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# On the existence of an optimal solution of the Mayer problem governed by 2D continuous counterpart of the Fornasini-Marchesini model

**Dorota Bors · Marek Majewski**

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**Abstract** In the paper the optimization problem described by some nonlinear hyperbolic equation being continuous counterpart of the Fornasini-Marchesini model is considered. A theorem on the existence of at least one solution to this hyperbolic PDE is proved and some properties of the set of all solutions are established. The existence of a solution to an optimization problem under appropriate assumptions is the main result of this paper. Some application of the obtained results to the process of gas filtration is also presented.

**Keywords** Mayer problem · Continuous counterpart of the Fornasini-Marchesini model · Existence of optimal solutions

## 1 Introduction

In this paper we consider an optimal control problem governed by system of hyperbolic equations of the form

$$\frac{\partial^2 z}{\partial x \partial y}(x, y) = f\left(x, y, \frac{\partial z}{\partial x}(x, y), \frac{\partial z}{\partial y}(x, y), z(x, y), u(x, y)\right) \tag{1}$$

for almost every $(x, y) \in P := [0, 1] \times [0, 1]$ with the cost indicator

$$J(z) = \int_0^1 F\left(t, \varphi'(t), \varphi''(t), \psi'(t), \psi''(t)\right) dt + g\left(\varphi(0), \varphi'(0), \psi'(0)\right),$$

where $\varphi(t) = z(t, 0)$ and $\psi(t) = z(0, t)$ for every $t \in [0, 1]$.

D. Bors · M. Majewski (✉)
Faculty of Mathematics and Computer Science, University of Lodz, Lodz, Poland
e-mail: marmaj@math.uni.lodz.pl

D. Bors
e-mail: bors@math.uni.lodz.pl

System (1) can be viewed as a continuous nonlinear version of the Fornasini-Marchesini model (cf. Fornasini and Marchesini 1978/1979; Kaczorek 1985; Klamka 1991), which is well known in the theory of discrete multidimensional systems. It should be underlined that such discrete systems play an important role in the theory of automatic control (cf. Fornasini and Marchesini 1976). Moreover, continuous systems of the form specified by (1) can be used for modelling of the process of gas absorption (cf. Idczak et al. 1994; Tikhonov and Samarski 1990) for which some numerical results can be found in Rehbock et al. (1998). For related results on Fornasini-Marchesini models one can see Cheng et al. (2011), Yang et al. (2007), Idczak (2008).

Furthermore, it should be noted that system (1) was investigated in many papers apart from the aforementioned ones. Specifically, the problem of the existence and uniqueness of solutions to (1) with boundary conditions $\varphi(t) = z(t, 0)$ and $\psi(t) = z(0, t)$ has been proved for the linear case in Idczak and Walczak (2000) and for the nonlinear case in Idczak and Walczak (1994). Moreover, some results establishing the existence of optimal solutions for the problem governed by (1) can be found in Idczak and Walczak (1994) for the case of the Lagrange problem with controls with bounded variation, in Idczak et al. (1994) for the case of the problem with the cost of rapid variation of control, and in Majewski (2006) for the case of the Lagrange problem with integrable controls. It should be underlined that both in Idczak and Walczak (2000) and Idczak and Walczak (1994) zero initial conditions were considered. While in this paper the problem with general initial conditions are treated. Our considerations involve the minimization of the cost functional which depends on the boundary values of the solutions to the PDE. The situation in which the boundary data appear in the cost functional is referred to as the classical Mayer problem for ODEs. Our extension can be seen as a new contribution towards the Mayer problem governed by PDEs which can be useful in many practical applications.

The paper is organized as follows. In Sect. 2, the optimization problem is formulated and the space of solutions is defined. Section 3 is devoted to formulation of the required assumptions. Next, in Sect. 4, the theorem on the existence of a solution to the system (1) is proved and some properties of the set of all solutions are stated. Subsequently, the main result of the paper can be proved, namely the theorem stating that under some assumptions optimal control problem possesses at least one solution. Finally, in Sect. 5, an application of the obtained results to the process of gas filtration is presented.

