Two Groups in a Curie-Weiss Model

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Abstract

We analyse a Curie-Weiss model with two disjoint groups of spins with homogeneous coupling. We show that similarly to the single-group Curie-Weiss model a bivariate law of large numbers holds for the normed sums of both groups' spin variables. We also show central limit theorem in the high temperature regime.

Keywords Curie-Weiss model · Central limit theorem · Statistical physics

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1 Introduction

The Curie-Weiss model is probably the easiest model of magnetism which shows a phase transition between a diamagnetic and a ferromagnetic phase. In this model the spins can take values in $\{-1, 1\}$ (or up/down), each spin interacts with all the others in the same way. More precisely, for finitely many spins $(X_1, X_2, \dots, X_N) \in \{-1, 1\}^N$ the energy of the spins is given by

$$H = H(X_1, \dots, X_N) := -\frac{1}{2N} \left(\sum_{j=1}^N X_j \right)^2. \quad (1)$$

Consequently, in the 'canonical ensemble' with inverse temperature $\beta \geq 0$ the probability of a spin configuration is given by

$$\mathbb{P}(X_1 = x_1, \dots, X_N = x_N) := Z^{-1} e^{-\beta H(x_1, \dots, x_N)} \quad (2)$$

where $x_i \in \{-1, 1\}$ and Z is a normalization constant which depends on N and β .

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The quantity

$$S_N = \sum_{j=1}^N X_j \quad (3)$$

is called the (total) magnetization. It is well known (see e. g. Ellis [4] or [10]) that the Curie-Weiss model has a phase transition at $\beta = 1$ in the following sense

$$\frac{1}{N} S_N \implies \frac{1}{2} (\delta_{-m(\beta)} + \delta_{m(\beta)}) \quad (4)$$

where $\implies$ denotes convergence in distribution, δ_x the Dirac measure in x .

For $\beta \leq 1$ we have $m(\beta) = 0$ which is the unique solution of

$$\tanh(\beta x) = x \quad (5)$$

for this case.

If $\beta > 1$ (5) has exactly three solutions and $m(\beta)$ is the unique positive one.

Equation (4) is a substitute for the law of large numbers for i.i.d. random variables.

Moreover, for $\beta < 1$ there is a central limit theorem, i. e.

$$\frac{1}{\sqrt{N}} S_N \implies \mathcal{N}\left(0, \frac{1}{1-\beta}\right) \quad (6)$$

For $\beta = 1$ there is no such central limit theorem. In fact, the random variables

$$\frac{1}{N^{3/4}} S_N \quad (7)$$

converge in distribution to a limit which is not a normal distribution.

The Curie-Weiss model is also called the Husimi-Temperley model. It was first introduced by Husimi [7] and Temperley [15]. Subsequently it was discussed by Kac [9], Thompson [16], and Ellis [4]. More recently, the Curie-Weiss model has been used in the context of social and political interactions. See e.g. [3] and [11].

We mention that the Curie-Weiss model is also used to describe the behaviour of voters who have the choice to vote ‘Yea’ (spin=1, say) or ‘Nay’ (spin=-1) (see [11]).

In this paper, we partition the set of all N Curie-Weiss spins into two disjoint groups $X_1, \dots, X_{N_1}$ and $Y_1, \dots, Y_{N_2}$ with $N_1 + N_2 = N$. We let N_1 and N_2 depend on N in such a way that both N_1 and N_2 go to infinity as N does. We consider the asymptotic behaviour of the two-dimensional random variables

$$\left(\sum_{i=1}^{N_1} X_i, \sum_{j=1}^{N_2} Y_j \right) \quad (8)$$

as N goes to infinity and prove results similar to the one-group case considered above. We will use the method of moments which was used in [12] for the one-group case. We rely on the method developed there.

Note that all spins are coupled by the same constant $\beta \geq 0$. We could also define the model using a homogeneous coupling matrix

$$J = \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

For an analysis of heterogeneous coupling matrices see [13].

