On the Self-Adjointness of $H+A^*+A$

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Abstract

Let $H : \text{dom}(H) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$ be self-adjoint and let $A : \text{dom}(H) \rightarrow \mathfrak{F}$ (playing the role of the annihilation operator) be H -bounded. Assuming some additional hypotheses on A (so that the creation operator A^* is a singular perturbation of H), by a twofold application of a resolvent Kreĭn-type formula, we build self-adjoint realizations $\widehat{H}$ of the formal Hamiltonian $H + A^* + A$ with $\text{dom}(H) \cap \text{dom}(\widehat{H}) = \{0\}$. We give an explicit characterization of $\text{dom}(\widehat{H})$ and provide a formula for the resolvent difference $(-\widehat{H}+z)^{-1} - (-H+z)^{-1}$. Moreover, we consider the problem of the description of $\widehat{H}$ as a (norm resolvent) limit of sequences of the kind $H + A_n^* + A_n + E_n$, where the A_n 's are regularized operators approximating A and the E_n 's are suitable renormalizing bounded operators. These results show the connection between the construction of singular perturbations of self-adjoint operators by Kreĭn's resolvent formula and nonperturbative theory of renormalizable models in Quantum Field Theory; in particular, as an explicit example, we consider the Nelson model.

Keywords Singular perturbations · Selfadjoint operators · Kreĭn's resolvent formula · Renormalizable QFT models

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1 Introduction

In the last few years several works appeared where questions about the characterization of the self-adjointness domains of some renormalizable quantum fields Hamiltonians and their spectral properties were addressed (see [7], [8], [6], [14], [13], [11], [12], [23], [24]). In such papers (see also [17], [27], [29] for some antecedent works considering simpler models) the operator theoretic framework much resembles the one involved in the construction of singular perturbations of self-adjoint operators (a.k.a. self-adjoint extensions of symmetric restrictions) by Kreĭn's type resolvent formulae (see [19], [22] and references therein). The correspondence is

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exact as regards the Fermi polaron model considered in [6] (see the remark following [6, Corollary 4.3] and our Remark 2.19); instead, as regards the Nelson model studied in [13] (this paper was our main source of inspiration), the self-adjointness domain of the Nelson Hamiltonian H_{Nelson} there provided does not correspond, even if it has a similar structure, to the domain of a singular perturbation of the non-interacting Hamiltonian H_{free} . Indeed, if that were so, then, by [19, Remark 2.10] (see also (2.11) below), the domain of H_{Nelson} should be given by

$$\{\Psi \in \mathfrak{F} : \Psi_0 := \Psi + (AH_{\text{free}}^{-1})^* \Phi \in \text{dom}(H_{\text{free}}), A\Psi_0 = \Theta\Phi, \Phi \in \text{dom}(\Theta)\},$$

for some self-adjoint operator Θ (here A denotes the annihilation operator) while, by [13],

$$\text{dom}(H_{\text{Nelson}}) = \{\Psi \in \mathfrak{F} : \Psi + (AH_{\text{free}}^{-1})^* \Psi \in \text{dom}(H_{\text{free}})\}.$$

These two domain representations would coincide whenever $\Theta = A - A(AH_{\text{free}}^{-1})^*$, which, beside containing the ill-defined term $A(AH_{\text{free}}^{-1})^*$, is not even formally symmetric. The lack of a direct correspondence between the two approaches apparently prevents the writing of a formula for the resolvents difference $(-H_{\text{Nelson}} + z)^{-1} - (-H_{\text{free}} + z)^{-1}$. Such a kind of resolvent formula can help the study, beside of the spectrum, of the scattering theory (see [16] and references therein).

Our main aim here is to show that H_{Nelson} can be still obtained using the theory of singular perturbations (thus providing a resolvent formula) by applying Kreĭn's formula twice: at first one singularly perturbs H_{free} obtaining an intermediate Hamiltonian (related to a different physical model, see Remark 2.19) and then one singularly perturbs the latter obtaining the Nelson Hamiltonian (such a strategy is suggested by the use of an abstract Green-type formula, see Lemma 3.1); since for both the two operators Kreĭn's resolvent formula holds, by inserting the resolvent of the first operator in the resolvent formula for the second one, re-arranging and using operator block matrices, at the end one obtains a final formula for the resolvent difference $(-H_{\text{Nelson}} + z)^{-1} - (-H_{\text{free}} + z)^{-1}$ only containing the resolvent of H_{free} and the extension parameter (which is a suitable operator in Fock space), see (3.32).

We consider also the problem of the description of H_{Nelson} as a (norm resolvent) limit of sequences of the kind $H_n := H_{\text{free}} + A_n^* + A_n + E_n$, where the A_n 's are the regularized annihilation operators corresponding with an ultraviolet cutoff and the E_n 's are suitable renormalizing constants. We approach this problem by employing the resolvent formula for H_{Nelson} here obtained and an analogous one for the approximating H_n ; this shows the role of the ever-present term of the kind $A_n H_{\text{free}}^{-1} A_n^*$: it is due to the difference between the so-called Weyl functions (see (2.4)) in the resolvents of the H_n 's and the limit one. The Weyl function of H_{Nelson} contains $A((-AH_{\text{free}}^{-1})^* - (A(-H_{\text{free}} + \bar{z})^{-1})^*)$ and $(-AH_{\text{free}}^{-1})^*$ plays the role of a regularizing term: indeed the operator difference $(-AH_{\text{free}}^{-1})^* - (A(-H_{\text{free}} + \bar{z})^{-1})^*$ has range in the domain of A while the ranges of the single terms never are. Contrarily, the Weyl function of H_n contains $-A_n(-H_{\text{free}} + z)^{-1} A_n^*$ only, without the need, being a bounded operator, of adding the balancing term $-A_n H_{\text{free}}^{-1} A_n^*$. This explain why one has to take into account such an addendum (and also a renormalizing counterterm E_n since $A_n H_{\text{free}}^{-1} A_n^*$ does not converge when the ultraviolet cutoff is removed) in order to approximate H_{Nelson} in norm resolvent sense (see Theorem 3.10 and Subsection 3.1).

In the present paper we embed the previous discussion in an abstract framework; thus we consider a general self-adjoint operators H (playing the role of the free Hamiltonian H_{free}) in an abstract Hilbert space $\mathfrak{F}$ (playing the role of the Fock space) and an abstract annihilation operators A . In Section 2 we provide a self-contained presentation (with some simplifications and generalizations) of (parts of) our previous results contained in the papers [19], [20], [21], [22] that we will need later and give a results of the approximation (in norm resolvent sense) by regular perturbations of the singular perturbations here provided. In particular, in Subsection 2.1, we consider the problem of the construction, by providing their resolvents, of the self-adjoint extensions of the symmetric restriction $S := H|_{\ker(\Sigma)}$, where $\Sigma : \text{dom}(H) \rightarrow \mathfrak{X}$ is bounded with respect to the graph norm in $\text{dom}(H)$ and $\mathfrak{X}$ is an auxiliary Hilbert space. Successively, in Section 3, we apply the previous results to the case where $\mathfrak{X} = \mathfrak{F}$ and $\Sigma = A$. This provides a family H_T of self-adjoint extension of S , where the parameterizing operator T is self-adjoint in $\mathfrak{F}$. Then, we apply again the results in Subsection 2.1 now to the case where $H = H_T$ and $\Sigma = 1 - A_*$, A_* a suitable left inverse of $(A(-H + \bar{z})^{-1})^*$. The final self-adjoint operator $\widehat{H}_T$ is the one we were looking for: it can be represented as $\widehat{H}_T = \overline{H} + A^* + A_T$, where $\overline{H}$ is a (no more $\mathfrak{F}$ -valued) suitable closure of H such that $\overline{H} + A^*$ is $\mathfrak{F}$ -valued when restricted to $\text{dom}(S^*)$ and A_T is an extension of the abstract annihilation operator A . By inserting the resolvent Kreĭn formula for H_T into the one for $\widehat{H}_T$, one gets a Kreĭn resolvent formula for the difference $(-\widehat{H}_T + z)^{-1} - (-H + z)^{-1}$ which contains only the resolvent of H and the operator T (see (3.11)) in Theorem 3.4). Since A_T has the additive representation $A_T = A_0 + T$, where A_0 corresponds to the case $T = 0$, T enters in an additive way in the definition of $\widehat{H}_T$, i.e., $\widehat{H}_T = \widehat{H}_0 + T$ and so one can relax the self-adjointness request on T , and suppose that T is symmetric and $\widehat{H}_0$ -bounded with relative bound $\widehat{a} < 1$, see Theorems 3.9 and 3.13. The same resolvent formula holds also in this case, see (3.26). Notice that this does not contradict the usual parameterization of self-adjoint extensions by self-adjoint operators; indeed the true parameterizing operator turns out to be a (T -dependent) 2×2 block operator matrix which is always self-adjoint, even in the case T is merely symmetric (see Remark 3.16). In Theorem 3.10 we address the problem of the approximation of $\widehat{H}_T$ by a sequence of regular perturbations on H . Finally, in Subsection 3.1, we show how, by the suitable choice $T = T_{\text{Nelson}}$ provided in [13], one obtains $\widehat{H}_{T_{\text{Nelson}}} = H_{\text{Nelson}}$, where the self-adjoint Hamiltonian H_{Nelson} is the one constructed in the seminal paper [18]; the same kind of analysis can be applied to other renormalizable quantum field models.

1.1 Notations

- $\text{dom}(L)$, $\ker(L)$, $\text{ran}(L)$ denote the domain, kernel and range of the linear operator L respectively;
- $\varrho(L)$ denotes the resolvent set of L ;
- $L|_V$ denotes the restriction of L to the subspace $V \subset \text{dom}(L)$;
- $\mathcal{B}(X, Y)$ denotes the set of bounded linear operators on the Banach space X to the Banach space Y , $\mathcal{B}(X) := \mathcal{B}(X, X)$;
- $\|\cdot\|_{X,Y}$ denotes the norm in $\mathcal{B}(X, Y)$;

- $\|\cdot\|_{\text{dom}(L), Y}$ denotes the norm in $\mathcal{B}(\text{dom}(L), Y)$, where $L : \text{dom}(L) \subset X \rightarrow Y$ is a closed linear operator and $\text{dom}(L)$ is equipped with the graph norm;
- $\mathbb{C}_{\pm} := \{z \in \mathbb{C} : \pm \text{Im}(z) > 0\}$.

2 Singular Perturbations and Kreĭn-Type Resolvent Formulae.

2.1 Singular Perturbations

For convenience of the reader, in this subsection we provide a compact (almost) self-contained presentation (with some simplifications and generalizations) of parts of the results from papers [19], [20], [21], [22] that we will need in the next section; we also refer to papers [21] and [22] for the comparison with other formulations (mainly with boundary triple theory, see, e.g., [5, Section 7.3], [2, Chapter 2]) which produce some similar outcomes.

Let

$$H : \text{dom}(H) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$$

be a self-adjoint operator in the Hilbert space $\mathfrak{F}$ with scalar product $\langle \cdot, \cdot \rangle$; just in order to simplify the exposition, we suppose that $\varrho(H) \cap \mathbb{R} \neq \emptyset$ (without this hypothesis some formulae become a bit longer). We introduce the following definition:

$\mathfrak{H}_1$ denotes the Hilbert space given by $\text{dom}(H)$ endowed with the scalar product $\langle \cdot, \cdot \rangle_1$,

$$\langle \psi_1, \psi_2 \rangle_1 := \langle (H^2 + 1)^{1/2} \psi_1, (H^2 + 1)^{1/2} \psi_2 \rangle;$$

$\mathfrak{H}_1$ coincides, as a Banach space, with $\text{dom}(H)$ equipped with the graph norm. Given a bounded linear map

$$\Sigma : \mathfrak{H}_1 \rightarrow \mathfrak{X},$$

$\mathfrak{X}$ an auxiliary Hilbert space with scalar product $(\cdot, \cdot)$, for any $z \in \varrho(H)$ we define the linear bounded operator

$$G_z : \mathfrak{X} \rightarrow \mathfrak{F}, \quad G_z := (\Sigma R_z^*)^*, \quad (2.1)$$

where

$$R_z : \mathfrak{F} \rightarrow \mathfrak{H}_1, \quad R_z := (-H + z)^{-1}.$$

In the most typical situations, whenever H is an 2nd order, elliptic differential operator, Σ is the Dirichlet (Neumann) trace operator on the boundary of a subset of $\mathbb{R}^n$ and G_z is a single (double) layer operator (see, e.g., [22, Example 5.5], [15] and references therein).

We pick $\lambda_0 \in \varrho(H) \cap \mathbb{R}$ and set

$$R := R_{\lambda_0}, \quad G := G_{\lambda_0}. \quad (2.2)$$

By first resolvent identity one has

$$(z - w)R_w G_z = G_w - G_z = (z - w)R_z G_w. \quad (2.3)$$

Hence

$$\text{ran}(G_w - G_z) \subseteq \mathfrak{H}_1,$$

and the linear operator (playing the role of what is called a Weyl operator-valued function in boundary triple theory, see [21], [5, Section 7.3], [2, Chapter 2])

$$M_z := \Sigma(G - G_z) : \mathfrak{X} \rightarrow \mathfrak{X} \quad (2.4)$$

is well defined and bounded; by (2.3) it can be re-written as

$$M_z = (z - \lambda_o)G^*G_z = (z - \lambda_o)G_z^*G. \quad (2.5)$$

By (2.5) one gets the relations

$$M_z^* = M_{\bar{z}}, \quad M_z - M_w = (z - w)G_w^*G_z. \quad (2.6)$$

Given $\Theta : \text{dom}(\Theta) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ self-adjoint, we define

$$Z_{\Sigma, \Theta} := \{z \in \varrho(H) : \Theta + M_z \text{ has inverse in } \mathcal{B}(\mathfrak{X})\}. \quad (2.7)$$

Remark 2.1 By $(\Theta + M_z)^* = \Theta + M_{\bar{z}}$ and by [9, Theorem 5.30, Chap. III], one has

$$z \in Z_{\Sigma, \Theta} \Rightarrow \bar{z} \in Z_{\Sigma, \Theta}.$$

Theorem 2.2 Let $\Sigma : \mathfrak{H}_1 \rightarrow \mathfrak{X}$ be bounded and let $\Theta : \text{dom}(\Theta) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ be self-adjoint. Suppose that

$$Z_{\Sigma, \Theta} \text{ is not empty} \quad (2.8)$$

and define

$$(-H_{\Theta} + z)^{-1} := (-H + z)^{-1} + G_z(\Theta + M_z)^{-1}G_z^*, \quad z \in Z_{\Sigma, \Theta}. \quad (2.9)$$

If

$$\ker(G) = \{0\}, \quad \text{ran}(G) \cap \mathfrak{H}_1 = \{0\}, \quad (2.10)$$

then (2.9) is the resolvent of a self-adjoint operator H_{Θ} and $Z_{\Sigma, \Theta} = \varrho(H) \cap \varrho(H_{\Theta})$; moreover

$$\text{dom}(H_{\Theta}) = \{\psi \in \mathfrak{F} : \exists \phi \in \text{dom}(\Theta) \text{ s.t. } \psi_0 := \psi - G\phi \in \mathfrak{H}_1 \text{ and } \Sigma\psi_0 = \Theta\phi\} \quad (2.11)$$

and

$$(-H_{\Theta} + \lambda_o)\psi = (-H + \lambda_o)\psi_0.$$

Proof At first let us notice that, by $\text{ran}(G - G_z) \subseteq \mathfrak{H}_1$, (2.10) implies that the same relations hold for G_z for any $z \in \varrho(H)$. By (2.6), the operator family on the righthand side of (2.9) (here denoted by $\check{R}_z$) is a pseudo-resolvent (i.e., it satisfies the first resolvent identity) and $\check{R}_z^* = \check{R}_{\bar{z}}$ (see [19, page 115]). Moreover, if $\psi \in \ker(\check{R}_z)$ then $(-H + z)^{-1}\psi = -G_z(\Theta + M_z)^{-1}G_z^*\psi = -G_z(\Theta + M_z)^{-1}\Sigma(-H + z)^{-1}\psi$; this gives $\psi = 0$ by (2.10) and so $\ker(\check{R}_z) = \{0\}$. Hence, by [26, Theorems 4.10 and 4.19], $\check{R}_z$ is the resolvent of a self-adjoint operator $\check{H}$ defined by

$$\text{dom}(\check{H}) := \text{ran}(\check{R}_z) = \{\psi = \psi_z + G_z(\Theta + M_z)^{-1}\Sigma\psi_z, \psi_z \in \mathfrak{H}_1\},$$

$$(-\check{H} + z)\psi := \check{R}_z^{-1}\psi = (-H + z)\psi_z.$$

Let us now show that $\check{H} = H_\Theta$. Posing $\phi_z := (\Theta + M_z)^{-1} \Sigma \psi_z \in \text{dom}(\Theta)$, since the definition of $\check{H}$ is z -independent, $\psi \in \text{dom}(\check{H})$ if and only if, for any $z \in Z_{\Sigma, \Theta}$, there exists $\psi_z \in \mathfrak{H}_1$, $\Sigma \psi_z = (\Theta + M_z)\phi_z$, such that $\psi = \psi_z + G_z \phi_z$. Then, by (2.3),

$$\psi_z - \psi_w = G_w \phi_w - G_z \phi_z = G_z(\phi_w - \phi_z) + (z - w)R_z G_w \phi_w.$$