## 2 Formulation of the problem

The problem under consideration is as follows:

Find a minimum of the functional

$$J(z) = \int_0^1 F\left(t, \varphi'(t), \varphi''(t), \psi'(t), \psi''(t)\right) dt + g\left(\varphi(0), \varphi'(0), \psi'(0)\right), \tag{2}$$

subject to

$$\frac{\partial^2 z}{\partial x \partial y}(x, y) = f\left(x, y, \frac{\partial z}{\partial x}(x, y), \frac{\partial z}{\partial y}(x, y), z(x, y), u(x, y)\right)$$
$$\text{for a.e. } (x, y) \in P := [0, 1] \times [0, 1] \tag{3}$$

where $\varphi(t) = z(t, 0)$ and $\psi(t) = z(0, t)$ for $t \in [0, 1]$,

$$z \in \mathcal{Z} := \left\{ z \in AC\left(P, \mathbb{R}^N\right) : z(\cdot, 0), z(0, \cdot) \in H^2\left([0,1], \mathbb{R}^N\right) \right\}, \tag{4}$$

$$u \in \mathcal{U} := \left\{ u : P \to \mathbb{R}^M : u \text{ is measurable and } u(x, y) \in \mathbb{U} \text{ for a.e. } (x, y) \in P \right\} \tag{5}$$

where $\mathbb{U} \subset \mathbb{R}^M$ is a given compact set.

In the definition of $\mathcal{Z}$ given in (4), $AC\left(P, \mathbb{R}^N\right)$ denotes the set of absolutely continuous functions of two variables defined on $P$. A function $z : P \to \mathbb{R}$ is said to be absolutely continuous on $P$ if

1. the associated function $\mathcal{F}_z$ of an interval defined by the formula

$$\mathcal{F}_z([x_1, x_2] \times [y_1, y_2]) = z(x_2, y_2) - z(x_1, y_2) + z(x_1, y_1) - z(x_2, y_1)$$

for all intervals $[x_1, x_2] \times [y_1, y_2] \subset P$ is an absolutely continuous function of an interval (see Łojasiewicz (1988) for details),
2. the functions $z(\cdot, 0)$ and $z(0, \cdot)$ are absolutely continuous on $[0, 1]$.

A function $z = (z_1, \ldots, z_N) : P \to \mathbb{R}^N$ is said to be absolutely continuous on $P$ if all coordinates functions $z_i$ are absolutely continuous on $P$ for $i = 1, \ldots N$. In the paper Walczak (1987), the author proved that a function $z : P \to \mathbb{R}^N$ is absolutely continuous if and only if there exist functions $l_z \in L^1\left(P, \mathbb{R}^N\right)$, $l_z^1, l_z^2 \in L^1\left([0,1], \mathbb{R}^N\right)$, and a constant $c \in \mathbb{R}^N$ such that

$$z(x, y) = \int_0^x \int_0^y l_z(s, t)\, ds dt + \int_0^x l_z^1(s)\, ds + \int_0^y l_z^2(t)\, dt + c \tag{6}$$

for all $(x, y) \in P$. Moreover, an absolutely continuous function $z$ having the representation (6) possesses, in the classical sense, the partial derivatives

$$\frac{\partial z}{\partial x}(x, y) = \int_0^y l_z(x, t)\, dt + l_z^1(x),$$

$$\frac{\partial z}{\partial y}(x, y) = \int_0^x l_z(s, y)\, ds + l_z^2(y),$$

$$\frac{\partial^2 z}{\partial x \partial y}(x, y) = l_z(x, y)$$

for a.e. $(x, y) \in P$.

It is obvious that $z \in \mathcal{Z}$ if and only if it has the following representation

$$z(x, y) = \int_0^x \int_0^y l(s, t)\, ds dt + \varphi(x) + \psi(y) - z(0, 0) \text{ for } (x, y) \in P, \tag{7}$$

where $l \in L^1\left(P, \mathbb{R}^N\right), \varphi, \psi \in H^2\left([0,1], \mathbb{R}^N\right)$ and $\varphi(0) = \psi(0)$. Furthermore, we have that $\varphi(x) = z(x, 0)$, $\psi(y) = z(0, y)$ for $x, y \in [0, 1]$ and $z$ possesses derivatives $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x \partial y}(x, y) = l(x, y)$, $\frac{\partial z}{\partial x}(x, y) = \int_0^y \frac{\partial^2 z}{\partial x \partial y}(x, t)\, dt + \varphi'(x)$, $\frac{\partial z}{\partial y}(x, y) = \int_0^x \frac{\partial^2 z}{\partial x \partial y}(s, y)\, ds + \psi'(y)$ for a.e. $(x, y) \in P$.