We prove

Theorem 1 (Law of Large Numbers) *If $N_1, N_2 \rightarrow \infty$ as $N \rightarrow \infty$, then we have for all β*

$$\left(\frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right) \xrightarrow[N \rightarrow \infty]{} \frac{1}{2} (\delta_{(-m(\beta), -m(\beta))} + \delta_{(m(\beta), m(\beta))}). \quad (9)$$

Above ‘ $\Rightarrow$ ’ denotes convergence in distribution of the 2-dimensional random variable on the left hand side and $m(\beta)$ is again the largest solution of (5), in particular $m(\beta) = 0$ for $\beta \leq 1$ and $m(\beta) > 0$ for $\beta > 1$.

Remark 2 If we consider a model without interaction between the groups X_i and Y_j then the limit in (9) is

$$\frac{1}{4} (\delta_{(-m(\beta), -m(\beta))} + \delta_{(-m(\beta), m(\beta))} + \delta_{(m(\beta), -m(\beta))} + \delta_{(m(\beta), m(\beta))})$$

For $\beta < 1$ we also have a central limit theorem. The covariance of the limiting normal distribution depends on the growth rate of N_1 and N_2 . We set

$$\alpha_1 = \lim_{N \rightarrow \infty} \frac{N_1}{N} \quad \alpha_2 = \lim_{N \rightarrow \infty} \frac{N_2}{N} \quad (10)$$

and assume that these limits exist.

Theorem 3 (Central Limit Theorem) *If $\beta < 1$, then*

$$\left(\frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right) \xrightarrow[N \rightarrow \infty]{} \mathcal{N}((0, 0), C), \quad (11)$$

where the covariance matrix C is given by

$$C = \begin{bmatrix} 1 + \alpha_1 \frac{\beta}{1-\beta} & \sqrt{\alpha_1 \alpha_2} \frac{\beta}{1-\beta} \\ \sqrt{\alpha_1 \alpha_2} \frac{\beta}{1-\beta} & 1 + \alpha_2 \frac{\beta}{1-\beta} \end{bmatrix} \quad (12)$$

In particular, for sublinear growth of either N_1 or N_2 , i. e. if $\alpha_1 = 0$ or $\alpha_2 = 0$, the standardized sums in (11) are asymptotically independent.

In the proof of both results we employ the moment method (see e. g. [1] or [10]). In [12] the method of moments is used to prove limit theorems for the (one-group) Curie-Weiss Model.

Thus, to show the convergence in distribution of a sequence (X_n, Y_n) of two-dimensional random variables to a measure μ on $\mathbb{R}^2$ we prove that

$$\mathbb{E} \left(X_n^K \cdot Y_n^L \right) \longrightarrow \int x^K y^L \mu(dx, dy) \quad (13)$$

for all $K, L \in \mathbb{N}$.

Equation (13) implies convergence in distribution if the moments of μ grow only moderately, namely if for some constants A and C and all K, L

$$\int |x|^K |y|^L \mu(dx, dy) \leq A C^{K+L} (K+L)! \quad (14)$$

holds. For the multidimensional case of the moment method we refer to [14].

After publishing the first version of this paper on arXiv, we became aware of the articles [5] and [6] which contain the above results as special cases. The methods used by those authors are different from ours. In [6], the authors show a central limit theorem based on the assumption that a certain function has one or more global minima. The article [5] shows how for two groups, under asymmetric scaling of the sums of spins, a central limit theorem can be proved. We are grateful to Francesca Collet for drawing our attention to the papers [2], [5] and [6].

2 Preparation

We have to evaluate terms of the form

$$\begin{aligned} & \mathbb{E} \left(\left(\sum_{i=1}^{N_1} X_i \right)^K \left(\sum_{j=1}^{N_2} Y_j \right)^L \right) \\ &= \sum_{i_1, i_2, \dots, i_K=1}^{N_1} \sum_{j_1, j_2, \dots, j_L=1}^{N_2} \mathbb{E} (X_{i_1} \cdot X_{i_2} \cdot \dots \cdot X_{i_K} \cdot Y_{j_1} \cdot \dots \cdot Y_{j_L}) \end{aligned} \quad (15)$$

