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Involutions of Halphen Pencils of Index 2 and Discrete Integrable Systems

Kangning Wei[1]

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Abstract
We constructed involutions for a Halphen pencil of index 2, and proved that the birational mapping corresponding to the autonomous reduction of the elliptic Painlevé equation for the same pencil can be obtained as the composition of two such involutions.

Keywords Elliptic Painlevé equation · Halphen pencil · Discrete integrable system · Manin involution

Mathematics Subject Classification 37J70

1 Introduction

The discrete elliptic Painlevé equation, defined in [8], as a discrete dynamical system, can be described by a birational mapping (depending on parameters), which we shall refer to as the elliptic Painlevé mapping. This mapping is often referred to as the non-autonomous version of the QRT mapping. The QRT mapping, often defined on $\mathbb{P}^1 \times \mathbb{P}^1$, preserves a pencil of biquadratic curves, which corresponds to a pencil of cubic curves on $\mathbb{P}^2$ under the canonical birational equivalence of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Similarly, the elliptic Painlevé mapping also preserves a pencil of cubic curves at each step of the time evolution, but this pencil of cubic curves changes from one step to another. It is in this sense that we say the elliptic Painlevé mapping is the non-autonomous version of the QRT mapping.

An elliptic Painlevé mapping comes with nine parameter points, whose configuration determines its dynamical properties [5,8]. For some special configurations

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✉ Kangning Wei
weikangning12@gmail.com

1 Department of Mathematics, Technische Universität Berlin, Str. des 17. Juni, Berlin, Germany

of the parameter points, one can find autonomous reductions of the elliptic Painlevé mapping. More precisely, if we start with nine parameter points which are the base points of a index k Halphen pencil, i.e. they are the base points of a pencil of degree $3k$ curves, each having multiplicity k, then the kth iteration of the elliptic Painlevé mapping becomes autonomous and preserves the Halphen pencil [4].

In the QRT case, the QRT mapping is written as the composition of a horizontal and a vertical switch. For a pencil of cubic curves, one can define Manin involutions in terms of the group law on cubic curves. It is shown that the QRT mapping can also be realized as the composition of two generalized Manin involutions [9]. In [7], the geometric construction of Manin involutions is generalized to certain pencils of elliptic curves of degree 4 and degree 6, which are birationally equivalent to a pencil of cubics and comes naturally from Kahan discretizations.

By definition, pencils of cubic curves are Halphen pencils of index 1. In this paper, we construct involutions on Halphen pencils of index 2, which are pencils of sextic curves with nine double base points. These involutions are known as Bertini involutions in the literature [3, p. 127]. Our main result is that the autonomous reduction of the elliptic Painlevé mapping for a index 2 Halphen pencil can be realized as the composition of two Bertini involutions.

2 Elliptic Painlevé Mapping

There are several different ways to define the elliptic Painlevé mapping in the literature. Here we follow the geometric description from [4], without mentioning the rather involved construction using root systems on the Picard lattice of the blowing up surface [5,8]. The statements in this section can also be proved using the algebraic geometry tools [5]. To make this paper self-contained, we give proofs entirely based on the geometric descriptions.

Let $P_1, \ldots, P_9$ be nine points on the projective plane $\mathbb{CP}^2$ in generic position, i.e. there is a unique cubic curve C_0 passing through the nine points $P_1, \ldots, P_9$. They are considered as parameters, and $P_{10} \in \mathbb{CP}^2$ is the dependent variable. Then the elliptic Painlevé mapping

$$T_{ij} : \{P_1, \ldots, P_{10}\} \mapsto \{\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_{10}\}$$

has the following geometric description:

- The transformation of the parameters $P_1, \ldots, P_9$ under T_{ij} is determined by

$$\bar{P}_k = P_k \quad for\ k \neq i, j, \tag{1}$$

$$P_1 + \cdots + P_{j-1} + \bar{P}_j + P_{j+1} + \cdots + P_9 = 0, \tag{2}$$

$$\bar{P}_i + \bar{P}_j = P_i + P_j, \tag{3}$$

where the addition is taken as the group law on the cubic curve C_0.

Remark 1 Equation (2) means the nine points $P_1, \ldots, P_{j-1}, \bar{P}_j, P_{j+1}, \ldots, P_9$ are the base points of a pencil of cubic curves.

