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Stability analysis and control design for 2-D fuzzy systems via basis-dependent Lyapunov functions

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Abstract This paper investigates the problem of stability analysis and stabilization for two-dimensional (2-D) discrete fuzzy systems. The 2-D fuzzy system model is established based on the Fornasini–Marchesini local state-space model, and a control design procedure is proposed based on a relaxed approach in which basis-dependent Lyapunov functions are used. First, nonquadratic stability conditions are derived by means of linear matrix inequality (LMI) technique. Then, by introducing an additional instrumental matrix variable, the stabilization problem for 2-D fuzzy systems is addressed, with LMI conditions obtained for the existence of stabilizing controllers. Finally, the effectiveness and advantages of the proposed design methods based on basis-dependent Lyapunov functions are shown via two examples.

Keywords 2-D system · Basis-dependent Lyapunov function · Control design · Fuzzy system · Stability analysis

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1 Introduction

As is well known, many practical systems can be modeled as two-dimensional (2-D) systems (Chen et al. 1999; Kaczorek 1985), such as those in image data processing and transmission, thermal processes, gas absorption and water stream heating. During the last few decades, the investigation of 2-D systems in the control and signal processing fields has attracted considerable attention and many important results have been reported to the literature. Among these results, the stability problem of 2-D systems has been investigated in Du and Xie (1999), Hinamoto (1997), Liu et al. (1998), Lu et al. (1994). Du et al. investigated the stability problem and gave some stability conditions obtained by the Lyapunov function for 2-D systems (Du and Xie 1999). They showed that the stability of 2-D discrete systems can be guaranteed if there are some matrices satisfying a certain linear matrix inequality (LMIs). The controller and filter design problems have been addressed in Du et al. (2000), Gao et al. (2004), Liu et al. (1998), Liu and Zhang (2003), Lu and Antoniou (1992), Xie et al. (2002), Lin et al. (2001), Wu et al. (2007, 2008). Gao et al. (2004) addressed the controller and filter design problems of controllers and filters for 2-D systems. They extended the results obtained for from one-dimensional (1-D) Markovian jump systems to investigate the problems of stabilization and H_∞ control for two-dimensional (2-D) systems with Markovian jump parameters. In addition, the model reduction of 2-D systems has also been solved in Du et al. (2001), Xu et al. (2005).

However, it is disappointing that many basic issues of 2-D systems still remain. Among them, the issue of 2-D nonlinear system is quite typical as no systematic and effective approach can handle its problem up to now. One of the main reasons might be the difficulty in modeling the nonlinearity. It is noticeable that, in the one-dimensional (1-D) case, the Takagi-Sugeno (T-S) fuzzy model (Jadbabaie 1999; Tanaka and Wang 2001; Tanaka et al. 2001; Zhou and Li 2005) has shed some light on this difficult problem, based on the fact that the T-S fuzzy model can approximate the smoothly nonlinear system on a compact set. This model formulates the 1-D nonlinear systems into a framework consisting of a set of local models which are smoothly connected by some membership functions. Based on the local linearities, the stability and performance analysis approaches for 1-D linear systems can be fully developed for 1-D nonlinear systems in this framework. In virtue of this advantage, a number of important issues in 1-D nonlinear fuzzy control systems have been well studied. Among these results, stability analysis has been studied in Jadbabaie (1999), Kim and Kim (2002), Kim and Lee (2000), systematic design procedures have been proposed in Wang et al. (1996), robustness and optimality have been investigated in Liu et al. (2005), Lu and Doyle (1995), Yoneyama (2006), Zhou and Li (2005), the problems of stability analysis and stabilization for a class of discrete-time T-S fuzzy systems with time-varying state delay has been studied in Wu et al. (2011), the robust fault detection problem for T-S fuzzy Ito stochastic systems has been tackled in Wu and Ho (2009).

On the other hand, it is noted that the aforementioned research efforts have been focused on the use of a single quadratic Lyapunov function (de Oliveira et al. 2002; Haddad and Bernstein 1995), which tends to yield more conservative conditions. More recently, there appeared a number of results on stability analysis and control synthesis of 1-D dynamic systems based on basis-dependent Lyapunov functions (Choi and Park 2003; Gao et al. 2009; Guerra and Vermeiren 2004; Zhou et al. 2007). It is shown that, with the use of a basis-dependent Lyapunov function, less conservative results can be obtained than those with the use of a single Lyapunov quadratic function. Examples of reduced conservative conditions based on basis-dependent Lyapunov functions can be found in Choi and Park (2003), Guerra and Vermeiren (2004), Lam and Zhou (2007).

As explained above, although many problems on 2-D linear systems have been studied, the synthesis problems for 2-D nonlinear systems have not been fully investigated. On the other hand, basis-dependent Lyapunov function has not been used in the study of 2D nonlinear systems. This motivates our study. In this paper, we represent the 2-D nonlinear systems using the T-S fuzzy model and thus solve the problems of 2-D nonlinear fuzzy control systems with the use of basis-dependent Lyapunov functions. In detail, the 2-D fuzzy system model is established based on the Fornasini–Marchesini local state-space (FMLSS) model (Chen et al. 1999; Xie et al. 2002), and the controller design procedure is presented based on a relaxed approach in which basis-dependent Lyapunov functions are used. First, nonquadratic stability is derived by means of linear matrix inequality (LMI) technique (Boyd et al. 1994). Then, by introducing an additional instrumental matrix variable, the stabilization problem for 2-D fuzzy systems is addressed, with LMI conditions obtained for the existence of stabilizing controllers. Finally, two illustrative examples are provided to show the effectiveness and it is shown that the results based on basis-dependent Lyapunov functions are less conservative than those based on basis-independent Lyapunov functions.

The rest of the paper is organized as follows. The problem under consideration is formulated in Sect. 2. Stability analysis is given in Sect. 3, based on which controller designs are presented in Sect. 4. Illustrative examples are given in Sect. 5 to demonstrate the effectiveness of the results. Finally, the paper is concluded in Sect. 6.

