi, t, \bar{a})$ is the solution of (26) and $w(\chi, t)$ is defined by (33) and (32). The equivalent formulation of (36) is the following:

$$J_{II}(u) = \int_0^\infty \left[\sum_{j=1}^{\mathcal{J}} y_j^2(t, \bar{a}) + \vartheta_1\, w_j^2(t) + \vartheta_2\, u_j^2(t)\right] dt, \tag{37}$$

where $\mathcal{J} = \infty$ for the strict equivalence of (37) and (36) (see also Remark 9). In (37) $u_j(t)$ is defined as follows $u_j(t) = \int_\Omega U(\chi, t)\, v_j(\chi)\, d\chi,\ j = 1, 2, \ldots$. Then, $y_j(t)$'s and $w_j(t)$'s are the solutions of the following equations:

$$\dot{y}_j(t, \bar{a}) = -\lambda_j(\bar{a})\, y_j(t, \bar{a}) + u_j(t), \qquad y_j(0) = \int_\Omega \varphi(\chi)\, v_j(\chi)\, d\chi, \tag{38}$$

$$\dot{w}_j(t) = -\lambda_j(\bar{a})\, w_j(t) - b_j^{(\alpha)}\, y_j(t, \bar{a}), \qquad w_j(0) = 0. \tag{39}$$

Notice that the minimization of (37) under constraints (38) and (39) can be done separately for each $j = 1, 2, \ldots \mathcal{J}$ and we obtain.

Corollary 2 *The optimal control in Case II can be obtained as follows.*

(a) Solve the following sequence of problems: find the square integrable function $u_j^*(t)$ *that minimizes*

$$\int_0^\infty \left[y_j^2(t, \bar{a}) + \vartheta_1\, w_j^2(t) + \vartheta_2\, u_j^2(t)\right] dt, \quad j = 1, 2, \ldots \mathcal{J}. \tag{40}$$

under constraints (38) *and* (39)*. To this end apply Theorem 1. Note that the stabilizability of* (38) *is ensured by the fact that* $\lambda_j(\bar{a}) > 0$*,*

(b) Form the optimal solution as $U^*(\chi, t) = \sum_{j=1}^{\mathcal{J}} u_j^*(t)\, v_j(\chi)$*.*

Remark 10 Notice that if one would like to realize the optimal control for (26) as described in Corollarys 1 and 2, but in the closed loop, then the access to observations of $y_1(t, \bar{a}), y_2(t, \bar{a}), \ldots, y_{\mathcal{J}}(t, \bar{a})$ is necessary. These signals can be available, if one can observe $q(\chi, t, \bar{a})$ for $\chi \in \Omega$, e.g., using an infra-red camera (Rafajłowicz and Rafajłowicz 2010). Otherwise, $y_j(t, \bar{a}) = \int_\Omega q(\chi, t, \bar{a})\, v_j(\chi)\, d\chi$ has to be approximated by a quadrature rule from observations $q(\chi_n, t, \bar{a})$, provided by pointwise sensors (e.g., thermocouples) located at spatial points $\chi_n,\ n = 1, 2, \ldots N$.

In order to apply Corollary 2 the following procedure can be used.

Step 1 Verify conditions (A1)–(A3) for $A_\chi(\bar{a})$ [condition (A3)] is easy to verify, (A1) and (A2) usually require the necessity of finding appropriate results in mathematical books). Calculate eigenfunctions $v_j(\chi)$ (for typical second and fourth order differential operators they are known)
Step 2 Calculate elements $b_j^{(i)}$ of $\bar{b}_j$ as eigenvalues of $P_\chi^{(i)}\, v_j(\chi) = b_j^{(i)}\, v_j(\chi)$. Select a parameter α among the elements of $\bar{a}$ for which our system should be less sensitive and denote the corresponding $b_j^{(i)}$ as b_j^α.
Step 3 Select the largest number of eigenfunctions $\mathcal{J}$ that are taken into account and nominal values of parameters $\bar{a}^0$. Solve the sequence of optimal control problems described in Corollary 2 (use the well known tools, e.g., the Matlab package `lqry`). As a result, the optimal gains θ_j^* for y_j's and τ_j^* for w_j^* of modal controllers are obtained.
Step 4 For the synthesis of the optimal closed loop controller use the hints contained in Remark 10. At each time instant observe $q(\chi_n, t, \bar{a}),\ n = 1, 2, \ldots N$ and for each $j = 1, 2, \ldots, \mathcal{J}$ calculate approximations $\hat{y}_j(t, \bar{a})$ of $y_j(t, \bar{a})$ as $\hat{y}_j(t, \bar{a}) = \sum_{n=1}^{N} \omega_n\, q(\chi_n, t, \bar{a})\, v_j(\chi_n)$, where ω_n's are weights of the selected quadrature.
Step 5 Solve Eq. (39), substituting $\hat{y}_j(t, \bar{a})$ instead of $y_j(t, \bar{a})$.
Step 6 Calculate the input for the system as $\sum_{j=1}^{\mathcal{J}} (\theta_j^*\, \hat{y}_j(t, \bar{a}) + \tau_j^*\, w_j(t))\, v_j(\chi)$.