To shorten notation we set

$$\begin{aligned} \underline{i} &= (i_1, \dots, i_K) \in \{1, 2, \dots, N_1\}^K \\ \underline{j} &= (j_1, \dots, j_L) \in \{1, 2, \dots, N_2\}^L \end{aligned}$$

and $W_{Q,M} := \{1, 2, \dots, M\}^Q$. We also denote for $\underline{i} \in W_{K,N_1}$

$$X(\underline{i}) := X_{i_1} \cdot X_{i_2} \cdot \dots \cdot X_{i_K}$$

and similar for $Y(\underline{j})$. So, (15) reads

$$\sum_{\underline{i} \in W_{K,N_1}} \sum_{\underline{j} \in W_{L,N_2}} \mathbb{E} (X(\underline{i}) Y(\underline{j})) \quad (16)$$

The energy function H and hence the probability measure $\mathbb{P}$ are invariant with respect to permutation of indices. Thus we observe that

$$\mathbb{E} (X_{i_1} \cdot \dots \cdot X_{i_K} \cdot Y_{j_1} \cdot \dots \cdot Y_{j_L}) = \mathbb{E} (X_1 \cdot \dots \cdot X_K \cdot Y_1 \cdot \dots \cdot Y_L)$$

whenever both $i_1, \dots, i_K$ and $j_1, \dots, j_L$ are pairwise distinct. Since

$$X_i^{2m} = 1, \quad X_i^{2m+1} = X_i$$

and similar for Y_j , we conclude that

$$\mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) = \mathbb{E} \left(X_1 \cdot \dots \cdot X_{\rho(\underline{i})} \cdot Y_1 \cdot \dots \cdot Y_{\rho(\underline{j})} \right) \quad (17)$$

where $\rho(\underline{i})$ denotes the number of indices i_K which occur an odd number of times in $\underline{i}$. So, to compute sums of the form (15) we need good estimates of expectations ('correlations') as in (17). Such estimates are provided in [12] (see also [11]).

Proposition 4 *For $K + L$ even we have*

1) *for $\beta < 1$:*

$$\mathbb{E} (X_1 \cdot X_2 \cdot \dots \cdot X_K \cdot Y_1 \cdot Y_2 \cdot \dots \cdot Y_L) \approx (K + L - 1)!! \left(\frac{\beta}{1 - \beta} \right)^{\frac{K+L}{2}} N^{-\frac{K+L}{2}}$$

2) *for $\beta = 1$:*

$$\mathbb{E} (X_1 \cdot X_2 \cdot \dots \cdot X_K \cdot Y_1 \cdot \dots \cdot Y_L) \approx \frac{\int t^{K+L} e^{-\frac{1}{12}t^4} dt}{\int e^{-\frac{1}{12}t^4} dt} N^{-\frac{K+L}{4}}$$

3) *for $\beta > 1$:*

$$\mathbb{E} (X_1 \cdot X_2 \cdot \dots \cdot X_K \cdot Y_1 \cdot \dots \cdot Y_L) \approx m(\beta)^{K+L},$$

where $m(\beta)$ is the unique (strictly) positive solution of $t = \tanh(\beta t)$. For $K + L$ odd:

$$\mathbb{E} (X_1 \cdot X_2 \cdot \dots \cdot X_K \cdot Y_1 \cdot \dots \cdot Y_L) = 0$$

Above $a_n \approx b_n$ means $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ and $(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 3) \cdot (2n - 1)$. For a proof of Proposition 4 see [12].

3 Laws of Large Numbers

To prove Theorem 1 we consider

$$\frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{\underline{i} \in W_{K,N_1}} \sum_{\underline{j} \in W_{L,N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \quad (18)$$

We assume $\beta \leq 1$ first.

By U_{K,N_1} we denote the set of those $\underline{i} \in W_{K,N_1}$ for which the i_v are pairwise distinct. Note that for $\underline{i} \in U_{K,N_1}$ and $\underline{j} \in W_{L,N_2}$ we have

$$\mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \leq C N^{-\frac{K}{4}}$$

by Proposition 4. We single out the following Lemma (where $|M|$ denotes the cardinality of the set M).

Lemma 5

$$1) \quad |U_{K, N_1}| = \frac{N_1!}{(N_1 - K)!} \approx N_1^K \quad (19)$$

$$2) \quad |W_{K, N_1} \setminus U_{K, N_1}| \leq K! N_1^{K-1} \quad (20)$$

Proof 1) is obvious.