Notation: The notation used throughout the paper is standard. The superscript T stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and the notation $P > 0$ means that P is real symmetric and positive definite; The notation $|\cdot|$ refers to the Euclidean vector norm; and $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalues of the corresponding matrix respectively. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and diag $\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2 Problem formulation

Let us first recall the well known 2-D discrete FMLSS model (Fornasini and Marchesini 1978) given by:

$$\mathscr{F}: x_{i+1,j+1} = A_1 x_{i,j+1} + A_2 x_{i+1,j} + B_1 u_{i,j+1} + B_2 u_{i+1,j}, \tag{1}$$

where $x_{i,j} \in \mathbb{R}^n$ is the local state vector and $u_{i,j} \in \mathbb{R}^m$ is the input; A_1, A_2, B_1, B_2 are system matrices. Similar to the well-established fuzzy model of 1-D system (Gahinet et al. 1995), we consider 2-D discrete fuzzy model based on a suitable choice of a set of linear subsystems, according to rules associated with some physical knowledge and some linguistic characterization of the properties of the system. The linear subsystems describe, at least locally, the behavior of the nonlinear system for a pre-defined region of the state space. The T-S model for the nonlinear system is given by the following IF-THEN rules:

Model Rule k: IF $\theta_{1(i,j)}$ is μ_{k1} and $\theta_{2(i,j)}$ is μ_{k2} and ... and $\theta_{p(i,j)}$ is μ_{kp}, THEN

$$x_{i+1,j+1} = A_{1k} x_{i,j+1} + A_{2k} x_{i+1,j} + B_{1k} u_{i,j+1} + B_{2k} u_{i+1,j},$$

where $\mu_{k1}, \ldots, \mu_{kp}$ are fuzzy sets; A_{1k}, B_{1k}, A_{2k}, B_{2k} are constant matrices; r is the number of IF-THEN rules and $\theta_{i,j} = \begin{bmatrix} \theta_{1(i,j)}, & \theta_{2(i,j)}, & \ldots, & \theta_{p(i,j)} \end{bmatrix}$ is the premise variable vector. Throughout the paper, it is assumed that the premise variables do not depend on the input variables explicitly. Then, the final output of the fuzzy system is inferred as

$$\mathscr{S} : x_{i+1,j+1} = \sum_{k=1}^{r} h_k(\theta_{i,j}) \left\{ A_{1k} x_{i,j+1} + A_{2k} x_{i+1,j} + B_{1k} u_{i,j+1} + B_{2k} u_{i+1,j} \right\}, \quad (2)$$

where

$$h_k\left(\theta_{i,j}\right) = \omega_k(\theta_{i,j}) \Big/ \sum_{k=1}^{r} \omega_k(\theta_{i,j}),$$
$$\omega_k(\theta_{i,j}) = \prod_{l=1}^{p} \mu_{kl}(\theta_{l(i,j)}),$$

with $\mu_{kl}(\theta_l(i, j)) \in [0, 1]$ representing the grade of membership of $\theta_l(i, j)$ in μ_{kl}. We have

$$\sum_{k=1}^{r} \omega_k(\theta_{i,j}) > 0,$$
$$\omega_k(\theta_{i,j}) \geq 0, \quad k = 1, 2, \ldots, r,$$

for all i, j. Therefore, for all i, j we have

$$\sum_{k=1}^{r} h_k(\theta_{i,j}) = 1,$$
$$h_k(\theta_{i,j}) \geq 0, \quad k = 1, 2, \ldots, r.$$

The boundary conditions are defined by

$$X_0^h = \left[x_{0,1}^T \quad x_{0,2}^T \quad \ldots \quad x_{0,M}^T \right]^T,$$
$$X_0^v = \left[x_{1,0}^T \quad x_{2,0}^T \quad \ldots \quad x_{N,0}^T \right]^T.$$

Denote

$$X_r = \sup \left\{ \left| x_{i,j} \right| : i + j = r, \quad i, j \in \mathbb{Z} \right\}.$$

Assumption 1 The boundary condition is assumed to satisfy

$$\lim_{N \to \infty} \left\{ \sum_{\eta=1}^{N} \left(\left| x_{0,\eta} \right|^2 + \left| x_{\eta,0} \right|^2 \right) \right\} < \infty.$$

Then we give the following stability definition which will be used in the paper.

Definition 1 The 2-D discrete fuzzy system $\mathscr{S}$ in (2) is said to be asymptotically stable if $\lim_{r \to \infty} X_r = 0$ under the zero input and any boundary conditions such that $X_0 < \infty$.

3 Stability analysis

In this section, we are concerned with the stability analysis of the 2-D discrete fuzzy systems and we will give some stability conditions obtained with the use of a basis-dependent Lyapunov function. Before presenting Theorem 1, we first introduce the following lemma which will be used in the proof of Theorem 1.

Lemma 1 *For matrices $P \geq 0$, A, B with appropriate dimensions, the following matrix inequality holds.*

$$A^T PB + B^T PA \leq A^T PA + B^T PB. \tag{3}$$

Then, the following theorem shows that the stability of 2-D discrete fuzzy systems can be guaranteed if there exist some matrices satisfying certain LMIs.

Theorem 1 *Consider the 2-D fuzzy system $\mathscr{S}$ in (2) under Assumption 1. The 2-D discrete fuzzy system $\mathscr{S}$ in (2) is asymptotically stable if there exist matrices $X_k > 0$, $Y_k \geq 0$ and $Z_k > 0$ satisfying*

$$\begin{bmatrix} -X_m & -Y_m & A_{1m}^T Q_k \\ * & -Z_m & A_{2m}^T Q_k \\ * & * & -Q_k \end{bmatrix} < 0, \tag{4}$$

$$\begin{bmatrix} -X_m - X_n & -Y_m - Y_n & A_{1m}^T Q_k & A_{1n}^T Q_k \\ * & -Z_m - Z_n & A_{2n}^T Q_k & A_{2m}^T Q_k \\ * & * & -Q_k & 0 \\ * & * & * & -Q_k \end{bmatrix} < 0, \tag{5}$$

where $k = 1, 2, \ldots, r$; $1 \leq m < n \leq r$ and $Q_k := X_k + 2Y_k + Z_k$.

Proof To establish the stability of system $\mathscr{S}$, assume $u_{i,j} = 0$. Then the system $\mathscr{S}$ in (2) can be represented by

$$x_{i+1,j+1} = \sum_{k=1}^{r} h_k(\theta_{i,j}) \left\{ A_{1k} x_{i,j+1} + A_{2k} x_{i+1,j} \right\}.$$

First, by Schur complement equivalence (Boyd et al. 1994), LMIs (4) and (5) are equivalent to

$$\begin{bmatrix} A_{1m}^T Q_k A_{1m} - X_m & A_{1m}^T Q_k A_{2m} - Y_m \\ * & A_{2m}^T Q_k A_{2m} - Z_m \end{bmatrix} < 0, \tag{6}$$

$$\begin{bmatrix} A_{1m}^T Q_k A_{1m} + A_{1n}^T Q_k A_{1n} - X_m - X_n & A_{1m}^T Q_k A_{2n} + A_{1n}^T Q_k A_{2m} - Y_m - Y_n \\ * & A_{2m}^T Q_k A_{2m} + A_{2n}^T Q_k A_{2n} - Z_m - Z_n \end{bmatrix} < 0. \tag{7}$$

