Multimedia Tools and Applications (2023) 82:41767–41790
https://doi.org/10.1007/s11042-023-14532-9

Einstein exponential operation laws of spherical fuzzy sets and aggregation operators in decision making

D. Ajay[1] · Ganeshsree Selvachandran[2,3] · J. Aldring[1,4] · Pham Huy Thong[5] · Le Hoang Son[5] · Bui Cong Cuong[6]

Received: 24 March 2021 / Revised: 21 May 2021 / Accepted: 31 January 2023 /
Published online: 12 April 2023

Abstract
The spherical fuzzy set (SFS) model is one of the newly developed extensions of fuzzy sets (FS) for the purpose of dealing with uncertainty or vagueness in decision making. The aim of this paper is to define new exponential and Einstein exponential operational laws for spherical fuzzy sets and their corresponding aggregation operators. We introduce the operational laws for exponential and Einstein exponential SFSs in which the base values are crisp numbers and the exponents (weights) are spherical fuzzy numbers. Some of the properties and characteristics of the proposed operations are then discussed. Based on these operational laws, some new aggregation operators for the SFS model, namely Spherical Fuzzy Weighted Exponential Averaging (SFWEA) and Spherical Fuzzy Einstein Weighted Exponential Averaging (SFEWEA) operators are introduced. Finally, a decision-making algorithm based on these newly introduced aggregation operators is proposed and applied to a multi-criteria decision making (MCDM) problem related to ranking different types of psychotherapy.

Keywords Spherical fuzzy set · Exponential operational laws · Einstein exponential operational laws · Aggregate operator · Decision making

1 Introduction

The concept of fuzzy set theory [40] was developed by Zadeh in 1965 by assigning membership grade values to each elements of the set in the interval [0, 1] and it is utilized for describing situations where results are imprecise. This traditional fuzzy set has been applied in several fields like decision making, clustering analysis, pattern recognition and medical diagnosis. Unfortunately, this traditional fuzzy set theory deals only with positive membership degree of elements. To overcome this limitation, Atanassov introduced negative membership or non-membership degree function to cover the gaps in the fuzzy set theory and the resulting set is called intuitionistic fuzzy set [12]. Therefore, the notion of

✉ Bui Cong Cuong
cuongbuicong46@gmail.com

Extended author information available on the last page of the article.

IFS theory is an extension of FS theory. Atanassov handled both the degree of membership ($\xi_{\widetilde{\mathcal{A}}}$) and non-membership ($\psi_{\widetilde{\mathcal{A}}}$) where the sum of both values is less than or equal to one ($\xi_{\widetilde{\mathcal{A}}} + \psi_{\widetilde{\mathcal{A}}} \leq 1$). In some cases, the decision maker's may provide their preference values like $\xi_{\widetilde{\mathcal{A}}} = 0.7$ and $\psi_{\widetilde{\mathcal{A}}} = 0.6$, then clearly it violates the condition of intuitionistic fuzzy set as the sum of these values is more than 1. Therefore to deal with such issues, Yager developed the concept of Pythagorean fuzzy set [37] with the condition that $\xi^2_{\widetilde{\mathcal{A}}} + \psi^2_{\widetilde{\mathcal{A}}} \leq 1$. Definitely, the PFSs handle uncertainties more effectively than IFSs and hence Pythagorean fuzzy set theory became more important and interesting research area. Many aggregation operators have been introduced by Yager and Abbasov [38] to handle MCDM problems in pythagorean fuzzy environment. Neutrosophic set [36] is yet another important generalization of classical fuzzy set and it is further extended to neutrosophic cubic sets [27]. From the literature, it is evident that many contributions have been made on neutrosophic sets (NSs) and neutrosophic cubic sets (NCSs) with their aggregation operators. The theory of NCSs have been discussed by Alia et al. [7] and applied in pattern recognition. Also, Je [39] developed operations and aggregation operations for NCSs. Ajay et al. [2] utilized NCSs to multi criteria decision making (MCDM) with the help of geometric Bonferroni mean operators. More recently, Atta et al. [13] utilized the notion of NSs in an advanced image steganography based on exploiting modification direction.

The concept of spherical fuzzy set (SFS) and its accompanying theory was introduced by Gundogdu et al. [22], and this model is one of the new extensions of fuzzy set theory, which is characterized by triple membership structure that consists of a membership, non-membership and hesitancy function, and their sum of squares is equal to or less than one. The SFS model has the ability to handle uncertainty, imprecision and vagueness in a more efficient way compared to PFSs. A recent review of some of the latest literature shows an increasing trend in studies concerning SFSs. Ashraf et al. [8] developed sequences of aggregation operators in a spherical fuzzy environment, Ashraf et al. [10] introduced a grey method (GRA) based on the novel concept of spherical linguistic fuzzy Choquet integrals, while the logarithmic operator for SFSs have been developed and applied to decision support systems by Jin et al. [26]. Rafiq et al. [31] proposed a cosine similarity measure for the SFS model to enable decision-making in the context of vague and imprecise data, whereas Ashraf et al. [11] introduced a group decision making method for the spherical fuzzy environment and applied this in solving a multi-criteria group decision-making (MCGDM) problem. Gundogdu et al. [24] extended the well-known VIKOR method for the SFS model and applied this to a MCDM problem in a spherical fuzzy environment. Acharjya and Rathi [1] proposed an integrated decision-making method which integrates the fuzzy rough set and genetic algorithm models and applied this in a MCDM problem related to smart agriculture, Sharaff et al. [33, 34] studied a fuzzy-based text summarization extraction method, and proposed a document classification method using fuzzy clustering approach in [33] and [34], respectively. Gou et al. [20] defined the exponential operational laws for IFSs and introduced several new aggregation operators for the IFS model, while Garg [18] introduced new exponential operational laws for the PFS model and aggregation operators based on these newly defined exponential operational laws to better handle the uncertainties, impreciseness and vagueness of information. Furthermore, exponential operational laws of PFSs have been used to form projection models for decision making in Borg et al. [16], while Haque et al. [25] applied the concept of exponential operational laws to generalized SFSs.

Akram et al. [4] studied the concept of spherical fuzzy graphs and introduced some results on the symmetric difference, rejection, degree and total degrees for spherical fuzzy graphs, while Ashraf et al. [9] established a new integrated approach based on the well-known MCDM methods of Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) and Complex Proportional Assessment of Alternatives (COPRAS) methods, and applied this to a MCGDM problem related to emergency response to the COVID-19 pandemic. Quek et al. [30] developed new operational laws for the T-SFS model and introduced two types of Einstein aggregation operators for this models, and subsequently applied these to a multi-attribute multi-perception decision-making problem related to the degree of pollution in five major cities in China. Aydogdu and Gul [14] proposed a novel entropy measure for the SFS model and compared the performance of this measure with other existing measures in literature, while Shishavan et al. [35] introduced the Jaccard, exponential and square root cosine similarity measures in a spherical fuzzy environment and applied these measures to MCDM problems related to medical diagnosis and supplier selection. Ali et al. [6] introduced the novel concept of complex T-SFSs and their operational laws and went on to introduce two new aggregation operators for this model, whereas Garg et al. [19] proposed the concept of power aggregation operators for the T-SFS model and introduced a MCDM algorithm based on these aggregation operators. Liu et al. [28] on the other hand, defined the concept of linguistic T-spherical fuzzy numbers and proposed a weighted aggregation operator and two new MCDM algorithms for this concept, while Guleria and Bajaj [21] introduced the T-spherical fuzzy soft set model and its aggregation operators.

In terms of the application of the SFS-based models in MCDM methods, Sharaf and Khalil [32] extended the well-known MCDM method of Tomada de Decisao Interactive e Multicriterio (TODIM) to the spherical fuzzy environment to enable the hesitation degree of the decision-makers to be expressed independently while Mathew et al. [29] proposed a new decision-making framework that combines the well-known analytic hierarchy process (AHP) and TOPSIS methods in a spherical fuzzy environment. Gundogdu and Kahraman [23] introduced the concept of interval-valued SFSs (IV-SFS) and defined some important accompanying concepts for this model such as the score and accuracy functions, and the arithmetic and geometric mean operators. The authors also went on to propose an IV-SFS based TOPSIS method and applied this to solve a MCDM problem related to the selection of 3D printers. Barukab et al. [15] established an enhanced TOPSIS-based method for the SFS model for the handling of MCGDM problems and defined a generalized distance measure for SFSs based on spherical fuzzy entropy to compute the unknown weights of the criteria, whereas Farrokhizadeh et al. [17] expanded the original maximizing deviation method to the spherical fuzzy environment using single-valued and interval-valued SFSs to determine the weights of the criteria. Akram et al. [3] proposed four new aggregation operators for the complex SFS model and used these to extend the multi-criteria optimization and compromise solution (VIKOR) method to the complex spherical fuzzy environment, whereas Ali et al. [5] introduced a TOPSIS method based on the complex SFS model as well as two types of Bonferroni mean aggregation operators for the complex SFS model.

Motivated by the fast progressing studies related to the theory of SFSs as expounded above, the main objective of this paper is to define exponential and Einstein exponential operational laws for the SFS model which would yield results in the extended closed interval of one and two as well as to develop exponential and Einstein exponential spherical fuzzy aggregation operators.

The remaining part of the paper is organized as follows. Section 2 provides an overview of the important background knowledge pertaining to the SFS model and their operations, all of which are needed to facilitate the introduction of the new concepts of exponential operational laws and aggregation operators in the subsequent sections. The exponential operational laws for the SFS model, and the corresponding operations and properties are introduced in Section 3. This is followed by the introduction of new exponential spherical fuzzy aggregation operators. The Einstein exponential operation laws for the SFS model, their corresponding operations and properties, as well as the Einstein exponential spherical fuzzy aggregation operators are defined in Section 4. A MCDM algorithm based on the newly introduced spherical fuzzy aggregation operators is presented in Section 5. In Section 6, the applicability and utility of the newly introduced spherical fuzzy aggregation operators is demonstrated through the application of these operators in a MCDM problem related to the ranking of different types of psychotherapy that are available for the emotional problems faced by youngsters. To validate the proposed MCDM method, a comparative analysis is presented in Section 7 in which the results obtained using our proposed method is compared with the results obtained using fuzzy-based MCDM methods in the existing literature. Concluding remarks are presented in Section 8, followed by acknowledgements and the list of references.

2 Preliminaries

In this section, we discuss some of the basic concepts of fuzzy sets.

Definition 1 [40] A fuzzy set $\widetilde{\mathcal{F}}$ is defined on a universe of discourse U as the form:

$$\widetilde{\mathcal{F}} = \left\{ \left\langle \xi_{\widetilde{\mathcal{F}}}(\dot{x}) \right\rangle | \dot{x} \in U \right\},$$

where $\xi_{\widetilde{\mathcal{F}}}(\dot{x}) : U \to [0, 1]$. Here $\xi_{\widetilde{\mathcal{F}}}(\dot{x})$ denotes membership function to each $\dot{x}$.

Definition 2 [12] An IFS $\widetilde{\mathcal{A}}$ is defined as a set of ordered pairs over a universal set U given by

$$\widetilde{\mathcal{A}} = \left\{ \left\langle x, \left(\xi_{\widetilde{\mathcal{A}}}(x), \psi_{\widetilde{\mathcal{A}}}(x), \right) \right\rangle | x \in U \right\}$$

where $\xi_{\widetilde{\mathcal{A}}}(x) : U \to [0, 1]$, $\psi_{\widetilde{\mathcal{A}}}(x) : U \to [0, 1]$ and satisfy the condition $\xi_{\widetilde{\mathcal{A}}}(x) + \psi_{\widetilde{\mathcal{A}}}(x) \leq 1$ for each element $x \in U$. Here the membership and non-membership functions are denoted as $\xi_{\widetilde{\mathcal{A}}}(x)$ and $\psi_{\widetilde{\mathcal{A}}}(x)$ respectively.