Example: (2+1)-D system Consider the heat transfer equation a special case of (26) that describes one dimensional rod $\Omega = (0, \pi)$ with its end points kept at a constant temperature, assumed to be zero for simplicity. Then, (27) specializes to:

$$A_\chi(a)\, f(\chi) = a\, \frac{\partial^2 f(\chi)}{\partial \chi^2}, \qquad f(0) = f(\pi) = 0, \tag{41}$$

where $a > 0$ is constant. We refer the reader to Yosida (1981) Chap. XIV for a precise definition of the domain of operator (41) and for proofs concerning the fact that (A1), (A2) holds for it.

Let us note that the eigenfunctions of operator (41) have the form: $v_j(\chi) = \sqrt{2/\pi}\, \sin(j\, \chi)$, while the corresponding eigenvalues are given by $\lambda_j(a) = a\, j^2$, $j = 1, 2, \ldots$. Thus, in (39) $b_j^{(\alpha)} = j^2$, since in this case $\alpha = a$. Hence, (A3) also holds.

Parameter a is a material dependent constant and it is uncertain. Its nominal value is set to $a = 0.1$ in the simulations below. If a temperature of the rod falls below zero, it should be unfrozen, i.e., heated by sources with intensity $U(\chi, t)$ along the rod. That is we consider Case II.

Suppose that the rod is frozen at $t = 0$ and the initial condition for (26) has the form $\varphi(\chi) = 10\left(\chi - \frac{\pi}{2}\right)^4 - 60$. One can easily verify that for this initial condition $y_j(0) = \int_0^\pi \varphi(\chi)\, \sqrt{2/\pi}\, \sin(j\, \chi)\, d\chi = 0$ for $j = 2, 4, 6, \ldots$. Thus, only odd modes have to be controlled, since for j even we will have $y_j(t) \equiv 0$.

In order to calculate $U^*(\chi, t)$ the following steps have been performed.

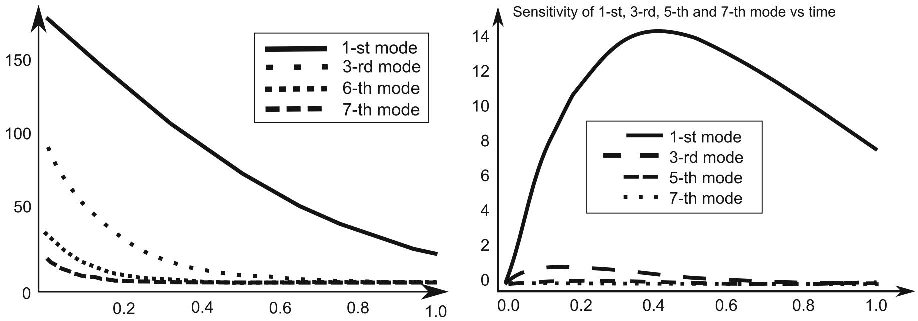

Fig. 4 Control signals (*left panel*) and state sensitivities versus time (*right panel*) for modes $k = 1, 3, 5, 7$ in an example of a (2+1)-D system

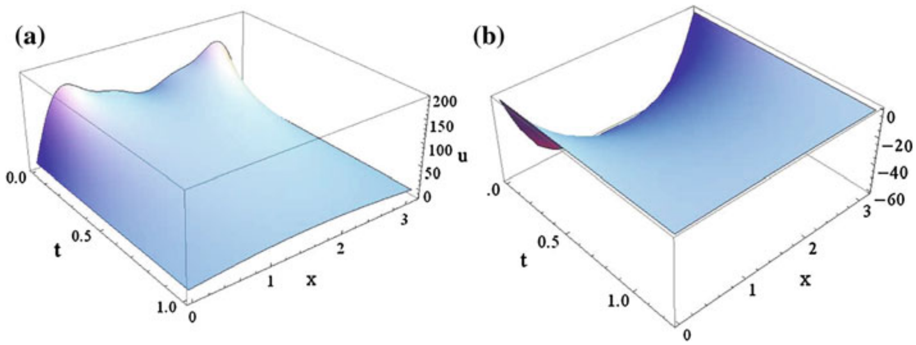

Fig. 5 Distributed control $U^*(\chi, t)$ (*left panel*) and the system state $q^*(\chi, t)$ (*right panel*) in example of a (2+1)-D system