Consider the following index

$$J := W_1 - W_2, \tag{8}$$

with

$$W_1 = \begin{bmatrix} x_{i+1,j+1}^T & x_{i+1,j+1}^T \end{bmatrix} \left(\sum_{k=1}^{r} h_k^+ P_k \right) \begin{bmatrix} x_{i+1,j+1} \\ x_{i+1,j+1} \end{bmatrix},$$

$$W_2 = \widetilde{x}^T \left(\sum_{k=1}^{r} h_k P_k \right) \widetilde{x},$$

where $h_k^+ = h_k\left(\theta_{i+1,j+1}\right)$, $\widetilde{x} = \begin{bmatrix} x_{i,j+1} \\ x_{i+1,j} \end{bmatrix}$, and $P_k := \begin{bmatrix} X_k & Y_k \\ * & Z_k \end{bmatrix} > 0$. By some algebraic manipulations, we have

$$\begin{aligned}
J &= x_{i+1,j+1}^T \begin{bmatrix} I & I \end{bmatrix} \left(\sum_{k=1}^r h_k^+ P_k \right) \begin{bmatrix} I \\ I \end{bmatrix} x_{i+1,j+1} - \widetilde{x}^T \left(\sum_{k=1}^r h_k P_k \right) \widetilde{x} \\
&= x_{i+1,j+1}^T \left(\sum_{k=1}^r h_k^+ Q_k \right) x_{i+1,j+1} - \widetilde{x}^T \left(\sum_{k=1}^r h_k P_k \right) \widetilde{x} \\
&= \widetilde{x}^T \left\{ \sum_{k=1}^r h_k^+ \left(\sum_{m=1}^r \sum_{n=1}^r h_m(\theta_{i,j}) h_n(\theta_{i,j}) M_1 \right) \right\} \widetilde{x} \\
&= \widetilde{x}^T \left\{ \sum_{k=1}^r h_k^+ \left(\begin{array}{c} \sum_{m=1}^r h_m^2(\theta_{i,j}) M_2 \\ + \sum_{m=1}^r \sum_{n>m} h_m(\theta_{i,j}) h_n(\theta_{i,j}) M_3 \end{array} \right) \right\} \widetilde{x} \\
&\leq \widetilde{x}^T \left\{ \sum_{k=1}^r h_k^+ \left(\begin{array}{c} \sum_{m=1}^r h_m^2(\theta_{i,j}) M_2 \\ + \sum_{m=1}^r \sum_{n>m} h_m(\theta_{i,j}) h_n(\theta_{i,j}) M_4 \end{array} \right) \right\} \widetilde{x} \\
&= \widetilde{x}^T \Psi(\theta_{i,j}, \theta_{i+1,j+1}) \widetilde{x},
\end{aligned} \tag{9}$$

where M_1, M_2, M_3, M_4 satisfying

$$\begin{aligned}
M_1 &= \begin{bmatrix} A_{1m}^T Q_k A_{1n} - X_m & A_{1m}^T Q_k A_{2n} - Y_m \\ A_{2m}^T Q_k A_{1n} - Y_m & A_{2m}^T Q_k A_{2n} - Z_m \end{bmatrix}, \\
M_2 &= \begin{bmatrix} A_{1m}^T Q_k A_{1m} - X_m & A_{1m}^T Q_k A_{2m} - Y_m \\ * & A_{2m}^T Q_k A_{2m} - Z_m \end{bmatrix}, \\
M_3 &= \begin{bmatrix} A_{1m}^T Q_k A_{1n} + A_{1n}^T Q_k A_{1m} - X_m - X_n & A_{1m}^T Q_k A_{2n} + A_{1n}^T Q_k A_{2m} - Y_m - Y_n \\ * & A_{2m}^T Q_k A_{2n} + A_{2n}^T Q_k A_{2m} - Z_m - Z_n \end{bmatrix}, \\
M_4 &= \begin{bmatrix} A_{1m}^T Q_k A_{1m} + A_{1n}^T Q_k A_{1n} - X_m - X_n & A_{1m}^T Q_k A_{2n} + A_{1n}^T Q_k A_{2m} - Y_m - Y_n \\ * & A_{2m}^T Q_k A_{2m} + A_{2n}^T Q_k A_{2n} - Z_m - Z_n \end{bmatrix}.
\end{aligned} \tag{10}$$

Hence, from the conditions in (6) and (7), we have $\Psi(\theta_{i,j}, \theta_{i+1,j+1}) < 0$. Then, for $\widetilde{x} \neq 0$, we have

$$\frac{W_1 - W_2}{W_2} = -\frac{\widetilde{x}^T \left(-\Psi(\theta_{i,j}, \theta_{i+1,j+1})\right) \widetilde{x}}{\widetilde{x}^T \left(\sum_{k=1}^r h_k(\theta_{i,j}) P_k\right) \widetilde{x}}. \tag{11}$$

It is noted that

$$-\Psi(\theta_{i,j},\theta_{i+1,j+1})=\sum_{k=1}^{r}h_k^{+}\left(\begin{array}{c}\sum_{m=1}^{r}h_m^2(\theta_{i,j})(-M_2(m,k))\\+\sum_{m=1}^{r}\sum_{n>m}h_m(\theta_{i,j})h_n(\theta_{i,j})(-M_4(m,n,k))\end{array}\right). \quad (12)$$

Since

$$\begin{aligned}
&\sum_{m=1}^{r}h_m^2(\theta_{i,j})(-M_2(m,k))+\sum_{m=1}^{r}\sum_{n>m}h_m(\theta_{i,j})h_n(\theta_{i,j})(-M_4(m,n,k))\\
&\geq\left[\lambda_{\min}(-M_2(m,k))\sum_{m=1}^{r}h_m^2(\theta_{i,j})+\lambda_{\min}(-M_4(m,n,k))\sum_{m=1}^{r}\sum_{n>m}h_m(\theta_{i,j})h_n(\theta_{i,j})\right]I\\
&\geq\left[\lambda_{\min}(-M_2(m,k))\sum_{m=1}^{r}h_m^2(\theta_{i,j})+\frac{1}{2}\lambda_{\min}(-M_4(m,n,k))\sum_{m=1}^{r}\sum_{n\neq m}^{r}h_m(\theta_{i,j})h_n(\theta_{i,j})\right]I\\
&\geq\left[\min\left\{\lambda_{\min}(-M_2(m,k)),\frac{1}{2}\lambda_{\min}(-M_4(m,n,k))\right\}\sum_{m=1}^{r}\sum_{n=1}^{r}h_m(\theta_{i,j})h_n(\theta_{i,j})\right]I\\
&=\left[\min\left\{\lambda_{\min}(-M_2(m,k)),\frac{1}{2}\lambda_{\min}(-M_4(m,n,k))\right\}\right]I, \qquad (13)
\end{aligned}$$

where $\lambda_{\min}(\cdot)$ is taken to mean the minimum eigenvalue out of all the matrices over the appropriate indices m, n and k, we have

$$\lambda_{\min}(-\Psi(\theta_{i,j},\theta_{i+1,j+1}))\geq\min\left\{\lambda_{\min}(-M_2(m,k)),\frac{1}{2}\lambda_{\min}(-M_4(m,n,k))\right\}. \quad (14)$$