Definition 3 [22] Let $\widetilde{\mathcal{A}}_s$ be a spherical fuzzy set in the universe of discourse U be defined by

$$\widetilde{\mathcal{A}}_s = \left\{ \left\langle x, \left(\xi_{\widetilde{\mathcal{A}}_s}(x), \psi_{\widetilde{\mathcal{A}}_s}(x), \pi_{\widetilde{\mathcal{A}}_s}(x) \right) \right\rangle | x \in U \right\} \quad (1)$$

where $\xi_{\widetilde{\mathcal{A}}_s}(x) : U \to [0, 1]$, $\psi_{\widetilde{\mathcal{A}}_s}(x) : U \to [0, 1]$, $\pi_{\widetilde{\mathcal{A}}_s}(x) : U \to [0, 1]$ and $0 \leq \xi^2_{\widetilde{\mathcal{A}}_s}(x) + \pi^2_{\widetilde{\mathcal{A}}_s}(x) + \pi^2_{\widetilde{\mathcal{A}}_s}(x) \leq 1$ for each x, the values $\xi_{\widetilde{\mathcal{A}}_s}(x), \psi_{\widetilde{\mathcal{A}}_s}(x), \pi_{\widetilde{\mathcal{A}}_s}(x)$ are membership, non-membership and hesitancy function of x in $\widetilde{\mathcal{A}}_s$ respectively.

Definition 4 [22] The score function and accuracy function of spherical fuzzy sets are defined respectively as $S\left(\widetilde{\mathcal{A}}_s(x)\right) = \left(\xi_{\widetilde{\mathcal{A}}_s}(x) - \pi_{\widetilde{\mathcal{A}}_s}(x)\right)^2 - \left(\psi_{\widetilde{\mathcal{A}}_s}(x) - \pi_{\widetilde{\mathcal{A}}_s}(x)\right)^2$ and $A\left(\widetilde{\mathcal{A}}_s(x)\right) = \xi^2_{\widetilde{\mathcal{A}}_s}(x) + \psi^2_{\widetilde{\mathcal{A}}_s}(x) + \pi^2_{\widetilde{\mathcal{A}}_s}(x)$

Definition 5 [22] The basic operations of spherical fuzzy numbers are as defined below.

$$\begin{aligned}\widetilde{\mathcal{A}}_s \oplus \widetilde{\mathcal{B}}_s &= \Big\{\sqrt{\xi^2_{\widetilde{\mathcal{A}}_s}(x) + \xi^2_{\widetilde{\mathcal{B}}_s}(x) - \xi^2_{\widetilde{\mathcal{A}}_s}(x)\xi^2_{\widetilde{\mathcal{A}}_s}(x)}, \quad \psi_{\widetilde{\mathcal{A}}_s}(x)\psi_{\widetilde{\mathcal{B}}_s}(x), \\ &\sqrt{\left(1-\xi^2_{\widetilde{\mathcal{B}}_s}(x)\right)\pi^2_{\widetilde{\mathcal{A}}_s}(x) + \left(1-\xi^2_{\widetilde{\mathcal{A}}_s}(x)\right)\pi^2_{\widetilde{\mathcal{B}}_s}(x) - \pi^2_{\widetilde{\mathcal{A}}_s}(x)\pi^2_{\widetilde{\mathcal{B}}_s}(x)}\Big\}\end{aligned} \quad (2)$$

$$\begin{aligned}\widetilde{\mathcal{A}}_s \otimes \widetilde{\mathcal{B}}_s &= \Big\{\xi_{\widetilde{\mathcal{A}}_s}(x).\xi_{\widetilde{\mathcal{B}}_s}(x), \sqrt{\psi^2_{\widetilde{\mathcal{A}}_s}(x) + \psi^2_{\widetilde{\mathcal{B}}_s}(x) - \psi^2_{\widetilde{\mathcal{A}}_s}(x)\psi^2_{\widetilde{\mathcal{A}}_s}(x)}, \\ &\sqrt{\left(1-\psi^2_{\widetilde{\mathcal{B}}_s}(x)\right)\pi^2_{\widetilde{\mathcal{A}}_s}(x) + \left(1-\psi^2_{\widetilde{\mathcal{A}}_s}(x)\right)\pi^2_{\widetilde{\mathcal{B}}_s}(x) - \pi^2_{\widetilde{\mathcal{A}}_s}(x)\pi^2_{\widetilde{\mathcal{B}}_s}(x)}\Big\}\end{aligned} \quad (3)$$

$$\lambda.\widetilde{\mathcal{A}}_s = \left\{\sqrt{1-\left(1-\xi^2_{\widetilde{\mathcal{A}}_s}(x)\right)^{\lambda}}, \quad \psi^{\lambda}_{\widetilde{\mathcal{A}}_s}(x), \quad \sqrt{\left(1-\xi^2_{\widetilde{\mathcal{A}}_s}(x)\right)^{\lambda} - \left(1-\xi^2_{\widetilde{\mathcal{A}}_s}(x) - \pi^2_{\widetilde{\mathcal{A}}_s}(x)\right)^{\lambda}}\right\}, \lambda > 0 \quad (4)$$

$$\widetilde{\mathcal{A}}^{\lambda}_s = \left\{\xi^{\lambda}_{\widetilde{\mathcal{A}}_s}(x), \quad \sqrt{1-\left(1-\psi^2_{\widetilde{\mathcal{A}}_s}(x)\right)^{\lambda}}, \quad \sqrt{\left(1-\psi^2_{\widetilde{\mathcal{A}}_s}(x)\right)^{\lambda} - \left(1-\psi^2_{\widetilde{\mathcal{A}}_s}(x) - \pi^2_{\widetilde{\mathcal{A}}_s}(x)\right)^{\lambda}}\right\}, \lambda > 0 \quad (5)$$

3 Exponential operational laws of SFSs

Here, this section defines new exponential operational laws of SFSs and their operations.

Definition 6 Let U be the universe of discourse, and $\widetilde{\beta}_s = \left\langle \xi_{\widetilde{\beta}_s}, \psi_{\widetilde{\beta}_s}, \pi_{\widetilde{\beta}_s} \right\rangle$ be a spherical fuzzy number (SFN), then the exponential operation of $\widetilde{\beta}_s$ is defined as

$$\lambda^{\widetilde{\beta}_s} = \begin{cases} \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_s}}}, \sqrt{1-\lambda^{2\pi_{\widetilde{\beta}_s}}} \right\rangle; \lambda \in (0,1) \\ \left\langle \left(\frac{1}{\lambda}\right)^{\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}, \sqrt{1-\left(\frac{1}{\lambda}\right)^{2\psi_{\widetilde{\beta}_s}}}, \sqrt{1-\left(\frac{1}{\lambda}\right)^{2\pi_{\widetilde{\beta}_s}}} \right\rangle; \lambda \geq 1 \end{cases} \quad (6)$$

Theorem 1 *For any SFN $\widetilde{\beta}_s$, the value of $\lambda^{\widetilde{\beta}_s}$ is an SFN in the extended interval* [1, 2].

Proof Let $\widetilde{\beta}_s = \left\langle \xi_{\widetilde{\beta}_s}, \psi_{\widetilde{\beta}_s}, \pi_{\widetilde{\beta}_s} \right\rangle$ be an SFN, where $\xi_{\widetilde{\beta}_s}$, $\psi_{\widetilde{\beta}_s}$ and $\pi_{\widetilde{\beta}_s}$ belong to [0, 1] with the condition that $0 \leq \xi^2_{\widetilde{\beta}_s} + \psi^2_{\widetilde{\beta}_s} + \pi^2_{\widetilde{\beta}_s} \leq 1$

Case(i): Let $\lambda \in (0,1)$, then the values of $\lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}$, $\sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_s}}}$ and $\sqrt{1-\lambda^{2\pi_{\widetilde{\beta}_s}}}$ lie in [0, 1] statisfying the condition that $1 \leq \left(\lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}\right)^2 + \left(\sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_s}}}\right)^2 + \left(\sqrt{1-\lambda^{2\pi_{\widetilde{\beta}_s}}}\right)^2 \leq 2$

Case(ii): When $\lambda \geq 1$ and $0 \leq \frac{1}{\lambda} \leq 1$ and it is obvious that $\lambda^{\widetilde{\beta}_s}$ is also an SFN.

Hence, based on the two cases, it follows that the values of $\lambda^{\widetilde{\beta}_s}$ are SFNs in the extended interval [1, 2]. □

Example 1 Let $\widetilde{\beta}_s = \langle 0.73, 0.28, 0.32\rangle$ be an SFN and $\lambda = 0.42$, then $\lambda^{\widetilde{\beta}_s} = 0.42^{\langle 0.73,0.28,0.32\rangle}$

$$\begin{aligned} 0.42^{\langle 0.73,0.28,0.32\rangle} &= \left\langle 0.42^{\sqrt{1-0.73^2}}, \sqrt{1-0.42^{2\times 0.28}}, \sqrt{1-0.42^{2\times 0.32}} \right\rangle \\ &= \langle 0.5527, 0.6203, 0.6527\rangle \end{aligned}$$

If $\lambda = 4$, it follows that $\left(\frac{1}{\lambda}\right)^{\widetilde{\beta}_s} = \left(\frac{1}{4}\right)^{\langle 0.73,0.28,0.32\rangle}$

$$\begin{aligned} \left(\frac{1}{4}\right)^{\langle 0.73,0.28,0.32\rangle} &= \left\langle \left(\frac{1}{4}\right)^{\sqrt{1-0.73^2}}, \sqrt{1-\left(\frac{1}{4}\right)^{2\times 0.28}}, \sqrt{1-\left(\frac{1}{4}\right)^{2\times 0.32}} \right\rangle \\ &= \langle 0.3877, 0.7348, 0.7669\rangle \end{aligned}$$

Further, we enumerate some of the basic operations on $\lambda^{\widetilde{\beta}_s}$.