2) If $\underline{i} \in W_{K, N_1} \setminus U_{K, N_1}$, then there are at most $K - 1$ distinct indices i_v belonging to $\underline{i}$. We have consequently at most N_1^{K-1} choices for the i_v . There are at most $K!$ ways to position them in the multiindex $\underline{i}$. $\square$

We estimate
(18)

$$\begin{aligned} &\leq \frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{\underline{i} \in U_{K, N_1}} \sum_{\underline{j} \in W_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\ &+ \frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{\underline{i} \in W_{K, N_1} \setminus U_{K, N_1}} \sum_{\underline{j} \in W_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\ &\leq \frac{1}{N_1^K} \frac{1}{N_2^L} N_1^K L! N_2^L C_K N_1^{-\frac{K}{4}} + \frac{1}{N_1^K} \frac{1}{N_2^L} K! N_1^{K-1} L! N_2^L \\ &\xrightarrow{N_1, N_2 \rightarrow \infty} 0 \end{aligned}$$

For $\beta > 1$ we prove in a similar way that

$$\frac{1}{N_1^K} \frac{1}{N_2^L} \left(\sum_{\underline{i} \in U_{K, N_1}} \sum_{\underline{j} \in W_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) + \sum_{\underline{i} \in W_{K, N_1} \setminus U_{K, N_1}} \sum_{\underline{j} \in W_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \right)$$

goes to zero.

Consequently

$$\begin{aligned} &\frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{\underline{i} \in W_{K, N_1} \setminus U_{K, N_1}} \sum_{\underline{j} \in W_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\ &\approx \frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{\underline{i} \in U_{K, N_1}} \sum_{\underline{j} \in U_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\ &\longrightarrow \begin{cases} m(\beta)^{K+L}, & \text{if } K + L \text{ even} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

which are the moments of the measure $\frac{1}{2} (\delta_{(-m(\beta), -m(\beta))} + \delta_{(+m(\beta), +m(\beta))})$.

4 The Central Limit Theorem

We start this section by computing the moments of a centred two-dimensional normal distribution. Let (Z_1, Z_2) be normally distributed with mean zero and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \bar{\sigma} \\ \bar{\sigma} & \sigma_2^2 \end{pmatrix} \quad (21)$$

Lemma 6 *We have*

$$\begin{aligned} \mathbb{E} \left(Z_1^{2K} Z_2^{2L} \right) &= \sum_{r=0}^{K \wedge L} \binom{2K}{2K-2r} \binom{2L}{2L-2r} (2r)! \\ &\quad (2K-2r-1)!! (2L-2r-1)!! \cdot (\sigma_1^2)^{K-r} \bar{\sigma}^{2r} (\sigma_2^2)^{L-r} \end{aligned} \quad (22)$$

and

$$\begin{aligned} \mathbb{E} \left(Z_1^{2K+1} Z_2^{2L+1} \right) &= \sum_{r=0}^{K \wedge L} \binom{2K+1}{2K-(2r+1)} \binom{2L+1}{2L-(2r+1)} (2r+1)! \\ &\quad (2K-2r-1)!! (2L-2r-1)!! \cdot (\sigma_1^2)^{K-r} \bar{\sigma}^{2r+1} (\sigma_2^2)^{L-r}. \end{aligned} \quad (23)$$

A proof is given in the [Appendix](#).

For this section we assume $\beta < 1$. We have to evaluate

$$\frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{\underline{i} \in W_{K, N_1}} \sum_{\underline{j} \in W_{L, N_2}} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right). \quad (24)$$

As for the law of large numbers, many multiindices $\underline{i}, \underline{j}$ are negligible. However, this time the selection of the leading terms is more subtle.