Similarly,

$$\sum_{k=1}^{r}h_k(\theta_{i,j})P_k\leq\sum_{k=1}^{r}P_k. \quad (15)$$

It follows from (14) and (15) that

$$\frac{W_1-W_2}{W_2}\leq-\frac{\min\left\{\lambda_{\min}(-M_2(m,k)),\frac{1}{2}\lambda_{\min}(-M_4(m,n,k))\right\}}{\lambda_{\max}\left(\sum_{k=1}^{r}P_k\right)}=\alpha-1.$$

Since

$$\frac{\min\left\{\lambda_{\min}(-M_2(m,k)),\frac{1}{2}\lambda_{\min}(-M_4(m,n,k))\right\}}{\lambda_{\max}\left(\sum_{k=1}^{r}P_k\right)}>0,$$

we have $\alpha<1$.
Obviously,

$$\alpha\geq\frac{W_1}{W_2}\geq 0,$$

that is, α belongs to [0, 1) and is independent of $\widetilde{x}$. Therefore, we have

$$W_1\leq\alpha W_2,$$

that is,

$$\begin{aligned}&\begin{bmatrix} x_{i+1,j+1}^T & x_{i+1,j+1}^T \end{bmatrix}\left(\sum_{k=1}^{r} h_k^+ P_k\right)\begin{bmatrix} x_{i+1,j+1} \\ x_{i+1,j+1} \end{bmatrix}\\ &\le \alpha \begin{bmatrix} x_{i,j+1}^T & x_{i+1,j}^T \end{bmatrix}\left(\sum_{k=1}^{r} h_k P_k\right)\begin{bmatrix} x_{i,j+1} \\ x_{i+1,j} \end{bmatrix}. \end{aligned} \tag{16}$$

Then, it can be established that

$$\begin{aligned}\begin{bmatrix} x_{\eta,1}^T & x_{\eta,1}^T \end{bmatrix}\left(\sum_{k=1}^{r} h_k^+ P_k\right)\begin{bmatrix} x_{\eta,1} \\ x_{\eta,1} \end{bmatrix} &\le \alpha \begin{bmatrix} x_{\eta-1,1}^T & x_{\eta,0}^T \end{bmatrix}\left(\sum_{k=1}^{r} h_k P_k\right)\begin{bmatrix} x_{\eta-1,1} \\ x_{\eta,0} \end{bmatrix}\\ &\le \alpha \sum_{k=1}^{r} h_k \left\{x_{\eta,0}^T\left(Y_k+Z_k\right)x_{\eta,0}+x_{\eta-1,1}^T\left(X_k+Y_k\right)x_{\eta-1,1}\right\}\\ &\le \alpha \sum_{k=1}^{r} h_k \left\{x_{\eta,0}^T Q_k x_{\eta,0}+x_{\eta-1,1}^T\left(X_k+Y_k\right)x_{\eta-1,1}\right\},\\ &\vdots\\ \begin{bmatrix} x_{1,\eta}^T & x_{1,\eta}^T \end{bmatrix}\left(\sum_{k=1}^{r} h_k^+ P_k\right)\begin{bmatrix} x_{1,\eta} \\ x_{1,\eta} \end{bmatrix} &\le \alpha \begin{bmatrix} x_{0,\eta}^T & x_{1,\eta-1}^T \end{bmatrix}\left(\sum_{k=1}^{r} h_k P_k\right)\begin{bmatrix} x_{0,\eta} \\ x_{1,\eta-1} \end{bmatrix}\\ &\le \alpha \sum_{k=1}^{r} h_k \left\{x_{1,\eta-1}^T\left(Y_k+Z_k\right)x_{1,\eta-1}+x_{0,\eta}^T Q_k x_{0,\eta}\right\}. \end{aligned}$$

Adding both sides of the above inequality system yields

$$\begin{aligned}\sum_{j=0}^{\eta+1} x_{\eta+1-j,j}^T\left(\sum_{k=1}^{r} h_k^+ Q_k\right)x_{\eta+1-j,j} &\le \alpha \sum_{j=0}^{\eta} x_{\eta-j,j}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{\eta-j,j}\\ &+x_{\eta+1,0}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{\eta+1,0}+x_{0,\eta+1}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{0,\eta+1}. \end{aligned}$$

Using this relationship iteratively, we can obtain

$$\begin{aligned}\sum_{j=0}^{\eta+1} x_{\eta+1-j,j}^T\left(\sum_{k=1}^{r} h_k^+ Q_k\right)x_{\eta+1-j,j} &\le \alpha^{\eta+1} x_{0,0}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{0,0}\\ &+\sum_{j=0}^{\eta}\alpha^j\left[x_{\eta+1-j,0}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{\eta+1-j,0}+x_{0,\eta+1-j}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{0,\eta+1-j}\right],\\ &\le \sum_{j=0}^{\eta+1}\alpha^j\left[x_{\eta+1-j,0}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{\eta+1-j,0}+x_{0,\eta+1-j}^T\left(\sum_{k=1}^{r} h_k Q_k\right)x_{0,\eta+1-j}\right]. \end{aligned}$$

Therefore, we have

$$\sum_{j=0}^{\eta+1}\left|x_{\eta+1-j,j}\right|^2 \le \mu \sum_{j=0}^{\eta+1}\alpha^j\left[\left|x_{\eta+1-j,0}\right|^2+\left|x_{0,\eta+1-j}\right|^2\right], \tag{17}$$

where

$$\mu := \frac{\lambda_{\max}\left(\sum_{k=1}^{r} h_k Q_k\right)}{\lambda_{\min}\left(\sum_{k=1}^{r} h_k^{+} Q_k\right)}.$$

We note that

$$\mu \leq \tau := \frac{\lambda_{\max}\left(\sum_{k=1}^{r} Q_k\right)}{\min_k(\lambda_{\min}(Q_k))}.$$

Now denote $\chi_\kappa := \sum_{j=0}^{\kappa} \left|x_{\kappa-j,j}\right|^2$, and then using the above inequality, we have

$$\begin{aligned}
\chi_0 &\leq \tau\left(\left|x_{0,0}\right|^2 + \left|x_{0,0}\right|^2\right),\\
\chi_1 &\leq \tau\left\{\alpha\left(\left|x_{0,0}\right|^2 + \left|x_{0,0}\right|^2\right) + \left(\left|x_{1,0}\right|^2 + \left|x_{0,1}\right|^2\right)\right\},\\
&\vdots\\
\chi_N &\leq \tau\left\{\alpha^N\left(\left|x_{0,0}\right|^2 + \left|x_{0,0}\right|^2\right) + \alpha^{N-1}\left(\left|x_{1,0}\right|^2 + \left|x_{0,1}\right|^2\right) + \cdots + \left(\left|x_{N,0}\right|^2 + \left|x_{0,N}\right|^2\right)\right\}.
\end{aligned}$$