By (2.10), this gives $\phi_z = \phi_w$, i.e., the definition of ϕ_z is z -independent. Thus, setting $\psi_0 := \psi_z + (G_z - G)\phi$, one has $\psi = \psi_0 + G\phi$, with $\psi_0 \in \mathfrak{H}_1$ and

$$\Sigma \psi_0 - \Theta \phi = \Sigma \psi_z - \Sigma(G - G_z)\phi - \Theta \phi = \Sigma \psi_z - (\Theta + M_z)\phi = 0.$$

Therefore $\text{dom}(\check{H}) \subseteq \text{dom}(H_\Theta)$. Conversely, given $\psi = \psi_0 + G\phi \in \text{dom}(H_\Theta)$, defining $\psi_z = \psi_0 + (G - G_z)\phi$, one has $\psi = \psi_z + G_z \phi$ and $\Sigma \psi_z = \Sigma \psi_0 + \Sigma(G - G_z)\phi = (\Theta + M_z)^{-1} \phi$, i.e. $\psi \in \text{dom}(\check{H})$; so $\text{dom}(H_\Theta) \subseteq \text{dom}(\check{H})$ and in conclusion $\text{dom}(\check{H}) = \text{dom}(H_\Theta)$. Then, by (2.3),

$$\begin{aligned} (-\check{H} + \lambda_o)\psi &= (-H + \lambda_o)\psi_z + (\lambda_o - z)(\psi - \psi_z) \\ &= (-H + \lambda_o)\psi_0 + (-H + \lambda_o)(\psi_z - \psi_0) + (\lambda_o - z)G_z \phi \\ &= (-H + \lambda_o)\psi_0 + (-H + \lambda_o)(G - G_z)\phi - (z - \lambda_o)G_z \phi \\ &= (-H + \lambda_o)\psi_0. \end{aligned}$$

Finally, [4, Theorem 2.19 and Remark 2.20] give $Z_{\Sigma, \Theta} \neq \emptyset \Rightarrow Z_{\Sigma, \Theta} = \varrho(H) \cap \varrho(H_\Theta)$. $\square$

Remark 2.3 Notice that, in order to prove that (2.9) is the resolvent of a self-adjoint operator, only the second of the two hypothesis in (2.10) is required; they both provide the domain representation (2.11). In particular, by $\psi - G\phi_1 - (\psi - G\phi_2) = G(\phi_1 - \phi_2) \in \mathfrak{H}_1$ and by (2.10), for any $\psi \in \text{dom}(H_\Theta)$ there is a unique $\phi \in \mathfrak{F}$ such that $\psi - G\phi \in \mathfrak{H}_1$. Hence the characterization of $\text{dom}(H_\Theta)$ in (2.11) is well defined.

Remarks 2.17, 3.5 and Theorems 2.15, 3.13 below show that one can still have a self-adjoint operator with a resolvent given by a formula like (2.9) even if hypothesis (2.10) does not hold true.

Remark 2.4 Obviously, if $0 \in \varrho(\Theta)$ then $\lambda_o \in Z_{\Sigma, \Theta}$. In this case, whenever (2.10) holds, $\lambda_o \in \varrho(H_\Theta)$ and

$$(-H_\Theta + \lambda_o)^{-1} = (-H + \lambda_o)^{-1} + G\Theta^{-1}G^*, \quad (2.12)$$

Regarding hypotheses (2.8) and (2.10), one has the following sufficient conditions:

Lemma 2.5 Let $\Sigma \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{X})$, let $\Theta : \text{dom}(\Theta) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ be self-adjoint and let G_z and $Z_{\Sigma, \Theta}$ be defined as in (2.1) and (2.7). Then

$$\text{ran}(\Sigma) \text{ dense in } \mathfrak{X} \quad \Leftrightarrow \quad \ker(G_z) = \{0\};$$

$$\ker(\Sigma) \text{ dense in } \mathfrak{F} \quad \Rightarrow \quad \text{ran}(G_z) \cap \mathfrak{H}_1 = \{0\};$$

$$\Sigma \text{ surjective onto } \mathfrak{X} \quad \Rightarrow \quad Z_{\Sigma, \Theta} \supseteq \mathbb{C} \setminus \mathbb{R}.$$

Proof 1) By $\ker(G_z) = \text{ran}(G_z^*)^\perp$, $\ker(G_z) = \{0\}$ if and only if $\text{ran}(G_z^*) = \text{ran}(\Sigma R_z) = \text{ran}(\Sigma)$ is dense.

2) Suppose $G_z\phi = R_z\psi$, equivalently $(-H + z)G_z\phi = \psi$. Then

$$\langle \psi, \varphi \rangle = \langle (-H + z)G_z\phi, \varphi \rangle = \langle \phi, G_z^*(-H + \bar{z})\varphi \rangle = \langle \phi, \Sigma\varphi \rangle = 0$$

for any $\varphi \in \ker(\Sigma) \subseteq \mathfrak{H}_1$. This gives $\psi = 0$ whenever $\ker(\Sigma)$ is dense in $\mathfrak{F}$.

3) Let $\phi \in \text{dom}(\Theta)$, $\|\phi\|_{\mathfrak{X}} = 1$; by (2.6) one gets

$$\|(\Theta + M_z)\phi\|^2 \geq |(\Theta + M_z)\phi, \phi|^2 \geq \text{Im}(z)^2 \|G_z\phi\|^4. \quad (2.13)$$

Since Σ is surjective, $G_z^* = \Sigma R_z^*$ has a closed range and so G_z has closed range as well by the closed range theorem. Therefore, since, by point 1), $\ker(G_z) = \{0\}$, there exists $\gamma_0 > 0$ such that $\|G_z\phi\| \geq \gamma_0 \|\phi\|$ (see [9, Thm. 5.2, Chap. IV]). Thus, by (2.13), $\Theta + M_z$ has a bounded inverse and, by [9, Thm. 5.2, Chap. IV], has a closed range. Therefore, by (2.13) again,

$$\text{dom}((\Theta + M_z)^{-1}) = \text{ran}(\Theta + M_z) = \ker(\Theta + M_z)^\perp = \{0\}^\perp = \mathfrak{X}$$

and so $(\Theta + M_z)^{-1} \in \mathcal{B}(\mathfrak{X})$. $\square$

Remark 2.6 Suppose that $\text{ran}(\Sigma) = \mathfrak{X}$. Then, $\text{ran}(G_z) \cap \mathfrak{H}_1 = \{0\}$ if and only if $\ker(\Sigma)$ is dense in $\mathfrak{F}$ (see [20, Lemma 2.1]).

In the following by *symmetric operator* we mean a (not necessarily densely defined) linear operator $S : \text{dom}(S) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$ such that $\langle S\psi_1, \psi_2 \rangle = \langle \psi_1, S\psi_2 \rangle$ for any ψ_1 and ψ_2 belonging to $\text{dom}(S)$; whenever S is densely defined, S^* denotes its adjoint.

Lemma 2.7 Let S be the symmetric operator $S := H|_{\ker(\Sigma)}$ and suppose that (2.10) holds true; define the linear operator

$$S^\times : \text{dom}(S^\times) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}, \quad (-S^\times + \lambda_0)\psi := (-H + \lambda_0)\psi_0,$$

$$\text{dom}(S^\times) := \{\psi \in \mathfrak{F} : \exists \phi \in \mathfrak{X} \text{ such that } \psi_0 := \psi - G\phi \in \mathfrak{H}_1\}.$$

If $\ker(\Sigma)$ is dense in $\mathfrak{F}$, then $S^\times \subseteq S^*$; if furthermore $\text{ran}(\Sigma) = \mathfrak{X}$, then $S^\times = S^*$. If (2.8) and (2.10) hold then H_Θ is a self-adjoint extension of S and $S \subseteq H_\Theta \subseteq S^\times$.

Proof Let $\psi \in \text{dom}(S^\times)$, $\psi = \psi_0 + G\phi$, and $\varphi \in \text{dom}(S) = \ker(\Sigma)$. Then, by $G^* = \Sigma R$,

$$\begin{aligned} \langle \psi, (-S + \lambda_0)\varphi \rangle &= \langle \psi, (-H + \lambda_0)\varphi \rangle = \langle \psi_0, (-H + \lambda_0)\varphi \rangle + \langle G\phi, (-H + \lambda_0)\varphi \rangle \\ &= \langle (-H + \lambda_0)\psi_0, \varphi \rangle + \langle \phi, G^*(-H + \lambda_0)\varphi \rangle \\ &= \langle (-H + \lambda_0)\psi_0, \varphi \rangle + \langle \phi, \Sigma\varphi \rangle \\ &= \langle (-H + \lambda_0)\psi_0, \varphi \rangle. \end{aligned}$$

Therefore $\psi \in \text{dom}(-S^* + \lambda_0) = \text{dom}(S^*)$ and $(-S^* + \lambda_0)\psi = (-H + \lambda_0)\psi_0 = (-S^\times + \lambda_0)\psi$. Hence $S^\times \subseteq S^*$. The equality $S^\times = S^*$ whenever $\text{ran}(\Sigma) = \mathfrak{X}$ is proven in [20, Theorem 4.1]. Finally, $\ker(\Sigma) \subseteq \text{dom}(H_\Theta)$ and $H_\Theta|_{\ker(\Sigma)} = H|_{\ker(\Sigma)}$ are immediate consequences of Theorem 2.2. $\square$

Lemma 2.8 Let $S^\times$ be defined as in Lemma 2.7. Then, for any $\psi, \varphi \in \text{dom}(S^\times)$, one has the abstract Green's identity

$$\langle S^\times \psi, \varphi \rangle - \langle \psi, S^\times \varphi \rangle = (\Sigma_* \psi, \Sigma_0 \varphi) - (\Sigma_0 \psi, \Sigma_* \varphi), \quad (2.14)$$

where, in case $\psi \in \text{dom}(S^\times)$ decomposes as $\psi = \psi_0 + G\phi$,

$$\Sigma_0 : \text{dom}(S^\times) \rightarrow \mathfrak{X}, \quad \Sigma_0 \psi := \Sigma \psi_0, \quad (2.15)$$

$$\Sigma_* : \text{dom}(S^\times) \rightarrow \mathfrak{X}, \quad \Sigma_* \psi := \phi. \quad (2.16)$$

Proof Let $\psi = \psi_0 + G\phi$, $\varphi = \varphi_0 + G\rho$. By the definition of $S^\times$ and by $G^* = \Sigma R$, one gets

$$\begin{aligned} \langle S^\times \psi, \varphi \rangle - \langle \psi, S^\times \varphi \rangle &= -(\langle (-S^\times + \lambda_o) \psi, \varphi \rangle - \langle \psi, (-S^\times + \lambda_o) \varphi \rangle) \\ &= -(\langle (-H + \lambda_o) \psi_0, \varphi_0 + G\rho \rangle - \langle \psi_0 + G\phi, (-H + \lambda_o) \varphi_0 \rangle) \\ &= -(\langle \psi_0, (-H + \lambda_o) \varphi_0 \rangle + (\Sigma \psi_0, \rho) - \langle \psi_0, (-H + \lambda_o) \varphi_0 \rangle - (\phi, \Sigma \varphi_0)) \\ &= (\Sigma_* \psi, \Sigma_0 \varphi) - (\Sigma_0 \psi, \Sigma_* \varphi). \end{aligned} \quad \square$$

Remark 2.9 By Lemma 2.8, whenever $\ker(\Sigma)$ is dense in $\mathfrak{F}$ and $\text{ran}(\Sigma) = \mathfrak{X}$, the triple $(\mathfrak{X}, \Sigma_*, \Sigma_0)$ is a boundary triple for S^* (see [21, Theorem 3.1], [22, Theorem 4.2]). Otherwise $(\mathfrak{X}, \Sigma_*, \Sigma_0)$ resembles a boundary triple of bounded type (see [5, Section 7.4], see also [3, Section 6.3] for the similar definition of quasi boundary triple).

Remark 2.10 Σ_* is a left inverse of G_z : since $\text{ran}(G_w - G_z) \subseteq \mathfrak{H}_1$, one has $\Sigma_* G_z \phi = \Sigma_* ((G_z - G)\phi + G\phi) = \phi$.

The operator $S^\times$ (and hence also H_Θ) has an alternative additive representation. At first, following [10, Section 9], we introduce a convenient scale of Hilbert spaces $\mathfrak{H}_s$, $s \in \mathbb{R}$, $\mathfrak{H}_t \hookrightarrow \mathfrak{H}_0 \equiv \mathfrak{F} \hookrightarrow \mathfrak{H}_u$, $t < 0 < u$. We define $\mathfrak{H}_s$ as (the completion of, whenever $s < 0$) $\text{dom}((H^2 + 1)^{s/2})$ endowed with the scalar product

$$\langle \psi_1, \psi_2 \rangle_s := \langle (H^2 + 1)^{s/2} \psi_1, (H^2 + 1)^{s/2} \psi_2 \rangle.$$

Notice that R_z extends to a bounded bijective map (which we denote by the same symbol) on $\mathfrak{H}_s$, $s < 0$, and $R_z \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{H}_{s+1})$ for any $z \in \varrho(H)$ and for any $s \in \mathbb{R}$; here we are in particular interested in the case $s = -1$. The linear operator H , being a densely defined bounded operator on $\mathfrak{F}$ to $\mathfrak{H}_{-1}$, extends to the bounded operator on the whole $\mathfrak{F}$ given by its closure: for any $\psi \in \mathfrak{F}$ and for any sequence $\{\psi_n\}_1^\infty \subseteq \mathfrak{H}_1$ such that $\psi_n \xrightarrow{\mathfrak{F}} \psi$

$$\overline{H} : \mathfrak{F} \rightarrow \mathfrak{H}_{-1}, \quad \overline{H} \psi := \mathfrak{H}_{-1} - \lim_{n \uparrow \infty} H \psi_n.$$

Let us denote by $\langle \cdot, \cdot \rangle_{-1,1} : \mathfrak{H}_{-1} \times \mathfrak{H}_1 \rightarrow \mathbb{C}$, the pairing obtained by extending the scalar product:

$$\langle \psi, \varphi \rangle_{-1,1} := \lim_{n \uparrow \infty} \langle \psi_n, \varphi \rangle, \quad \psi_n \xrightarrow{\mathfrak{H}_{-1}} \psi, \quad \psi_n \in \mathfrak{F}, \quad \varphi \in \mathfrak{H}_1. \quad (2.17)$$

Then we define $\Sigma^* : \mathfrak{X} \rightarrow \mathfrak{H}_{-1}$ by

$$\langle \Sigma^* \phi, \varphi \rangle_{-1,1} = (\phi, \Sigma \varphi), \quad \varphi \in \mathfrak{H}_1, \phi \in \mathfrak{X}. \quad (2.18)$$

Remark 2.11 Let us notice that $R_z : \mathfrak{H}_{-1} \rightarrow \mathfrak{F}$ is the adjoint, with respect the pairing $\langle \cdot, \cdot \rangle_{-1,1}$, of $R_z^* : \mathfrak{H}_1 \rightarrow \mathfrak{F}$ and it is the inverse of $(-\overline{H} + z) : \mathfrak{F} \rightarrow \mathfrak{H}_{-1}$; therefore $G_z = R_z \Sigma^*$ and

$$\ker(G) = \{0\} \Leftrightarrow \ker(\Sigma^*) = \{0\},$$

$$\text{ran}(G) \cap \mathfrak{H}_1 = \{0\} \Leftrightarrow \text{ran}(\Sigma^*) \cap \mathfrak{F} = \{0\}. \quad (2.19)$$

If $\Sigma_\circ : \mathfrak{H}_1 \subseteq \mathfrak{F} \rightarrow \mathfrak{X}$ denotes the densely defined, linear operator $\Sigma_\circ \psi := \Sigma \psi$, then $\Sigma_\circ^* : \text{dom}(\Sigma_\circ^*) \subseteq \mathfrak{X} \rightarrow \mathfrak{F}$ is the restriction of Σ^* to the subspace $\{\psi \in \mathfrak{X} : \Sigma^* \psi \in \mathfrak{F}\}$; therefore, by (2.19), $\text{ran}(G) \cap \mathfrak{H}_1 = \{0\}$ if and only if $\text{dom}(\Sigma_\circ^*) = \ker(\Sigma^*)$. Thus, if $\Sigma_\circ$ is closable, so that $\text{dom}(\Sigma_\circ^*)$ is dense, then the hypothesis $\text{ran}(G) \cap \mathfrak{H}_1 = \{0\}$ is violated (here we omit the trivial case $\Sigma \equiv 0$).