– The dependent variable P_{10} determines a cubic curve $C_{P_{10}}$ passing through the nine points $P_1, \ldots, \hat{P}_j, \ldots, P_{10}$, where $\hat{P}_j$ means P_j is deleted, and under T_{ij}, P_{10} transforms according to

$$\bar{P_{10}} + \bar{P}_j = P_{10} + P_i, \tag{4}$$

where the addition law is taken as the group law on the cubic curve $C_{P_{10}}$

Remark 2 The transformation of the parameters points $P_1, \ldots, P_9$ does not depend on the transformation of the variable P_{10}.

Remark 3 The definition does not depend on the choice of the zero element for the group law on the cubic curve. Since the Eq. (2) does not depend on the choice of zero, and the Eqs. (3) and (4) are both translations on a cubic curve, which also does not depend on the choice of zero.

Although we assumed that the nine parameter points $P_1, \ldots, P_9$ are in general position, the map T_{ij} is still well-defined even if $P_1, \ldots, P_9$ are base points of a pencil of cubic curves. In fact, in that case, the Eqs. (1), (2) and (3) imply that $P_1, \ldots, P_9$ are fixed by T_{ij}. So the map becomes autonomous.

More generally, we have the following statement:

Proposition 1 *If the nine parameter points $P_1, \ldots, P_9$ are the base points of a Halphen pencil of index k, i.e., in terms of the addition law on C_0, we have $k(P_1 + P_2 + \cdots + P_9) = 0$, then the kth iteration of the map T_{ij} is autonomous:*

$$T_{ij}^k(P_l) = P_l \quad for\ any\ l \in \{1, 2, \ldots, 9\}.$$

Proof We only need to prove for P_i and P_j. Denote by $\delta = P_1 + P_2 + \cdots + P_9$, then the Eq. (2) can be written as $\bar{P}_j - P_j = -\delta$, plugging into the Eq. (3), we obtain

$$\bar{P}_i = P_i + P_j - \bar{P}_j = P_i + \delta.$$

Since $P_i + P_j$ is invariant, so is δ, thus we have

$$T_{ij}^k(P_i) = P_i + k\delta = P_i.$$

The assertion for P_j holds since $P_i + P_j$ is invariant. □

3 Main Result

Now we focus on Halphen pencils of index 2, which are described by nine points $P_1, \ldots, P_9$ on $\mathbb{CP}^2$ satisfying $2(P_1 + P_2 + \cdots + P_9) = 0$, where the addition is taken on the unique cubic curve passing through the nine points. There is a one parameter family of sextic curves having double points at $P_1, \ldots, P_9$, which is the Halphen pencil of index 2 determined by the base points. The unique cubic curve C_0 passing through the base points $P_1, \dots, P_9$ is also contained in this pencil as a double curve. We denote this pencil by $\mathcal{H} = \mathcal{P}(6; P_1^2, \dots, P_9^2)$.

On a Halphen pencil of index 2, we can define the following involutions, which we denote by I_i, labeled by one of the base points P_i.

- For a generic point $P \in \mathbb{P}^2$, there is a unique sextic curve $H_P \in \mathcal{H}$ passing through P, and a unique cubic curve passing through $P_1, \dots, \hat{P}_i, \dots, P_9, P$, denoted by C_P^i.
- The curve H_P and C_P^i has 18 intersections, counted with multiplicity, 17 of them are given by the base points $P_1, \dots, \hat{P}_i, \dots, P_9$ with multiplicity 2, and P.
- We define $I_i(P)$ to be the unique other intersection of C_P^i and H_P.

Remark 4 The involutions I_is are known as Bertini involutions in the literature. We refer to [1] and [3] for details.

The following is our main result:

Theorem 1 *Let $P_1, \dots, P_9$ be the base points of a Halphen pencil of index 2. Then the autonomous map $T_{ij}^2 : \mathbb{P}^2 \to \mathbb{P}^2$, seen as a map of P_{10}, is equal to the composition of involutions*

$$T_{ij}^2 = I_j \circ I_i.$$

Since T_{ij} is not autonomous, in the expression T_{ij}^2, when considered as a map of P_{10}, the first and second step are actually different maps, we will denote the first step as $T_{ij}^{(1)}$, and the second step as $T_{ij}^{(2)}$, then the statement is written as $T_{ij}^{(2)} \circ T_{ij}^{(1)} = I_j \circ I_i$.