By $H^2\left([0,1], \mathbb{R}^N\right)$ we denote the space of absolutely continuous functions defined on $[0, 1]$ such that $x'$ is absolutely continuous and $x'' \in L^2\left([0,1], \mathbb{R}^N\right)$.

## 3 Basic assumptions

In the paper we shall use the following assumptions.

(A1) The function

$$P \ni (x, y) \mapsto f(x, y, z_1, z_2, z, u) \in \mathbb{R}^N$$

is measurable for $(z_1, z_2, z, u) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M$ and the function

$$\mathbb{R}^M \ni u \mapsto f(x, y, z_1, z_2, z, u) \in \mathbb{R}^N$$

is continuous for $(z_1, z_2, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $(x, y) \in P$.

(A2) There exists a constant $L > 0$ such that

$$|f(x, y, z_1, z_2, z, u) - f(x, y, w_1, w_2, w, u)| \le L(|z-w| + |z_1 - w_1| + |z_2 - w_2|)$$

for $(z_1, z_2, z), (w_1, w_2, w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, $u \in \mathbb{U}$ and a.e. $(x, y) \in P$.

(A3) There exists $b > 0$ such that

$$|f(x, y, 0, 0, 0, u)| \le b$$

for a.e. $(x, y) \in P$ and $u \in \mathbb{U}$.

(A4) The function

$$[0, 1] \ni t \mapsto F(t, v) \in \mathbb{R}^N$$

is measurable for every $v \in \mathbb{R}^{4N}$ and the function

$$\mathbb{R}^{4N} \ni v \mapsto F(t, v) \in \mathbb{R}^N$$

is continuous for a.e. $t \in [0, 1]$.

(A5) For every bounded set $B \subset \mathbb{R}^{4N}$ there is a function $\upsilon_B \in L^1\left([0, 1], \mathbb{R}^+\right)$ such that

$$F(t, v) \le \upsilon_B(t)$$

for a.e. $t \in [0, 1]$ and every $v \in B$.

(A6) There are positive constants $\alpha_i$ and functions $\beta_i \in L^2([0, 1], \mathbb{R})$, $\gamma_i \in L^1([0, 1], \mathbb{R})$, $i = 1, 2, 3, 4$ such that

$$F(t, v_1, v_2, v_3, v_4) \ge \sum_{i=1}^{4} \left(\alpha_i |v_i|^2 + \beta_i(t)|v_i| + \gamma_i(t)\right)$$

for a.e. $t \in [0, 1]$ and every $v_i \in \mathbb{R}^N$, $i = 1, 2, 3, 4$.

(A7) The function $g : \mathbb{R}^{3N} \to \mathbb{R}$ is lower semicontinuous and coercive, i.e. $g(v) \to \infty$ if $|v| \to \infty$.

## 4 Existence of solution and the main result

To begin with we shall prove the theorem on the existence of solution to the system (3). We also formulate some properties of the set of all solutions.

**Theorem 1** *Let assumptions (A1)–(A4) be satisfied. Then, for each control $u \in \mathcal{U}$, and each $\varphi, \psi \in H^2\left([0, 1], \mathbb{R}^N\right)$ such that $\varphi(0) = \psi(0)$ there exists a unique solution $z_{u,\varphi,\psi} \in \mathcal{Z}$ to (3) satisfying condition $z_{u,\varphi,\psi}(x, 0) = \varphi(x)$, and $z_{u,\varphi,\psi}(0, y) = \psi(y)$ for $x, y \in [0, 1]$.*

*Moreover, for any* $c > 0$ *there exists* $\rho > 0$ *such that if* $\varphi, \psi \in H^2([0,1], \mathbb{R}^N)$, $\varphi(0) = \psi(0)$ *and* $|\varphi(x)|, |\psi(x)|, |\varphi'(x)|, |\psi'(x)| \le c$ *for* $x \in [0,1]$, *then*

$$\left|\frac{\partial^2 z_{u,\varphi,\psi}}{\partial x \partial y}(x,y)\right|, \left|\frac{\partial z_{u,\varphi,\psi}}{\partial x}(x,y)\right|, \left|\frac{\partial z_{u,\varphi,\psi}}{\partial y}(x,y)\right|, \left|z_{u,\varphi,\psi}(x,y)\right| \le \rho$$

*for a.e.* $(x,y) \in P$ *and* $u \in \mathcal{U}$.