Lemma 2.12 *If $\psi \in \text{dom}(S^\times)$ then $\overline{H}\psi + \Sigma^* \Sigma_* \psi$ belongs to $\mathfrak{F}$ and it equals $S^\times \psi$:*

$$S^\times = (\overline{H} + \Sigma^* \Sigma_*)|_{\text{dom}(S^\times)}.$$

Proof Let $\psi \in \text{dom}(S^\times)$, $\psi = \psi_0 + G\phi$. Then

$$\begin{aligned} S^\times \psi &= -(-S^\times + \lambda_\circ)\psi + \lambda_\circ \psi = -(-H + \lambda_\circ)\psi_0 + \lambda_\circ \psi \\ &= -(-\overline{H} + \lambda_\circ)(\psi - G\phi) + \lambda_\circ \psi = \overline{H}\psi + (-\overline{H} + \lambda_\circ)G\phi. \end{aligned}$$

Noticing that, for any $\psi \in \mathfrak{F}$ and $\varphi \in \mathfrak{H}_1$, taking any sequence $\{\psi_n\}_1^\infty \subseteq \mathfrak{H}_1$ such that $\psi_n \xrightarrow{\mathfrak{F}} \psi$, one has

$$\begin{aligned} \langle (-\overline{H} + \lambda_\circ)\psi, \varphi \rangle_{-1,1} &= \lim_{n \uparrow \infty} \langle (-H + \lambda_\circ)\psi_n, \varphi \rangle_{-1,1} \\ &= \lim_{n \uparrow \infty} \langle \psi_n, (-H + \lambda_\circ)\varphi \rangle = \langle \psi, (-H + \lambda_\circ)\varphi \rangle, \end{aligned}$$

one gets

$$\begin{aligned} \langle (-\overline{H} + \lambda_\circ)G\phi, \varphi \rangle_{-1,1} &= \langle G\phi, (-H + \lambda_\circ)\varphi \rangle \\ &= (\phi, G^*(-H + \lambda_\circ)\varphi) = (\phi, \Sigma \varphi) = \langle \Sigma^* \phi, \varphi \rangle_{-1,1}. \end{aligned}$$

This gives $(-\overline{H} + \lambda_\circ)G\phi = \Sigma^* \phi = \Sigma^* \Sigma_* \psi$ and the proof is done. $\square$

Summing up, one gets the following

Theorem 2.13 *Given $\Sigma : \mathfrak{H}_1 \rightarrow \mathfrak{X}$ bounded and $\Theta : \text{dom}(\Theta) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ self-adjoint, suppose that hypotheses (2.8) and (2.10) hold. Then, setting*

$$\Sigma_\Theta : \text{dom}(\Sigma_\Theta) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}, \quad \Sigma_\Theta := \Sigma_\circ - \Theta \Sigma_*,$$

$$\text{dom}(\Sigma_\Theta) := \{\psi \in \text{dom}(S^\times) : \Sigma_* \psi \in \text{dom}(\Theta)\},$$

one has that $H_\Theta = S^\times|_{\ker(\Sigma_\Theta)}$ is a self-adjoint extension of $S = H|_{\ker(\Sigma)}$; moreover

$$H_\Theta = \overline{H} + \Sigma^* \Sigma_*$$

and

$$(-H_\Theta + z)^{-1} = (-H + z)^{-1} - G_z(\Sigma_\Theta G_z)^{-1} G_z^*, \quad z \in \varrho(H) \cap \varrho(H_\Theta). \quad (2.20)$$

Proof The thesis is consequence of Theorem 2.2, Lemmata 2.7, 2.8 and 2.12, noticing that, for any $\phi \in \text{dom}(\Theta)$,

$$(\Theta + M_z)\phi = \Theta\phi + \Sigma(G - G_z)\phi = -\Sigma_0((G_z - G)\phi + G\phi) + \Theta\phi = -(\Sigma_0 - \Theta\Sigma_*)G_z\phi.$$

□

Remark 2.14 Notice that if Θ has an inverse Λ then $\Sigma_*\psi = \Lambda\Sigma_0\psi$ for any $\psi \in \text{dom}(H_\Theta) = \ker(\Sigma_\Theta)$; therefore in this case one has, in Theorem 2.13,

$$H_\Theta = \overline{H} + \Sigma^* \Lambda \Sigma_0.$$

2.2 Approximations by Regular Perturbations

If Σ is a bounded operator on $\mathfrak{F}$, $\Sigma \in \mathcal{B}(\mathfrak{F}, \mathfrak{X})$, then $G_z = R_z \Sigma^*$ has values in $\mathfrak{H}_1$ and so hypothesis (2.10) does not hold; more generally, by Remark 2.11, hypothesis (2.10) is violated whenever Σ , as an operator in $\mathfrak{F}$ with domain $\mathfrak{H}_1$, is closable. A simple example of an analogue of resolvent formula (2.9) in the case of regular perturbations is provided in the following

Theorem 2.15 *Let $\Lambda : \text{dom}(\Lambda) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ be symmetric and let $\Sigma_o : \text{dom}(\Sigma_o) \subseteq \mathfrak{F} \rightarrow \mathfrak{X}$, $\text{dom}(\Sigma_o) \supseteq \mathfrak{H}_1$, be closable such that $\Sigma_o \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{X})$, $\Lambda\Sigma_o \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{X})$, $\Sigma_o^* \Lambda \Sigma_o \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{F})$ and $\Lambda\Sigma_o R\Sigma_o^* \in \mathcal{B}(\mathfrak{X})$. If*

$$\lim_{|\gamma| \uparrow \infty} \|\Sigma_o^* \Lambda \Sigma_o R_{i\gamma}\|_{\mathfrak{F}, \mathfrak{F}} = a < 1, \quad \lim_{|\gamma| \uparrow \infty} \|\Lambda \Sigma_o R_{i\gamma} \Sigma_o^*\|_{\mathfrak{X}, \mathfrak{X}} = b < 1, \quad \gamma \in \mathbb{R},$$

then $\tilde{H}_\Lambda := H + \Sigma_o^* \Lambda \Sigma_o$ is self-adjoint, with $\text{dom}(\tilde{H}_\Lambda) = \mathfrak{H}_1$ and resolvent given, whenever $|\gamma|$ is sufficiently large, by

$$(-\tilde{H}_\Lambda + i\gamma)^{-1} = R_{i\gamma} + (\Sigma_o R_{-i\gamma})^* (1 - \Lambda \Sigma_o R_{i\gamma} \Sigma_o^*)^{-1} \Lambda \Sigma_o R_{i\gamma}. \quad (2.21)$$

In the case $\Lambda = \Theta_o^{-1}$, $\Theta_o : \text{dom}(\Theta_o) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ self-adjoint with $0 \in \varrho(\Theta_o)$, one has

$$(-\tilde{H}_\Lambda + z)^{-1} = R_z + (\Sigma_o R_z)^* (\Theta_o - \Sigma_o R_z \Sigma_o^*)^{-1} \Sigma_o R_z, \quad z \in \varrho(H) \cap \varrho(\tilde{H}_\Lambda). \quad (2.22)$$

Proof At first notice that $\Lambda \Sigma_o R_z \Sigma_o^*$ is bounded for any $z \in \varrho(H)$ since both $\Lambda \Sigma_o R$ and $\Sigma_o R_z^*$ are and $\Lambda \Sigma_o R_z \Sigma_o^* = \Lambda \Sigma_o R \Sigma_o^* + \Lambda \Sigma_o (R_z - R) \Sigma_o^* = \Lambda \Sigma_o R \Sigma_o^* + (\lambda_o - z) \Lambda \Sigma_o R (\Sigma_o R_z^*)^*$. Since Σ_o is closable, $\Sigma_o^* \Lambda \Sigma_o$ is symmetric and, by our hypotheses, it is H -bounded with relative bound $a < 1$; thus, by the Rellich-Kato

theorem, $\tilde{H}_\Lambda$ is self-adjoint with domain $\text{dom}(\tilde{H}_\Lambda) = \mathfrak{H}_1$. For any $\gamma \in \mathbb{R}$ such that $\|\Sigma_o^* \Lambda \Sigma_o R_{i\gamma}\|_{\mathfrak{X}, \mathfrak{X}} < 1$ and $\|\Lambda \Sigma_o R_{i\gamma} \Sigma_o^*\|_{\mathfrak{F}, \mathfrak{F}} < 1$, one has

$$\begin{aligned} (-\tilde{H}_\Lambda + i\gamma)^{-1} &= R_{i\gamma}(1 - \Sigma_o^* \Lambda \Sigma_o R_{i\gamma})^{-1} = R_{i\gamma} + \sum_{n=1}^{\infty} R_{i\gamma}(\Sigma_o^* \Lambda \Sigma_o R_{i\gamma})^n \\ &= R_{i\gamma} + \sum_{n=1}^{\infty} \left((\Sigma_o R_{-i\gamma})^* (\Lambda \Sigma_o R_{i\gamma} \Sigma_o^*)^{n-1} \Lambda \Sigma_o R_{i\gamma} \right) \\ &= R_{i\gamma} + (\Sigma_o R_{-i\gamma})^* \left(\sum_{n=1}^{\infty} (\Lambda \Sigma_o R_{i\gamma} \Sigma_o^*)^{n-1} \right) \Lambda \Sigma_o R_{i\gamma} \\ &= R_{i\gamma} + (\Sigma_o R_{-i\gamma})^* (1 - \Lambda \Sigma_o R_{i\gamma} \Sigma_o^*)^{-1} \Lambda \Sigma_o R_{i\gamma}. \end{aligned}$$

Then, by $(1 - \Theta_o^{-1} \Sigma_o R_z \Sigma_o^*)^{-1} \Theta_o^{-1} = (\Theta_o(1 - \Theta_o^{-1} \Sigma_o R_z \Sigma_o^*))^{-1} = (\Theta_o - \Sigma_o R_z \Sigma_o^*)^{-1}$, one gets $(-\tilde{H}_\Lambda + z)^{-1} = R_z + (\Sigma_o R_z)^* (\Theta_o - \Sigma_o R_z \Sigma_o^*)^{-1} \Sigma_o R_z$ for $z = i\gamma$, $|\gamma|$ sufficiently large. Finally, such a resolvent formula holds for any $z \in \varrho(H) \cap \varrho(\tilde{H}_\Lambda)$ by [4, Theorem 2.19 and Remark 2.20]. $\square$

Remark 2.16 If $\mathfrak{X} = \mathfrak{F}$ and $\Sigma_o = 1$, then Theorem 2.15 is nothing else than the Rellich-Kato theorem for $H + \Lambda$. If $\mathfrak{X} = \mathfrak{F}$ and V is self-adjoint, then, taking $\Lambda = \text{sign}(V)$ and $\Sigma_o = |V|^{1/2}$, (2.21) provides the Konno-Kuroda formula (due to Kato) for the resolvent of $H + V$.

Remark 2.17 Since $\Theta_o - \Sigma_o R_z \Sigma_o^* = \Theta_o + \Sigma_o R \Sigma_o^* - \Sigma_o((\Sigma_o R)^* - (\Sigma_o R_z)^*)$ and $\Theta_o + \Sigma_o R \Sigma_o^*$ is self-adjoint, (2.22) coincides with (2.9) whenever $\Sigma = \Sigma_o$ and $\Theta = \Theta_o + \Sigma_o R \Sigma_o^*$. However resolvent formula (2.22) is not a consequence of Theorem 2.2; indeed, by $\text{dom}(\tilde{H}_\Lambda) = \mathfrak{H}_1$ and by (2.22), one has $\text{ran}((\Sigma_o R_z)^*) \cap \mathfrak{H}_1 \neq \{0\}$; this violates (2.10).

In the following we use the notations H_Θ and $\tilde{H}_\Lambda$ to indicate self-adjoint operators having resolvent given by formulae (2.9) and (2.21) (or (2.22)) respectively, this independently of the validity of (some of) the hypotheses required in Theorems 2.2 and 2.15.

Theorem 2.18 Let $\Theta : \text{dom}(\Theta) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$ be self-adjoint, let $\Sigma \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{X})$ and suppose that formula (2.9) provides the resolvent of a self-adjoint operator H_Θ . Further suppose that there exist a sequence of closable operators $\Sigma_n : \text{dom}(\Sigma_n) \subseteq \mathfrak{F} \rightarrow \mathfrak{X}$, $\text{dom}(\Sigma_n) \supseteq \mathfrak{H}_1$, and a sequence of self-adjoint operators $\Theta_n : \text{dom}(\Theta_n) \subseteq \mathfrak{X} \rightarrow \mathfrak{X}$, $\text{dom}(\Theta_n) \supseteq \text{dom}(\Theta)$, $0 \in \varrho(\Theta_n)$, such that $\Sigma_n \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{F})$, $\Sigma_n R \Sigma_n^* \in \mathcal{B}(\mathfrak{F}, \mathfrak{X})$ and $H + \Sigma_n^* \Lambda_n \Sigma_n$, $\Lambda_n := \Theta_n^{-1}$, is self-adjoint with resolvent given by (2.22). If

$$\lim_{n \uparrow \infty} \|\Sigma_n - \Sigma\|_{\mathfrak{H}_1, \mathfrak{X}} = 0, \quad (2.23)$$

$$\lim_{n \uparrow \infty} \|(\Theta_n - \Sigma_n R \Sigma_n^*) - \Theta\|_{\text{dom}(\Theta), \mathfrak{X}} = 0, \quad (2.24)$$

and, in the case of $\text{dom}(\Theta_n) \neq \text{dom}(\Theta)$, there exist a complex conjugate couple $z_{\pm} \in \mathbb{C}_{\pm}$ such that, for any $\phi \in \mathfrak{X}$,

$$\sup_{n \geq 1} \|(\Theta_n - \Sigma_n R_{z_{\pm}} \Sigma_n^*)^{-1} \phi\|_{\mathfrak{X}} < +\infty, \quad (2.25)$$

then

$$\lim_{n \uparrow \infty} (H + \Sigma_n^* \Lambda_n \Sigma_n) = H_{\Theta} \quad \text{in norm-resolvent sense.} \quad (2.26)$$

Proof Set $H_n := H + \Sigma_n^* \Lambda_n \Sigma_n$. Given $z \in \mathbb{C} \setminus \mathbb{R}$, by the resolvent formulae (2.9) and (2.22) one obtains

$$\begin{aligned} & (-H_n + z)^{-1} - (H_{\Theta} + z)^{-1} \\ &= (\Sigma_n R_z^*)^* (\Theta_n - \Sigma_n R_z \Sigma_n^*)^{-1} \Sigma_n R_z + G_z (\Sigma_{\Theta} G_z)^{-1} G_z^* \\ &= (\Sigma_n R_z^*)^* (\Theta_n - \Sigma_n R_z \Sigma_n^*)^{-1} (\Sigma_n R_z - G_z^*) + (G_z - (\Sigma_n R_z^*)^* (\Sigma_{\Theta} G_z)^{-1} G_z^* \\ &\quad + (\Sigma_n R_z^*)^* ((\Theta_n - \Sigma_n R_z \Sigma_n^*)^{-1} + (\Sigma_{\Theta} G_z)^{-1}) G_z^*. \end{aligned}$$

By the norm convergence of $(\Sigma_n R_z^*)^*$ and $\Sigma_n R_z$ to G_z and G_z^* respectively, the thesis amounts to show that

$$\lim_{n \uparrow \infty} \|(\Theta_n - \Sigma_n R_{z_{\pm}} \Sigma_n^*)^{-1} + (\Sigma_{\Theta} G_{z_{\pm}})^{-1}\|_{\mathfrak{F}, \mathfrak{F}} = 0. \quad (2.27)$$

By hypotheses (2.23), (2.24) and by the relation

$$\begin{aligned} & (\Theta_n - \Sigma_n R_z \Sigma_n^*) + \Sigma_{\Theta} G_z \\ &= \Theta_n - \Sigma_n R \Sigma_n^* - \Theta + \Sigma_n (R - R_z) \Sigma_n^* + \Sigma (G - G_z) \\ &= \Theta_n - \Sigma_n R \Sigma_n^* - \Theta + (z - \lambda_o) (\Sigma_n R (\Sigma_n R_z^*)^* - G^* G_z), \end{aligned}$$

one gets

$$\lim_{n \uparrow \infty} \|(\Theta_n - \Sigma_n R_z \Sigma_n^*) + \Sigma_{\Theta} G_z\|_{\text{dom}(\Theta), \mathfrak{X}} = 0.$$

Thus, by

$$\begin{aligned} & (\Theta_n - \Sigma_n R_{z_{\pm}} \Sigma_n^*)^{-1} + (\Sigma_{\Theta} G_{z_{\pm}})^{-1} \\ &= (\Theta_n - \Sigma_n R_{z_{\pm}} \Sigma_n^*)^{-1} ((\Theta_n - \Sigma_n R_{z_{\pm}} \Sigma_n^*) + \Sigma_{\Theta} G_{z_{\pm}}) (\Sigma_{\Theta} G_{z_{\pm}})^{-1}, \end{aligned}$$

by the estimate

$$\begin{aligned} & \|(\Sigma_{\Theta} G_{z_{\pm}})^{-1}\|_{\mathfrak{F}, \text{dom}(\Theta)} = \|(\Theta + M_z)^{-1}\|_{\mathfrak{F}, \text{dom}(\Theta)} \\ & \leq \|\Theta(\Theta + M_z)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} + \|(\Theta + M_z)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} \\ & \leq \|1 - M_z(\Theta + M_z)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} + \|(\Theta + M_z)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} < +\infty, \end{aligned}$$

and by (2.25) (together with uniform boundedness principle), (2.27) follows.