Definition 7 Let us set

$$\begin{aligned} W_{Q, M} &:= \{1, 2, \dots, M\}^Q, \\ W_{Q, M}(r) &:= \{\underline{i} \in W_{Q, M} \mid \text{exactly } r \text{ indices } i_v \text{ occur exactly once in } \underline{i}\}, \\ W_{Q, M}^+(r) &:= \{\underline{i} \in W_{Q, M}(r) \mid \text{at least one index } i_v \text{ occurs more than twice in } \underline{i}\}, \\ W_{Q, M}^0(r) &:= W_{Q, M}(r) \setminus W_{Q, M}^+(r). \end{aligned}$$

Moreover by $w_{Q, M}(r) \left(w_{Q, M}^+(r), w_{Q, M}^0(r) \right)$ we denote $|W_{Q, M}(r)| \left(|W_{Q, M}^+(r)|, |W_{Q, M}^0(r)| \right)$.

$W_{Q, M}^0(r)$ consists of those multiindices for which (exactly) r indices occur once and the remaining indices occur exactly twice. It follows that $W_{Q, M}^0(r) = \emptyset$ if $Q - r$ is odd.

We note the following combinatorial Proposition:

Proposition 8

$$\begin{aligned}
W_{Q,M}(Q) &= \frac{M!}{(M-Q)!} \approx M^Q, \\
W_{Q,M}(r) &\leq Q! M^{\frac{Q+r}{2}}, \\
W_{Q,M}^+(r) &\leq Q! M^{\frac{Q+r}{2}-\frac{1}{2}}, \\
W_{Q,M}^0(r) &= \begin{cases} \frac{M!}{\left(M-\frac{Q+r}{2}\right)!} \frac{Q!}{r! \left(\frac{Q-r}{2}\right)! 2^{\frac{Q-r}{2}}} & \text{if } Q-r \text{ is even} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For a proof see the [Appendix](#) or [12].

Now, we split the sum (24) in four parts.

(24)

$$\begin{aligned}
&= \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^K \sum_{\underline{i} \in W_{K,N_1}^0(r)} \sum_{s=0}^L \sum_{\underline{j} \in W_{L,N_2}^0(s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\
&\quad + \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^K \sum_{\underline{i} \in W_{K,N_1}^0(r)} \sum_{s=0}^L \sum_{\underline{j} \in W_{L,N_2}^+(s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\
&\quad + \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^K \sum_{\underline{i} \in W_{K,N_1}^+(r)} \sum_{s=0}^L \sum_{\underline{j} \in W_{L,N_2}^0(s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\
&\quad + \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^K \sum_{\underline{i} \in W_{K,N_1}^+(r)} \sum_{s=0}^L \sum_{\underline{j} \in W_{L,N_2}^+(s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \\
&= A_1 + A_2 + A_3 + A_4
\end{aligned}$$

Lemma 9 $A_2, A_3, A_4 \rightarrow 0$ as $N \rightarrow \infty$

Proof If $\underline{i} \in W_{K,N_1}^0(r)$, $\underline{j} \in W_{L,N_2}^+(s)$ then $\mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \leq C N^{-\frac{r+s}{2}}$ by Proposition 4. Moreover

$$\begin{aligned}
w_{K,N_1}^0(r) &\leq C_1 N_1^{\frac{K+r}{2}} \\
w_{L,N_2}^+(s) &\leq C_2 N_2^{\frac{L+r}{2}-\frac{1}{2}}
\end{aligned}$$

by Proposition 4. Thus

$$\frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{\underline{i} \in W_{K,N_1}^0(r)} \sum_{\underline{j} \in W_{L,N_2}^+(s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) \leq C' N^{-\frac{1}{2}}$$

It follows that $A_2 \rightarrow 0$. The proofs for A_3 and A_4 are similar. $\square$

We turn to the asymptotic computation of A_1 .

$$A_1 = \sum_{r=0}^K \sum_{\underline{i} \in W_{K, N_1}^0(r)} \sum_{s=0}^L \sum_{\underline{j} \in W_{L, N_2}^0(s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right)$$

First, we observe that for $K + L$ odd $\mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) = 0$ by Proposition 4. So, we may assume that K and L are both even or both odd. If K and L are even (odd) then both r and s have to be even (odd) otherwise $W_{K, N_1}^0(r) = \emptyset$ or $W_{L, N_2}^0(s) = \emptyset$. In the following we will treat the case K and L even, the other one being similar.