*Proof* For a fixed $u \in \mathcal{U}$ and $\varphi, \psi \in H^2([0,1], \mathbb{R}^N)$ such that $\varphi(0) = \psi(0)$, consider the operator $T : L^1(P, \mathbb{R}^N) \to L^1(P, \mathbb{R}^N)$ defined by

$$T(l)(x,y) = f\left(x, y, \int_0^y l(x,t)\,dt + \varphi'(x), \int_0^x l(s,y)\,ds + \psi'(y), \right.$$
$$\left. \int_0^x \int_0^y l(x_1,y_1)\,dx_1 dy_1 + \varphi(x) + \psi(y) - \varphi(0), u(x,y)\right).$$

It can be proved by applying the Banach Contraction Principle, in the same manner as in Idczak and Walczak (1994), that the operator $T$ possesses a unique fixed point $\tilde{l} \in L^1(P, \mathbb{R}^N)$ and consequently, if we define

$$z_{u,\varphi,\psi}(x,y) := \int_0^x \int_0^y \tilde{l}(s,t)\,ds dt + \varphi(x) + \psi(y) - \varphi(0), \qquad (x,y) \in P$$

we have that $z_{u,\varphi,\psi} \in \mathcal{Z}$ is the unique solution to (3) satisfying conditions $\varphi(x) = z_{u,\varphi,\psi}(x,0)$ and $\psi(y) = z_{u,\varphi,\psi}(0,y)$ for $x, y \in [0,1]$.

Moreover, from the proof of Banach Contraction Principle it follows that for $l_n := T^n(0)$, we get that $l_n \to \tilde{l}$ in $L^1(P, \mathbb{R}^N)$. Next, for $k \ge 2$, by (A2)-(A3), it is possible to show that $|l_k(x,y) - l_{k-1}(x,y)|$ is bounded by a sum of $3^{k-1}$ terms each of them is a product of $L^{k-1}$ and some multiple integral. In each of this multiple integral we have at least $\left[\frac{k-2}{2}\right]$ integrations with respect to variable which appears as the upper limit of the integration. Therefore, using the Cauchy formula for multiple integral we obtain

$$|l_k(x,y) - l_{k-1}(x,y)| \le (3L)^{k-1} \frac{c_1}{\left[\frac{k-2}{2}\right]!}$$

for a.e. $(x,y) \in P$ and $k \ge 2$, where $c_1$ is independent of $(x,y)$ and $k$. Passing then, if necessary, to a subsequence, we get the following estimate

$$\left|\tilde{l}(x,y)\right| \le \lim_{j\to\infty} \sum_{k=2}^{j} |l_k(x,y) - l_{k-1}(x,y)| + |l_1(x,y)| \le \sum_{k=2}^{\infty} c_1 \frac{(3L)^{k-1}}{\left[\frac{k-2}{2}\right]!} + c_2$$

for a.e. $(x,y) \in P$, where $c_2$ is independent of $(x,y)$ and $k$. Eventually,

$$\left|\frac{\partial^2 z_{u,\varphi,\psi}}{\partial x \partial y}(x,y)\right| \le \rho_1 < \infty,$$

$$\left|z_{u,\varphi,\psi}(x,y)\right| \le \int_0^x \int_0^y \left|\tilde{l}(s,t)\right| ds dt + |\varphi(x)| + |\psi(y)| + |\varphi(0)| \le \rho_1 + 3c := \rho,$$

$$\left|\frac{\partial z_{u,\varphi,\psi}}{\partial x}(x,y)\right| \leq \int_0^y \left|\tilde{l}(x,t)\right| dt + \left|\varphi'(x)\right| \leq \rho_1 + c \leq \rho$$

and

$$\left|\frac{\partial z_{u,\varphi,\psi}}{\partial y}(x,y)\right| \leq \int_0^x \left|\tilde{l}(s,y)\right| ds + \left|\psi'(y)\right| \leq \rho_1 + c \leq \rho$$

for a.e. $(x, y) \in P$, which completes the proof. □

Theorem 1 forms the basis for the proof of the main result of this paper.