The proof is concluded by showing that if $\text{dom}(\Theta_n) = \text{dom}(\Theta)$ then the hypothesis (2.25) is consequence of (2.23) and (2.24). By (2.28) and

$$\|\Sigma_{\Theta} G_z \phi\|_{\mathfrak{X}} \geq \|(\Sigma_{\Theta} G_z)^{-1}\|_{\mathfrak{X}, \mathfrak{X}}^{-1} \|\phi\|_{\mathfrak{X}}, \quad \phi \in \text{dom}(\Theta),$$

there exists $N > 0$ such that, for any $n > N$ and for any $\varphi \in \text{dom}(\Theta)$,

$$\begin{aligned} \|(\Theta_n - \Sigma_n R_z \Sigma_n^*)\varphi\|_{\mathfrak{X}} &\geq \|\Sigma_\Theta G_z \varphi\|_{\mathfrak{X}} - \|(\Theta_n - \Sigma_n R_z \Sigma_n^*)\varphi + \Sigma_\Theta G_z \varphi\|_{\mathfrak{X}} \\ &\geq \frac{1}{2} \|(\Sigma_\Theta G_z)^{-1}\|_{\mathfrak{X}, \mathfrak{X}}^{-1} \|\varphi\|_{\mathfrak{X}}. \end{aligned}$$

Therefore, choosing $\varphi = (\Theta_n - \Sigma_n R_z \Sigma_n^*)^{-1} \phi \in \text{dom}(\Theta_n) = \text{dom}(\Theta)$,

$$\|(\Theta_n - \Sigma_n R_z \Sigma_n^*)^{-1}\|_{\mathfrak{X}, \mathfrak{X}} \leq 2 \|(\Sigma_\Theta G_z)^{-1}\|_{\mathfrak{X}, \mathfrak{X}}. \quad \square$$

Remark 2.19 If in Theorem 2.18 one takes $\Theta_n = g_n^{-1}$, $g_n \in \mathbb{R} \setminus \{0\}$ such that hypotheses there hold for some self-adjoint Θ , then

$$\lim_{n \uparrow \infty} (H + g_n \Sigma_n^* \Sigma_n) = H_\Theta \quad \text{in norm-resolvent sense.}$$

In the case $\mathfrak{F}$ is the Fock space and Σ is the annihilation operator (as in the next section), this (and the obvious similar version where norm-resolvent convergence is replaced by strong-resolvent convergence) is our version of [6, Theorem 4.2]. It shows how the results provided in Subsection 2.1 can be used to define self-adjoint Hamiltonians describing a Fermi polaron model (see also the remark following [6, Corollary 4.3]) and, more generally, self-adjoint operators preserving the particles number.

3 Self-Adjointness of $H + A^* + A$

We start by applying the results in the previous section to the case

$$\mathfrak{X} = \mathfrak{F}, \quad \Sigma = A : \mathfrak{H}_1 \rightarrow \mathfrak{F}, \quad \Theta = -T : \text{dom}(T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F},$$

where $A \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{F})$ and T is self-adjoint. We suppose that hypotheses (2.8) and (2.10) hold and so, by Theorem 2.13, one gets a self-adjoint extension H_T of the symmetric operator $S = H|_{\ker(A)}$. Here A plays the role of an (abstract) annihilation operator; the change in notation is motivated by the fact that in this section we apply the previous results twice; at first with $\Sigma = A$ and then with Σ equal to a suitable left inverse of $((-H + \bar{z})^{-1}A)^*$.

Using here the notations

$$A_0 \equiv \Sigma_0, \quad A_* \equiv \Sigma_*,$$

one has (see (2.15) and (2.16)), whenever $\psi = \psi_0 + G\phi$,

$$A_0 : \text{dom}(S^\times) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}, \quad A_0 \psi := A \psi_0,$$

$$A_* : \text{dom}(S^\times) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}, \quad A_* \psi := \phi.$$

Defining then

$$A_T : \text{dom}(A_T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}, \quad A_T := A_0 + T A_*,$$

$$\text{dom}(A_T) := \{\psi \in \text{dom}(S^\times) : A_* \psi \in \text{dom}(T)\},$$

by Theorem 2.13,

$$H_T := S^\times|_{\ker(A_T)}$$

is self-adjoint,

$$(-H_T + z)^{-1} = (-H + z)^{-1} - G_z(A_T G_z)^{-1} G_z^*, \quad z \in \varrho(H) \cap \varrho(H_T) \quad (3.1)$$

and

$$H_T \psi = \overline{H} \psi + A^* A_* \psi, \quad (3.2)$$

where $A^* : \mathfrak{F} \rightarrow \mathfrak{H}_{-1}$ is defined as in (2.18).

The operator in (3.2) seems to be different from what we are looking for, i.e., an operator of the kind $\overline{H} + A^* + A$. However, the difference is not so big: by the definition of A_T and by Green's formula (2.14), for any $\psi, \varphi \in \text{dom}(A_T) \subseteq \text{dom}(S^\times)$ one has (here T symmetric would suffice)

$$\begin{aligned} & \langle A_T \psi, A_* \varphi \rangle - \langle A_* \psi, A_T \varphi \rangle \\ &= \langle A_0 \psi, A_* \varphi \rangle - \langle A_* \psi, A_0 \varphi \rangle + \langle T A_* \psi, A_* \varphi \rangle - \langle A_* \psi, T A_* \varphi \rangle \\ &= \langle \psi, S^\times \varphi \rangle - \langle S^\times \psi, \varphi \rangle. \end{aligned} \quad (3.3)$$

This gives the following

Lemma 3.1 *The linear operator $S_T^\times : \text{dom}(S_T^\times) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$, $\mathfrak{H}_1 \cap \text{dom}(S_T^\times) = \{0\}$, defined by*

$$\begin{aligned} \text{dom}(S_T^\times) &:= \{\psi \in \text{dom}(A_T) : A_* \psi = \psi\} = \{\psi \in \text{dom}(T) : \psi - G\psi \in \mathfrak{H}_1\}, \\ S_T^\times \psi &:= S^\times \psi + A_T \psi \equiv \overline{H} \psi + A^* \psi + A_T \psi \end{aligned} \quad (3.4)$$

is symmetric.

Proof By (3.3), for any $\psi, \varphi \in \text{dom}(S_T^\times)$ one has

$$\langle (S^\times + A_T) \psi, \varphi \rangle = \langle \psi, (S^\times + A_T) \varphi \rangle,$$

i.e., $S_T^\times$ is symmetric. Moreover

$$\mathfrak{H}_1 \cap \text{dom}(S_T^\times) = \{\psi \in \mathfrak{H}_1 \cap \text{dom}(T) : G\psi \in \mathfrak{H}_1\} = \{0\}. \quad \square$$

Since

$\text{dom}(H_T) \cap \text{dom}(S_T^\times) = \{\psi \in \text{dom}(H_T) : A_* \psi = \psi\} = \{\psi \in \ker(A_T) : A_* \psi = \psi\}$, by (3.2) and (3.4), one has

$$S_T^\times|_{\text{dom}(H_T) \cap \text{dom}(S_T^\times)} = H_T|_{\text{dom}(H_T) \cap \text{dom}(S_T^\times)},$$

i.e., $S_T^\times$ extends a restriction of a self-adjoint operator:

$$S_T^\times \supseteq \widehat{S} := H_T|_{\ker(\widehat{\Sigma}) \cap \text{dom}(H_T)},$$

where

$$\widehat{\Sigma} : \text{dom}(S^\times) \rightarrow \mathfrak{F}, \quad \widehat{\Sigma} := 1 - A_*.$$

Therefore we can try to apply the formalism recalled in Subsection 2.1 to the case $H = H_T$ and $\Sigma = \widehat{\Sigma}|_{\text{dom}(H_T)}$ in order to build self-adjoint extensions of $\widehat{S}$. If for some of such self-adjoint extensions $\widehat{H}$ one has $\widehat{H} \subseteq S_T^\times$, then, since $S_T^\times$ is symmetric by Lemma 3.1, $\widehat{H} = S_T^\times$ and so $S_T^\times$ itself is self-adjoint. To apply such a strategy, we need to check the validity of hypotheses in Theorem 2.2.

Since $\ker(A_*) = \mathfrak{H}_1 = \text{ran}(R_z)$ and A_* is a left inverse of G_z (see Remark 2.10), for any $z \in Z_{\Sigma, -T}$, one has

$$\begin{aligned}\widehat{\Sigma}(-H_T + z)^{-1} &= (-H_T + z)^{-1} - A_*((-H + z)^{-1} - G_z(A_T G_z)^{-1} G_z^*) \\ &= (-H_T + z)^{-1} + (A_T G_z)^{-1} G_z^*.\end{aligned}\quad (3.5)$$

Thus $\widehat{\Sigma} : \text{dom}(H_T) \rightarrow \mathfrak{F}$ is bounded w.r.t. the graph norm in $\text{dom}(H_T)$ and, for any $z \in \varrho(H_T)$ one can define the bounded operator

$$\widehat{G}_z : \mathfrak{F} \rightarrow \mathfrak{F}, \quad \widehat{G}_z := (\widehat{\Sigma}(-H_T + \bar{z})^{-1})^*.$$

By (3.5), for any $z \in Z_{\Sigma, -T}$, one has

$$\widehat{G}_z = (-H_T + z)^{-1} + G_z(A_T G_z)^{-1} = (-H + z)^{-1} + G_z(A_T G_z)^{-1}(1 - G_z^*). \quad (3.6)$$

This shows that

$$\text{ran}(\widehat{G}_z) \subseteq \text{dom}(S^\times)$$

and $\widehat{\Sigma}\widehat{G}_z$ is a well defined operator in $\mathcal{B}(\mathfrak{F})$:

$$\begin{aligned}\widehat{\Sigma}\widehat{G}_z &= \widehat{\Sigma}(-H_T + z)^{-1} + \widehat{\Sigma}G_z(A_T G_z)^{-1} \\ &= (-H_T + z)^{-1} + (A_T G_z)^{-1} G_z^* + G_z(A_T G_z)^{-1} - (A_T G_z)^{-1} \\ &= (-H + z)^{-1} - (1 - G_z)(A_T G_z)^{-1}(1 - G_z^*).\end{aligned}\quad (3.7)$$

Regarding the validity of hypothesis (2.10), one has the following:

Lemma 3.2 *For any $z \in \varrho(H) \cap \varrho(H_T)$, one has*

$$\ker(\widehat{G}_z) = \{0\} = \text{ran}(\widehat{G}_z) \cap \text{dom}(H_T).$$

Proof At first notice that, since $A_T(-H_T + z)^{-1} = 0$, $A_T\widehat{G}_z = 1$ by (3.6). Hence $\widehat{G}_z\phi = 0$ implies $0 = A_T\widehat{G}_z\phi = \phi$. Now suppose that $\widehat{G}_z\phi \in \text{dom}(H_T) = \ker(A_T)$. Then $0 = A_T\widehat{G}_z\phi = \phi$ and so $\widehat{G}_z\phi = 0$. $\square$

Now, let us suppose that $\mathbb{R} \cap \varrho(H) \cap \varrho(H_T)$ is not empty (this hypothesis is not necessary, it is used in order to simplify the exposition), pick $\hat{\lambda}_0$ there and set

$$\widehat{G} := \widehat{G}_{\hat{\lambda}_0}.$$

By Remark 2.4, one can take $\hat{\lambda}_0 = \lambda_0$ whenever $0 \in \varrho(T)$.

Define, as in Lemma 2.7, $\widehat{S}^\times : \text{dom}(\widehat{S}^\times) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$ by

$$\begin{aligned}\text{dom}(\widehat{S}^\times) &:= \{\psi \in \mathfrak{F} : \exists \phi \in \mathfrak{F} \text{ such that } \widehat{\psi}_0 := \psi - \widehat{G}\phi \in \text{dom}(H_T)\}, \\ (-\widehat{S}^\times + \widehat{\lambda}_0)\psi &:= (-H_T + \widehat{\lambda}_0)\widehat{\psi}_0.\end{aligned}$$

Then

Lemma 3.3 *One has $\text{dom}(\widehat{S}^\times) \subseteq \text{dom}(S^\times)$ and*

$$\widehat{S}^\times|_{\text{dom}(\widehat{S}^\times) \cap \ker(\widehat{\Sigma})} \subseteq S_T^\times.$$

Proof At first notice that, for any $\psi \in \text{dom}(\widehat{S}^\times)$ decomposed as $\psi = \widehat{\psi}_0 + \widehat{G}\phi$, where $\widehat{\psi}_0 \in \text{dom}(H_T)$ and $\phi \in \mathfrak{F}$, one has, since $\text{dom}(H_T) = \ker(A_T)$ and $A_T \widehat{G} = 1$ (see the proof of Lemma 3.2),

$$A_T \psi = A_T \widehat{\psi}_0 + A_T \widehat{G}\phi = \phi. \quad (3.8)$$

Since, by (3.6),

$$\psi = \widehat{\psi}_0 + \widehat{G}\phi = \widehat{\psi}_0 + (-H + \widehat{\lambda}_o)^{-1}\phi + G_{\widehat{\lambda}_o}(A_T G_{\widehat{\lambda}_o})^{-1}(1 - G_{\widehat{\lambda}_o}^*)\phi$$

and since $\text{ran}((A_T G_{\widehat{\lambda}_o})^{-1}) = \text{dom}(T)$, one gets

$$\text{dom}(\widehat{S}^\times) \subseteq \{\psi \in \text{dom}(S^\times) : A_*\psi \in \text{dom}(T)\} \subseteq \text{dom}(S^\times).$$

By $H_T \subseteq S^\times$, by $(-S^\times + \widehat{\lambda}_o)(-H + \widehat{\lambda}_o)^{-1} = 1$, by $\text{ran}(G_{\widehat{\lambda}_o}) = \ker(-S^\times + \widehat{\lambda}_o)$, by (3.6) and by (3.8), then one gets

$$\begin{aligned} \widehat{S}^\times \psi &= -(-H_T + \widehat{\lambda}_o)\widehat{\psi}_0 + \widehat{\lambda}_o \psi = -(-S^\times + \widehat{\lambda}_o)\widehat{\psi}_0 + \widehat{\lambda}_o \psi \\ &= -(-S^\times + \widehat{\lambda}_o)(\psi - \widehat{G}\phi) + \widehat{\lambda}_o \psi = S^\times \psi + (-S^\times + \widehat{\lambda}_o)\widehat{G}\phi \\ &= S^\times \psi + \phi = (S^\times + A_T)\psi. \end{aligned}$$

Hence, since

$$\text{dom}(\widehat{S}^\times) \cap \ker(\widehat{\Sigma}) \subseteq \{\psi \in \text{dom}(T) : \psi - G\psi \in \mathfrak{H}_1\} = \text{dom}(S_T^\times),$$

the proof is done. $\square$

Putting together the previous results, one gets the following

Theorem 3.4 *Let $T : \text{dom}(T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$ be self-adjoint and $A : \mathfrak{H}_1 \rightarrow \mathfrak{F}$ be bounded such that hypotheses (2.8) and (2.10) hold true. If there exists $z_o \in \varrho(H_T)$ such that $\widehat{\Sigma}\widehat{G}_{z_o}$ has a bounded inverse, then $\widehat{H}_T = S_T^\times$ is self-adjoint, $\text{dom}(H) \cap \text{dom}(\widehat{H}_T) = \{0\}$ and*

$$\text{dom}(\widehat{H}_T) = \{\psi \in \text{dom}(T) : \psi - G\psi \in \mathfrak{H}_1\}, \quad (3.9)$$

$$\widehat{H}_T = \overline{H} + A^* + A_T. \quad (3.10)$$

Moreover $\widehat{\Sigma}\widehat{G}_z$ has a bounded inverse for any $z \in \varrho(H_T) \cap \varrho(\widehat{H}_T)$ and

$$(-\widehat{H}_T + z)^{-1} = (-H_T + z)^{-1} - \widehat{G}_z(\widehat{\Sigma}\widehat{G}_z)^{-1}\widehat{G}_z^*;$$

if $z \in \varrho(H) \cap \varrho(H_T) \cap \varrho(\widehat{H}_T)$ then

$$(-\widehat{H}_T + z)^{-1} = (-H + z)^{-1} - \begin{bmatrix} G_z & R_z \end{bmatrix} \begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix}^{-1} \begin{bmatrix} G_z^* \\ R_z \end{bmatrix}. \quad (3.11)$$