Definition 7 Let $\widetilde{\beta}_{s_1}$ and $\widetilde{\beta}_{s_2}$ be two spherical fuzzy numbers. Then the basic exponential operational laws are as given below:

$$\lambda^{\widetilde{\beta}_{s_1}} \oplus \lambda^{\widetilde{\beta}_{s_2}} = \left\{ \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)},\ \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)}, \right.\right.$$
$$\left.\left. \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)-\left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}}-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)\left(\lambda^{2\pi_{\widetilde{\beta}_{s_2}}}-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)} \right\rangle \right\}$$

$$\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}} = \left\{ \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}+\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}}, \right.\right.$$
$$\left.\left. \sqrt{\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}-\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}-\lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}-\lambda^{2\pi_{\widetilde{\beta}_{s_2}}}\right)} \right\rangle \right\}$$

$$\mathcal{K}.\lambda^{\widetilde{\beta}_s} = \left\{ \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_s}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}\right)^{\mathcal{K}}-\left(\lambda^{2\pi_{\widetilde{\beta}_s}}-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_s}}}\right)^{\mathcal{K}}} \right\rangle \right\}$$

$$\left[\lambda^{\widetilde{\beta}_s}\right]^{\mathcal{K}} = \left\{ \left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}}\right)^{\mathcal{K}}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_s}}\right)^{\mathcal{K}}}, \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_s}}\right)^{\mathcal{K}}-\left(1-\lambda^{2\psi_{\widetilde{\beta}_s}}-\lambda^{2\pi_{\widetilde{\beta}_s}}\right)^{\mathcal{K}}} \right\rangle \right\}, \forall \mathcal{K} > 0.$$

Theorem 2 *Let* $\widetilde{\beta}_{s_1} = \left\langle \xi_{\widetilde{\beta}_{s_1}}, \psi_{\widetilde{\beta}_{s_1}}, \pi_{\widetilde{\beta}_{s_1}} \right\rangle$ *and* $\widetilde{\beta}_{s_2} = \left\langle \xi_{\widetilde{\beta}_{s_2}}, \psi_{\widetilde{\beta}_{s_2}}, \pi_{\widetilde{\beta}_{s_2}} \right\rangle$ *be spherical fuzzy numbers (SFNs) and* $\lambda \in (0, 1)$. *Then the following holds:*

(i) $\lambda^{\widetilde{\beta}_{s_1}} \oplus \lambda^{\widetilde{\beta}_{s_2}} = \lambda^{\widetilde{\beta}_{s_2}} \oplus \lambda^{\widetilde{\beta}_{s_1}}$
(ii) $\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}} = \lambda^{\widetilde{\beta}_{s_2}} \otimes \lambda^{\widetilde{\beta}_{s_1}}$

Theorem 3 *Let* $\widetilde{\beta}_{s_i} = \left\langle \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}} \right\rangle$ *for* $i = 1, 2, 3$ *be three spherical fuzzy numbers (SFNs) and* $\lambda \in (0, 1)$. *Then the following holds:*

(i) $\left(\lambda^{\widetilde{\beta}_{s_1}} \oplus \lambda^{\widetilde{\beta}_{s_2}}\right) \oplus \lambda^{\widetilde{\beta}_{s_3}} = \lambda^{\widetilde{\beta}_{s_1}} \oplus \left(\lambda^{\widetilde{\beta}_{s_2}} \oplus \lambda^{\widetilde{\beta}_{s_3}}\right)$

(ii) $\left(\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}}\right) \otimes \lambda^{\widetilde{\beta}_{s_3}} = \lambda^{\widetilde{\beta}_{s_1}} \otimes \left(\lambda^{\widetilde{\beta}_{s_2}} \otimes \lambda^{\widetilde{\beta}_{s_3}}\right)$

Proof The proof of Theorems 2 and 3 are straightforward from Definition 3. □

Theorem 4 *Let* $\widetilde{\beta}_{s_1} = \left\langle \xi_{\widetilde{\beta}_{s_1}}, \psi_{\widetilde{\beta}_{s_1}}, \pi_{\widetilde{\beta}_{s_1}} \right\rangle$ *and* $\widetilde{\beta}_{s_2} = \left\langle \xi_{\widetilde{\beta}_{s_2}}, \psi_{\widetilde{\beta}_{s_2}}, \pi_{\widetilde{\beta}_{s_2}} \right\rangle$ *be spherical fuzzy numbers (SFNs),* $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 > 0$ *Then be three real numbers, and* $\lambda, \lambda_1, \lambda_2 \in (0, 1)$*, then Then the following holds:*

(i) $\mathcal{K}(\lambda^{\widetilde{\beta}_{s_1}} \oplus \lambda^{\widetilde{\beta}_{s_2}}) = \mathcal{K}(\lambda^{\widetilde{\beta}_{s_1}}) \oplus \mathcal{K}(\lambda^{\widetilde{\beta}_{s_2}})$;
(ii) $(\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}})^{\mathcal{K}} = (\lambda^{\widetilde{\beta}_{s_1}})^{\mathcal{K}} \otimes (\lambda^{\widetilde{\beta}_{s_2}})^{\mathcal{K}}$;
(iii) $\mathcal{K}_1\lambda^{\widetilde{\beta}_{s_1}} \oplus \mathcal{K}_2\lambda^{\widetilde{\beta}_{s_1}} = (\mathcal{K}_1 + \mathcal{K}_2)\lambda^{\widetilde{\beta}_{s_1}}$;
(iv) $(\lambda^{\widetilde{\beta}_{s_1}})^{\mathcal{K}_1} \otimes (\lambda^{\widetilde{\beta}_{s_1}})^{\mathcal{K}_2} = (\lambda^{\widetilde{\beta}_{s_1}})^{\mathcal{K}_1+\mathcal{K}_2}$;
(v) $(\lambda_1)^{\widetilde{\beta}_{s_1}} \otimes (\lambda_2)^{\widetilde{\beta}_{s_1}} = (\lambda_1\lambda_2)^{\widetilde{\beta}_{s_1}}$.

Proof For two spherical fuzzy numbers $\widetilde{\beta}_{s_1}$ and $\widetilde{\beta}_{s_2}$ by Definition 3, we get

$$\lambda^{\widetilde{\beta}_{s_1}} = \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}}, \sqrt{1-\lambda^{2\pi_{\widetilde{\beta}_{s_1}}}} \right\rangle, \quad \lambda^{\widetilde{\beta}_{s_2}} = \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}}, \sqrt{1-\lambda^{2\pi_{\widetilde{\beta}_{s_2}}}} \right\rangle$$

and hence by using the exponential operational laws given in definition 4, we get

$$\lambda^{\widetilde{\beta}_{s_1}} \oplus \lambda^{\widetilde{\beta}_{s_2}} = \left\{ \left\langle \sqrt{1 - \left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)}, \quad \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)}, \right.\right.$$
$$\left.\left. \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right) - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)\left(\lambda^{2\pi_{\widetilde{\beta}_{s_2}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)} \right\rangle \right\}$$

$$\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}} = \left\{ \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}} + \sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}}, \right.\right.$$
$$\left.\left. \sqrt{\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\lambda^{2\psi_{\widetilde{\beta}_{s_2}}} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_2}}}\right)} \right\rangle \right\}$$

(i) For a real number $\mathcal{K} > 0$, we have

$$\mathcal{K}(\lambda^{\widetilde{\beta}_{s_1}} \oplus \lambda^{\widetilde{\beta}_{s_2}}) = \left\{ \left\langle \sqrt{1 - \left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}}\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}}}, \right.\right.$$
$$\left.\left. \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}}\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)^{\mathcal{K}} - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}}\left(\lambda^{2\pi_{\widetilde{\beta}_{s_2}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)^{\mathcal{K}}} \right\rangle \right\}$$

$$= \left\{ \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}} - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}}} \right\rangle \right.$$

$$\left. \oplus \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}}}, \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)^{\mathcal{K}} - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_2}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}\right)^{\mathcal{K}}} \right\rangle \right\}$$

$$= \mathcal{K}\lambda^{\widetilde{\beta}_{s_1}} \oplus \mathcal{K}\lambda^{\widetilde{\beta}_{s_2}}$$

(ii) For two spherical fuzzy numbers $\widetilde{\beta}_{s_1}$ and $\widetilde{\beta}_{s_2}$, and a real number $\mathcal{K} > 0$, we have

$$\left(\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}}\right)^{\mathcal{K}} = \left\{ \left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}} \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_2}}\right)}}\right)^{\mathcal{K}}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}\left(\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}}}, \right.\right.$$

$$\left.\left. \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}\left(\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}}} \right\rangle \right\}, \lambda > 0$$

$$\left(\lambda^{\widetilde{\beta}_{s_1}} \otimes \lambda^{\widetilde{\beta}_{s_2}}\right)^{\mathcal{K}} = \left\{ \left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}}, \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}}} \right\rangle \otimes \right.$$

$$\left. \left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_2}}\right)}}\right)^{\mathcal{K}}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}}}, \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_2}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_2}}}\right)^{\mathcal{K}}} \right\rangle \right\}$$

$$= \left(\lambda^{\widetilde{\beta}_{s_1}}\right)^{\mathcal{K}} \otimes \left(\lambda^{\widetilde{\beta}_{s_2}}\right)^{\mathcal{K}}$$

(iii) For a spherical fuzzy number $\widetilde{\beta}_{s_1} = \left\langle \xi_{\widetilde{\beta}_{s_1}}, \psi_{\widetilde{\beta}_{s_1}}, \pi_{\widetilde{\beta}_{s_1}} \right\rangle$, and real numbers $\mathcal{K}_1, \mathcal{K}_2 > 0$,

$$\mathcal{K}_1\lambda^{\widetilde{\beta}_{s_1}} \oplus \mathcal{K}_2\lambda^{\widetilde{\beta}_{s_1}} = \left\{ \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_1}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1}}, \right.\right.$$

$$\left. \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_1} - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_1}} \right\rangle \oplus$$

$$\left. \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_2}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2}}, \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_2} - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_2}} \right\rangle \right\}$$

$$= \left\{ \left\langle \sqrt{1-\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_1+\mathcal{K}_2}}, \sqrt{\left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1+\mathcal{K}_2}}, \right.\right.$$

$$\left.\left. \sqrt{\left(1-\lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_1+\mathcal{K}_2} - \left(\lambda^{2\pi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}\right)^{\mathcal{K}_1+\mathcal{K}_2}} \right\rangle \right\}$$

$$= (\mathcal{K}_1 + \mathcal{K}_2)\lambda^{\widetilde{\beta}_{s_1}}$$

(iv) For a SFN $\widetilde{\beta}_{s_1} = \left\langle \xi_{\widetilde{\beta}_{s_1}}, \psi_{\widetilde{\beta}_{s_1}}, \pi_{\widetilde{\beta}_{s_1}} \right\rangle$, and real numbers $\mathcal{K}_1, \mathcal{K}_2 > 0$.

$$\left(\lambda^{\widetilde{\beta}_{s_1}}\right)^{\mathcal{K}_1} = \left\{\left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}_1}, \sqrt{1-\left(\lambda^{\psi^2_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1}}, \sqrt{\left(\lambda^{\psi^2_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1}} \right\rangle\right\}, \forall \mathcal{K}_1 > 0.$$

$$\left(\lambda^{\widetilde{\beta}_{s_1}}\right)^{\mathcal{K}_2} = \left\{\left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}_2}, \sqrt{1-\left(\lambda^{\psi^2_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2}}, \sqrt{\left(\lambda^{\psi^2_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2}} \right\rangle\right\}, \forall \mathcal{K}_2 > 0.$$

and hence

$$\left(\lambda^{\widetilde{\beta}_{s_1}}\right)^{\mathcal{K}_1} \otimes \left(\lambda^{\widetilde{\beta}_{s}}\right)^{\mathcal{K}_2} = \left\{\left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}_1}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1}}, \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1}} \right\rangle\right.$$

$$\left. \otimes \left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}_2}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2}}, \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_2}} \right\rangle\right\}$$

$$= \left\{\left\langle \left(\lambda^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}\right)^{\mathcal{K}_1+\mathcal{K}_2}, \sqrt{1-\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1+\mathcal{K}_2}}, \sqrt{\left(\lambda^{2\psi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1+\mathcal{K}_2} - \left(1-\lambda^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda^{2\pi_{\widetilde{\beta}_{s_1}}}\right)^{\mathcal{K}_1+\mathcal{K}_2}} \right\rangle\right\}$$

$$= \left(\lambda^{\widetilde{\beta}_{s_1}}\right)^{\mathcal{K}_1+\mathcal{K}_2}.$$