**Theorem 2** *Assume (A1)–(A7). If the set*

$$Q(x,y,z) := \left\{(\zeta_1,\zeta_2,\zeta_3) \in \mathbb{R}^{3N} : \exists\, u \in \mathbb{U} \textit{ such that } \zeta_1 = f(x,y,\zeta_2,\zeta_3,z,u)\right\}$$

*is convex for a.e.* $(x, y) \in P$ *and any* $z \in \mathbb{R}^N$, *then problem (2–5) possesses at least one solution.*

*Proof* Let $\{z_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for $J$. By (A6–A7), there is a constant $\bar{c} > 0$ such that

$$\int_0^1 \left|\varphi_n'(x)\right|^2 dx, \int_0^1 \left|\psi_n'(x)\right|^2 dx, \int_0^1 \left|\varphi_n''(x)\right|^2 dx, \int_0^1 \left|\psi_n''(x)\right|^2 dx,$$
$$\left|\varphi_n(0)\right|, \left|\varphi_n'(0)\right|, \left|\psi_n(0)\right|, \left|\psi_n'(0)\right| \leq \bar{c},$$

for $n \in \mathbb{N}$, where $\varphi_n(t) = z_n(t, 0)$ and $\psi_n(t) = z_n(0, t)$. Therefore,

$$\left|\varphi_n(x)\right| \leq \int_0^x \left|\varphi_n'(s)\right| ds + \left|\varphi_n(0)\right| \leq \int_0^1 \left|\varphi_n'(s)\right| ds + \bar{c} \leq \sqrt{\int_0^1 \left|\varphi_n'(s)\right|^2 ds} + \bar{c} \leq c,$$

where $c > 0$ and similarly $\left|\psi_n(x)\right|, \left|\varphi_n'(x)\right|, \left|\psi_n'(x)\right| \leq c$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. By virtue of Theorem 1, we have

$$\left|\frac{\partial^2 z_n}{\partial x \partial y}(x,y)\right|, \left|\frac{\partial z_n}{\partial x}(x,y)\right|, \left|\frac{\partial z_n}{\partial y}(x,y)\right|, \left|z_n(x,y)\right| \leq \rho \tag{8}$$

for a.e. $(x, y) \in P$ and $n \in \mathbb{N}$, thus $\left\{\frac{\partial^2 z_n}{\partial x \partial y}\right\}_{n\in\mathbb{N}}$, $\left\{\frac{\partial z_n}{\partial x}(\cdot, 0)\right\}_{n\in\mathbb{N}}$, $\left\{\frac{\partial z_n}{\partial y}(0, \cdot)\right\}_{n\in\mathbb{N}}$ are equiabsolutely integrable and therefore $\{z_n\}_{n\in\mathbb{N}}$ is equiabsolutely continuous (see Idczak and Walczak 2000, Th. 3.3).

Next, applying the Arzelà-Ascoli theorem (see Idczak and Walczak 2000, Th. 3.4) and the Dunford-Pettis theorem (see Cesari 1983, Th. 10.3.i), we may assume that $z_n \rightrightarrows z_0 \in \mathcal{Z}$ on $P$ uniformly, and $\frac{\partial^2 z_n}{\partial x \partial y} \rightharpoonup \frac{\partial^2 z_0}{\partial x \partial y}$, $\frac{\partial z_n}{\partial x} \rightharpoonup \frac{\partial z_0}{\partial x}$, $\frac{\partial z_n}{\partial y} \rightharpoonup \frac{\partial z_0}{\partial y}$ weakly in $L^1\left(P, \mathbb{R}^N\right)$ as $n \to \infty$. Since $z_n$ is a solution to (3), then

$$\left(\frac{\partial^2 z_n}{\partial x \partial y}(x,y), \frac{\partial z_n}{\partial x}(x,y), \frac{\partial z_n}{\partial y}(x,y)\right) \in Q\left(x, y, z_n(x,y)\right)$$