Proof By Lemma 3.3, since $S_T^\times$ is symmetric, if $\widehat{H}_T := \widehat{S}^\times|_{\text{dom}(\widehat{S}^\times) \cap \ker(\widehat{\Sigma})}$ is self-adjoint then $\widehat{H}_T = S_T^\times$. This holds by Lemma 3.2, by Theorem 2.2 and Theorem 2.13 applied to the case

$$H = H_T, \quad \Sigma = \widehat{\Sigma}|_{\text{dom}(H_T)}, \quad \Theta = -\widehat{\Sigma}\widehat{G}.$$

Notice that, by these choices,

$$\Sigma_\Theta \psi = \widehat{\Sigma}\widehat{\psi}_0 + \widehat{\Sigma}\widehat{G}\phi = \widehat{\Sigma}\psi$$

and so (3.9) holds. By (2.10), $G\psi \in \mathfrak{H}_1$ if and only if $\psi = 0$; hence

$$\{\psi \in \text{dom}(T) : \psi - G\psi \in \mathfrak{H}_1\} \cap \mathfrak{H}_1 = \{0\}.$$

Then to conclude we only need to prove (3.11). By (2.20), (3.1), (3.6) and (3.7), one gets

$$\begin{aligned} (-\widehat{H}_T + z)^{-1} &= (-H_T + z)^{-1} - \widehat{G}_z(\widehat{\Sigma}\widehat{G}_z)^{-1}\widehat{G}_z^* \\ &= (-H + z)^{-1} - G_z(A_T G_z)^{-1}G_z^* \\ &\quad - \left((-H + z)^{-1} + G_z(A_T G_z)^{-1}(1 - G_z^*)\right)(\widehat{\Sigma}\widehat{G}_z)^{-1} \\ &\quad \left((-H + z)^{-1} + (1 - G_z)(A_T G_z)^{-1}G_z^*\right) \\ &= (-H + z)^{-1} - \begin{bmatrix} G_z & R_z \end{bmatrix} \mathbb{M} \begin{bmatrix} G_z^* \\ R_z \end{bmatrix}, \end{aligned}$$

where $\mathbb{M}$ is the block operator matrix $\mathbb{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ with entries

$$\begin{aligned} M_{11} &= (A_T G_z)^{-1} + (A_T G_z)^{-1}(1 - G_z^*)(\widehat{\Sigma}\widehat{G}_z)^{-1}(1 - G_z)(A_T G_z)^{-1} \\ &= (A_T G_z)^{-1} + (A_T G_z)^{-1}(1 - G_z^*) \\ &\quad \left((-H + z)^{-1} - (1 - G_z)(A_T G_z)^{-1}(1 - G_z^*)\right)^{-1} \\ &\quad (1 - G_z)(A_T G_z)^{-1}, \end{aligned}$$

$$\begin{aligned} M_{12} &= (A_T G_z)^{-1}(1 - G_z^*)(\widehat{\Sigma}\widehat{G}_z)^{-1} \\ &= (A_T G_z)^{-1}(1 - G_z^*) \left((-H + z)^{-1} - (1 - G_z)(A_T G_z)^{-1}(1 - G_z^*)\right)^{-1}, \end{aligned}$$

$$\begin{aligned} M_{21} &= (\widehat{\Sigma}\widehat{G}_z)^{-1}(1 - G_z)(A_T G_z)^{-1} \\ &= \left((-H + z)^{-1} - (1 - G_z)(A_T G_z)^{-1}(1 - G_z^*)\right)^{-1} (1 - G_z)(A_T G_z)^{-1}, \end{aligned}$$

$$M_{22} = (\widehat{\Sigma}\widehat{G}_z)^{-1} = \left((-H + z)^{-1} - (1 - G_z)(A_T G_z)^{-1}(1 - G_z^*)\right)^{-1}.$$

Finally, one checks that

$$\mathbb{M} \begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix} = \begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix} \mathbb{M} = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

i.e.,

$$\mathbb{M} = \begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix}^{-1}$$

and the proof is done. $\square$

In the next remark and below, we use the notations introduced in the previous section with letters in blackboard bold style to denote block matrix operators.

Remark 3.5 Let the hypotheses in Theorem 3.4 hold. Noticing that

$$\begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix} = -(\Theta_T + \Sigma(\mathbb{G} - \mathbb{G}_z)) \equiv \Sigma_{\Theta_T} \mathbb{G}_z,$$

where

$$\begin{aligned} \Sigma : \mathfrak{H}_1 &\rightarrow \mathfrak{F} \oplus \mathfrak{F}, & \Sigma \psi &:= A\psi \oplus \psi, \\ \mathbb{G}_z : \mathfrak{F} \oplus \mathfrak{F} &\rightarrow \mathfrak{F}, & \mathbb{G}_z &:= (\Sigma R_z^*)^*, \quad \mathbb{G} := \mathbb{G}_{\lambda_0}, \end{aligned}$$

and

$$\Theta_T : \text{dom}(T) \oplus \mathfrak{F} \subseteq \mathfrak{F} \oplus \mathfrak{F} \rightarrow \mathfrak{F} \oplus \mathfrak{F}, \quad \Theta_T := \begin{bmatrix} -T & 1 - G^* \\ 1 - G & -R \end{bmatrix},$$

one gets

$$\widehat{H}_T = H_{\Theta_T}$$

and (3.11) is rewritten as (compare with (2.9))

$$(-\widehat{H}_T + z)^{-1} \equiv (-H_{\Theta_T} + z)^{-1} = (-H + z)^{-1} - \mathbb{G}_z(\Sigma_{\Theta_T} \mathbb{G}_z)^{-1} \mathbb{G}_z^*. \quad (3.12)$$

Since $\mathbb{G}(\psi_1 \oplus \psi_2) = G\psi_1 + R\psi_2$, one has $\text{ran}(\mathbb{G}) \cap \mathfrak{H}_1 = \mathfrak{H}_1$; this shows that (2.9) can still represent the resolvent of a self-adjoint operator even if hypotheses (2.10) in Theorem 2.2 does not hold.

In order to apply Theorem 3.4 one needs to show that there exists at least one $z_0 \in \varrho(H)$ such that $\widehat{\Sigma} \widehat{G}_{z_0}$ has a bounded inverse. A simple criterion is provided in the next Lemma.

Lemma 3.6 *Let $A \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{F})$ for some $s \in (0, 1)$. Then*

$$(1 - G_{\pm i\gamma})^{-1} \in \mathcal{B}(\mathfrak{F}), \quad (1 - G_{\pm i\gamma}^*)^{-1} \in \mathcal{B}(\mathfrak{F})$$

whenever $\gamma \in \mathbb{R}$ and $|\gamma|$ is sufficiently large.

Further suppose that $T : \text{dom}(T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$ is self-adjoint such that $Z_{A, -T} \neq \emptyset$ and $\text{dom}(T) \supseteq \mathfrak{H}_t$, $T|_{\mathfrak{H}_t} \in \mathcal{B}(\mathfrak{H}_t, \mathfrak{F})$ for some $t \in [0, 1 - s)$. Then

$$(\widehat{\Sigma} \widehat{G}_{\pm i\gamma})^{-1} \in \mathcal{B}(\mathfrak{F})$$

whenever $\gamma \in \mathbb{R}$ and $|\gamma|$ is sufficiently large.

Proof Let us take $|\gamma| \geq 1$. By

$$\|(-H \pm i\gamma)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} \leq \frac{1}{|\gamma|}, \quad \|(-H \pm i\gamma)^{-1}\|_{\mathfrak{F}, \mathfrak{H}_1} \leq 1,$$

since interpolating theorems hold for Hilbert scales of the kind $\mathfrak{H}_s$, $s \in \mathbb{R}$, (see [10, Section 9]), one gets

$$\|(-H \pm i\gamma)^{-1}\|_{\mathfrak{F}, \mathfrak{H}_r} \leq \frac{1}{|\gamma|^{1-r}}, \quad 0 \leq r \leq 1.$$

Thus,

$$\|(-H \pm i\gamma)^{-1}\|_{\mathfrak{H}_u, \mathfrak{H}_t} = \|(-H \pm i\gamma)^{-1}\|_{\mathfrak{F}, \mathfrak{H}_{t-u}} \leq \frac{1}{|\gamma|^{1-(t-u)}}, \quad 0 \leq t - u \leq 1. \quad (3.13)$$

Hence

$$\|G_{\mp i\gamma}^*\|_{\mathfrak{H}_t, \mathfrak{F}} \leq \|A\|_{\mathfrak{H}_s, \mathfrak{F}} \|(-H \pm i\gamma)^{-1}\|_{\mathfrak{H}_t, \mathfrak{H}_s} \leq \frac{\|A\|_{\mathfrak{H}_s, \mathfrak{F}}}{|\gamma|^{1-(s-t)}}$$

and

$$\begin{aligned} \|G_{\pm i\gamma}\|_{\mathfrak{H}_t, \mathfrak{H}_t} &\leq \|(-H \pm i\gamma)^{-1}\|_{\mathfrak{H}_{-s}, \mathfrak{H}_t} \|A^*\|_{\mathfrak{H}_t, \mathfrak{H}_{-s}} \\ &= \|(-H \pm i\gamma)^{-1}\|_{\mathfrak{H}_{-s}, \mathfrak{H}_t} \|A\|_{\mathfrak{H}_s, \mathfrak{H}_{-t}} \leq \frac{\|A\|_{\mathfrak{H}_s, \mathfrak{F}}}{|\gamma|^{1-(t+s)}}. \end{aligned}$$

This shows that both $1 - G_{\pm i\gamma} : \mathfrak{H}_t \rightarrow \mathfrak{H}_t$ and $1 - G_{\mp i\gamma}^* : \mathfrak{F} \rightarrow \mathfrak{F}$ have bounded inverses whenever $|\gamma|$ is sufficiently large.

Since $Z_{A, -T} \neq \emptyset$, by [4, Theorem 2.19 and Remark 2.20], $A_T G_z$ has a bounded inverse for any $z \in \varrho(H) \cap \varrho(H_T) \subseteq \mathbb{C} \setminus \mathbb{R}$ and so

$$\begin{aligned} \widehat{\Sigma} \widehat{G}_{\pm i\gamma} &= (1 - G_{\pm i\gamma})(A_T G_{\pm i\gamma})^{-1} (A_T G_{\pm i\gamma} (1 - G_{\pm i\gamma})^{-1} \\ &\quad (-H \pm i\gamma)^{-1} (1 - G_{\mp i\gamma}^*)^{-1} - 1) (1 - G_{\mp i\gamma}^*)^{-1}. \end{aligned}$$

Since

$$\begin{aligned} \|(1 - G_{\pm i\gamma}^*)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} &\leq \sum_{n=0}^{\infty} \|G_{\pm i\gamma}^*\|_{\mathfrak{F}, \mathfrak{F}}^n = \frac{1}{1 - |\gamma|^{s-1} \|A\|_{\mathfrak{H}_s, \mathfrak{F}}} \leq c_0, \\ \|(1 - G_{\mp i\gamma})^{-1}\|_{\mathfrak{H}_t, \mathfrak{H}_t} &\leq \sum_{n=0}^{\infty} \|G_{\mp i\gamma}\|_{\mathfrak{H}_t, \mathfrak{H}_t}^n = \frac{1}{1 - |\gamma|^{t+s-1} \|A\|_{\mathfrak{H}_s, \mathfrak{F}}} \leq c_t \end{aligned}$$

and

$$\begin{aligned} \|A_T G_{\pm i\gamma}\|_{\mathfrak{H}_t, \mathfrak{F}} &\leq \|T\|_{\mathfrak{H}_t, \mathfrak{F}} + \|M_{\pm i\gamma}\|_{\mathfrak{H}_t, \mathfrak{F}} \\ &\leq \|T\|_{\mathfrak{H}_t, \mathfrak{F}} + |\pm i\gamma - \lambda_0| \|G^*\|_{\mathfrak{H}_t, \mathfrak{F}} \|G_{\pm i\gamma}\|_{\mathfrak{H}_t, \mathfrak{H}_t} \\ &\leq \|T\|_{\mathfrak{H}_t, \mathfrak{F}} + \frac{|\lambda_0| + |\gamma|}{|\gamma|^{2(1-s)}} \|A\|_{\mathfrak{H}_s, \mathfrak{F}}^2 \leq \kappa_{t,s} \left(1 + \frac{|\lambda_0| + |\gamma|}{|\gamma|^{2(1-s)}}\right) \end{aligned}$$

one has

$$\begin{aligned} &\|A_T G_{\pm i\gamma} (1 - G_{\pm i\gamma})^{-1} (-H \pm i\gamma)^{-1} (1 - G_{\mp i\gamma}^*)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} \\ &\leq \|A_T G_{\pm i\gamma}\|_{\mathfrak{H}_t, \mathfrak{F}} \|(1 - G_{\pm i\gamma})^{-1}\|_{\mathfrak{H}_t, \mathfrak{H}_t} \|(-H \pm i\gamma)^{-1}\|_{\mathfrak{F}, \mathfrak{H}_t} \|(1 - G_{\mp i\gamma}^*)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} \\ &\leq \kappa_{t,s} c_0 c_t \left(1 + \frac{|\lambda_0| + |\gamma|}{|\gamma|^{2(1-s)}}\right) \frac{1}{|\gamma|^{1-t}} < 1 \end{aligned}$$

whenever $|\gamma|$ is sufficiently large. Hence, whenever $|\gamma|$ is sufficiently large, $\widehat{\Sigma} \widehat{G}_{\pm i\gamma}$ has a bounded inverse given by

$$\begin{aligned} (\widehat{\Sigma} \widehat{G}_{\pm i\gamma})^{-1} &= (1 - G_{\mp i\gamma}^*)^{-1} (A_T G_{\pm i\gamma} (1 - G_{\pm i\gamma})^{-1} (-H \pm i\gamma)^{-1} \\ &\quad (1 - G_{\mp i\gamma}^*)^{-1} - 1)^{-1} A_T G_{\pm i\gamma} (1 - G_{\pm i\gamma})^{-1}. \end{aligned} \quad \square$$

Corollary 3.7 *Let A and T satisfy the hypotheses in Lemma 3.6 and further suppose that both $\ker(A|_{\mathfrak{H}_1})$ and $\text{ran}(A|_{\mathfrak{H}_1})$ are dense in $\mathfrak{F}$. Then $\widehat{H}_T = \overline{H} + A^* + A_T$ is self-adjoint with domain $\text{dom}(\widehat{H}_T) = \{\psi \in \mathfrak{H}_{1-s} : \psi - G\psi \in \mathfrak{H}_1\}$ and resolvent given by formula (3.11).*

Proof Theorem 3.4 and Lemmata 3.6 and 2.5 give the thesis with $\text{dom}(\widehat{H}_T) = \{\psi \in \mathfrak{H}_1 : \psi - G\psi \in \mathfrak{H}_1\}$. Then, by using the $\mathfrak{H}_{-s}$ - $\mathfrak{H}_s$ duality, $G = RA^*$, where $A^* \in \mathcal{B}(\mathfrak{F}, \mathfrak{H}_{-s})$ and $R \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{H}_{1-s})$; therefore $G \in \mathcal{B}(\mathfrak{F}, \mathfrak{H}_{1-s})$ and the proof is concluded noticing that $\psi \in \text{dom}(\widehat{H}_T)$ belongs to $\mathfrak{H}_{1-s}$ if and only if $G\psi \in \mathfrak{H}_{1-s}$. $\square$

Remark 3.8 As the proof of previous Lemma 3.6 shows, if H is semibounded then the same conclusions there hold with $\pm i\gamma$ replaced by $\lambda \in \mathbb{R}$ sufficiently far away from $\sigma(H)$.