(v) For a SFN $\widetilde{\beta}_{s_1} = \left\langle \xi_{\widetilde{\beta}_{s_1}}, \psi_{\widetilde{\beta}_{s_1}}, \pi_{\widetilde{\beta}_{s_1}} \right\rangle$, and real numbers $\lambda_1, \lambda_2 > 0$,

$$(\lambda_1)^{\widetilde{\beta}_{s_1}} \otimes (\lambda_2)^{\widetilde{\beta}_{s_1}} = \left\{\left\langle (\lambda_1)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}, \sqrt{1-{\lambda_1}^{\psi^2_{\widetilde{\beta}_{s_1}}}}, \sqrt{1-{\lambda_1}^{\pi^2_{\widetilde{\beta}_{s_1}}}} \right\rangle\right.$$

$$\left. \otimes \left\langle (\lambda_2)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}, \sqrt{1-{\lambda_2}^{\psi^2_{\widetilde{\beta}_{s_1}}}}, \sqrt{1-{\lambda_2}^{\pi^2_{\widetilde{\beta}_{s_1}}}} \right\rangle\right\}$$

$$= \left\{\left\langle (\lambda_1\lambda_2)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}}, \sqrt{1-\left({\lambda_1}^{\psi^2_{\widetilde{\beta}_{s_1}}}\right)\left({\lambda_2}^{\psi^2_{\widetilde{\beta}_{s_1}}}\right)}, \right.\right.$$

$$\left.\left. \sqrt{{\lambda_1}^{2\psi_{\widetilde{\beta}_{s_1}}}{\lambda_2}^{2\psi_{\widetilde{\beta}_{s_1}}} - \left(1-{\lambda_1}^{2\psi_{\widetilde{\beta}_{s_1}}} - {\lambda_1}^{2\pi_{\widetilde{\beta}_{s_1}}}\right)\left(1-{\lambda_2}^{2\psi_{\widetilde{\beta}_{s_1}}} - {\lambda_2}^{2\pi_{\widetilde{\beta}_{s_1}}}\right)} \right\rangle\right\}$$

$$= (\lambda_1\lambda_2)^{\widetilde{\beta}_{s_1}}$$

□

Theorem 5 *Let* $\widetilde{\beta}_s = \left\langle \xi_{\widetilde{\beta}_s}, \psi_{\widetilde{\beta}_s}, \pi_{\widetilde{\beta}_s} \right\rangle$ *be an SFN, and* $\lambda_1, \lambda_2 > 0$*. When* $\lambda_1 \geq \lambda_2$*, we can obtain* $(\lambda_1)^{\widetilde{\beta}_s} \geq (\lambda_2)^{\widetilde{\beta}_s}$ *for* $\lambda_1, \lambda_2 \in (0, 1)$ *and* $(\lambda_1)^{\widetilde{\beta}_s} \leq (\lambda_2)^{\widetilde{\beta}_s}$ *for* $\lambda_1, \lambda_2 \geq 1$.

Proof If $\lambda_1 \geq \lambda_2$ and $\lambda_1, \lambda_2 \in (0, 1)$, then by the exponential operational laws of SFN, we have

$$\left\{\begin{array}{l}(\lambda_1)^{\widetilde{\beta}_s} = \left\langle (\lambda_1)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}}, \sqrt{1-\lambda_1^{\psi^2_{\widetilde{\beta}_s}}}, \sqrt{1-\lambda_1^{\pi^2_{\widetilde{\beta}_s}}} \right\rangle \\ (\lambda_2)^{\widetilde{\beta}_s} = \left\langle (\lambda_2)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}}, \sqrt{1-\lambda_2^{\psi^2_{\widetilde{\beta}_s}}}, \sqrt{1-\lambda_2^{\pi^2_{\widetilde{\beta}_s}}} \right\rangle \end{array}\right\}$$

The score values of $(\lambda_1)^{\widetilde{\beta}_s}$ and $(\lambda_2)^{\widetilde{\beta}_s}$ is denoted respectively as $S((\lambda_1)^{\widetilde{\beta}_s})$ and $S((\lambda_2)^{\widetilde{\beta}_s})$ are defined by

$$S((\lambda_1)^{\widetilde{\beta}_s}) = \left((\lambda_1)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}} - \sqrt{1-\lambda_1^{\pi^2_{\widetilde{\beta}_s}}} \right)^2 - \left(\sqrt{1-\lambda_1^{\psi^2_{\widetilde{\beta}_s}}} - \sqrt{1-\lambda_1^{\pi^2_{\widetilde{\beta}_s}}} \right)^2 \quad (7)$$

$$S((\lambda_2)^{\widetilde{\beta}_s}) = \left((\lambda_2)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}} - \sqrt{1-\lambda_2^{\pi^2_{\widetilde{\beta}_s}}} \right)^2 - \left(\sqrt{1-\lambda_2^{\psi^2_{\widetilde{\beta}_s}}} - \sqrt{1-\lambda_2^{\pi^2_{\widetilde{\beta}_s}}} \right)^2 \quad (8)$$

The membership values of $\widetilde{\beta}_s \in [0, 1]$ which means that the degree of membership $\left(\xi_{\widetilde{\beta}_s}\right)$, the degree of non membership $\left(\psi_{\widetilde{\beta}_s}\right)$ and the degree of hesistancy $\left(\pi_{\widetilde{\beta}_s}\right)$ values lie in [0, 1]. Since $\lambda_1 \geq \lambda_2$, $(\lambda_1)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}} \geq (\lambda_2)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}}$, $1-\lambda_1^{\psi^2_{\widetilde{\beta}_s}} \leq 1-\lambda_2^{\psi^2_{\widetilde{\beta}_s}}$ and $1-\lambda_1^{\pi^2_{\widetilde{\beta}_s}} \leq 1-\lambda_2^{\pi^2_{\widetilde{\beta}_s}}$, and hence $S((\lambda_1)^{\widetilde{\beta}_s}) \geq S((\lambda_2)^{\widetilde{\beta}_s})$. The following two cases arise:

(i) If $S((\lambda_1)^{\widetilde{\beta}_s}) > S((\lambda_2)^{\widetilde{\beta}_s})$, then $(\lambda_1)^{\widetilde{\beta}_s} > (\lambda_2)^{\widetilde{\beta}_s}$

(ii) If $S((\lambda_1)^{\widetilde{\beta}_s}) = S((\lambda_2)^{\widetilde{\beta}_s})$, then $(\lambda_1)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}} = (\lambda_2)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_s}\right)}}$, $1-\lambda_1^{\psi^2_{\widetilde{\beta}_s}} = 1-\lambda_2^{\psi^2_{\widetilde{\beta}_s}}$ and
$1-\lambda_1^{\pi^2_{\widetilde{\beta}_s}} = 1-\lambda_2^{\pi^2_{\widetilde{\beta}_s}}$, which implies that $H((\lambda_1)^{\widetilde{\beta}_s}) = H((\lambda_2)^{\widetilde{\beta}_s})$ and hence $(\lambda_1)^{\widetilde{\beta}_s} = (\lambda_2)^{\widetilde{\beta}_s}$.

Thus, by combining these two cases, we get $(\lambda_1)^{\widetilde{\beta}_s} \geq (\lambda_2)^{\widetilde{\beta}_s}$. Suppose $\lambda_1, \lambda_2 \geq 1$ and $\lambda_1 \geq \lambda_2$, we get $0 \leq \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq 1$. Similarly we can obtain $(\lambda_1)^{\widetilde{\beta}_s} \leq (\lambda_2)^{\widetilde{\beta}_s}$. □

3.1 Exponential aggregation operator for SFNs

Definition 8 Let $\widetilde{\beta}_{s_i} = \left\langle \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}} \right\rangle$ be a collection of SFNs and λ_i, $(i = 1, 2, \ldots, 3)$ be a collection of real numbers, then $SFWEA : S^n \to S$, called the spherical fuzzy weighted exponential averaging operator, is given by

$$SFWEA\left(\lambda_i^{\widetilde{\beta}_{s1}}, \lambda_i^{\widetilde{\beta}_{s2}}, \ldots, \lambda_i^{\widetilde{\beta}_{sn}}\right) = \lambda_i^{\widetilde{\beta}_{s1}} \otimes \lambda_i^{\widetilde{\beta}_{s2}} \otimes \cdots \lambda_i^{\widetilde{\beta}_{sn}} \quad (9)$$

where S is the collection of SFNs and $\widetilde{\beta}_{s_i}$ are the exponential weights of λ_i, $(i = 1, 2, \ldots, n)$.

Theorem 6 *Let* $\widetilde{\beta}_{s_i} = \left\langle \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}} \right\rangle$ *be a collection of SFNs. The aggregated value by using the SFWEA operator is also an SFN in the extended interval* $[1, 2]$*, where*

$$SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_n}\right) = \begin{cases} \left\langle \prod_{i=1}^n \lambda_i^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \quad \sqrt{1-\prod_{i=1}^n \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}, \right. \\ \left. \sqrt{\prod_{i=1}^n \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^n \left(1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle; & \lambda_i \in (0,1); \\ \left\langle \prod_{i=1}^n \left(\frac{1}{\lambda_i}\right)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \quad \sqrt{1-\prod_{i=1}^n \left(\frac{1}{\lambda_i}\right)^{2\psi_{\widetilde{\beta}_{s_i}}}}, \right. \\ \left. \sqrt{\prod_{i=1}^n \left(\frac{1}{\lambda_i}\right)^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^n \left(1-\left(\frac{1}{\lambda_i}\right)^{2\psi_{\widetilde{\beta}_{s_i}}} - \left(\frac{1}{\lambda_i}\right)^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle; & \lambda_i \geq 1. \end{cases} \tag{10}$$

and $\widetilde{\beta}_{s_i}$ *are the exponential weights of* λ_i, $(i = 1, 2, \ldots, n)$.

Proof We prove the above aggregation operator $SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_n}\right)$ by mathematical induction on n. Let $\lambda_i \in (0, 1)$. Since $\widetilde{\beta}_{s_i}$ is SFN for each i, $0 \leq \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}} \leq 1$ and $\xi^2_{\widetilde{\beta}_{s_i}} + \psi^2_{\widetilde{\beta}_{s_i}} + \pi^2_{\widetilde{\beta}_{s_i}} \leq 1$.

Step 1: When n=2, we can see that

$$\lambda_1^{\widetilde{\beta}_{s_1}} = \left\langle \lambda_1^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_1}}}}, \sqrt{1-\lambda_1^{2\psi_{\widetilde{\beta}_{s_1}}}}, \sqrt{1-\lambda_1^{2\pi_{\widetilde{\beta}_{s_1}}}} \right\rangle,$$
$$\lambda_2^{\widetilde{\beta}_{s_2}} = \left\langle \lambda_2^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_2}}}}, \sqrt{1-\lambda_2^{2\psi_{\widetilde{\beta}_{s_2}}}}, \sqrt{1-\lambda_2^{2\pi_{\widetilde{\beta}_{s_2}}}} \right\rangle.$$

are SFNs. Then

$$SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}\right) = \lambda_1^{\widetilde{\beta}_{s_1}} \otimes \lambda_2^{\widetilde{\beta}_{s_2}}$$
$$= \left\{ \left\langle \lambda_1^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_1}}\right)}} \lambda_2^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_2}}\right)}}, \quad \sqrt{1-\lambda_1^{2\psi_{\widetilde{\beta}_{s_1}}} \lambda_2^{2\psi_{\widetilde{\beta}_{s_2}}}}, \right. \right.$$
$$\left. \left. \sqrt{\lambda_1^{2\psi_{\widetilde{\beta}_{s_1}}} \lambda_2^{2\psi_{\widetilde{\beta}_{s_2}}} - \left(1-\lambda_1^{2\psi_{\widetilde{\beta}_{s_1}}} - \lambda_1^{2\pi_{\widetilde{\beta}_{s_1}}}\right)\left(1-\lambda_2^{2\psi_{\widetilde{\beta}_{s_2}}} - \lambda_2^{2\pi_{\widetilde{\beta}_{s_2}}}\right)} \right\rangle \right\}$$
$$= \left\{ \left\langle \prod_{i=1}^{2} \lambda_i^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \quad \sqrt{1-\prod_{i=1}^{2} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}, \quad \sqrt{\prod_{i=1}^{2} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^{2}\left(1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle \right\}$$

is also an SFN in the extended interval [1,2].