for a.e. $(x, y) \in P$. Consequently, by Filippov's Lemma (Cesari 1983, Th. 10.6.i), we have

$$\left( \frac{\partial^2 z_0}{\partial x \partial y}(x, y), \frac{\partial z_0}{\partial x}(x, y), \frac{\partial z_0}{\partial y}(x, y) \right) \in Q(x, y, z_0(x, y))$$

for a.e. $(x, y) \in P$. Furthermore, from the implicit function theorem (Kisielewicz 1991, Th. 3.12) we infer that there exist a control $u_0 \in \mathcal{U}$ such that

$$\frac{\partial^2 z_0}{\partial x \partial y}(x, y) = f\left( x, y, \frac{\partial z_0}{\partial x}(x, y), \frac{\partial z_0}{\partial y}(x, y), z_0(x, y), u_0(x, y) \right) \qquad \text{for a.e. } (x, y) \in P.$$

Moreover, since $z_n \rightrightarrows z_0$, then $\varphi_n(\cdot) = z_n(\cdot, 0) \rightrightarrows z_0(\cdot, 0) =: \varphi_0(\cdot)$ and $\psi_n(\cdot) = z_n(0, \cdot) \rightrightarrows z_0(0, \cdot) =: \psi_0(\cdot)$. Finally, by (A4), (A5), and invoking the Lebesgue dominated convergence theorem, we get that $z_0$ is optimal, which completes the proof. □

## 5 Example of application

Consider a gas filter in the form of a pipe filled up with an appropriate absorbent. A mixture of gas and air is pressed through the filter with a speed $v(x, t) > a > 0$, where $x$ is a distance from the inlet of the pipe, $t$ is a time. Let $z(x, t)$ be the concentration of the gas in the pores of the absorbent. If we assume that the speed $v$ is sufficiently large to neglect the process of diffusion then the process of gas absorption can be described by the following equation

$$\frac{\partial^2 z}{\partial x \partial t}(x, t) + \frac{\beta}{v(x, t)} \frac{\partial z}{\partial t}(x, t) + \beta\gamma \frac{\partial z}{\partial x}(x, t) = 0,$$

where $\beta, \gamma$ are some physical quantities characterizing the given gas. For more details concerning the derivation of the equation we refer the reader to Rehbock et al. (1998), Tikhonov and Samarski (1990).

Let $\varphi(x) = z(x, 0)$ be the concentration of the gas at a distance $x$ at the time $t = 0$ and $\psi(t) = z(0, t)$ be the concentration of a gas at the time $t$ at the inlet of a pipe. Without loss of generality, we may assume that $(x, t) \in [0, 1] \times [0, 1]$. Suppose that we can control the process of gas absorption by changing the speed $v(x, t) \in [a, v_{\max}]$ to minimize the following cost indicator

$$J(z) = \int_0^1 F\left(\tau, \varphi'(\tau), \varphi''(\tau), \psi'(\tau), \psi''(\tau)\right) d\tau + g\left(\varphi(0), \varphi'(0), \psi'(0)\right),$$

where $F$ and $g$ are chosen to satisfy assumptions (A3)–(A7). The quantity $\varphi'$ can be interpreted as a change of gas concentration per unit of distance $x$ at the time $t = 0$ and $\psi'$ can be interpreted as a change of gas concentration per unit of time at the inlet. Consequently, $\varphi''$ and $\psi''$ are rates of speed of such changes.

It is easy to check that the assumptions (A1)–(A2) are satisfied. Moreover, since the equation is linear, the convexity assumption required by Theorem 2 is also satisfied. To sum up, there is an optimal speed $v(x, t)$ which minimizes the functional $J$.

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## Author Biographies

**Dorota Bors** is an assistant professor at the Faculty of Mathematics and Computer Science, University of Lodz. She received her Ph. D. degree in 2001 from the University of Lodz. Her research interests focus on variational methods in the theory of differential equations, optimal control problems governed by ordinary and partial differential equations and continuous 2D control systems.

**Marek Majewski** is an assistant professor at the Faculty of Mathematics and Computer Science, University of Lodz, Poland. He received his Ph.D. degree in 2003 from the University of Lodz. His research interests are optimal control problems described by ordinary and partial differential equations, stability and sensitivity of solutions, continuous 2D control systems and fractional calculus.