Since the operator T enters as an additive perturbation in the definition of $\widehat{H}_T$, one can eventually avoid the self-adjointness hypothesis on it and work with $\widehat{H}_0$ alone:

Theorem 3.9 *Let $A \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{F})$ for some $0 < s < 1$ and such that both $\ker(A|\mathfrak{H}_1)$ and $\text{ran}(A|\mathfrak{H}_1)$ are dense in $\mathfrak{F}$. Then $\widehat{H}_0 := \overline{H} + A^* + A_0$ is self-adjoint with domain*

$$\text{dom}(\widehat{H}_0) = \{\psi \in \mathfrak{H}_{1-s} : \psi - G\psi \in \mathfrak{H}_1\}$$

and resolvent given, for any $z \in \mathbb{C}$ such that $\mu + z \in \varrho(H) \cap \varrho(\widehat{H}_0)$, $\mu \in \mathbb{R} \setminus \{0\}$, by

$$(-\widehat{H}_0 + z)^{-1} = (-H + \mu + z)^{-1} - [G_{\mu+z} \ R_{\mu+z}] \begin{bmatrix} A_{\mu} G_{\mu+z} & G_{\mu+z}^* - 1 \\ G_{\mu+z} - 1 & R_{\mu+z} \end{bmatrix}^{-1} \begin{bmatrix} G_{\mu+z}^* \\ R_{\mu+z} \end{bmatrix}. \quad (3.14)$$

If $T : \text{dom}(T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$, $\text{dom}(T) \supseteq \text{dom}(\widehat{H}_0)$, is symmetric and $\widehat{H}_0$ -bounded with relative bound $\widehat{a} < 1$ then $\widehat{H}_T := \overline{H} + A^ + A_T$ is self-adjoint, has domain $\text{dom}(\widehat{H}_T) = \text{dom}(\widehat{H}_0)$ and resolvent*

$$(-\widehat{H}_T + z)^{-1} = (-\widehat{H}_0 + z)^{-1} + (-\widehat{H}_0 + z)^{-1} (1 - T(-\widehat{H}_0 + z)^{-1})^{-1} T(-\widehat{H}_0 + z)^{-1}. \quad (3.15)$$

Proof By Remark 2.4 and Lemma 2.5, hypotheses (2.8) and (2.10) are satisfied with $\Theta = -T = -\mu \neq 0$. Hence, by Lemma 3.6 and Theorem 3.4, $\widehat{H}_\mu$ (i.e., $\widehat{H}_T$ with $T = \mu$) is selfadjoint with domain $\text{dom}(\widehat{H}_\mu) = \{\psi \in \mathfrak{F} : \psi - G\psi \in \mathfrak{H}_1\}$ and resolvent $(-\widehat{H}_\mu + z)^{-1} = (-H_\mu + z)^{-1} - \widehat{G}_z(\widehat{\Sigma}\widehat{G}_z)^{-1}\widehat{G}_z^*$. Therefore $\widehat{H}_0 = \widehat{H}_\mu - \mu$ is self-adjoint with domain $\text{dom}(\widehat{H}_0) = \text{dom}(\widehat{H}_\mu)$ and resolvent $(-\widehat{H}_0 + z)^{-1} = (-\widehat{H}_\mu + \mu + z)^{-1}$. Since $A \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{F})$, one gets $\text{dom}(\widehat{H}_0) \subseteq \mathfrak{H}_{1-s}$ by the same arguments as in the proof of Corollary 3.7. Formula (3.15) is consequence of $\widehat{H}_T = \widehat{H}_0 + T$ and Remark 2.16. $\square$

The next result shows how to obtain $\widehat{H}_T$ as limits of regular perturbations of H .

Theorem 3.10 *Suppose that the operator*

$$\widehat{H}_0 := \overline{H} + A^* + A_0, \quad \text{dom}(\widehat{H}_0) = \{\psi \in \mathfrak{F} : \psi - G\psi \in \mathfrak{H}_1\}$$

is self-adjoint with resolvent given by (3.14) for some $\mu \in \mathbb{R}$. Let $A_n : \text{dom}(A_n) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$ be a sequence of closable operators such that, for some $s \in [0, \frac{1}{2}]$,

$$\text{dom}(A_n) \supseteq \mathfrak{H}_s, \quad A_n|_{\mathfrak{H}_s} \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{F}),$$

and

$A_n^* + A_n$ is H -bounded with relative bound $a < 1$;

further suppose, whenever $s = \frac{1}{2}$, that H is semi-bounded and $\mu = 0$.

Let

$$\begin{aligned} \widehat{H}_n : \mathfrak{H}_1 &\subseteq \mathfrak{F} \rightarrow \mathfrak{F}, & H_n &:= H + A_n^* + A_n, \\ \widetilde{H}_n : \mathfrak{H}_1 &\subseteq \mathfrak{F} \rightarrow \mathfrak{F}, & \widetilde{H}_n &:= H_n - A_n R A_n^*. \end{aligned}$$

If

$$\lim_{n \uparrow \infty} \|A_n - A\|_{\mathfrak{H}_1, \mathfrak{F}} = 0, \quad (3.16)$$

then

$$\lim_{n \uparrow \infty} \widetilde{H}_n = \widehat{H}_0 \quad \text{in norm-resolvent sense.} \quad (3.17)$$

Let $T : \text{dom}(T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$, $\text{dom}(T) \supseteq \text{dom}(\widehat{H}_0)$, be symmetric and $\widehat{H}_0$ -bounded with relative bound $\widehat{a} < 1$; let $\widehat{H}_T$ be the self-adjoint operator $\widehat{H}_T := \widehat{H} + A^* + A_T$, $\text{dom}(\widehat{H}_T) = \text{dom}(\widehat{H}_0)$. If, alongside with (3.16), there exist a sequence $\{E_n\}_1^\infty$ of bounded symmetric operators in $\mathfrak{F}$ such that

$$A_n R A_n^* + E_n \text{ is } \widetilde{H}_n\text{-bounded with } n\text{-independent relative bound } \widetilde{a} < 1 \quad (3.18)$$

and

$$\lim_{n \uparrow \infty} \|A_n R A_n^* + E_n - T\|_{\text{dom}(T), \mathfrak{F}} = 0, \quad (3.19)$$

then

$$\lim_{n \uparrow \infty} (H_n + E_n) = \widehat{H}_T \quad \text{in norm-resolvent sense.}$$

Proof By Remark 3.5, one has $\widehat{H}_\mu = H_\Theta$, where

$$\Theta := \begin{bmatrix} -\mu & 1 - G^* \\ 1 - G & -R \end{bmatrix}.$$

Let

$$\Sigma_n : \mathfrak{F} \rightarrow \mathfrak{F} \oplus \mathfrak{F}, \quad \Sigma_n \psi := A_n \psi \oplus \psi,$$

and

$$\Theta_n := \begin{bmatrix} A_n R A_n^* - \mu & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that, by $A_n \in \mathcal{B}(\mathfrak{H}_{1/2}, \mathfrak{F})$ and $R \in \mathcal{B}(\mathfrak{H}_{-1/2}, \mathfrak{H}_{1/2})$, $A_n R A_n^* \in \mathcal{B}(\mathfrak{F})$; therefore Θ_n is bounded with bounded inverse given by

$$\Lambda_n := \Theta_n^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & \mu - A_n R A_n^* \end{bmatrix}$$

and, by the Rellich-Kato theorem, $H + \Sigma_n^* \Lambda_n \Sigma_n = \widetilde{H}_n + \mu$ is self-adjoint with domain $\text{dom}(\widetilde{H}_n) = \mathfrak{H}_1$ ($A_n^* + A_n + A_n R A_n^*$ is symmetric since A_n is closable).

If $0 \leq s < \frac{1}{2}$, then, by (3.13) and

$$\Sigma_n R_z \Sigma_n^* = \begin{bmatrix} A_n R_z A_n^* & A_n R_z \\ R_z A_n^* & R_z \end{bmatrix},$$

one gets $\|\Lambda_n \Sigma_n R_{\pm i\gamma} \Sigma_n^*\|_{\mathfrak{F} \oplus \mathfrak{F}, \mathfrak{F} \oplus \mathfrak{F}} \rightarrow 0$ as $|\gamma| \uparrow \infty$; so, by Theorem 2.15, $\widetilde{H}_n + \mu$ has resolvent given by formula (2.22).

Suppose now $s = \frac{1}{2}$, H semi-bounded and $\mu = 0$. Since $(A_n R)^*$ and $A_n R$ norm converge to G and G^* respectively and, by Remark 3.8, $1 - G$ and $1 - G^*$ have bounded inverses whenever λ_o is chosen sufficiently far away from $\sigma(H)$ (see next Remark 3.14 as regard the freedom to choose the value of λ_o), $1 - RA_n^*$ and $1 - A_n R$ have bounded inverses as well whenever n is sufficiently large. Hence, by the relation

$$-\tilde{H}_n + \lambda_o = (1 - A_n R)(-H + \lambda_o)(1 - RA_n^*),$$

one gets

$$\begin{aligned} (-H + \lambda_o)^{-1} &= (1 - RA_n^*)(-\tilde{H}_n + \lambda_o)^{-1}(1 - A_n R) \\ &= (-\tilde{H}_n + \lambda_o)^{-1} - [RA_n^* R] \begin{bmatrix} (-\tilde{H}_n + \lambda_o)^{-1} & (1 - RA_n^*)^{-1} \\ (1 - A_n R)^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_n R \\ R \end{bmatrix} \\ &= (-\tilde{H}_n + \lambda_o)^{-1} - [RA_n^* R] \begin{bmatrix} 0 & 1 - A_n R \\ 1 - RA_n^* & -R \end{bmatrix}^{-1} \begin{bmatrix} A_n R \\ R \end{bmatrix} \\ &= (-\tilde{H}_n + \lambda_o)^{-1} - [RA_n^* R] \left(\begin{bmatrix} A_n RA_n^* & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} A_n RA_n^* & A_n R \\ RA_n^* & R \end{bmatrix} \right)^{-1} \begin{bmatrix} A_n R \\ R \end{bmatrix} \\ &= (-\tilde{H}_n + \lambda_o)^{-1} - R \Sigma_n (\Theta_n - \Sigma_n R \Sigma_n^*)^{-1} \Sigma_n R. \end{aligned}$$

This, together with [4, Theorem 2.19 and Remark 2.20], gives the resolvent formula (2.22) for $\tilde{H}_n$.

Once we get formula (2.22) for $(-\tilde{H}_n + z)^{-1}$ and for any $s \in [0, \frac{1}{2}]$, since

$$\lim_{n \uparrow \infty} \|\Sigma - \Sigma_n\|_{\mathfrak{H}_1, \mathfrak{F} \oplus \mathfrak{F}} = \lim_{n \uparrow \infty} \|A - A_n\|_{\mathfrak{H}_1, \mathfrak{F}} = 0,$$

$$\lim_{n \uparrow \infty} \|(\Theta_n - \Sigma_n R \Sigma_n^*) - \Theta\|_{\mathfrak{F} \oplus \mathfrak{F}, \mathfrak{F} \oplus \mathfrak{F}} = \lim_{n \uparrow \infty} \left\| \begin{bmatrix} 0 & G^* - A_n R \\ G - RA_n^* & 0 \end{bmatrix} \right\|_{\mathfrak{F} \oplus \mathfrak{F}, \mathfrak{F} \oplus \mathfrak{F}} = 0,$$

and $\text{dom}(\Theta_n) = \text{dom}(\Theta) = \mathfrak{F} \oplus \mathfrak{F}$, by Theorem 2.18, one gets

$$\lim_{n \uparrow \infty} (\tilde{H}_n + \mu) = \lim_{n \uparrow \infty} (H + \Sigma_n^* \Lambda_n \Sigma_n) = H_\Theta = \hat{H}_\mu \quad \text{in norm-resolvent sense.}$$

Equivalently,

$$\lim_{n \uparrow \infty} \tilde{H}_n = \hat{H}_0 \quad \text{in norm-resolvent sense.} \quad (3.20)$$

Now, let us consider the relations, which hold for $\gamma \in \mathbb{R}$, $|\gamma|$ sufficiently large,

$$\begin{aligned} -(H_n + E_n) \pm i\gamma)^{-1} &= (-\tilde{H}_n + T_n) \pm i\gamma)^{-1} \\ &= (1 - (-\tilde{H}_n \pm i\gamma)^{-1} T_n)^{-1} (-\tilde{H}_n \pm i\gamma)^{-1}, \end{aligned}$$

where $T_n := A_n RA_n^* + E_n$, and

$$(-(\hat{H}_0 + T) \pm i\gamma)^{-1} = (-\hat{H}_0 \pm i\gamma)^{-1} (1 - T(-\hat{H}_0 \pm i\gamma)^{-1})^{-1}.$$

We also use the relation

$$\begin{aligned} &(-\tilde{H}_n \pm i\gamma)^{-1} - (\hat{H}_0 \pm i\gamma)^{-1} \\ &= [(-\tilde{H}_n \pm i\gamma)^{-1} \tilde{H}_n](-\hat{H}_0 \pm i\gamma)^{-1} - (-\tilde{H}_n \pm i\gamma)^{-1} \hat{H}_0 (-\hat{H}_0 \pm i\gamma)^{-1} \end{aligned}$$

(here and below we use the brackets [...] to group maps which provide bounded operators defined on the whole $\mathfrak{F}$). Therefore one gets

$$\begin{aligned}
 & (- (H_n + E_n) \pm i\gamma)^{-1} - (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \\
 &= [(- (\tilde{H}_n + T_n) \pm i\gamma)^{-1} (\tilde{H}_n + T_n)] (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \\
 &\quad - (- (\tilde{H}_n + T_n) \pm i\gamma)^{-1} (\hat{H}_0 + T) (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \\
 &= (1 - (- \tilde{H}_n \pm i\gamma)^{-1} T_n)^{-1} [(- \tilde{H}_n \pm i\gamma)^{-1} (\tilde{H}_n + T_n)] (- \hat{H}_0 \pm i\gamma)^{-1} (1 - T (- \hat{H}_0 \pm i\gamma)^{-1})^{-1} \\
 &\quad - (1 - (- \tilde{H}_n \pm i\gamma)^{-1} T_n)^{-1} (- \tilde{H}_n \pm i\gamma)^{-1} (\hat{H}_0 + T) (- \hat{H}_0 \pm i\gamma)^{-1} (1 - T (- \hat{H}_0 \pm i\gamma)^{-1})^{-1} \\
 &= (1 - (- \tilde{H}_n \pm i\gamma)^{-1} T_n)^{-1} ((-\tilde{H}_n \pm i\gamma)^{-1} - (-\hat{H}_0 \pm i\gamma)^{-1}) (1 - T (-\hat{H}_0 \pm i\gamma)^{-1})^{-1} \\
 &\quad + (- (\tilde{H}_n + T_n) \pm i\gamma)^{-1} (T_n - T) (- (\hat{H}_0 + T) \pm i\gamma)^{-1}
 \end{aligned}$$

and so,

$$\begin{aligned}
 & \| (- (H_n + E_n) \pm i\gamma)^{-1} - (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \\
 &\leq \| (1 - T (- \hat{H}_0 \pm i\gamma)^{-1})^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \| (1 - (- \tilde{H}_n \pm i\gamma)^{-1} T_n)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \\
 &\quad \| (- \tilde{H}_n \pm i\gamma)^{-1} - (- \hat{H}_0 \pm i\gamma)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \\
 &\quad + \frac{1}{|\gamma|} \| (T_n - T) (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \|_{\mathfrak{F}, \mathfrak{F}}.
 \end{aligned}$$

By (3.18),

$$\sup_{n \geq 1} \| (1 - (- \tilde{H}_n \pm i\gamma)^{-1} T_n)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \leq \frac{1}{1 - \tilde{a}}$$

and, since T is $\hat{H}_0$ -bounded,

$$\| (- \hat{H}_0 \pm i\gamma)^{-1} \|_{\mathfrak{F}, \text{dom}(T)} \leq \| T (- \hat{H}_0 \pm i\gamma)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} + (\| (- \hat{H}_0 \pm i\gamma)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} < +\infty.$$

Then, by (3.19),

$$\begin{aligned}
 & \lim_{n \uparrow \infty} \| (T_n - T) (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \\
 &\leq \| (- (\hat{H}_0 + T) \pm i\gamma)^{-1} \|_{\mathfrak{F}, \text{dom}(T)} \lim_{n \uparrow \infty} \| T_n - T \|_{\text{dom}(T), \mathfrak{F}} \\
 &\leq \| (1 - T (- \hat{H}_0 \pm i\gamma)^{-1})^{-1} \|_{\mathfrak{F}, \mathfrak{F}} \| (- \hat{H}_0 \pm i\gamma)^{-1} \|_{\mathfrak{F}, \text{dom}(T)} \lim_{n \uparrow \infty} \| T_n - T \|_{\text{dom}(T), \mathfrak{F}} = 0.
 \end{aligned}$$

Hence, by (3.20), the sequence $H_n + E_n$ converges in norm-resolvent sense to $\hat{H}_T$ as $n \uparrow \infty$. $\square$

Remark 3.11 Previous Theorem 3.10 suggests that if the sequence $A_n R A_n^*$ were convergent then one could take $E_n = 0$ and $T = AG \equiv ARA^*$. However ARA^* is ill-defined in presence of strongly singular interactions and E_n 's role is to compensate the divergence of $A_n R A_n^*$ as $n \rightarrow +\infty$ so that $A_n R A_n^* + E_n$ converges to some regularized version of ARA^* ; see next subsection for the case of quantum fields models.