Step 2: Assume that the aggregation operator $SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_n}\right)$ holds for $n = \mathcal{K}$. Then

$$SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_\mathcal{K}}\right) = \left\{\left\langle \prod_{i=1}^{\mathcal{K}} \lambda_i^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \sqrt{1-\prod_{i=1}^{\mathcal{K}} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}, \sqrt{\prod_{i=1}^{\mathcal{K}} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^{\mathcal{K}}\left(1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle\right\}$$

and the aggregated value is an SFN.

Step 3: When $n = \mathcal{K}+1$, we have

$$SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_\mathcal{K}}\right) = \lambda_1^{\widetilde{\beta}_{s_1}} \otimes \lambda_2^{\widetilde{\beta}_{s_2}} \otimes \cdots \otimes \lambda_{\mathcal{K}}^{\widetilde{\beta}_{s_\mathcal{K}}} \otimes \lambda_{\mathcal{K}+1}^{\widetilde{\beta}_{s_{\mathcal{K}+1}}}$$
$$= \left\{\left\langle \prod_{i=1}^{\mathcal{K}} \lambda_i^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \sqrt{1-\prod_{i=1}^{\mathcal{K}} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}, \sqrt{\prod_{i=1}^{\mathcal{K}} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^{\mathcal{K}}\left(1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle \otimes \right.$$
$$\left. \left\langle \lambda_{\mathcal{K}+1}^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_{\mathcal{K}+1}}}\right)}}, \sqrt{1-\lambda_{\mathcal{K}+1}^{2\psi_{\widetilde{\beta}_{s_{\mathcal{K}+1}}}}}, \sqrt{\lambda_{\mathcal{K}+1}^{2\psi_{\widetilde{\beta}_{s_{\mathcal{K}+1}}}} - \left(1-\lambda_{\mathcal{K}+1}^{2\psi_{\widetilde{\beta}_{s_{\mathcal{K}+1}}}} - \lambda_{\mathcal{K}+1}^{2\pi_{\widetilde{\beta}_{s_{\mathcal{K}+1}}}}\right)} \right\rangle \right\}$$
$$= \left\{\left\langle \prod_{i=1}^{\mathcal{K}+1} \lambda_i^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \sqrt{1-\prod_{i=1}^{\mathcal{K}+1} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}, \sqrt{\prod_{i=1}^{\mathcal{K}+1} \lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^{\mathcal{K}+1}\left(1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}} - \lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle\right\}$$

whose aggregated value is alos an SFN in the extended $[1, 2]$. Therefore, (9) holds. On the other hand, When $\lambda_i \geq 1$, and $0 \leq \frac{1}{\lambda_i} \leq 1$, we can also get

$$SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_\mathcal{K}}\right) = \left\{\left\langle \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{\sqrt{1-\left(\xi^2_{\widetilde{\beta}_{s_i}}\right)}}, \sqrt{1-\prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{2\psi_{\widetilde{\beta}_{s_i}}}}, \right.\right.$$
$$\left.\left. \sqrt{\prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{2\psi_{\widetilde{\beta}_{s_i}}} - \prod_{i=1}^{n}\left(1-\left(\frac{1}{\lambda_i}\right)^{2\psi_{\widetilde{\beta}_{s_i}}} - \left(\frac{1}{\lambda_i}\right)^{2\pi_{\widetilde{\beta}_{s_i}}}\right)} \right\rangle\right\}$$

and the aggregated value is SFN in the extended interval $[1, 2]$. Hence the proof. □

4 Einstein exponential operational laws of SFSs

In this section, we introduce the Einstein exponential operational laws of sphericalfor the SFS model alongwith some of its properties.

Definition 9 Let $\lambda^{\widetilde{\mathcal{A}}_s} = \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}, \sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}} \right\rangle$ and $\lambda^{\widetilde{\mathcal{B}}_s} = \left\langle \lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{B}}_s}}}, \sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{B}}_s}}}, \sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{B}}_s}}} \right\rangle$ be two families of exponential spherical fuzzy numbers, for all $\lambda \in (0,1)$, where $\widetilde{\mathcal{A}}_s = \left\langle \xi_{\widetilde{\mathcal{A}}_s}, \psi_{\widetilde{\mathcal{A}}_s}, \pi_{\widetilde{\mathcal{A}}_s} \right\rangle$, $\widetilde{\mathcal{B}}_s = \left\langle \xi_{\widetilde{\mathcal{B}}_s}, \psi_{\widetilde{\mathcal{B}}_s}, \pi_{\widetilde{\mathcal{B}}_s} \right\rangle$ are spherical fuzzy numbers. Then the basic Einstein exponential operational laws of spherical fuzzy operators are as given below:

$$\lambda^{\widetilde{\mathcal{A}}_s} \oplus_{\mathcal{E}} \lambda^{\widetilde{\mathcal{B}}_s} = \left\{ \left\langle \frac{\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}} + \lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{B}}_s}}}}{1+\left(\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right).\left(\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{B}}_s}}}\right)}, \frac{\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}.\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{B}}_s}}}}{1+\left(1-\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right).\left(1-\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{B}}_s}}}\right)}, \frac{\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}.\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{B}}_s}}}}{1+\left(1-\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right).\left(1-\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{B}}_s}}}\right)} \right\rangle \right\} \tag{11}$$

$$\lambda^{\widetilde{\mathcal{A}}_s} \otimes_{\mathcal{E}} \lambda^{\widetilde{\mathcal{B}}_s} = \left\{ \left\langle \frac{\left(\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right).\left(\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{B}}_s}}}\right)}{1+\left(1-\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right).\left(1-\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{B}}_s}}}\right)}, \frac{\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}+\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{B}}_s}}}}{1+\left(\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right).\left(\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{B}}_s}}}\right)}, \frac{\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}+\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{B}}_s}}}}{1+\left(\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right).\left(\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{B}}_s}}}\right)} \right\rangle \right\} \tag{12}$$

$$\mathcal{K}._{\mathcal{E}}\left(\lambda^{\widetilde{\mathcal{A}}_s}\right) = \left\{ \left\langle \frac{\left[1+\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}} - \left[1-\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}{\left[1+\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}} + \left[1-\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}, \frac{2\left[\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}{\left[2-\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}+\left[\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}, \frac{2\left[\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}{\left[2-\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}+\left[\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]}^{\mathcal{K}} \right\rangle \right\} \tag{13}$$

$$\left[\lambda^{\widetilde{\mathcal{A}}_s}\right]^{\mathcal{K}} = \left\{ \left\langle \frac{2\left[\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}\right]^{\mathcal{K}}}{\left[2-\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}\right]^{\mathcal{K}}+\left[\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}\right]^{\mathcal{K}}}, \frac{\left[1+\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}-\left[1-\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}{\left[1+\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}+\left[1-\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}, \frac{\left[1+\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}-\left[1-\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}}{\left[1+\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}+\left[1-\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right]^{\mathcal{K}}} \right\rangle \right\} \tag{14}$$

Theorem 7 *Let $\lambda^{\widetilde{\beta}_s}$, $\lambda^{\widetilde{\beta}_{s1}}$ and $\lambda^{\widetilde{\beta}_{s2}}$ be a family of three exponential spherical fuzzy numbers of $\widetilde{\beta}_s = \left\langle \xi_{\widetilde{\beta}_s}, \psi_{\widetilde{\beta}_s}, \pi_{\widetilde{\beta}_s} \right\rangle$, $\widetilde{\beta}_{s_1} = \left\langle \xi_{\widetilde{\beta}_{s_1}}, \psi_{\widetilde{\beta}_{s_1}}, \pi_{\widetilde{\beta}_{s_1}} \right\rangle$ and $\widetilde{\beta}_{s_2} = \left\langle \xi_{\widetilde{\beta}_{s_2}}, \psi_{\widetilde{\beta}_{s_2}}, \pi_{\widetilde{\beta}_{s_2}} \right\rangle$ respectively, $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 > 0$ be three real numbers, and $\lambda \in (0,1)$, then we have the following:*

(i) $\lambda^{\widetilde{\beta}_{s1}} \oplus_{\mathcal{E}} \lambda^{\widetilde{\beta}_{s2}} = \lambda^{\widetilde{\beta}_{s2}} \oplus_{\mathcal{E}} \lambda^{\widetilde{\beta}_{s1}}$;
(ii) $\mathcal{K}._{\mathcal{E}}(\lambda^{\widetilde{\beta}_{s1}} \oplus_{\mathcal{E}} \lambda^{\widetilde{\beta}_{s2}}) = \mathcal{K}._{\mathcal{E}}(\lambda^{\widetilde{\beta}_{s1}}) \oplus_{\mathcal{E}} \mathcal{K}._{\mathcal{E}}(\lambda^{\widetilde{\beta}_{s2}})$;
(iii) $\mathcal{K}_1._{\mathcal{E}}(\lambda^{\widetilde{\beta}_s}) \oplus_{\mathcal{E}} \mathcal{K}_2._{\mathcal{E}}(\lambda^{\widetilde{\beta}_s}) = (\mathcal{K}_1 + \mathcal{K}_1)._{\mathcal{E}}(\lambda^{\widetilde{\beta}_s})$;
(v) $(\mathcal{K}_1.\mathcal{K}_2)._{\mathcal{E}}\lambda^{\widetilde{\beta}_s} = \mathcal{K}_1._{\mathcal{E}}(\mathcal{K}_2._{\mathcal{E}}\lambda^{\widetilde{\beta}_s})$.

Proof The proof is similar to the proof of Theorem (4), and therefore is omitted. □

4.1 Einstein exponential aggregation operator for SFNs

In this part, a new Einstein exponential aggregation operator with spherical fuzzy information is developed namely Spherical Fuzzy Einstein Weighted Exponential Averaging Operator is developed to aggregate spherical fuzzy information.

Definition 10 Let $\lambda_i^{\widetilde{\beta}_{s_i}}$, $\langle i=1,2,\ldots,n\rangle$ be a family of exponential of SFNs with respect to $\widetilde{\beta}_{s_i}=\left\langle \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}}\right\rangle$, where λ_i are real numbers. Let $\mathcal{K}_i=\langle \mathcal{K}_1,\mathcal{K}_2\ldots,\mathcal{K}_n\rangle^T$ be the weighting vector of $\lambda^{\widetilde{\beta}_{s_i}}$ $\langle i=1,2,\ldots,n\rangle$, such that $\mathcal{K}_i\in[0,1]$ and $\sum_{i=1}^n \mathcal{K}_i=1$; then, a SFEWEA operator of n is a mapping $SFEWEA:(\lambda^{\widetilde{\beta}_s})^*\rightarrow\lambda^{\widetilde{\beta}_s}$, and

$$SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s1}},\lambda_2^{\widetilde{\beta}_{s2}},\ldots,\lambda_n^{\widetilde{\beta}_{sn}}\right)=\mathcal{K}_1\cdot_{\mathcal{E}}\lambda_1^{\widetilde{\beta}_{s1}}\oplus_{\mathcal{E}}\mathcal{K}_2\cdot_{\mathcal{E}}\lambda_2^{\widetilde{\beta}_{s2}}\oplus_{\mathcal{E}}\ldots\oplus_{\mathcal{E}}\mathcal{K}_1\cdot_{\mathcal{E}}\lambda_n^{\widetilde{\beta}_{sn}} \tag{15}$$

Theorem 8 *Let* $\lambda_i^{\widetilde{\beta}_{s_i}}$, $\langle i=1,2,\ldots,n\rangle$ *be a family of exponential of SFNs with respect to* $\widetilde{\beta}_{s_i}=\left\langle \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}}\right\rangle$*, then the aggregated value by using the SFEWEA operator is also an SFN in the extended interval* $[1,2]$*, and*

$$SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s1}},\lambda_2^{\widetilde{\beta}_{s2}},\ldots,\lambda_n^{\widetilde{\beta}_{sn}}\right)=\begin{cases}\left\langle\dfrac{\prod_{i=1}^n\left(1+\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}-\prod_{i=1}^n\left(1-\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^n\left(1+\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}+\prod_{i=1}^n\left(1-\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}},\right.\\[2ex] \dfrac{2\prod_{i=1}^n\left(\sqrt{1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^n\left(2-\sqrt{1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}+\prod_{i=1}^n\left(\sqrt{1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}},\\[2ex] \left.\dfrac{2\prod_{i=1}^n\left(\sqrt{1-\lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^n\left(2-\sqrt{1-\lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}+\prod_{i=1}^n\left(\sqrt{1-\lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}\right\rangle; \quad \lambda_i\in(0,1).\end{cases} \tag{16}$$

where $\mathcal{K}_i=\langle \mathcal{K}_1,\mathcal{K}_2\ldots,\mathcal{K}_n\rangle^T$ *is the weighting vector of* $\widetilde{\beta}_{s_i}=\left\langle \xi_{\widetilde{\beta}_{s_i}}, \psi_{\widetilde{\beta}_{s_i}}, \pi_{\widetilde{\beta}_{s_i}}\right\rangle$ *such that* $\mathcal{K}_i\in[0,1]$, $\langle i=1,2,\ldots n\rangle$ *and* $\sum_{i=1}^n\mathcal{K}_i=1$.