Adding both sides of the above inequality system yields

$$\sum_{\eta=0}^{N} \chi_\eta \leq \tau \frac{1-\alpha^{N+1}}{1-\alpha} \sum_{k=0}^{N} \left\{\left|x_{k,0}\right|^2 + \left|x_{0,k}\right|^2\right\}.$$

Then from Assumption 1, the right side of the above inequality is bounded, which means: $\lim_{\eta\to\infty}\chi_\eta = 0$, that is, $\left|x_{i,j}\right|^2 \to 0$ as $i + j \to \infty$, hence $\lim_{r\to\infty} X_r = 0$ and then the proof is completed. □

Remark 1 Theorem 1 provides the LMI based conditions for the asymptotic stability of 2-D fuzzy systems, which can be solved efficiently by employing standard numerical software (Gahinet et al. 1995). Actually from the proof of Theorem 1, we see that the conditions $M_2 < 0$ and $M_3 < 0$ can be used for the stability analysis of system $\mathscr{S}$. However, it is noticed that the product terms between the system matrices and the matrix Q_k can not be eliminated in this case. Therefore, the conditions $M_2 < 0$ and $M_3 < 0$ is not powerful for controller synthesis.

If the basis-dependent Lyapunov functions reduce to a common quadratic Lyapunov function, by following similar lines as in the proof of Theorem 1, we obtain the following corollary.

Corollary 1 *Consider the fuzzy system $\mathscr{S}$ in (2) with Assumption 1. The 2-D discrete fuzzy system $\mathscr{S}$ in (2) is asymptotically stable if there exist matrices $X > 0$, $Y \geq 0$ and $Z > 0$ satisfying*

$$\begin{bmatrix} -X & -Y & A_{1m}^T Q \\ * & -Z & A_{2m}^T Q \\ * & * & -Q \end{bmatrix} < 0,$$

$$\begin{bmatrix} -X & -Y & A_{1m}^T Q & A_{1n}^T Q \\ * & -Z & A_{2n}^T Q & A_{2m}^T Q \\ * & * & -Q & 0 \\ * & * & * & -Q \end{bmatrix} < 0. \tag{18}$$

where $m, n = 1, 2, \ldots, r$; $m < n \leq r$ and $Q := X + 2Y + Z$.

Remark 2 From Corollary 1, we can find that the basis-independent result is a special case of basis-dependent result. Thus Theorem 1 is less conservative than that based on Corollary 1.

Remark 3 From the proof of Theorem 1, we see that when the systems are linear time-invariant and the basis-dependent Lyapunov functions become basis-independent Lyapunov functions, $M_1 \equiv M_2$ and M_3, M_4 disappear. Therefore, LMIs (4) and (5) become

$$\begin{bmatrix} -X & -Y & A_1^T Q \\ * & -Z & A_2^T Q \\ * & * & -Q \end{bmatrix} < 0, \tag{19}$$

which has been obtained in Tuan et al. (2002). From this point of view, Theorem 1 and Corollary 1 can be seen as an extension of Tuan et al. (2002) to 2-D fuzzy systems.

Since Theorem 1 is derived from Tuan's results (Tuan et al. 2002), in the following we will show that we can also establish the asymptotic stability on the basis of another elegant stability result for 2-D systems proposed in Xie et al. (2002). As the proof is analogous to that of Theorem 2 in Gao et al. (2005), it is omitted for brevity.

Theorem 2 *LMIs* (4) *and* (5) *in Theorem* 1 *hold if and only if there exist matrices* $R_k > 0$ *and* $T_k > 0$ *satisfying*

$$\begin{bmatrix} T_m - R_n & 0 & A_{1m}^T R_k \\ * & -T_m & A_{2m}^T R_k \\ * & * & -R_k \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} T_m - R_m + T_n - R_n & 0 & A_{1m}^T R_k & A_{1n}^T R_k \\ * & -T_m - T_n & A_{2n}^T R_k & A_{2m}^T R_k \\ * & * & -R_k & 0 \\ * & * & * & -R_k \end{bmatrix} < 0, \tag{21}$$

where $k, m, n = 1, 2, \ldots, r; m < n \leq r$.

Remark 4 Similar to Remark 3, when the systems are linear time-invariant and the basis-dependent Lyapunov functions become common quadratic Lyapunov functions, LMIs (20) and (21) will reduce to

$$\begin{bmatrix} T - R & 0 & A_1^T R \\ * & -T & A_2^T R \\ * & * & -R \end{bmatrix} < 0, \tag{22}$$

which has been obtained in Xie et al. (2002).

Remark 5 Theorem 2 is in fact equivalent to Theorem 1 [please refer to Theorem 3 in Gao et al. (2005)]. In the following, we will only present the stabilization results based on Theorem 1, and equivalent results based on Theorem 2 can be readily obtained by employing similar arguments.

4 Stabilization of 2-D fuzzy systems

In this section, we shall deal with the problem of stabilization for systems via a parallel distributed compensation (PDC) fuzzy controller. More specifically, we are interested in finding a PDC fuzzy controller such that the closed-loop system with this controller is asymptotically stable.

In the PDC design, each control rule is designed from the corresponding rule of a T-S fuzzy model. The designed fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts. For the fuzzy models in (2), we construct the following fuzzy controller via the PDC:

Control Rule k: IF $\theta_{1(i,j)}$ is μ_{k1} and $\theta_{2(i,j)}$ is μ_{k2} and ... and $\theta_{p(i,j)}$ is μ_{kp}, THEN

$$u_{i,j} = -F_k x_{i,j}.$$

The overall fuzzy controller is represented by

$$u_{i,j} = -\sum_{k=1}^{r} h_k(\theta_{i,j}) F_k x_{i,j}.$$

First, the closed-loop system with the PDC fuzzy controller can be given by

$$x_{i+1,j+1} = \sum_{p=1}^{r}\sum_{q=1}^{r} h_p(\theta_{i,j}) h_q(\theta_{i,j}) \left\{ \left(A_{1p} - B_{1p}F_q\right) x_{i,j+1} + \left(A_{2p} - B_{2p}F_q\right) x_{i+1,j} \right\}.$$

Before stating the main result of this section, we present the following proposition first, which is useful in establishing our results.