In order to prove the statement, we introduce another involution, defined via the intersection of the following two pencils:

- The pencil of cubic curves passing through $P_1, \dots, \hat{P}_i, \dots, P_9$, denoted by $\mathcal{C}^i$,
- The pencil of cubic curves passing through $P_1, \dots, \hat{P}_j, \dots, P_9$, denoted by $\mathcal{C}^j$.

Generic curves of the two pencils will have nine intersection points, with seven of them being the base points $P_1, \dots, \hat{P}_i, \dots, \hat{P}_j, \dots, P_9$, so we define the involution J_{ij} by the following:

- For a generic point $P \in \mathbb{P}^2$, there is a unique curve of the pencil $\mathcal{C}^i$ passing through P, denoted by C_P^i, and a unique curve of the pencil $\mathcal{C}^j$ passing through P, denoted by C_P^j.
- The curve C_P^i and C_P^j have nine intersections, counted with multiplicities, with eight of them given by the seven base points $P_1, \dots, \hat{P}_i, \dots, \hat{P}_j, \dots, P_9$ and P.
- We define $J_{ij}(P)$ to be the unique other intersection of the two curves C_P^i and C_P^j.

Notice that both of the involution J_{ij} and the Halphen involution I_j are involutions preserving the pencil $\mathcal{C}^j$, while the first step elliptic Panilevé map $T_{ij}^{(1)}$ is a translation on the pencil $\mathcal{C}^j$. We claim the following:

Proposition 2 *The map $T_{ij}^{(1)}$ (as a map of P_{10}) is the composition of the two involutions J_{ij} and I_j:*

$$T_{ij}^{(1)} = J_{ij} \circ I_j.$$

Proof The elliptic Painlevé mapping $T_{ij}^{(1)}$, as a birational map of $P = P_{10}$ on $\mathbb{P}^2$, is given by

$$\bar{P} = T_{ij}^{(1)}(P) = P + P_i - \bar{P}_j,$$

where the group law is taken on the cubic curve C_P^j of the pencil $\mathcal{C}^j$. In other words, for any given P, $T_{ij}^{(1)}$ is a translation on the curve $\mathcal{C}_P^j$ by $P_i - \bar{P}_j$.
To prove the statement, it is then sufficient to show that the composition $J_{ij} \circ I_j$ is also the same translation. Since both J_{ij} and I_j are involutions preserving the pencil $\mathcal{C}^j$, the composition is automatically a translation on the pencil. It suffices to prove that $J_{ij} \circ I_j(\bar{P}_j) = P_i$. □

So the proposition is proved if we can prove the following two statements

Claim 1 *$I_j(\bar{P}_j)$ is well defined and $I_j(\bar{P}_j) = \bar{P}_j$.*

Claim 2 *$J_{ij}(\bar{P}_j)$ is well defined and $J_{ij}(\bar{P}_j) = P_i$.*

To prove the first claim, we observe that $\bar{P}_j$ is a base point of the pencil $\mathcal{C}^j$, but it is not a base point of the Halphen pencil $\mathcal{H}$. It follows that there is a unique curve $H_{\bar{P}_j}$ of the Halphen pencil $\mathcal{H}$ passing through $\bar{P}_j$, which is exactly the double curve of the cubic curve C_0 which passes through $P_1, \ldots, P_9$ (also passing through $\bar{P}_i$ and $\bar{P}_j$ by definition of the elliptic Painlevé mapping). So for any generic curve of the pencil $\mathcal{C}^j$, its intersections with $H_{\bar{P}_j}$ are given by the nine points $P_1, \ldots, \bar{P}_j, \ldots, P_9$, each with multiplicity 2. It follows from definition that $I_j(\bar{P}_j) = \bar{P}_j$.
To prove the second claim, we observe that $\bar{P}_j$ is not a base point of the pencil $\mathcal{C}^i$, there is one unique curve $C^i_{\bar{P}_j}$ of the pencil $\mathcal{C}^i$ passing through $\bar{P}_j$, which is exactly the cubic curve C_0 which passes through $P_1, \ldots, P_9$ and $\bar{P}_i$, $\bar{P}_j$.So for any generic curve of the pencil $\mathcal{C}^j$, its intersections with $C^i_{\bar{P}_j}$ are the nine points $P_1, \ldots, \bar{P}_j, \ldots, P_9$. It follows from the definition that $J_{ij}(\bar{P}_j) = P_i$. Similarly, we have the following proposition