Proof We will prove (14) by mathematical induction. It is obviously true that (14) holds for $n=1$. Now assume that (14) holds for $n=\delta$, i.e.,

$$SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}},\lambda_2^{\widetilde{\beta}_{s_2}},\ldots,\lambda_\delta^{\widetilde{\beta}_{s_\delta}}\right)=\left\{\left\langle\frac{\prod_{i=1}^{\delta}\left(1+\lambda_i^{\sqrt{1-\xi_{\widetilde{\beta}_{s_i}}^2}}\right)^{\mathcal{K}_i}-\prod_{i=1}^{\delta}\left(1-\lambda_i^{\sqrt{1-\xi_{\widetilde{\beta}_{s_i}}^2}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^{\delta}\left(1+\lambda_i^{\sqrt{1-\xi_{\widetilde{\beta}_{s_i}}^2}}\right)^{\mathcal{K}_i}+\prod_{i=1}^{\delta}\left(1-\lambda_i^{\sqrt{1-\xi_{\widetilde{\beta}_{s_i}}^2}}\right)^{\mathcal{K}_i}},\right.\right.$$
$$\frac{2\prod_{i=1}^{\delta}\left(\sqrt{1-\lambda_i^{2\psi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^{\delta}\left(2-\sqrt{1-\lambda_i^{2\psi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}+\prod_{i=1}^{\delta}\left(\sqrt{1-\lambda_i^{2\psi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}},$$
$$\left.\left.\frac{2\prod_{i=1}^{\delta}\left(\sqrt{1-\lambda_i^{2\pi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^{\delta}\left(2-\sqrt{1-\lambda_i^{2\pi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}+\prod_{i=1}^{\delta}\left(\sqrt{1-\lambda_i^{2\pi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}}\right\rangle;\quad \lambda_i\in(0,1).\right.$$

Let $a_1=\prod_{i=1}^{\delta}\left(1+\lambda_i^{\sqrt{1-\xi_{\widetilde{\beta}_{s_i}}^2}}\right)^{\mathcal{K}_i}$, $b_1=\prod_{i=1}^{\delta}\left(1-\lambda_i^{\sqrt{1-\xi_{\widetilde{\beta}_{s_i}}^2}}\right)^{\mathcal{K}_i}$, $c_1=\prod_{i=1}^{\delta}\left(\sqrt{1-\lambda_i^{2\psi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}$ $d_1=\prod_{i=1}^{\delta}\left(\sqrt{1-\lambda_i^{2\pi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}$, $e_1=\prod_{i=1}^{\delta}\left(2-\sqrt{1-\lambda_i^{2\psi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}$, $f_1=\prod_{i=1}^{\delta}\left(2-\sqrt{1-\lambda_i^{2\pi\widetilde{\beta}_{s_i}}}\right)^{\mathcal{K}_i}$

$$SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}},\lambda_2^{\widetilde{\beta}_{s_2}},\ldots,\lambda_\delta^{\widetilde{\beta}_{s_\delta}}\right)=\left\langle\frac{a_1-b_1}{a_1+b_1},\frac{2c_1}{e_1+c_1},\frac{2d_1}{f_1+d_1}\right\rangle$$

Then if $n=\delta+1$, we have

$$\begin{aligned}SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}},\lambda_2^{\widetilde{\beta}_{s_2}},\ldots,\lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right)&=\mathcal{K}_1\cdot_{\mathcal{E}}\left(\lambda_1^{\widetilde{\beta}_{s_1}}\right)\oplus_{\mathcal{E}}\cdots\oplus_{\mathcal{E}}\mathcal{K}_\delta\cdot_{\mathcal{E}}\left(\lambda_\delta^{\widetilde{\beta}_{s_\delta}}\right)\oplus_{\mathcal{E}}\mathcal{K}_{\delta+1}\cdot_{\mathcal{E}}\left(\lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right)\\&=SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}},\lambda_2^{\widetilde{\beta}_{s_2}},\ldots,\lambda_\delta^{\widetilde{\beta}_{s_\delta}}\right)\oplus_{\mathcal{E}}\mathcal{K}_{\delta+1}\cdot_{\mathcal{E}}\left(\lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right)\end{aligned}$$

Let $a_2=\left(1+\lambda_{\delta+1}^{\sqrt{1-\xi_{\widetilde{\beta}_{s_{\delta+1}}}^2}}\right)^{\mathcal{K}_{\delta+1}}$, $b_2=\left(1-\lambda_{\delta+1}^{\sqrt{1-\xi_{\widetilde{\beta}_{s_{\delta+1}}}^2}}\right)^{\mathcal{K}_{\delta+1}}$, $c_2=\left(\sqrt{1-\lambda_{\delta+1}^{2\psi\widetilde{\beta}_{s_{\delta+1}}}}\right)^{\mathcal{K}_{\delta+1}}$ $d_2=\left(\sqrt{1-\lambda_{\delta+1}^{2\pi\widetilde{\beta}_{s_{\delta+1}}}}\right)^{\mathcal{K}_{\delta+1}}$, $e_2=\left(2-\sqrt{1-\lambda_{\delta+1}^{2\psi\widetilde{\beta}_{s_{\delta+1}}}}\right)^{\mathcal{K}_{\delta+1}}$, $f_2=\left(2-\sqrt{1-\lambda_{\delta+1}^{2\pi\widetilde{\beta}_{s_{\delta+1}}}}\right)^{\mathcal{K}_{\delta+1}}$;

Then,

$$\mathcal{K}_{\delta+1}\cdot_{\mathcal{E}}\left(\lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right)=\left\langle\frac{a_2-b_2}{a_2+b_2},\frac{2c_2}{e_2+c_2},\frac{2d_2}{f_2+d_2}\right\rangle;$$

thus, by the Einstein operational law, we have

$$
\begin{aligned}
SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}}, \lambda_2^{\widetilde{\beta}_{s_2}}, \ldots, \lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right) &= SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}}, \lambda_2^{\widetilde{\beta}_{s_2}}, \ldots, \lambda_{\delta}^{\widetilde{\beta}_{s_\delta}}\right) \oplus_{\mathcal{E}} \mathcal{K}_{\delta+1} \cdot_{\mathcal{E}} (\lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}) \\
&= \left\langle \frac{a_1 - b_1}{a_1 + b_1}, \frac{2c_1}{e_1 + c_1}, \frac{2d_1}{f_1 + d_1} \right\rangle \oplus_{\mathcal{E}} \left\langle \frac{a_2 - b_2}{a_2 + b_2}, \frac{2c_2}{e_2 + c_2}, \frac{2d_2}{f_2 + d_2} \right\rangle \\
&= \left\langle \frac{a_1 a_2 - b_1 b_2}{a_1 a_2 + b_1 b_2}, \frac{2c_1 c_2}{e_1 e_2 + c_1 c_2}, \frac{2d_1 d_2}{f_1 f_2 + d_1 d_2} \right\rangle
\end{aligned}
$$

$$
SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}}, \lambda_2^{\widetilde{\beta}_{s_2}}, \ldots, \lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right) = \left\{
\begin{array}{l}
\left\langle \dfrac{\prod_{i=1}^{\delta+1}\left(1+\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i} - \prod_{i=1}^{\delta+1}\left(1-\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^{\delta+1}\left(1+\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i} + \prod_{i=1}^{\delta+1}\left(1-\lambda_i^{\sqrt{1-\xi^2_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}, \right. \\[2ex]
\dfrac{2\prod_{i=1}^{\delta+1}\left(\sqrt{1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^{\delta+1}\left(2-\sqrt{1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i} + \prod_{i=1}^{\delta+1}\left(\sqrt{1-\lambda_i^{2\psi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}, \\[2ex]
\left. \dfrac{2\prod_{i=1}^{\delta+1}\left(\sqrt{1-\lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}}{\prod_{i=1}^{\delta+1}\left(2-\sqrt{1-\lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i} + \prod_{i=1}^{\delta+1}\left(\sqrt{1-\lambda_i^{2\pi_{\widetilde{\beta}_{s_i}}}}\right)^{\mathcal{K}_i}} \right\rangle; \quad \lambda_i \in (0, 1).
\end{array}
\right.
$$

Hence, (14) is true for $n = \delta + 1$. Therefore, (14) holds for all n, which completes the proof of the theorem. □

Similarly, we can easily get the $SFEWEA\left(\lambda_1^{\widetilde{\beta}_{s_1}}, \lambda_2^{\widetilde{\beta}_{s_2}}, \ldots, \lambda_{\delta+1}^{\widetilde{\beta}_{s_{\delta+1}}}\right)$ operator, when the value of $\lambda_i \geq 1$, and $0 \leq \frac{1}{\lambda_i} \leq 1$ and also the aggregated value is SFN in the extended interval [1, 2].

5 Decision-making algorithm based on the proposed aggregation operators

In this section, we propose an MCDM approach based on the proposed operators, which involves the following steps:

Step 1: Consider a decision-making problem in which there are n alternatives $\widetilde{\mathcal{A}}_i \langle i = 1, 2, \ldots, n \rangle$ and m attributes $\widetilde{\mathcal{B}}_j \langle j = 1, 2, \ldots, m \rangle$ whose spherical fuzzy weight vector values are $\widetilde{\beta}_{s_j} = \left\langle \widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_m} \right\rangle \langle j = 1, 2, \ldots, m \rangle$ such that $\widetilde{\beta}_{s_j} = \left\langle \xi_{\widetilde{\beta}_{s_j}}, \psi_{\widetilde{\beta}_{s_j}}, \pi_{\widetilde{\beta}_{s_j}} \right\rangle; 0 \leq \xi_{\widetilde{\beta}_{s_j}}, \psi_{\widetilde{\beta}_{s_j}}, \pi_{\widetilde{\beta}_{s_j}} \leq 1$ and $0 \leq \xi^2_{\widetilde{\beta}_{s_j}} + \psi^2_{\widetilde{\beta}_{s_j}} + \pi^2_{\widetilde{\beta}_{s_j}} \leq 1$. Then the given alternatives are evaluated under a set of attributes by experts and they give their preference values under the fuzzy information denoted by $\lambda_{ij} \langle i = 1, 2, \ldots, n \rangle \langle j = 1, 2, \ldots, m \rangle$ and $0 \leq \lambda_{ij} \leq 1$. Generally, the attributes are of two types; the first one being the benefit type (B_1) and the other one the cost type (B_2). If the attributes of the MCDM are of the same type, then the preference values do not need normalization. If the attributes are of different type, we can use the following formula to convert the benefit type preference values into cost type.

$$\lambda_{ij} = \begin{cases} \lambda_{ij}; & j \in B_1 \\ \lambda_{ij}^{c}; & j \in B_2 \end{cases}$$

Then, the decision matrix is presented based on the preference values.

$$D = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \ddots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nm} \end{bmatrix}$$

Step 2: Utilize the aggregation operators such as SFWEA and SFEWEA to aggregate the different preference values of each alternative into collective values $a_i \langle i = 1, 2, \dots, n \rangle$.