Proposition 1 *Consider the 2-D fuzzy system $\mathscr{S}$ in (2) with given boundary condition. The 2-D discrete fuzzy system $\mathscr{S}$ in (2) is asymptotically stable if there exist matrices $X_k > 0$, $Y_k \geq 0$, $Z_k > 0$ and V_k satisfying*

$$\begin{bmatrix} -X_m & -Y_m & A_{1m}^T V_k \\ * & -Z_m & A_{2m}^T V_k \\ * & * & Q_k - V_k - V_k^T \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} -X_m - X_n & -Y_m - Y_n & A_{1m}^T V_k & A_{1n}^T V_k \\ * & -Z_m - Z_n & A_{2n}^T V_k & A_{2m}^T V_k \\ * & * & Q_k - V_k - V_k^T & 0 \\ * & * & * & Q_k - V_k - V_k^T \end{bmatrix} < 0, \tag{24}$$

where $k = 1, 2, \ldots, r$; $1 \leq m < n \leq r$ and $Q_k := X_k + 2Y_k + Z_k$.

Proof If LMIs (23) and (24) hold, we have $V_k + V_k^T - Q_k > 0$. From the conditions $X_k > 0$, $Y_k \geq 0$, $Z_k > 0$, we have $Q_k > 0$, so that V_k is nonsingular. In addition, we have $\left(Q_k - V_k^T\right) Q_k^{-1} \left(Q_k - V_k\right) \geq 0$, which implies

$$-V_k^T Q_k^{-1} V_k \leq Q_k - V_k - V_k^T. \tag{25}$$

Therefore, we can conclude from (23) and (24) that

$$\begin{bmatrix} -X_m & -Y_m & A_{1m}^T V_k \\ * & -Z_m & A_{2m}^T V_k \\ * & * & -V_k^T Q_k^{-1} V_k \end{bmatrix} < 0, \tag{26}$$

$$\begin{bmatrix} -X_m - X_n & -Y_m - Y_n & A_{1m}^T V_k & A_{1n}^T V_k \\ * & -Z_m - Z_n & A_{2n}^T V_k & A_{2m}^T V_k \\ * & * & -V_k^T Q_k^{-1} V_k & 0 \\ * & * & * & -V_k^T Q_k^{-1} V_k \end{bmatrix} < 0. \tag{27}$$

Performing a congruence transformation to (26) and (27) by diag $\left\{I, I, V_k^{-1} Q_k\right\}$ and diag $\left\{I, I, V_k^{-1} Q_k, V_k^{-1} Q_k\right\}$ yields (4) and (5), and then the proof is completed. □

Based on Proposition 1, we are in a position to establish conditions to the stabilization problem for system $\mathscr{S}$ in (2).

Theorem 3 *The 2-D fuzzy system $\mathscr{S}$ in (2) can be stabilized via a PDC fuzzy controller if there exist matrices $\overline{X_k} > 0$, $\overline{Y_k} \geq 0$, $\overline{Z_k} > 0$, $\overline{F_l}$ and G_l satisfying*

$$\begin{bmatrix} -\overline{X_m} & -\overline{Y_m} & G_l^T A_{1m}^T - \overline{F_l}^T B_{1m}^T \\ * & -\overline{Z_m} & G_l^T A_{2m}^T - \overline{F_l}^T B_{2m}^T \\ * & * & \overline{Q_k} - G_l^T - G_l \end{bmatrix} < 0, \tag{28}$$

$$\begin{bmatrix} -\overline{X_m} - \overline{X_p} & -\overline{Y_m} - \overline{Y_p} & G_l^T A_{1m}^T - \overline{F_l}^T B_{1m}^T & G_l^T A_{1p}^T - \overline{F_l}^T B_{1p}^T \\ * & -\overline{Z_m} - \overline{Z_p} & G_l^T A_{2p}^T - \overline{F_l}^T B_{2p}^T & G_l^T A_{2m}^T - \overline{F_l}^T B_{2m}^T \\ * & * & \overline{Q_k} - G_l^T - G_l & 0 \\ * & * & * & \overline{Q_k} - G_l^T - G_l \end{bmatrix} < 0. \tag{29}$$

Moreover, if the above conditions have feasible solutions, the controller gain matrices are given by

$$F_l = \overline{F_l} G_l^{-1}, \tag{30}$$

where $\overline{Q_k} \triangleq \overline{X_k} + 2\overline{Y_k} + \overline{Z_k}$; $k, l = 1, 2, \ldots, r$; $1 \leq m \leq p \leq r$.

Proof According to Proposition 1, the closed-loop system is asymptotically stable if there exist matrices X_k, Y_k, Z_k and V_l satisfying

$$\begin{bmatrix} -X_m & -Y_m & (A_{1m} - B_{1m} F_l)^T V_l \\ * & -Z_m & (A_{2m} - B_{2m} F_l)^T V_l \\ * & * & Q_k - V_l - V_l^T \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} -X_m - X_p & -Y_m - Y_p & (A_{1m} - B_{1m} F_l)^T V_l & \left(A_{1p} - B_{1p} F_l\right)^T V_l \\ * & -Z_m - Z_p & \left(A_{2p} - B_{2p} F_l\right)^T V_l & (A_{2m} - B_{2m} F_l)^T V_l \\ * & * & Q_k - V_l - V_l^T & 0 \\ * & * & * & Q_k - V_l - V_l^T \end{bmatrix} < 0. \tag{32}$$

The congruence transformations to (31) and (32) by diag $\left\{V_l^{-1}, V_l^{-1}, V_l^{-1}\right\}$ and diag $\left\{V_l^{-1}, V_l^{-1}, V_l^{-1}, V_l^{-1}\right\}$ together with a change of variables by

$$\begin{aligned}\overline{X_m} &\triangleq V_l^{-T} X_m V_l^{-1},\ \overline{Y_m} \triangleq V_l^{-T} Y_m V_l^{-1},\\ \overline{Z_m} &\triangleq V_l^{-T} Z_m V_l^{-1},\ G_l \triangleq V_l^{-1},\ \overline{F_l} \triangleq F_l V_l^{-1}\end{aligned} \tag{33}$$

yield LMIs (28) and (29). In addition, we know that if LMIs (28) and (29) are feasible, the control law can be given by (30) and the proof is completed. □

Remark 6 Theorem 3 solves the stabilization problem on the basis of Proposition 1. It should be pointed out that the LMI conditions in Theorem 1 contain product terms between the system matrices and the matrix Q_k (which is a substitute for $X_k + 2Y_k + Z_k$). Therefore, it is not an easy task to solve the stabilization problem based on Theorem 1. On the other hand, by introducing the slack variable V_k, Proposition 1 eliminates the product terms involving the matrix Q_k. In such a way, the dilated LMI conditions in Proposition 1 are not only preferable for stability analysis of the systems, but also powerful for controller synthesis.

5 Illustrative examples

In this section, we will use two examples to illustrate the applicability of the approach proposed in this paper.