Proposition 3 *The map $T_{ij}^{(2)}$ (as a map of P_{10}) is the composition of the two involutions I_i and J_{ij}:*

$$T_{ij}^{(2)} = I_i \circ J_{ij}.$$

Proof The map $T_{ij}^{(2)}$,as a birational map on $\mathbb{P}^2$, is defined by

$$T_{ij}^{(2)}(P) = P + \bar{P}_i - P_j,$$

where the group law is taken on the cubic curve C^i_P of the pencil $\mathcal{C}^i$. In other words, for any given P,$T^{(2)}_{ij}$ is a translation on the curve $\mathcal{C}^i_P$ by $\bar{P}_i - P_j$. Similarly, the proposition follows from the following two statements: □

Claim 3 *$J_{ij}(P_j)$ is well defined and $J_{ij}(P_j) = \bar{P}_i$.*

Claim 4 *$I_i(\bar{P}_i)$ is well defined and $I_i(\bar{P}_i) = \bar{P}_i$.*

The first claim is proved in exactly the same fashion as before, by observing that P_j is not a base point of the pencil $\mathcal{C}^j$ and the unique cubic curve of the pencil $\mathcal{C}^j$ passing through P_j is the cubic C_0 passing through $P_1, \ldots, P_9$ and $\bar{P}_i, \bar{P}_j$. So for any generic curve of the pencil $\mathcal{C}^i$, its intersections with C_0 are the nine points $P_1, \ldots, \bar{P}_i, \ldots, P_j, \ldots, P_9$, and by definition $J_{ij}(P_j) = \bar{P}_i$.
The second claim also follows similarly from the observation that $\bar{P}_i$ is not a base point of the Halphen pencil $\mathcal{H}$ and the unique Halphen curve $H_{\bar{P}_i}$ of the pencil $\mathcal{H}$ passing through $\bar{P}_i$ is the double curve of the cubic curve C_0. So any generic curve of the pencil $\mathcal{C}^i$ intersects with the curve $H_{\bar{P}_i}$ at the nine points $P_1, \ldots, \bar{P}_i, \ldots, P_9$, each with multiplicity 2. It follows from the definition that $I_i(\bar{P}_i) = \bar{P}_i$.

Combining the propositions, the proof of the theorem now follows easily as

$$T^{(2)}_{ij} \circ T^{(1)}_{ij} = I_i \circ J_{ij} \circ J_{ij} \circ I_j = I_i \circ I_j$$

since J_{ij} is a involution.

Remark 5 We remark here that it does not seem to be possible to define similar involutions for Halphen pencil of index higher than 2. The involutions we defined make use of another pencil of curves whose intersection with the Halphen pencil generically having exactly 2 points besides the base points. Or in more precise terms, the divisor classes of these two pencils have intersection number 2. However, for Halphen pencil with higher index, such pencil does not exist as the divisor class of the Halphen pencil will be of the form $-nK$ with $n \geq 3$ and K being the canonical divisor.

4 Example: HKY Mapping

This example is taken from [6], known as an HKY mapping. In [2], a symmetric version of this map is considered, and is classified in the (i − 2) class, i.e., mappings which preserve a index 2 Halphen pencil.

The map is defined by the recurrence relation:

$$y_n y_{n-1} = \frac{x_n^2 - t^2}{s x_n - 1}, \tag{5}$$

$$x_{n+1} x_n = \frac{y_n^2 - t^2}{y_n / s - 1}. \tag{6}$$

The map $\Phi_1 : (x_n, y_n) \mapsto (x_{n+1}, y_n)$ and $\Phi_2 : (x_{n+1}, y_n) \mapsto (x_{n+1}, y_{n+1})$ are both involutions.

Remark 6 In the symmetric case, i.e., s=1, we denote $X_{2n} = x_n$, $X_{2n-1} = y_n$, then the map $\Phi : (X_k, X_{k+1}) \mapsto (X_{k+1}, X_{k+2})$ is the composition of Φ_1 or Φ_2 with a symmetry switch. In comparison with the QRT case, the two involutions Φ_1 and Φ_2 corresponds to the horizontal and vertical switches, while in the symmetric case the map Φ corresponds to the QRT root.

Rewrite the two maps in homogeneous coordinates:

$$\Phi_1 : [x : y : z] \mapsto [s(y^2 - t^2z^2)z : xy(y - sz) : xz(y - sz)], \tag{7}$$
$$\Phi_2 : [x : y : z] \mapsto [xy(sx - z) : (x^2 - t^2z^2)z : yz(sx - z)]. \tag{8}$$

The HKY mapping is the composition $\Phi_2 \circ \Phi_1$.

This map preserves an index 2 Halphen pencil. To see this, one can performs nine blowing ups at the following points to lift the map to a surface automorphism.

$$\begin{aligned} P_1 &= [0 : -t : 1], \\ P_2 &= [0 : t : 1], \\ P_3 &= [-t : 0 : 1], \\ P_4 &= [t : 0 : 1], \\ P_5 &= [1 : 0 : 0], \\ P_6 &= [0 : 1 : 0], \\ P_7 &= [s^2 : 1 : 0]. \end{aligned} \tag{9}$$

- The point P_8 is infinitely close to P_5, given by the direction $\{y = sz\}$,
- The point P_9 is infinitely close to P_6, given by the direction $\{z = sx\}$.

Denoting the corresponding divisors coming from the blowing up at P_i by E_i.

Remark 7 This example is a degeneration of the generic case which we discussed in previous sections, where the base points are considered as distinct points on $\mathbb{P}^2$. Here we have infinitely close points. One can unify the formulation by describing the index 2 Halphen pencil in terms of divisor classes on the blowing up surface. In both of the cases (either generic or degenerate), one obtains a surface with an elliptic fibration by blowing up the nine base points, and the index 2 Halphen pencil corresponds to the divisor class $-2K_X$, where K_X is the canonical class of the surface.

Then the singularity confinement pattern of Φ_1 and Φ_2 is summarized as follows:

$$
\begin{aligned}
&\overline{\{y = sz\}} \xrightarrow{\Phi_1} E_8 \xrightarrow{\Phi_2} E_7 \xrightarrow{\Phi_1} E_9 \xrightarrow{\Phi_2} \overline{\{z = sx\}},\\
&\overline{\{x = 0\}} \xrightarrow{\Phi_1} E_5 - E_8 \xrightarrow{\Phi_2} \overline{\{z = 0\}} \xrightarrow{\Phi_1} E_6 - E_9 \xrightarrow{\Phi_2} \overline{\{y = 0\}},\\
&\overline{\{y = tz\}} \xrightarrow{\Phi_1} E_2 \xrightarrow{\Phi_2} E_2 \xrightarrow{\Phi_1} \overline{\{y = tz\}},\\
&\overline{\{y = -tz\}} \xrightarrow{\Phi_1} E_1 \xrightarrow{\Phi_2} E_1 \xrightarrow{\Phi_1} \overline{\{y = -tz\}},\\
&\overline{\{x = tz\}} \xrightarrow{\Phi_2} E_4 \xrightarrow{\Phi_1} E_4 \xrightarrow{\Phi_2} \overline{\{x = tz\}},\\
&\overline{\{x = -tz\}} \xrightarrow{\Phi_2} E_3 \xrightarrow{\Phi_1} E_3 \xrightarrow{\Phi_2} \overline{\{x = -tz\}}.
\end{aligned}
\tag{10}
$$

Then the mappings Φ_1 and Φ_2 are exactly the involutions $I_4 = I_3$ and $I_1 = I_2$, which we defined for an index 2 Halphen pencil.

5 Conclusion

In this paper, we defined a certain type of involutions on a index 2 Halphen pencil. And we showed that the compositions of these involutions are twice iterations of elliptic Painlevé mapping whose nine parameter points being the base points of the same Halphen pencil. In this sense, the involutions we defined for index 2 Halphen pencils play the same role as the Manin involutions defined on a cubic pencil, where compositions of the involutions give rise to the QRT mapping.

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