Step 3: Compute the score values of aggregated SFNs $a_i \langle i = 1, 2, \dots, n \rangle$ using Eq.(7) and rank the alternatives according to their score values.

The above mentioned approach will be demonstrated using a real life numerical example in Section 6.

6 Illustrative example

Youngsters who face a lot of emotional problems go for counselling and treatment. Therefore, for decision-making problem, the decision maker (counsellor) has chosen different types of psychotherapy for treatment and he listed them as alternatives as follows:

$\mathcal{A}_1$- Cognitive Behavioral Therapy
$\mathcal{A}_2$- Dialectical Behaviour Therapy
$\mathcal{A}_3$- Exposure Therapy
$\mathcal{A}_4$- Interpersonal Therapy
$\mathcal{A}_5$- Psychodynamic Psychotherapy
$\mathcal{A}_6$- Therapy Pets

Here the aim of this analysis is to find the best suitable psychotherapy for youngsters based on the problems they face. For this, we have considered seven major problems and listed them as following attributes:

$\mathcal{B}_1$- Erosion of national pride/Identity
$\mathcal{B}_2$- Poverty
$\mathcal{B}_3$- Educational disparity
$\mathcal{B}_4$- Obesity
$\mathcal{B}_5$- Materialism
$\mathcal{B}_6$- Violence in schools
$\mathcal{B}_7$- Drug/Alcohol Abuse

The exponential spherical fuzzy weights of these seven attributes are expressed as SFNs $\widetilde{\beta}_{s_j} = \langle j=1,2,\dots,7 \rangle$: $(\mathcal{B}_1 = \langle 0.1, 0.9, 0.1 \rangle)$, $(\mathcal{B}_2 = \langle 0.9, 0.1, 0.1 \rangle)$, $(\mathcal{B}_3 = \langle 0.8, 0.2, 0.2 \rangle)$, $(\mathcal{B}_4 = \langle 0.4, 0.6, 0.4 \rangle)$, $(\mathcal{B}_5 = \langle 0.2, 0.8, 0.2 \rangle)$, $(\mathcal{B}_6 = \langle 0.6, 0.4, 0.4 \rangle)$, $(\mathcal{B}_7 = \langle 0.5, 0.5, 0.5 \rangle)$ on the basis of the preference of the decision-makers. Also, let $\mathcal{K} = \langle 0.05, 0.25, 0.2,$

$0.1, 0.1, 0.15, 0.1, 0.1, 0.15, 0.15\rangle^T$ be the single valued weight vector to factors that are associated with the attributes. The following steps have been executed to get the most recommended alternative(s):

Step 1: The experts' opinions are expressed in the form of a decision matrix $D = \left[\lambda_{ij}\right]_{6\times 7}$ whose elements indicate the evaluation values of all the alternatives $\mathcal{A}_i \langle 1, 2, \ldots, 6\rangle$ against each attribute $\mathcal{B}_i \langle 1, 2, \ldots, 7\rangle$ under fuzzy environment.

$$D = \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.5 & 0.5 & 0.6 & 0.6 \\ 0.4 & 0.4 & 0.4 & 0.7 & 0.6 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0.7 & 0.4 & 0.5 & 0.6 & 0.6 \\ 0.9 & 0.9 & 0.9 & 0.6 & 0.8 & 0.8 & 0.8 \\ 0.8 & 0.4 & 0.2 & 0.6 & 0.6 & 0.7 & 0.7 \\ 0.06 & 0.05 & 0.05 & 0.7 & 0.1 & 0.7 & 0.2 \end{bmatrix}_{6\times 7}$$

Step 2: Using the aggregation operator SFWEA given in (10) to aggregate the different preference values of each factor, we obtain the following:

$$\begin{aligned} a_i &= SFWEA\left(\widetilde{\beta}_{s_1}, \widetilde{\beta}_{s_2}, \ldots, \widetilde{\beta}_{s_{\mathcal{K}}}\right) \\ a_1 &= \langle 0.055585, 0.987763, 0.156011\rangle \\ a_2 &= \langle 0.033179, 0.992547, 0.122215\rangle \\ a_3 &= \langle 0.051742, 0.988096, 0.152444\rangle \\ a_4 &= \langle 0.280116, 0.896102, 0.482993\rangle \\ a_5 &= \langle 0.042852, 0.981404, 0.192536\rangle \\ a_6 &= \langle 0.000039, 0.999999, 0.034176\rangle \end{aligned}$$

Step 3: The score values of these aggregated SFNs $a_i \langle i = 1, 2, \ldots, n\rangle$ in the extended interval $[1, 2]$ are calculated using Eq.(7) and these are as given below:

$$\begin{aligned} S(a_1) &= -0.6817, S(a_2) = -0.7496, S(a_3) = -0.6882, \\ S(a_4) &= -0.1295, S(a_5) = -0.5999, S(a_6) = -0.9316. \end{aligned}$$

The alternatives are ranked based on the highest score values and the rank order is

$$S(a_4) > S(a_5) > S(a_1) > S(a_3) > S(a_2) > S(a_6) : \mathcal{A}_4 > \mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_6.$$

Similarly, the SFEWEA operator (16) are used to find the score values and the values are represented as follows:

$$\begin{aligned} S(a_1) &= 0.3245, S(a_2) = 0.2543, S(a_3) = 0.3237, S(a_4) = 0.5325, S(a_5) = 0.2490, \\ S(a_6) &= 0.0897. \end{aligned}$$

The rank order is

$$S(a_4) > S(a_1) > S(a_3) > S(a_2) > S(a_5) > S(a_6) : \mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6$$

7 Comparative analysis

In this section, the results obtained through the proposed model is compared with the existing fuzzy models such as fuzzy TOPSIS, Weighted Sum Model (WSM) and Weighted Product Model (WPM). In order to find the results using these existing models, we have used the following Matlab codes:

```
Xval=length(X(:,1));
Y = zeros([Xval,length(W)]);
%% Step 1. calculating the normalized matrix
for j=1:length(W)
    for i=1:Xval
    Y(i,j)=X(i,j)/sqrt(sum((X(:,j).^2)));
    end
end
Normalized_Matrix = num2str([Y])
%% Step 2. calculating the weighted mormalized matrix
for j=1:length(W)
    for i=1:Xval
        Yw(i,j)=Y(i,j).*W(j);
    end
end
Weighted_Normalized_Matrix = num2str([Yw])
%% Step 3. calculating the positive and negative best
for j=1:length(W)
    if Wcriteria(1,j)== 0
        Vp(1,j)= min(Yw(:,j));
        Vn(1,j)= max(Yw(:,j));
    else
        Vp(1,j)= max(Yw(:,j));
        Vn(1,j)= min(Yw(:,j));
    end
end
 Positive_best = num2str([Vp])
 Negative_best = num2str([Vn])
%% Step 4. Euclidean distance from Ideal Best and Worst
for j=1:length(W)
    for i=1:Xval
        Sp(i,j)=((Yw(i,j)-Vp(j)).^2);
        Sn(i,j)=((Yw(i,j)-Vn(j)).^2);
    end
end
for i=1:Xval
    Splus(i)=sqrt(sum(Sp(i,:)));
    Snegative(i)=sqrt(sum(Sn(i,:)));
end
%% Step 5. calculating the performance score
P=zeros(Xval,1);
for i=1:Xval
    P(i)=Snegative(i)/(Splus(i)+Snegative(i));
end
Performance_Score =  num2str([P])
```

Algorithm 1 Fuzzy TOPSIS method.

Note: Using the above mentioned matlab code the following results are arrived. Here, we have taken the following inputs from the illustrative problem; $X = D = \left[\lambda_{ij}\right]_{6\times 7}$ is the decision Matrix; $W = \mathcal{K} = \langle 0.05, 0.25, 0.2, 0.1, 0.1, 0.15, 0.15\rangle^T$ is the single valued weights of the criteria.

Step 2. The weighted normalized matrix is calculated

$$NWM = \begin{bmatrix} 0.0211 & 0.1204 & 0.0992 & 0.0344 & 0.0366 & 0.0548 & 0.0583 \\ 0.0121 & 0.0688 & 0.0567 & 0.0482 & 0.0439 & 0.0548 & 0.0681 \\ 0.0241 & 0.1204 & 0.0992 & 0.0275 & 0.0366 & 0.0548 & 0.0583 \\ 0.0272 & 0.1548 & 0.1275 & 0.0413 & 0.0585 & 0.0730 & 0.0778 \\ 0.0241 & 0.0688 & 0.0283 & 0.0413 & 0.0439 & 0.0639 & 0.0681 \\ 0.0018 & 0.0086 & 0.0071 & 0.0482 & 0.0073 & 0.0639 & 0.0194 \end{bmatrix}_{6\times 7}$$

Step 3. The positive and negative best values are obtained

$$Positive_{+best} = \langle 0.027168, 0.1548, 0.12752, 0.04819, 0.058502, 0.07303, 0.077784\rangle$$

$$Negative_{-best} = \langle 0.0018112, 0.0086003, 0.0070844, 0.027537, 0.0073127, 0.054772, 0.019446\rangle$$

Step 5. Finally, the performance score values of the alternatives are given as follows;

$$\text{Performance Score} = \langle 0.72543, 0.46842, 0.71977, 0.96789, 0.41014, 0.098582\rangle$$

```
Xval=length(X(:,1));
for i=1:Xval
for j= 1:length(W)
if Wcriteria(1,j)== 0
Y(i,j)=min(X(:,j))/X(i,j);
else
Y(i,j)=X(i,j)/max(X(:,j));
end
end
end
for i=1:Xval
PWSM(i,1)=sum(Y(i,:).*W);
PWPM(i,1)=prod(Y(i,:).^W);
end
Preference_Score_of_Weighted_Sum_Model = num2str([PWSM])
Preference_Score_of_Weighted_Product_Model= num2str([PWPM])
```

Algorithm 2 Weighted Sum Model (WSM) and Weighted Product Model (WPM).

The preference score of the alternatives using the WSM and WPM models are as given below:

$$WSM = 0.74782, 0.64097, 0.73909, 0.98571, 0.62321, 0.30958$$

$$WPM = 0.74631, 0.6081, 0.73473, 0.9847, 0.55229, 0.15382$$

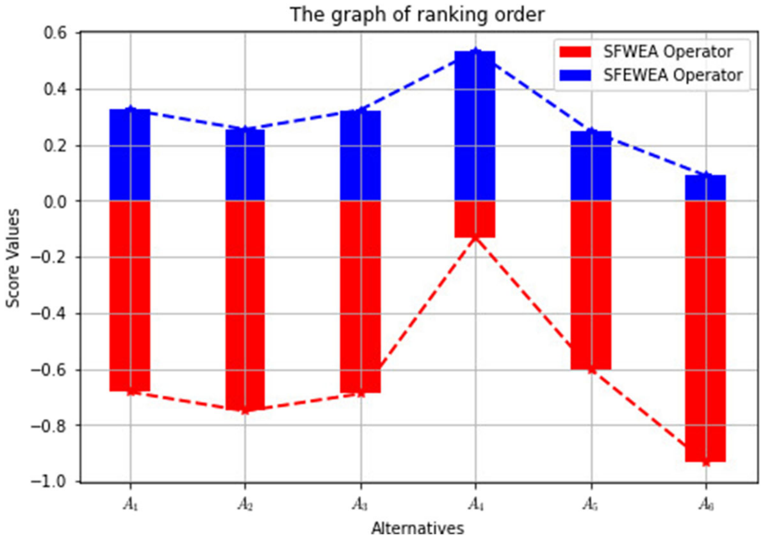

Fig. 1 Score values using the proposed aggregation operators

7.1 Sensitivity analysis

The results obtained from the proposed spherical fuzzy aggregation operators and the existing well known fuzzy MCDM models are shown in Table 1, and we find that the alternative $\mathcal{A}_4$ is recommended as the best choice.