- Set $\vartheta_1 = 4,\ \vartheta_2 = 1$ in (37) and apply Corollary 2.
- Solve the sequence of the Riccati equations (see Theorem 1) for $j = 1, 3, 5, 7$, corresponding to (40), (38) and (39). To this end, `lqry MATLAB` function has been applied to the extended systems (38) and (39), separately for $j = 1, 3, 5, 7$.
- Form $u_j^*(t),\ j = 1, 3, 5, 7$, according to (21) in Theorem 1 and calculate $U^*(\chi, t)$ as described in Corollary 2 b).

Obtained $u_j^*(t)$'s are shown in Fig. 4 (left panel). The corresponding $U^*(\chi, t)$ is presented in Fig. 5 (left panel).

In order to verify how the control system behaves when instead of nominal parameter value $a = 0.1$ we have $a = 0.05$, the system (26) with $a = 0.05$ in (41) was solved and its solution $q(\chi, t)$ is shown in Fig. 5 (right panel). The corresponding sensitivities $w_j(t),\ j = 1, 3, 5, 7$ are depicted in Fig. 4 (right panel). The following conclusions can be drawn:

(1) the control system with reduced sensitivity that was designed for $a = 0.1$ is able to undertake a proper control action when the uncertain parameter change is 50 % (to $a = 0.05$),
(2) it suffices to use several first modes, since both the control signals and the sensitivities of higher modes are low and decay quickly,
(3) after a short time there are no essential changes of the spatial shape of $U^*(\chi, t)$, thus one can consider a one zone control for this system by designing it as described in Case I and selecting $g(\chi)$ as $U^*(\chi, 0.5)$.

Remark 11 If in the above example we consider a plate with homogeneous boundary conditions of the first kind, then instead of operator (41) we use operator (28). Its eigenfunctions have the form: $\sin(j_1\,\chi_1)\,\sin(j_2\,\chi_2),\ j_1,\ j_2 = 1,\ 2, \ldots$. The corresponding eigenvalues are given by: $-a_1\,j_1^2 - a_2\,j_2^2 + a_3,\ j_1,\ j_2 = 1,\ 2, \ldots$ and the rest of a controller design is the same as above.

5 Concluding remarks

The proposed extension of an *nD* system with uncertain parameters, considered as additional independent variables, may have a large number of applications, including the selection of the most sensitizing inputs in order to increase the accuracy of the system identification. Another possibility, namely selecting input signals that are as much insensitive as possible to parameter changes, taking into account also the control goal, has been discussed in this paper in more detail.

The results for systems described by ODE's are similar to those in Kreindler (1968). They were stated for one parameter for the sake of simplicity, but they can be smoothly extended to several uncertain parameters and to more than 1-st derivative of sensitivity functions, but then the number of the optimal controller inputs grows linearly with the number of uncertain parameters. There are no essential difficulties to obtain similar results for systems with several inputs. The only formal difference is in the necessity of rewriting the Riccati equations in Theorem 1 to the multi input case.

The main contribution of this paper is in extending the approach for ODE's from Kreindler (1968) to the desensitization of systems described by a class of PDE's. It is proved that the optimal controller is also linear in additional sensitivity signals and it can be designed by solving a sequence of properly formulated control tasks for ODE's. It is also possible to extend these results to a certain class of *nD* systems described by difference equations.

The next class of systems that allows for a generalization of our results consists of linear systems with delays that are considered as uncertain parameters. It is known that variability of delays can essentially influence optimal control systems (see Styczeń 1989). However, one should notice that the sensitivity equations for uncertain delay parameters require different derivations than presented in this paper.

It would also be of interest to apply a feedback from sensitivity signals in control problems for repetitive processes (see Rogers et al. 2007; Rabenstein and Steffen 2012)—in which uncertainties can be caused, e.g., by varying parameters of raw materials—and for iterative learning control problems (Cichy et al. 2012). In the former important class of problems one can take additional advantage of attenuating uncertainties along a pass at hand by using additional information on uncertainties gained from a previous pass (or previous passes).

Furthermore, our main idea can be generalized to nonlinear systems described by PDE's, with the added bonus of linearity of sensitivity equations, but then the modal expansion cannot always be used.

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