So we assume $K = 2k$ and $L = 2\ell$ and rewrite A_1 :

$$\sum_{r=0}^k \sum_{s=0}^{\ell} \sum_{\underline{i} \in W_{2k, N_1}(2r)} \sum_{\underline{j} \in W_{2\ell, N_2}(2s)} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right)$$

If $\underline{i} \in W_{2k, N_1}^0(2r)$ and $\underline{j} \in W_{2\ell, N_2}^0(2s)$ then by Proposition 4

$$\begin{aligned} \mathbb{E} \left(X(\underline{i}) Y(\underline{j}) \right) &\approx (2r + 2s - 1)!! \left(\frac{\beta}{1-\beta} \right)^{r+s} N^{-(r+s)} \\ &= (2r + 2s - 1)!! \left(\frac{\alpha_1 \beta}{1-\beta} \right)^r \left(\frac{\alpha_2 \beta}{1-\beta} \right)^s N_1^{-r} N_2^{-s} \end{aligned}$$

using Proposition 8 we get:

$$\begin{aligned} A_1 &\approx \frac{1}{N_1^k} \frac{1}{N_2^\ell} \sum_{r=0}^k \sum_{s=0}^{\ell} \frac{N_1!}{(N_1! - k - r)!} \frac{N_2!}{(N_2! - \ell - s)!} \frac{2k!}{2r! (k-r)! 2^{k-r}} \frac{2\ell!}{2s! (\ell-s)! 2^{\ell-s}} \\ &\quad \cdot (2r + 2s - 1)!! \left(\frac{\alpha_1 \beta}{1-\beta} \right)^r \left(\frac{\alpha_2 \beta}{1-\beta} \right)^s N_1^{-r} N_2^{-s} \\ &\approx \sum_{r=0}^k \sum_{s=0}^{\ell} \frac{2k!}{2r! (k-r)! 2^{k-r}} \frac{2\ell!}{2s! (\ell-s)! 2^{\ell-s}} \cdot (2r + 2s - 1)!! (\alpha_1 \bar{\beta})^r (\alpha_2 \bar{\beta})^s \quad (25) \end{aligned}$$

where we set $\bar{\beta} := \frac{\beta}{1-\beta}$.

This proves that all moments converge, hence $\left(\frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right)$ converge in distribution. In the following we have to identify the limit measure.

First we consider the case that $\alpha_1 = 0$ or $\alpha_2 = 0$. Let's assume $\alpha_2 = 0$. Then (25) simplifies to

$$\begin{aligned} &\sum_{r=0}^k \frac{(2k)!}{(2r)! (k-r)! 2^{k-r}} \frac{(2\ell)!}{\ell! 2^\ell} \frac{(2r)!}{r! 2^r} (\alpha_1 \bar{\beta})^r \\ &= \frac{(2k)!}{k! 2^k} \frac{(2\ell)!}{\ell! 2^\ell} \sum_{r=0}^k \binom{k}{r} (\alpha_1 \bar{\beta})^r \\ &= (K-1)!! (L-1)!! (1 + \alpha_1 \bar{\beta})^{\frac{K}{2}} \end{aligned}$$

The last expression is the (K, L) -moment of a centred normal distribution with covariance matrix

$$\begin{pmatrix} 1 + \frac{\alpha_1 \beta}{1 - \beta} & 0 \\ 0 & 1 \end{pmatrix}. \quad (26)$$

Now, we turn to the general case ($\alpha_2 \geq 0$).

We use the following combinatorial identity which is Corollary 13 in the [Appendix](#). A proof is given there.

$$(2r + 2s - 1)!! = \sum_{t=0}^{r \wedge s} \frac{2r!}{(2r - 2t)! 2t!} \frac{2s!}{(2s - 2t)!} \frac{(2r - 2t)!}{(r - t)! 2^{r-t}} \frac{(2s - 2t)!}{(s - t)! 2^{s-t}}. \quad (27)$$