Remark 3.12 Suppose that the operator $\hat{H}_0$ is self-adjoint with resolvent given by (3.14) for some $\mu \in \mathbb{R}$ and let $A_n \in \mathcal{B}(\mathfrak{F})$ defined by $A_n := n i A R_{n i}$, where $A \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{F})$ satisfies the hypotheses in Lemma 3.6. Since $R_z A_n^*$ and $A_n R_z$ norm converge to G_z and G_z^* respectively and since $1 - G_{\pm i\gamma}$ and $1 - G_{\mp i\gamma}^*$ have bounded

inverses whenever $|\gamma| \gg 1$ (see Lemma 3.6), $1 - R_{\pm i\gamma} A_n^*$ and $1 - A_n R_{\pm i\gamma}$ have bounded inverses as well whenever n is sufficiently large; moreover $(1 - R_{\pm i\gamma} A_n^*)^{-1}$ and $(1 - A_n R_{\pm i\gamma})^{-1}$ norm converge to $(1 - G_{\pm i\gamma})^{-1}$ and $(1 - G_{\mp i\gamma}^*)^{-1}$ respectively. Hence

$$\lim_{n \uparrow \infty} \|(1 - R_{\pm i\gamma} A_n^*)^{-1} R_{\pm i\gamma} (1 - A_n R_{\pm i\gamma})^{-1} - (1 - G_{\pm i\gamma})^{-1} R_{\pm i\gamma} (1 - G_{\mp i\gamma}^*)^{-1}\|_{\mathfrak{F}, \mathfrak{F}} = 0. \quad (3.21)$$

Since

$$(1 - A_n R_z)(-H + z)(1 - R_z A_n^*) = (-\tilde{H}_n + z) + (\lambda_o - z) A_n R R_z A_n^*,$$

one has

$$\begin{aligned} & (-\tilde{H}_n \pm i\gamma)^{-1} \\ &= ((1 - A_n R_{\pm i\gamma})(-H + z)(1 - R_{\pm i\gamma} A_n^*) + (\pm i\gamma - \lambda_o) A_n R R_{\pm i\gamma} A_n^*)^{-1} \end{aligned}$$

and so, by (3.21) and (3.17), one gets

$$(-\hat{H}_0 \pm i\gamma)^{-1} = \left((1 - G_{\mp i\gamma}^*)(-H \pm i\gamma)(1 - G_{\pm i\gamma}) + (\pm i\gamma - \lambda_o) G^* G_{\pm i\gamma} \right)^{-1}.$$

Hence

$$-\hat{H}_0 \pm i\gamma = (1 - G_{\mp i\gamma}^*)(-H \pm i\gamma)(1 - G_{\pm i\gamma}) + (\pm i\gamma - \lambda_o) G^* G_{\pm i\gamma}$$

which, by (2.3), is equivalent to (compare with [13, equation (15)])

$$-\hat{H}_0 + \lambda_o = (1 - G^*)(-H + \lambda_o)(1 - G). \quad (3.22)$$

Our next aim is to show that the two resolvent formulae (3.15) and (3.11) (equivalently (3.12)) coincide. At first, let us come back to Remark 3.5: the map $\mathbb{Z}\psi := A\psi \oplus \psi$ there obviously belongs to $\mathcal{B}(\mathfrak{H}_1, \mathfrak{F} \oplus \mathfrak{H}_1)$; hence, using the $\mathfrak{H}_{-1}$ - $\mathfrak{H}_1$ duality induced by the dense embeddings $\mathfrak{H}_1 \hookrightarrow \mathfrak{F} \hookrightarrow \mathfrak{H}_{-1}$ (i.e., by the pairing $\langle \cdot, \cdot \rangle_{-1,1}$ defined in (2.17)), one gets the bounded operator

$$\mathbb{C}_z : \mathfrak{F} \oplus \mathfrak{H}_{-1} \rightarrow \mathfrak{F}, \quad \mathbb{C}_z := (\mathbb{Z} R_z^*)^*.$$

This also gives $\mathbb{C}_z^* \in \mathcal{B}(\mathfrak{F}, \mathfrak{F} \oplus \mathfrak{H}_1)$ and so (3.12) is well defined whenever $(\mathbb{Z} \oplus_T \mathbb{C}_z)^{-1} \in \mathcal{B}(\mathfrak{F} \oplus \mathfrak{H}_1, \mathfrak{F} \oplus \mathfrak{H}_{-1})$, where now $\oplus_T$ is regarded as an operator from $\mathfrak{F} \oplus \mathfrak{H}_{-1}$ to $\mathfrak{F} \oplus \mathfrak{H}_1$. Let us remark that in this setting (3.12) still conforms with the framework in [19] (also see [16], [4]); indeed there the dual couple $\mathfrak{X}^*$ - $\mathfrak{X}$ (here given by $(\mathfrak{F} \oplus \mathfrak{H}_{-1})$ -($\mathfrak{F} \oplus \mathfrak{H}_1$)) comes into play.

Because, by (3.22), $(1 - G)\phi \in \mathfrak{H}_1$ whenever $\psi \in \text{dom}(\hat{H}_0)$ and supposing that $\text{dom}(\hat{H}_0) \subseteq \text{dom}(T)$, the block operator matrix

$$\oplus_T = \begin{bmatrix} -T & 1 - G^* \\ 1 - G & -R \end{bmatrix} : \text{dom}(\hat{H}_0) \oplus \mathfrak{F} \subseteq \mathfrak{F} \oplus \mathfrak{H}_{-1} \rightarrow \mathfrak{F} \oplus \mathfrak{H}_1 \quad (3.23)$$

is well defined. Analogously

$$\mathbb{Z}_{\Theta_T} G_z = \begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix} : \text{dom}(\widehat{H}_0) \oplus \mathfrak{F} \subseteq \mathfrak{F} \oplus \mathfrak{H}_{-1} \rightarrow \mathfrak{F} \oplus \mathfrak{H}_1$$

is well defined as well. Since the unbounded operator $-R : \mathfrak{F} \subseteq \mathfrak{H}_{-1} \rightarrow \mathfrak{H}_1$ has the bounded inverse $H - \lambda_o : \mathfrak{H}_1 \rightarrow \mathfrak{F}$, by (3.22) and the first Schur complement, the candidate for the inverse of Θ_T is

$$\begin{aligned} & \begin{bmatrix} \widehat{R}_T & \widehat{R}_T(-\widehat{H}_0 + \lambda_o)(1 - G)^{-1} \\ (1 - G^*)^{-1}(-\widehat{H}_0 + \lambda_o)\widehat{R}_T & (1 - G^*)^{-1}(-\widehat{H}_0 + \lambda_o)\widehat{R}_T(-\widehat{H}_0 + \lambda_o)(1 - G)^{-1} - (-H + \lambda_o) \end{bmatrix} \\ &= \begin{bmatrix} \widehat{R}_0(1 - T\widehat{R}_0)^{-1} & (1 - \widehat{R}_0 T)^{-1}(1 - G)^{-1} \\ (1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1} & ((1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1}(1 - G^*) - 1)(-H + \lambda_o) \end{bmatrix}, \end{aligned} \quad (3.24)$$

where for brevity we set

$$\widehat{R}_T := (-\widehat{H}_T + \lambda_o)^{-1} = \widehat{R}_0(1 - T\widehat{R}_0)^{-1} = (1 - \widehat{R}_0 T)^{-1}\widehat{R}_0.$$

In the following, as regards Θ_T , we use the notion of self-adjointness for a linear operator acting between dual pairs: a densely defined $L : \text{dom}(L) \subseteq \mathfrak{F} \oplus \mathfrak{H}_{-1} \rightarrow \mathfrak{F} \oplus \mathfrak{H}_1$ is said to be self-adjoint whenever $L^* = L$, where $L^* : \text{dom}(L^*) \subseteq \mathfrak{F} \oplus \mathfrak{H}_{-1} \rightarrow \mathfrak{F} \oplus \mathfrak{H}_1$ is the dual of L with respect to the duality induced by the pairing $\langle \cdot, \cdot \rangle_{-1,1}$.

If H is semibounded then Theorem 3.9 conforms with Theorem 3.4:

Theorem 3.13 *Let H be semibounded; let $A \in \mathcal{B}(\mathfrak{H}_s, \mathfrak{F})$ for some $0 < s < 1$ and such that both $\ker(A|_{\mathfrak{H}_1})$ and $\text{ran}(A|_{\mathfrak{H}_1})$ are dense in $\mathfrak{F}$. Then*

$$\widehat{H}_0 := \overline{H} + A^* + A_0$$

is self-adjoint, semibounded with domain

$$\text{dom}(\widehat{H}_0) = \{\psi \in \mathfrak{H}_{1-s} : \psi - G\psi \in \mathfrak{H}_1\}$$

and resolvent given, for any $z \in \varrho(H) \cap \varrho(\widehat{H}_0)$, by

$$(-\widehat{H}_0 + z)^{-1} = (-H + z)^{-1} - \begin{bmatrix} G_z & R_z \end{bmatrix} \begin{bmatrix} A_0 G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix}^{-1} \begin{bmatrix} G_z^* \\ R_z \end{bmatrix}. \quad (3.25)$$

If $T : \text{dom}(T) \subseteq \mathfrak{F} \rightarrow \mathfrak{F}$, $\text{dom}(T) \supseteq \text{dom}(\widehat{H}_0)$, is symmetric and $\widehat{H}_0$ -bounded with relative bound $\widehat{a} < 1$ then

$$\widehat{H}_T := \overline{H} + A^* + A_T$$

is self-adjoint and semibounded, with domain $\text{dom}(\widehat{H}_T) = \text{dom}(\widehat{H}_0)$ and resolvent given, for any $z \in \varrho(H) \cap \varrho(\widehat{H}_T)$, by

$$(-\widehat{H}_T + z)^{-1} = (-H + z)^{-1} - \begin{bmatrix} G_z & R_z \end{bmatrix} \begin{bmatrix} A_T G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix}^{-1} \begin{bmatrix} G_z^* \\ R_z \end{bmatrix}. \quad (3.26)$$

The block operator matrix inverse in (3.26) exists as a bounded operator from $\mathfrak{F} \oplus \mathfrak{H}_1$ to $\mathfrak{F} \oplus \mathfrak{H}_{-1}$; whenever $T = 0$, the same inverse exists also as a bounded operator in $\mathfrak{F} \oplus \mathfrak{F}$.

Proof By [28, Theorem 2.2.18], Θ_T defined in (3.23) is closed and, by [28, Theorem 2.3.3], it has a bounded inverse Θ_T^{-1} given by the block operator matrix in (3.24).

If H is semibounded then, by (3.22), $\widehat{H}_0$ and hence (by Rellich-Kato theorem) $\widehat{H}_T$ are semibounded as well.

By Remark 3.8 and (3.22), taking $\lambda_o \in \mathbb{R}$ sufficiently far away from $\sigma(H)$ in the definition (2.2), one has $\lambda_o \in \varrho(H) \cap \varrho(\widehat{H}_0)$ and

$$(-\widehat{H}_0 + \lambda_o)^{-1} = (1 - G)^{-1}(-H + \lambda_o)^{-1}(1 - G^*)^{-1}, \quad (3.27)$$

i.e.,

$$\begin{aligned} (-H + \lambda_o)^{-1} &= (1 - G)(-\widehat{H}_0 + \lambda_o)^{-1}(1 - G^*) \\ &= (-\widehat{H}_0 + \lambda_o)^{-1} - [G \ R] \begin{bmatrix} (-\widehat{H}_0 + \lambda_o)^{-1} (1 - G)^{-1} & 0 \\ (1 - G^*)^{-1} & 0 \end{bmatrix} \begin{bmatrix} G^* \\ R \end{bmatrix} \\ &= (-\widehat{H}_0 + \lambda_o)^{-1} - [G \ R] \begin{bmatrix} 0 & 1 - G^* \\ 1 - G & -R \end{bmatrix}^{-1} \begin{bmatrix} G^* \\ R \end{bmatrix}. \end{aligned}$$

This gives the resolvent formula

$$(-\widehat{H}_0 + \lambda_o)^{-1} = (-H + \lambda_o)^{-1} + \mathbb{G} \mathbb{G}_0^{-1} \mathbb{G}^*. \quad (3.28)$$

Therefore $\mathbb{G} \mathbb{G}_0^{-1} \mathbb{G}^* = \widehat{R}_0 - R$ is symmetric. By $\text{ran}(A|\mathfrak{H}_1)$ dense, $\text{ran}(G^*) = \text{ran}(AR)$ is dense and so $\text{ran}(\mathbb{G}^*) = \text{ran}(G^*) \oplus \mathfrak{H}_1$ is dense as well. Thus $\mathbb{G}_0^{-1}$ is symmetric (both as an operator in $\mathfrak{F} \oplus \mathfrak{F}$ and as an operator from $\mathfrak{F} \oplus \mathfrak{H}_1$ to $\mathfrak{F} \oplus \mathfrak{H}_{-1}$); hence it is self-adjoint since bounded. So, by [9, Theorem 5.30, Chap. III], $\mathbb{G}_0$ is self-adjoint. Then, since $\mathbb{Z} \mathbb{G}_0 \mathbb{G}_z = -(\mathbb{G}_0 + \mathbb{Z}(\mathbb{G} - \mathbb{G}_z))$ and $(\mathbb{Z}(\mathbb{G} - \mathbb{G}_z))^* = \mathbb{Z}(\mathbb{G} - \mathbb{G}_z^*)$, by [9, Theorem 5.30, Chap. III] again, $((\mathbb{Z} \mathbb{G}_0 \mathbb{G}_z^{-1})^* = (\mathbb{Z} \mathbb{G}_0 \mathbb{G}_z^{-1})^{-1}$ for any complex conjugate couple for which the inverses exist. Therefore, by [4, Theorem 2.19 and Remark 2.20], the existence of the bounded inverse $\mathbb{G}_0^{-1}$ implies that the resolvent formula

$$(-\widehat{H}_0 + z)^{-1} = (-H + z)^{-1} - \mathbb{G}_z (\mathbb{Z} \mathbb{G}_0 \mathbb{G}_z)^{-1} \mathbb{G}_z^*$$

holds for any $z \in \varrho(H) \cap \varrho(\widehat{H}_0)$. The latter is equivalent to (3.25).

By the same kind of reasonings as above, to prove the resolvent formula (3.26) it suffices to show that it holds in the case $z = \lambda_o$, i.e., that

$$(-\widehat{H}_T + \lambda_o)^{-1} = R + \mathbb{G} \mathbb{G}_T^{-1} \mathbb{G}^*. \quad (3.29)$$

By (3.24), (3.22) and (3.15), one gets

$$\begin{aligned} R + \mathbb{G} \mathbb{G}_T^{-1} \mathbb{G}^* &= R - [G \ R] \begin{bmatrix} 0 & 0 \\ 0 & -H + \lambda_o \end{bmatrix} \begin{bmatrix} G^* \\ R \end{bmatrix} \\ &+ [G \ R] \begin{bmatrix} \widehat{R}_0(1 - T\widehat{R}_0)^{-1} & (1 - \widehat{R}_0T)^{-1}(1 - G)^{-1} \\ (1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1} & (1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1}(1 - G^*)(-H + \lambda_o) \end{bmatrix} \begin{bmatrix} G^* \\ R \end{bmatrix} \\ &= [G \ R] \begin{bmatrix} \widehat{R}_0(1 - T\widehat{R}_0)^{-1} & (1 - \widehat{R}_0T)^{-1}\widehat{R}_0 \\ (1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1} & (1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1} \end{bmatrix} \begin{bmatrix} G^* \\ 1 - G^* \end{bmatrix} \\ &= [G \ R] \begin{bmatrix} \widehat{R}_0(1 - T\widehat{R}_0)^{-1} \\ (1 - G^*)^{-1}(1 - T\widehat{R}_0)^{-1} \end{bmatrix} = (G\widehat{R}_0 + R(1 - G^*)^{-1})(1 - T\widehat{R}_0)^{-1} \\ &= \widehat{R}_0(1 - T\widehat{R}_0)^{-1} = (-\widehat{H}_0 + z)^{-1} + (-\widehat{H}_0 + z)^{-1}(1 - T(-\widehat{H}_0 + z)^{-1})T(-\widehat{H}_0 + z)^{-1} \\ &= (-\widehat{H}_T + \lambda_o)^{-1}. \end{aligned}$$

□

Remark 3.14 In the proof of Theorem 3.13 we took $|\lambda_o|$ sufficiently large so that $1 - G$ has a bounded inverse; hence the validity such a theorem seems to depend on

the value of λ_o . This is not the case. Indeed, given λ such that $1 - G_\lambda$ has a bounded inverse, let $A_T^{(\lambda)} G_z := T + A(G_\lambda - G_z)$; then $A_T^{(\lambda)} G_z = A_{T+T_\lambda} G_z$, where $T_\lambda := A(G - G_\lambda) = (\lambda - \lambda_o) G^* G_\lambda = (\lambda - \lambda_o) G_\lambda^* G \in \mathcal{B}(\mathfrak{F})$ and so, by Theorem 3.13, $\widehat{H}_T^{(\lambda)} := \overline{H} + A^* + A_{T+T_\lambda}$ is self-adjoint with domain $\text{dom}(\widehat{H}_{T+T_\lambda}) = \{\psi \in \mathfrak{H}_{1-s} : \psi - G_\lambda \psi \in \mathfrak{H}_1\}$ and resolvent given by formula (3.26) with T replaced by $T + T_\lambda$. Since $\text{ran}(G - G_\lambda) \subseteq \mathfrak{H}_1$, $\text{dom}(\widehat{H}_{T+T_\lambda}) = \{\psi \in \mathfrak{H}_{1-s} : \psi - G\psi \in \mathfrak{H}_1\}$ and, since T_λ is symmetric and bounded, $\widehat{H}_T := \widehat{H}_T^{(\lambda)} - T_\lambda = \overline{H} + A^* + A_0 + T = \overline{H} + A^* + A_T$ is self-adjoint with $\text{dom}(\widehat{H}_T) = \text{dom}(\widehat{H}_{T+T_\lambda})$. By $\widehat{H}_T = \widehat{H}_0^{(\lambda)} + (T - T_\lambda)$ and by Theorem 3.13, the resolvent of $\widehat{H}_T$ is given by formula (3.26) with $A_T G_z$ replaced by $A_{T-T_\lambda} G_z$, i.e., since $A_{T-T_\lambda}^{(\lambda)} G_z = A_T G_z$, by (3.26) itself.