Example 1 We consider a 2-D system with the following system matrices:

$$\begin{aligned}A_1 &= \begin{bmatrix} -0.15 + \gamma \sin(x_{1(i,j+1)}) & 0.5 \\ -0.15 & 0.0025 \end{bmatrix},\\ A_2 &= \begin{bmatrix} 0.06 & -0.5 \\ -0.3 & 0.005 \end{bmatrix},\end{aligned} \tag{34}$$

where $x_{i,j}$ is the local state of coordinates (i, j) and

$$x_{i,j} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We have the fuzzy model for the nonlinear system

$$x_{i+1,j+1} = \sum_{k=1}^{2} h_k(\theta_{i,j}) \left\{ A_{1k} x_{i,j+1} + A_{2k} x_{i+1,j} + B_{1k} u_{i,j+1} + B_{2k} u_{i+1,j} \right\},$$

with

$$\begin{aligned}A_{11} &= \begin{bmatrix} \gamma - 0.15 & 0.5 \\ -0.15 & 0.0025 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -\gamma - 0.15 & 0.5 \\ -0.15 & 0.0025 \end{bmatrix},\\ A_{21} &= \begin{bmatrix} 0.06 & -0.5 \\ -0.3 & 0.005 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.06 & -0.5 \\ -0.3 & 0.005 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}h_1(\theta_{i,j}) &= \frac{1 + \sin(x_{1(i,j+1)})}{2},\\ h_2(\theta_{i,j}) &= \frac{1 - \sin(x_{1(i,j+1)})}{2}.\end{aligned}$$

Fig. 1 State variable x_1 of open-loop system in Example 1 ($\gamma = 0.5$)

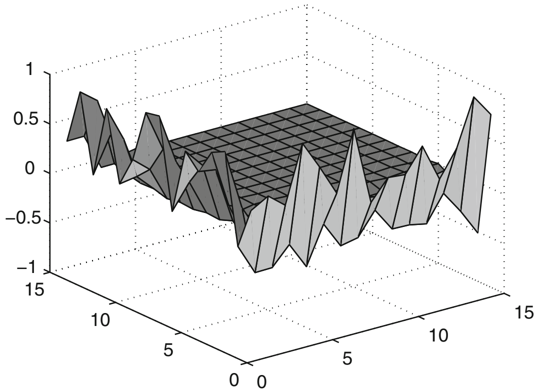
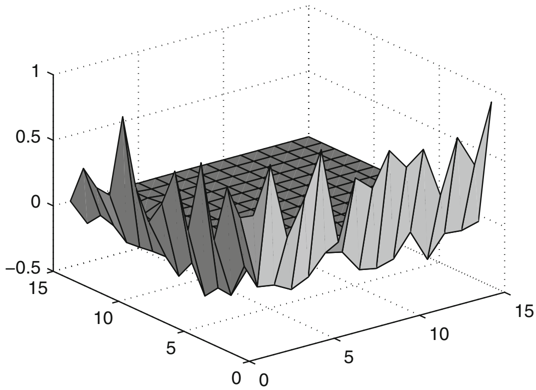


Fig. 2 State variable x_2 of open-loop system in Example 1 ($\gamma = 0.5$)



When $\gamma = 0.5$, using LMI Control Toolbox to solve LMIs (18) in Corollary 1, we obtain the following feasible solutions:

$$X=\begin{bmatrix} 66.8992 & -26.1721 \\ -26.1721 & 70.1356 \end{bmatrix}, \quad Y=\begin{bmatrix} 28.8264 & -9.1054 \\ -9.1054 & 38.7482 \end{bmatrix}, \quad Z=\begin{bmatrix} 48.2373 & -5.7484 \\ -5.7484 & 73.1166 \end{bmatrix}.$$

By solving LMIs (4) and (5) in Theorem 1, our results give the following feasible solutions:

$$X_1 = \begin{bmatrix} 0.7461 & -0.0484 \\ -0.0484 & 0.8671 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.4919 & -0.0227 \\ -0.0227 & 0.5640 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.2556 & -0.0262 \\ -0.0262 & 0.2920 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.1607 & -0.0070 \\ -0.0070 & 0.1291 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} 0.6641 & -0.0257 \\ -0.0257 & 0.8673 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.4298 & -0.0059 \\ -0.0059 & 0.5641 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.9214 & -0.1265 \\ -0.1265 & 2.3185 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1.2431 & -0.0427 \\ -0.0427 & 1.3862 \end{bmatrix}.$$

Figures 1 and 2 show the state variables of the above system. It shows that both the basis-dependent and basis-independent results can guarantee stability for the system $\mathscr{S}$ in (2).

Fig. 3 State variable x_1 of open-loop system in Example 1 ($\gamma = 0.8$)

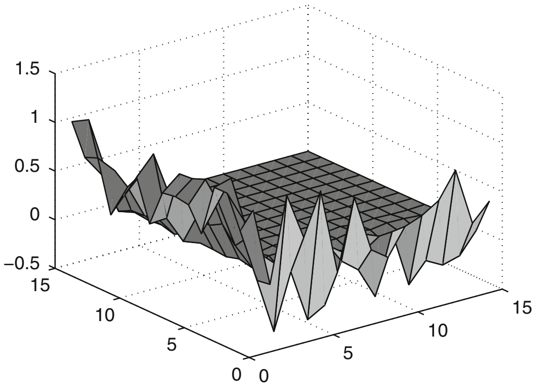
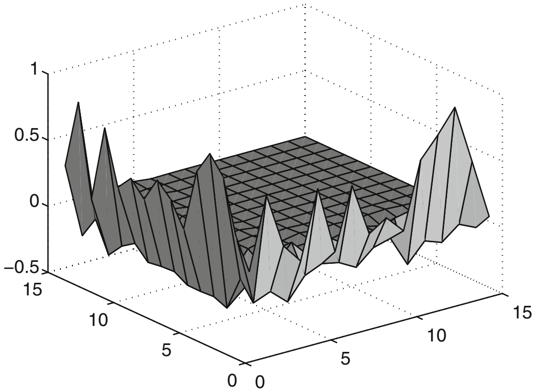


Fig. 4 State variable x_2 of open-loop system in Example 1 ($\gamma = 0.8$)



When $\gamma = 0.8$, using LMI Control Toolbox to solve LMIs (18) in Corollary 1, the LMIs are infeasible. However, by solving LMIs (4) and (5) in Theorem 1, our results give the following feasible solutions:

$$X_1 = \begin{bmatrix} 1.2577 & -0.4595 \\ -0.4595 & 1.0741 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.7389 & -0.2237 \\ -0.2237 & 0.7193 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.5106 & -0.1041 \\ -0.1041 & 0.4019 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.2872 & -0.0292 \\ -0.0292 & 0.2010 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} 0.7286 & -0.0713 \\ -0.0713 & 1.1435 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.5048 & -0.0417 \\ -0.0417 & 0.7484 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 3.0074 & -0.7390 \\ -0.7390 & 3.0215 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1.8181 & -0.3237 \\ -0.3237 & 1.8696 \end{bmatrix}.$$

Figures 3 and 4 show the state variables of the above system. It shows that the basis-dependent results can guarantee stability for the system $\mathscr{S}$ in (2), while basis-independent results cannot. Therefore results based on basis-dependent Lyapunov functions are less conservative than those based on single quadratic Lyapunov functions.