From Table 1, we can see that the ranking order based on the SFWEA Operator alone slightly differs from the other methods due to its nature. The SFEWEA operator deals with spherical fuzzy numbers in addition to the weights of attributes. So the score values have been arrived in positive numbers using the SFEWEA operator, whereas without considering the weights of the attributes the SFWEA operator yields negative values. These are presented as a graphical representation in Fig. 1 for a better understanding of the proposed models. This analysis deliberates the validity of the methods and the proposed operators.

Table 1 Rank of the proposed methods

Methods	Ranking order
SFWEA operator	$\mathcal{A}_4 > \mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_6$
SFEWEA operator	$\mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6$
TOPSIS method	$\mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6$
WSM method	$\mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6$
WPM method	$\mathcal{A}_4 > \mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6$

8 Conclusions

The main contributions of this paper can be summarized as follows:

1. The exponential and Einstein exponential operational lawsfor the SFS modelare defined, and their properties are discussed in detail. The main finding in exponential operational laws of SFSs is that when we have spherical fuzzy number with the condition $0 \leq \xi^2_{\widetilde{\mathcal{A}}_s}(x) + \psi^2_{\widetilde{\mathcal{A}}_s}(x) + \pi^2_{\widetilde{\mathcal{A}}_s}(x) \leq 1$, then the exponential operation on spherical fuzzy numbers yield values in the extended interval $[1, 2]$ with the condition that
$$1 \leq \left(\lambda^{\sqrt{1-\xi^2_{\widetilde{\mathcal{A}}_s}}}\right)^2 + \left(\sqrt{1-\lambda^{2\psi_{\widetilde{\mathcal{A}}_s}}}\right)^2 + \left(\sqrt{1-\lambda^{2\pi_{\widetilde{\mathcal{A}}_s}}}\right)^2 \leq 2.$$
2. New aggregation operators such as the SFWEA and SFEWEA are introduced in a spherical fuzzy environment and examined for their properties. Finally,the applicability and utility of the proposed aggregation operators were demonstrated using a real life application.

In future, more aggregation operators such as Einstein geometric aggregation operator, Bonferroni mean aggregation operator, and Yager ordered weighted average (OWA) aggregation operatoris plan tobe developed using the exponential operational lawsand Einstein exponential operational laws for the SFS models.

Acknowledgements This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.05-2018.02and the Fundamental Research Grant Scheme (FRGS) given by the Ministry of Higher Education, Malaysia under grant number FRGS/1/2020/STG06/UCSI/02/1.

References

1. Acharjya DP, Rathi R (2021) An integrated fuzzy rough set and real coded genetic algorithm approach for crop identification in smart agriculture. Multimed Tools Appl, pp 1–26. https://doi.org/10.1007/s11042-021-10518-7
2. Ajay D, Broumi S, Aldring J (2020) An MCDM Method under Neutrosophic Cubic Fuzzy Sets with Geometric Bonferroni Mean Operator. Neutrosophic Sets Syst 32(1):187–202. https://doi.org/10.5281/zenodo.3723621
3. Akram M, Kahraman C, Zahid K (2021) Group decision-making based on complex spherical fuzzy VIKOR approach. Knowl-Based Syst 216:106793
4. Akram M, Saleem D, Al-Hawary T (2020) Spherical fuzzy graphs with application to decision-making. Math Comput Appl 25(1):8
5. Ali Z, Mahmood T, Yang MS (2020) TOPSIS method based on complex spherical fuzzy sets with Bonferroni mean operators. Mathematics 8:1739
6. Ali Z, Mahmood T, Yang MS (2020) Complex T-spherical fuzzy aggregation operators with application to multi-attribute decision making. Symmetry 12(8):1311
7. Alia M, Delib I, Smarandache F (2016) The theory of neutrosophic cubic sets and their applications in pattern recognition. J Intell Fuzzy Syst 30:1–7
8. Ashraf S, Abdullah S (2019) Spherical aggregation operators and their application in multi-attribute group decision-making. Int J Intell Syst 34(3):493–523
9. Ashraf S, Abdullah S, Almagrabi AO (2020) A new emergency response of spherical intelligent fuzzy decision process to diagnose of COVID19. Soft Comput. https://doi.org/10.1007/s00500-020-05287-8
10. Ashraf S, Abdullah S, Mahmood T (2018) GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems. Math Sci 12(4):263–275

11. Ashraf S, Abdullah S, Mahmood T (2020) Spherical fuzzy Dombi aggregation operators and their application in group decision making problems. Journal of Ambient Intelligence and Humanized Computing 11:2731–2749
12. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96
13. Atta R, Ghanbari M, Elnahry I (2021) Advanced image steganography based on exploiting modification direction and neutrosophic set. Multimed Tools and Appl, pp 1–19. https://doi.org/10.1007/s11042-021-10784-5
14. Aydogdu A, Gul S (2020) A novel entropy proposition for spherical fuzzy sets and its application in multiple attribute decision-making. Int J Intell Syst 35(9):1354–1374
15. Barukab O, Abdullah S, Ashraf S, Arif M, Khan SA (2019) A new approach to fuzzy TOPSIS method based on entropy measure under spherical fuzzy information. Entropy 21(12):1231
16. Borg SJ, Ajay D, Aldring J (2020) Exponential operational laws of pythagorean fuzzy projection models for decision making. Adv Math Sci J 9(8):6107–6114. https://doi.org/10.37418/amsj.9.8.78
17. Farrokhizadeh E, Shishavan SAS, Gundogdu FK, Donyatalab Y, Kahraman C, Seifi SH (2021) A spherical fuzzy methodology integrating maximizing deviation and TOPSIS methods. Eng Appl Artif Intell 101:104212
18. Garg H (2018) New exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multicriteria decision-making. Int J Intell Syst. https://doi.org/10.1002/int.21966
19. Garg H, Ullah K, Mahmood T, Hassan N, Jan N (2021) T-spherical fuzzy power aggregation operators and their applications in multi-attribute decision making. J Ambient Intell Humanized Comput. https://doi.org/10.1007/s12652-020-02600-z
20. Gou XJ, Xu ZS, Liao HC (2016) Exponential operations of interval-valued intuitionistic fuzzy numbers. J Mach Learn Cyb 7(3):501–518
21. Guleria A, Bajaj RK (2021) T-spherical fuzzy soft sets and its aggregation operators with application in decision-making. Sci Iran 28(2):1014–1029
22. Gundogdu FK, Kahraman C (2019) Spherical fuzzy sets and spherical fuzzy TOPSIS method. J Intell Fuzzy Syst 36(1):337–352. https://doi.org/10.3233/JIFS-181401
23. Gundogdu FK, Kahraman C (2019) A novel fuzzy TOPSIS method using emerging interval-valued spherical fuzzy sets. Eng Appl Artif Intell 85:307–323
24. Gundogdu FK, Kahraman C, Karasan A (2020) Spherical Fuzzy VIKOR Method and Its Application to Waste Management. https://doi.org/10.1007/978-3-030-23756-1_118
25. Haque TS, Chakraborty A, Mondal AP, Alam S (2020) Approach to solve multi-criteria group decision-making problems by exponential operational law in generalised spherical fuzzy environment. CAAI Trans Intell Technol 5(2):106–114
26. Jin H, Ashraf S, Abdullah S (2019) Spherical Fuzzy Logarithmic Aggregation Operators Based on Entropy and Their Application in Decision Support Systems. Entropy 21(7):628
27. Jun YB, Smarandache F, Kim CS (2017) Neutrosophic Cubic Sets. New Math Nat Comput 13:41–54
28. Liu P, Zhu B, Wang P, Shen M (1032) An approach based on linguistic spherical fuzzy sets for public evaluation of shared bicycles in China. Eng Appl Artif Intell 87:95
29. Mathew M, Chakrabortty RK, Ryan MJ (1039) A novel approach integrating AHP and TOPSIS under spherical fuzzy sets for advanced manufacturing system selection. Eng Appl Artif Intell 96:88
30. Quek SG, Selvachandran G, Munir M, Mahmood T, Ullah K, Son LH, Thong PH, Kumar R, Priyadarshini I (2019) Multi-attribute multi-perception decision-making based on generalized T-spherical fuzzy weighted aggregation operators on neutrosophic sets. Mathematics 7(9):780
31. Rafiq M, Ashraf S, Abdullah S, Mahmood T, Muhammad S (2019) The cosine similarity measures of spherical fuzzy sets and their applications in decision making. J Intell Fuzzy Syst 36:6059–6073
32. Sharaf IM, Khalil EAH (2021) A spherical fuzzy TODIM approach for green occupational health and safety equipment supplier selection. Int J Manag Sci Eng Manag 16(1):1–13. https://doi.org/10.1080/17509653.2020.1788467
33. Sharaff A, Khaire AS, Sharma D (2019) Analysing Fuzzy Based Approach for Extractive Text Summarization. In: International Conference on Intelligent Computing and Control Systems (ICCS). p 906–910. https://doi.org/10.1109/ICCS45141.2019.9065722
34. Sharaff A, Verma A, Shrawgi H (2018) Generic Document Classification Using Clustering, Centrality, and Voting. In: Mandal J, Saha G, Kandar D, Maji A (eds) Proceedings of the International Conference on Computing and Communication Systems. Lecture Notes in Networks and Systems, vol 24. Springer, Singapore. https://doi.org/10.1007/978-981-10-6890-4-8
35. Shishavan SAS, Gundogdu FK, Farrokhizadeh E, Donyatalab Y, Kahraman C (2020) Novel similarity measures in spherical fuzzy environment and their applications. Eng Appl Artif Intell 94:103837
36. Smarandache F (1999) A Unifying Field in Logics, Neutrosophic Logic, Neutrosophy, Neutrosophic Set and Neutrosophic Probabilty, 4th (eds). American Research Press, Rehoboth

37. Yager RR (2013) Pythagorean fuzzy subsets. In 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS) (pp. 57–61). IEEE.
38. Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers, and decision making. Int J Intell Syst 28(5):436–452
39. Ye J (2018) Operations and aggregation method of neutrosophic cubic numbers for multiple attribute decision-making. Soft Comput. https://doi.org/10.1007/s00500-018-3194-x
40. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353

Affiliations

D. Ajay[1] · Ganeshsree Selvachandran[2,3] · J. Aldring[1,4] · Pham Huy Thong[5] · Le Hoang Son[5] · Bui Cong Cuong[6]

D. Ajay
dajaypravin@gmail.com

Ganeshsree Selvachandran
Ganeshsree@ucsiuniversity.edu.my

J. Aldring
jaldring24@gmail.com

Pham Huy Thong
thongph@vnu.edu.vn

Le Hoang Son
sonlh@vnu.edu.vn

1 Department of Mathematics, Sacred Heart College, Tamilnadu, India

2 Institute of Actuarial Science and Data Analytics, UCSI University, Jalan Menara Gading, 56000 Cheras, Kuala Lumpur, Malaysia

3 Symbiosis Institute of Technology Symbiosis International University, Pune, 412115, India

4 Panimalar Engineering College, Department of Mathematics, Chennai - 600 123, Tamil Nadu, India

5 VNU Information Technology Institute, Vietnam National University, Hanoi, Vietnam

6 Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam