Correction to: Height Fluctuations for the Stationary KPZ Equation

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Theorem 2.1 in the original paper provides a Fredholm determinant formula for the Laplace transform of $\mathbf{Z}^{N,M}(\tau)$ which is the partition function of the semi-discrete directed random polymer with boundary sources. The kernel of the Fredholm determinant is given by

$$K_u(v, v') = \frac{1}{2\pi i} \int_{\mathcal{D}_v} ds \Gamma(-s) \Gamma(1+s) \prod_{n=1}^N \frac{\Gamma(v - a_n)}{\Gamma(s + v - a_n)} \prod_{m=1}^M \frac{\Gamma(\alpha_m - v - s)}{\Gamma(\alpha_m - v)} \frac{u^s e^{v\tau s + \tau s^2/2}}{v + s - v'}$$

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for $v, v' \in \mathcal{C}_{a;\alpha;\varphi}$ where the contours $\mathcal{C}_{a;\alpha;\varphi}$ and $\mathcal{D}_v$ are given in Definition 2.4.

The contour $\mathcal{D}_v$ should be taken such that, $v + \mathcal{D}_v$ has the following properties: (a) it goes from $\eta - i\infty$ to $\eta + i\infty$ (for a constant η independent of v), (b) it stays to the right of $\mathcal{C}_{a;\alpha;\varphi}$, (c) it passes to the left of the poles $\{v + 1, v + 2, \dots\}$, (d) it passes to the left of the poles $\cup_{1 \leq m \leq M} \{\alpha_m, \alpha_m + 1, \dots\}$, see Fig. 1.

The issue with the Definition 2.4 of the original paper is that the contour $\mathcal{D}_v$ that appears there satisfies conditions (a)–(c) and satisfies (d) when $\min(\alpha) - \max(a) > 1$ but violates condition (d) when $\min(\alpha) - \max(a) < 1$ and if v is close to μ due to the local deformation made to satisfy property (c). However this is not an issue, since for that case the vertical contour without the local modification already satisfy (c). Thus, the Definition 2.4 should be modified as follows.

Definition Let $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ and $\alpha = (\alpha_1, \dots, \alpha_M) \in (\mathbb{R}_{>0})^M$ be such that $\alpha_m - a_n > 0$ for all $1 \leq n \leq N$ and $1 \leq m \leq M$. Set $\mu = \frac{1}{2} \max(a) + \frac{1}{2} \min(\alpha)$ and $\eta = \frac{1}{4} \max(a) + \frac{3}{4} \min(\alpha)$. Then, for all $\varphi \in (0, \pi/4)$, we define the contour $\mathcal{C}_{a;\alpha;\varphi} = \{\mu + e^{i(\pi+\varphi)}y\}_{y \in \mathbb{R}_+} \cup \{\mu + e^{i(\pi-\varphi)}y\}_{y \in \mathbb{R}_+}$. The contours are oriented so as to have increasing imaginary part. For every $v \in \mathcal{C}_{a;\alpha;\varphi}$ we choose $R = -\operatorname{Re}(v) + \eta$, $d > 0$, and define a contour $\mathcal{D}_v$ depending on the value of R as follows. If $R \leq 1/2$, then $\mathcal{D}_v$ is the vertical line $R + i\mathbb{R}$; if $R > 1/2$, then $\mathcal{D}_v$ goes by straight lines from $R - i\infty$, to $R - id$, to $1/2 - id$, to $1/2 + id$, to $R + id$, to $R + i\infty$. The parameter d is taken small enough so that $v + \mathcal{D}_v$ does not intersect $\mathcal{C}_{a;\alpha;\varphi}$. See Fig. 1 for an illustration.

The original definition changes only in the case when $R \leq 1/2$. The statement of Theorem 2.1 remains valid change with the new definition of the contour $\mathcal{D}_v$.

Fig. 1 Contours for the case $R > 1/2$. Left: the contour $\mathcal{C}_{a;\alpha;\varphi}$ (dashed) where the black dots symbolize the set of singularities of $K_u(v, v')$ in v at $\cup_{1 \leq n \leq N} \{a_n, a_n - 1, \dots\}$ coming from the factors $\Gamma(v - a_n)$. The contour $v + \mathcal{D}_v$ is the solid line. Right: the contour $\mathcal{D}_v$ where the light gray dots are the singularities at $\{1, 2, \dots\}$ and the dark gray dots are those at $\cup_{1 \leq m \leq M} \{\alpha_m - v, \alpha_m + 1 - v, \dots\}$ coming from $\Gamma(\alpha_m - v - s)$. For $R \leq 1/2$ the contour $\mathcal{D}_v$ is simply the vertical line

We validate the modifications which we propose in two steps: first we explain why Theorem 2.1 is true with the new choice of contours, in the second step we argue that further results in the original paper are not affected by these modifications. In particular, Theorems 2.9 and 2.13 about the characterization of the one point distribution of the SHE/KPZ equation with two-sided Brownian and stationary initial data, as well as Theorem 2.17 about the long time asymptotic of the rescaled height remain valid.

In what follows, we point out which steps of the proof in Theorem 2.1 use the new definition of contours and that the proof remains valid.

Definition 3.2 in the original paper changes as follows.

Definition For $\tilde{\alpha}$, $\tilde{\alpha}$ and $\tilde{\beta}$ as in Definition 3.1 and an angle $\varphi \in (0, \pi/2]$ define $\tilde{C}_{\tilde{\alpha};\tilde{\alpha},\tilde{\beta};\varphi} = \{\tilde{\mu} + e^{-i\varphi \operatorname{sgn}(y)} y, y \in \mathbb{R}\}$ (oriented so as to have decreasing imaginary part) where $\tilde{\mu} = \frac{1}{2} \max(\tilde{\alpha} \cup \tilde{\beta}) + \frac{1}{2} \min(\tilde{\alpha}^{-1})$. For $w \in \tilde{C}_{\tilde{\alpha};\tilde{\alpha},\tilde{\beta};\varphi}$, we choose $R > 0$ so that it satisfies the equation $|w|q^R = \frac{3}{4} \max(\tilde{\alpha} \cup \tilde{\beta}) + \frac{1}{4} \min(\tilde{\alpha}^{-1})$. We choose $d > 0$ and the contour $\tilde{D}_w$ as follows. If $R \leq 1/2$, the $\tilde{D}_w$ is the vertical line $R + i\mathbb{R}$; if $R > 1/2$, then $\tilde{D}_w$ goes by straight lines from $R - i\infty$, to $R - id$, to $1/2 - id$, to $1/2 + id$, to $R + id$, to $R + i\infty$. We choose R and d such that the following holds: For all $s \in \tilde{D}_w$, $q^s w$ lies to the left of $\tilde{C}_{\tilde{\alpha};\tilde{\alpha},\tilde{\beta};\varphi}$ and encloses all $\tilde{\alpha}$ and $\tilde{\beta}$; and for $|w|$ large, $R \approx \ln |w|$ and $d \approx |w|^{-1}$ (here $\approx$ means up to a positive constant bounded from zero and infinity).

Theorem 3.3, the central result of this section, remains formally unchanged with the new definition of the contour $\tilde{D}_w$. The proof of Theorem 3.3 consists of several steps and substeps. The argument depends on the choice of the contour $\tilde{D}_w$ only in Steps 1e and 1f.

Step 1e involves proving the general formula (3.14), which rewrites a summation as a contour integral. The formula follows by the application of Lemma 3.10 with the original choice $C_{1,2,\dots} = \tilde{D}_w$ and the contours C_k can be obtained by the same procedure from $C_{1,2,\dots}$. The condition which has to be satisfied is that for large k , the integral over the symmetric difference of $C_{1,2,\dots}$ and C_k goes to 0 as $k \rightarrow \infty$. This condition is satisfied by our modified definition of $\tilde{D}_w$ since it depends only on the decay of the integrand at distance at least k from the origin which is worked out in the rest of Step 1e in the original paper. The condition is not affected by a local modification of the contour $\tilde{D}_w$ around 0.

In the analytic continuation argument in Step 1f, a convergent upper bound is given on the Fredholm determinant series in formula (3.16). Here we use the notation $s = x + iy$. Once $\tilde{\alpha}$ and $\tilde{\beta}$ are fixed, the contour $\tilde{D}_w$ is such that x is uniformly positive and s has a uniformly positive distance from the singularities of the functions which appear in the integrand of the kernel. Hence all further upper bounds given in the rest of Step 1f remain valid, possibly with different constants. The exact value of constant prefactors, however, does not play a role at this point. Step 2 in Section 3 in the original paper remains unchanged.

The choice of $\mathcal{D}_v$ also plays a role in Sections 5.3–5.5 in the original paper. In Section 5.3, we develop the upper bound (5.9) on the kernel $K_u(v, v')$ with v, v' along $\mathcal{C}_{\alpha;\alpha;\varphi}$. It relies on the large $|v|$ behaviour of the integrand in the kernel $K_u(v, v')$

which is at most a quantity exponentially small in $|v|$. The real part $\operatorname{Re}(s) = x$ also appears in the exponent in some of the bounds, but it is uniformly positive for fixed $\tilde{\alpha}, \tilde{\beta}$. The constant upper bound in (5.10) depends on the positive distance of the contour $\mathcal{D}_v$ from the poles of the gamma function and by the fact that $\operatorname{Re}(s)$ is bounded away from 0. The same applies for the argument in Section 5.5. In Section 5.4, Lemma 5.10 follows from the pointwise convergence (5.5)–(5.8) along $\mathcal{D}_v$. The rest of the section deals with the contribution coming from the vertical part of $\mathcal{D}_v$ only. Hence we can conclude that Theorem 2.1 remains true with the new definition of $\mathcal{D}_v$.

Section 6 is devoted to the proof of Theorem 6.3 which is the essential input to Theorem 2.9 and 2.13 about the characterization of the SHE/KPZ equation with two-sided Brownian and stationary initial data. The first step in the proof of Theorem 6.3 is a change of contours $\mathcal{C}_{a;\alpha;\varphi}$ and $\mathcal{D}_v$ to $\mathcal{C}_v$ and $\mathcal{C}_{\tilde{z}}$ defined around (6.17)–(6.18). The contour $\mathcal{C}_{\tilde{z}}$ consists of a vertical line and possibly the union of small circles around a finite number of poles at $v + 1, v + 2, \dots, v + \ell$. In the first part of the proof of Theorem 6.3, condition (6.16) is assumed in which case the original definition of $\mathcal{D}_v$ can be used.

It is explained in the paragraphs following (6.39) how the assumption (6.16) can be relaxed. The only case which we have to deal with is when $\beta - b < 1$. In this case, we locally modify the contours $\mathcal{C}_v$ and $\mathcal{C}_{\tilde{z}}$ around θ and consequently $\mathcal{C}_w$ and $\mathcal{C}_z$ around 0. As long as w belongs to the vertical part of the contour $\mathcal{C}_w$, the part of $\mathcal{C}_z$ consisting of small circles is not present since the vertical part of $\mathcal{C}_z$ is close enough to w . Hence the change in the definition of the contour $\mathcal{D}_v$ does not influence the bounds in Section 6 for large z and/or large w which mainly rely on the decay shown by Lemma 6.6. Once Theorem 2.9 is verified, the rest of the paper remains unchanged.

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