Similar considerations apply to the proof of Theorem 3.10 (in the case $s = \frac{1}{2}$). Defining $\widetilde{H}_n^{(\lambda)} := H_n - A_n R_\lambda A_n^*$ with $|\lambda|$ so large that $1 - G_\lambda$ has a bounded inverse, by Theorem 3.10, $\widetilde{H}_n^{(\lambda)}$ converges in norm resolvent sense to $\widehat{H}_0^{(\lambda)}$. Then, since $\widetilde{H}_n = \widetilde{H}_n^{(\lambda)} - A_n R A_n^* + A_n R_\lambda A_n^* = \widetilde{H}_n^{(\lambda)} - (\lambda - \lambda_o) A_n R (A_n R_\lambda)^*$, $\widehat{H}_0 = \widehat{H}_0^{(\lambda)} - T_\lambda$ and $\|(\lambda - \lambda_o) A_n R (A_n R_\lambda)^* - T_\lambda\|_{\mathfrak{F}, \mathfrak{F}} \rightarrow 0$, one gets that $\widetilde{H}_n$ converges in norm resolvent sense to $\widehat{H}_0$ regardless of the value of λ_o .

Remark 3.15 By the same kind of reasonings as in the proof of Theorem 3.13, if $\widehat{H}_T$ is a self-adjoint operator with a resolvent given by (3.12) with $(\Sigma_{\Theta_T} \mathbb{G}_z)^{-1} \in \mathcal{B}(\mathfrak{F} \oplus \mathfrak{H}_1, \mathfrak{F} \oplus \mathfrak{H}_{-1})$ and $\text{ran}(A|\mathfrak{H}_1)$ dense, then, by $(\mathbb{G}_z(\Sigma_{\Theta_T} \mathbb{G}_z)^{-1} \mathbb{G}_z^*)^* = \mathbb{G}_z(\Sigma_{\Theta_T} \mathbb{G}_z)^{-1} \mathbb{G}_z^*$, by $\Theta_T = -(\Sigma_{\Theta_T} \mathbb{G}_z + \Sigma(\mathbb{G} - \mathbb{G}_z))$ and $(\Sigma(\mathbb{G} - \mathbb{G}_z))^* = \Sigma(\mathbb{G} - \mathbb{G}_z)$, one infers that Θ_T is self-adjoint.

Remark 3.16 Regarding Theorem 3.13, people working in extension theory could be puzzled by the fact that the family of self-adjoint operators $\widehat{H}_T$, coming out from the self-adjoint extensions of the symmetric $S = H|_{\ker(A)}$, is parameterized by symmetric ($\widehat{H}_0$ -bounded) operators T which, unlike what is requested in Theorem 3.4, are not necessarily self-adjoint. However, looking at the Kreĭn-type resolvent formula (3.26) (equivalently (3.12)), the true parameterizing operator turns out to be Θ_T in (3.23) which is self-adjoint (relatively to the dual couple $\mathfrak{F} \oplus \mathfrak{H}_{-1} - \mathfrak{F} \oplus \mathfrak{H}_1$), even when T is merely symmetric (see Remark 3.15).

Remark 3.17 Notice that, unlike Theorem 3.4, in Theorem 3.13 one does not need $A_T G_z$ to have a bounded inverse, i.e., one does not need hypothesis (2.8). Indeed, in (3.11) (equivalently (3.12)) the inverse $(\Sigma_{\Theta_T} \mathbb{G}_z)^{-1}$ is regarded as an operator in $\mathfrak{F} \oplus \mathfrak{F}$ and so, since $R_z : \mathfrak{F} \rightarrow \mathfrak{F}$ has no bounded inverse, one uses the second Schur complement, which requires $(A_T G_z)^{-1} \in \mathcal{B}(\mathfrak{F})$; on the contrary, in (3.26) the same inverse block operator matrix is regarded as an operator from $\mathfrak{F} \oplus \mathfrak{H}_1$ to $\mathfrak{F} \oplus \mathfrak{H}_{-1}$ and so, since $R_z : \mathfrak{H}_{-1} \subseteq \mathfrak{F} \rightarrow \mathfrak{H}_1$ has a bounded inverse, one can use the first Schur complement. Also notice that, by (3.24), the only case where one can show that $(\Sigma_{\Theta_T} \mathbb{G}_z)^{-1} \in \mathcal{B}(\mathfrak{F} \oplus \mathfrak{F})$ without requiring $(A_T G_z)^{-1} \in \mathcal{B}(\mathfrak{F})$ is the one given by the choice $T = 0$.

Remark 3.18 The strategy employed in Theorem 3.13 can be also applied to cases where T is not $\widehat{H}_0$ -bounded. For example, one can consider the case where $T = T_1 + T_2$, with T_1 such that $H_{(1)} := H + T_1$ is self-adjoint semibounded and T_2 is $\widehat{H}_{(1)}$ -bounded with relative bound less than one, where $\widehat{H}_{(1)}$ is constructed in the same way as $\widehat{H}_0$, replacing H with $H_{(1)}$. This is what was done for the QFT model studied in [12]. If A and T_1 self-adjoint satisfy the hypotheses in Corollary 3.7, and T_2 is $\widehat{H}_{T_1}$ -bounded with relative bound less than one, then $\widehat{H}_T = \widehat{H}_{T_1} + T_2$ is self-adjoint with domain $\text{dom}(\widehat{H}_T) = \{\psi \in \mathfrak{H}_{1-s} : \psi - G\psi \in \mathfrak{H}_1\}$.

3.1 Renormalizable QFT Models.

Here we show, using results contained in [13], how the 3-D Nelson model [18] fits to our abstract framework; similar consideration apply to the other renormalizable models considered in [13] (2-D polaron-type model with point interactions), [23] (the 3-D Eckmann and 2-D Gross models), [24] (the massless 3-D Nelson model) and [12] (the Bogoliubov-Fröhlich model).

We take

$$\mathfrak{F} = L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b \equiv \bigoplus_{n=0}^{\infty} \left(L^2(\mathbb{R}^{3N}) \otimes L^2_{\text{sym}}(\mathbb{R}^{3n}) \right), \quad (3.30)$$

where $\mathfrak{F}_b := \Gamma_b(L^2(\mathbb{R}^3))$ denotes the boson Fock space over $L^2(\mathbb{R}^3)$, and $H = H_{\text{free}}$, where H_{free} is the positive self-adjoint operator

$$H_{\text{free}} := -\Delta_{(3N)} \otimes 1 + 1 \otimes d\Gamma_b((-\Delta_{(3)} + 1)^{1/2}).$$

Here $\Delta_{(d)} : H^2(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ denotes the Laplace operator in $L^2(\mathbb{R}^d)$ with self-adjointness domain the Sobolev space $H^2(\mathbb{R}^d)$ and $d\Gamma_b(L)$ denotes the boson second quantization of L (see, e.g., [1, Chapter 5]). Since $0 \in \varrho(H_{\text{free}})$, we can take $\lambda_0 = 0$ in the definition (2.2) of G , so that $G = -(AH_{\text{free}}^{-1})^*$. In order to define the appropriate annihilation operator A we use the identification $L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b \equiv L^2(\mathbb{R}^{3N}; \mathfrak{F}_b)$ which maps $\psi \otimes \Phi$ to $\mathbf{x} \mapsto \Psi(\mathbf{x}) := \psi(\mathbf{x})\Phi$. Given $v := (-\Delta_{(3)} + 1)^{-1/4}\delta$, δ denoting the Dirac delta distribution supported at $0 \in \mathbb{R}^3$, we define

$$(A\Psi)(\mathbf{x}) := a(v_{\mathbf{x}})\Psi(\mathbf{x}), \quad (3.31)$$

where

$$v_{\mathbf{x}}(y) := g \sum_{j=1}^N v(x_j - y), \quad g \in \mathbb{R}, \quad \mathbf{x} \equiv (x_1, \dots, x_N) \in \mathbb{R}^{3N}$$

and $a(v)$ denotes the annihilation operator in $\mathfrak{F}_b$ with test vector v (see, e.g. [1, Section 5.7]). By [13, Corollary 3.2], one has $G \in \mathcal{B}(L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b, \text{dom}(H_{\text{free}}^s))$ for any $s < \frac{1}{4}$, equivalently $A \in \mathcal{B}(\text{dom}(H_{\text{free}}^s), L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b)$ for any $s > \frac{3}{4}$. By [13, Lemma 2.2], $\ker(A|_{\text{dom}(H_{\text{free}})})$ is dense in $L^2(\mathbb{R}^{3N}; \mathfrak{F}_b)$. The proof that $\text{ran}(A|_{\text{dom}(H_{\text{free}})})$ is dense in $L^2(\mathbb{R}^{3N}; \mathfrak{F}_b)$ follows the same kind of reasonings as in the proof of [13, Lemma 2.2]: let $D_{\mathbf{x}} \subset H^2(\mathbb{R}^3)$ be a $L^2(\mathbb{R}^3)$ -dense set of $\mathbf{x}$ -smooth functions $f_{\mathbf{x}}$ such that $\langle v_{\mathbf{x}}, f_{\mathbf{x}} \rangle \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the $H^{-2}(\mathbb{R}^3)$ - $H^2(\mathbb{R}^3)$ duality; then, given $\Phi(f_{\mathbf{x}})$, the coherent state generated by $f_{\mathbf{x}}$, one uses

the relation $a(v_x)\Phi(f_x) = \langle v_x, f_x \rangle \Phi(f_x)$ and the denseness of the linear span of $\{\Phi(f_x), f_x \in D_x\}$ (see [14, Proposition 6.2]).

Hence Theorem 3.13 applies and defines a self-adjoint operator $\widehat{H}_T$ for any symmetric operator T which is $\widehat{H}_0$ -bounded with relative bound $\widehat{a} < 1$. By Remark 3.11, T should be a suitable regularization of the ill-defined operator $-AH_{\text{free}}^{-1}A^*$; for A given in (3.31), the right choice, consisting in a regularization of the diagonal (with respect to the direct sum structure of $\mathfrak{F}$ in (3.30)) part of $-AH_{\text{free}}^{-1}A^*$, is provided in [13, equations (29)–(32)]. Here we denote such an operator by $T = T_{\text{Nelson}}$; it is infinitesimally $\widehat{H}_0$ -bounded by [13, Lemma 3.10] (let us notice that, by (3.22), our $\widehat{H}_0$ coincides with the operator there defined as $(1 - G^*)L(1 - G)$).

Given the sequence $v_n \in L^2(\mathbb{R}^3)$ with Fourier transform $\widehat{v}_n = \chi_n \widehat{v}$, where χ_n denotes the characteristic function of a ball of radius $R = n$ (this provides an ultraviolet cutoff on the boson momenta), let us denote by A_n the sequence of operators in $L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b$ defined as A in (3.31) with v replaced by v_n . One has that A_n is closed, $A_n \in \mathcal{B}(\text{dom}(H_{\text{free}}^{1/2}), L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b)$ and $A_n^* + A_n$ is infinitesimally H_{free} -bounded (see, e.g., [1, Section 14.5.1], [8, Appendix B]) and so such A_n 's fit to the hypotheses in Theorem 3.10. By [23, Proposition 3.2] (see also the proof of [13, Theorem 1.4]), one has $\|(A_n H_{\text{free}}^{-1})^* - (A H_{\text{free}}^{-1})^*\|_{\mathfrak{F}, \mathfrak{F}} \rightarrow 0$, which is equivalent to (3.16). Let E_n be the sequence of bounded symmetric operators in $L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b$ corresponding to the multiplication by the real constant given by (minus) the leading order term in the expansion in the coupling constant g of the the ground state energy at zero total momentum of the regularized Hamiltonian $H_{\text{free}} + A_n^* + A_n$ (see, e.g., [25, Section 19.2]):

$$\begin{aligned} E_n &:= g^2 N \left\| \left(-\Delta_{(3)} + (-\Delta_{(3)} + 1)^{1/2} \right)^{-1/2} v_n \right\|_{L^2(\mathbb{R}^3)}^2 \\ &= g^2 N \int_{\mathbb{R}^3} \frac{|\widehat{v}_n(\kappa)|^2}{|\kappa|^2 + (|\kappa|^2 + 1)^{1/2}} d\kappa. \end{aligned}$$

Defining then

$$T_n := E_n - A_n H_{\text{free}}^{-1} A_n^*,$$

by [23, Proposition 3.1] (see also the proof of Theorem 1.4 in [13]), one has $T_n \rightarrow T_{\text{Nelson}}$ in norm as operators in $\mathcal{B}(\text{dom}(T_{\text{Nelson}}), L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b)$; thus hypothesis (3.19) holds. Hypothesis (3.18) holds since the estimates in [13] with $\widehat{v}$ replaced by $\widehat{v}_n$ are bounded by the integrals with $\widehat{v}$ (see in particular the arguments given in the proof of [13, Theorem 1.4]). Therefore, by Theorem 3.10,

$$\lim_{n \uparrow \infty} (H_{\text{free}} + A_n^* + A_n + E_n) = H_{\text{Nelson}} := \overline{H}_{\text{free}} + A^* + A T_{\text{Nelson}} \quad \text{in norm resolvent sense}$$

and so the self-adjoint Hamiltonian H_{Nelson} provided by Theorem 3.13 with $T = T_{\text{Nelson}}$ coincides with the one given by Nelson in [18] (this is our version of [13, Theorem 1.4]; see also [23, Proposition 2.4]).

By Theorem 3.13,

$$\text{dom}(H_{\text{Nelson}}) = \{\Psi \in \text{dom}(H_{\text{free}}^{1-s}) : \Psi + (A H_{\text{free}}^{-1})^* \Psi \in \text{dom}(H_{\text{free}})\}, \quad s > \frac{3}{4},$$

and

$$(-H_{\text{Nelson}} + z)^{-1} = (-H_{\text{free}} + z)^{-1} - [G_z \ R_z] \begin{bmatrix} A_{T_{\text{Nelson}}} G_z & G_z^* - 1 \\ G_z - 1 & R_z \end{bmatrix}^{-1} \begin{bmatrix} G_z^* \\ R_z \end{bmatrix}, \quad (3.32)$$

where $R_z := (-H_{\text{free}} + z)^{-1}$, $G_z := (AR_z)^*$ and $A_{T_{\text{Nelson}}} G_z = T_{\text{Nelson}} - A(G - G_z)$. Notice that, since the operator sequence $A_n(H_{\text{free}}^{-1}A_n^* + (-H_{\text{free}} + z)^{-1}A_n^*)$ converges to $-A(G - G_z)$ in $\mathcal{B}(L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b)$, one has that $A_{T_{\text{Nelson}}} G_z$ is the limit of $E_n + A_n(-H_{\text{free}} + z)^{-1}A_n^*$ as operators in $\mathcal{B}(\text{dom}(T_{\text{Nelson}}), L^2(\mathbb{R}^{3N}) \otimes \mathfrak{F}_b)$.

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