Example 2 Consider a 2-D system with two variables $x_{i,j+1}$, $x_{i+1,j}$ and the following system matrices

$$A_1 = \begin{bmatrix} 0.5\sin(x_{1(i,j+1)}) & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}. \tag{35}$$

For simplicity, we assume that

$$x_{(i,j)} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We have the fuzzy model for the nonlinear system

$$x_{i+1,j+1} = \sum_{k=1}^{2} h_k(\theta_{i,j}) \left\{ A_{1k}x_{i,j+1} + A_{2k}x_{i+1,j} + B_{1k}u_{i,j+1} + B_{2k}u_{i+1,j} \right\}, \tag{36}$$

with

$$A_{11} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$A_{21} = A_{22} = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_{11} = B_{12} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix},$$
$$B_{21} = B_{22} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

where

$$h_1(\theta_{i,j}) = \frac{1 + \sin(x_{1(i,j+1)})}{2},$$
$$h_2(\theta_{i,j}) = \frac{1 - \sin(x_{1(i,j+1)})}{2}.$$

Figures 5 and 6 show the state variables of the above system. It can be seen that the open-loop system is not asymptotically stable. Our purpose now is to design a controller

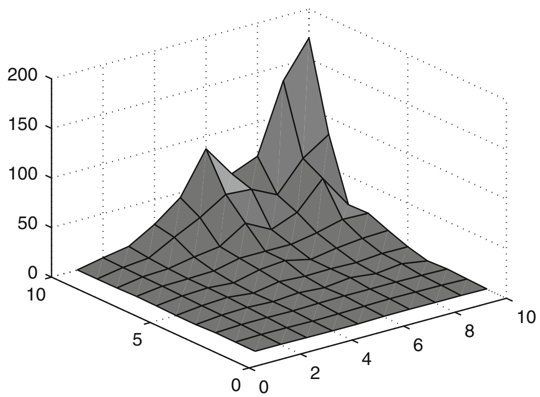


Fig. 5 State variable x_1 of open-loop system in Example 2

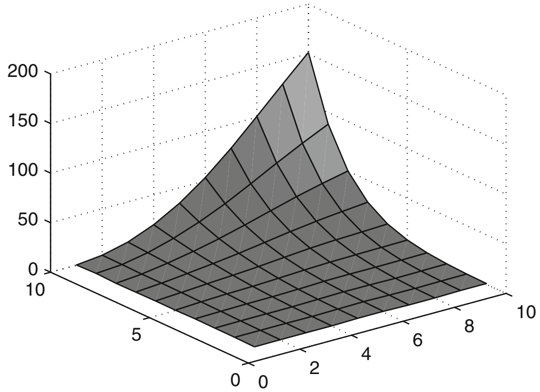


Fig. 6 State variable x_2 of open-loop system in Example 2

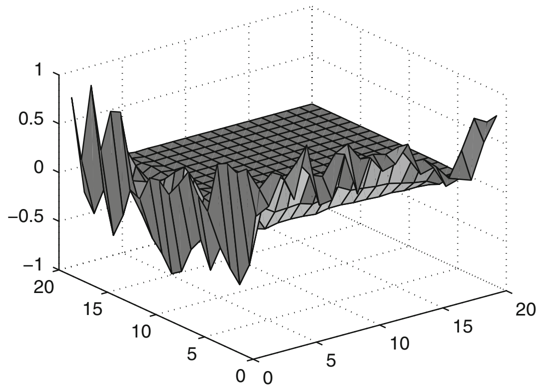


Fig. 7 State variable x_1 of closed-loop system in Example 2

such that the closed-loop system is asymptotically stable. By solving LMIs (28) and (29) in Theorem 3, we can obtain a feasible solution with

$$X_1 = \begin{bmatrix} 0.1349 & -0.0277 \\ -0.0277 & 0.0980 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.0634 & 0.0172 \\ 0.0172 & 0.0689 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 0.0363 & 0.0109 \\ 0.0109 & 0.0389 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.0545 & 0.0145 \\ 0.0145 & 0.0451 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} 0.0530 & 0.0156 \\ 0.0156 & 0.0677 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0590 & 0.0094 \\ 0.0094 & 0.0683 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.2605 & 0.0098 \\ 0.0098 & 0.2436 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.2314 & 0.0557 \\ 0.0557 & 0.2275 \end{bmatrix}.$$

Then, from (30), the corresponding controller gain matrices are given by

$$F_1 = [\,0.0192 \quad 1.0082\,],$$
$$F_2 = [\,0.1642 \quad 1.0791\,].$$

Figures 7 and 8 show that the state variables of the closed-loop system converge to zero. This shows that the PDC fuzzy controller designed in the paper can stabilize the originally unstable system.

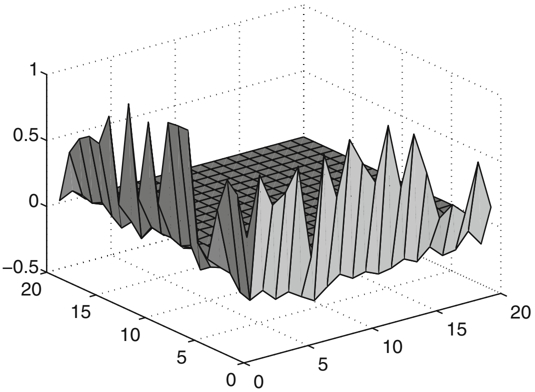


Fig. 8 State variable x_2 of closed-loop system in Example 2

6 Conclusions

In this paper, we have investigated the problem of stability analysis and stabilization for 2-D fuzzy discrete systems. The 2-D fuzzy system model is established based on the FMLSS model, based on which nonquadratic stability conditions are derived. Then by introducing an additional instrumental matrix variable, a control design approach is proposed based on basis-dependent Lyapunov functions. Both the stability and controller existence conditions are expressed as LMI conditions, which can be solved efficiently. Two illustrative examples have been used to show the advantage and effectiveness of the obtained results.

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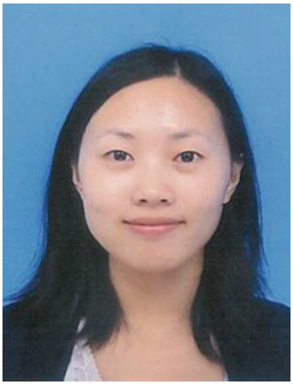

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