Substituting this expression into (25) we get:

$$\begin{aligned} A_1 &\approx \sum_{r=0}^k \sum_{s=0}^{\ell} \alpha_1^r \alpha_2^s \frac{2k!}{2r! (k-r)! 2^{k-r}} \frac{2\ell!}{2s! (\ell-s)! 2^{\ell-s}} \\ &\quad \cdot \sum_{t=0}^{r \wedge s} \frac{2r!}{(2r - 2t)! 2t!} \frac{2s!}{(2s - 2t)!} \frac{(2r - 2t)!}{(r - t)! 2^{r-t}} \frac{(2s - 2t)!}{(s - t)! 2^{s-t}} \bar{\beta}^{r+s} \\ &= \sum_{r=0}^k \sum_{s=0}^{\ell} \sum_{t=0}^{r \wedge s} \alpha_1^r \alpha_2^s \frac{2k!}{(k-r)! 2^{k-r}} \frac{2\ell!}{(\ell-s)! 2^{\ell-s}} \\ &\quad \cdot \frac{1}{2t!} \frac{1}{(r-t)! 2^{r-t}} \frac{1}{(s-t)! 2^{s-t}} \bar{\beta}^{r+s} \\ &= \sum_{t=0}^{k \wedge \ell} \sum_{r=t}^k \sum_{s=t}^{\ell} \alpha_1^r \alpha_2^s \frac{2k!}{(k-r)! 2^{k-r}} \frac{2\ell!}{(\ell-s)! 2^{\ell-s}} \frac{1}{2t!} \frac{1}{(r-t)! 2^{r-t}} \frac{1}{(s-t)! 2^{s-t}} \bar{\beta}^{r+s} \end{aligned}$$

We set $m := r - t$ and $n := s - t$ and continue:

$$\begin{aligned} &= \sum_{t=0}^{k \wedge \ell} \sum_{m=0}^{k-t} \sum_{n=0}^{\ell-t} \frac{2k!}{(k-m-t)! 2^{k-m-t}} \frac{2\ell!}{(\ell-n-t)! 2^{\ell-n-t}} \frac{1}{2t!} \frac{1}{m! 2^m} \frac{1}{n! 2^n} \bar{\beta}^{m+n+2t} \\ &= \sum_{t=0}^{k \wedge \ell} \frac{2k!}{2^{k-t}} \frac{2\ell!}{2^{\ell-t}} \frac{1}{2t!} (\sqrt{\alpha_1 \alpha_2} \bar{\beta})^{2t} \\ &\quad \cdot \sum_{m=0}^{k-t} \frac{1}{(k-m-t)! m!} (\alpha_1 \bar{\beta})^m \cdot \sum_{n=0}^{\ell-t} \frac{1}{(\ell-n-t)! n!} (\alpha_2 \bar{\beta})^n \\ &= \sum_{t=0}^{k \wedge \ell} \frac{2k!}{(2k-2t)! 2t!} \frac{2\ell!}{(2\ell-2t)!} \frac{(2k-2t)!}{(k-t)! 2^{k-t}} \frac{(2\ell-2t)!}{(\ell-t)! 2^{\ell-t}} \\ &\quad \cdot (1 + \alpha_1 \bar{\beta})^{k-t} (\sqrt{\alpha_1 \alpha_2} \bar{\beta})^{2t} (1 + \alpha_2 \bar{\beta})^{\ell-t}. \end{aligned}$$

The last expression is, according to Lemma 6, the moment $\mathbb{E}(Z_1^{2k} Z_2^{2\ell})$ of a bivariate normal distribution with zero mean and covariance matrix

$$\begin{pmatrix} 1 + \alpha_1 \bar{\beta} & \sqrt{\alpha_1 \alpha_2} \bar{\beta} \\ \sqrt{\alpha_1 \alpha_2} \bar{\beta} & 1 + \alpha_2 \bar{\beta} \end{pmatrix}.$$

This concludes the proof of the Central Limit Theorem 3.

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Appendix

In this appendix we discuss some combinatorics in connection with the moment method and the Theorem of Isserlis.

Let us denote by $\mathcal{P}_L(r)$ the set of all partitions $\Pi = \{\pi_1, \pi_2, \dots, \pi_\ell\}$ of $\{1, 2, \dots, L\}$ with r sets π_i with $|\pi_i| = 1$ and $\ell - r$ sets π_j with $|\pi_j| = 2$. In particular, $\mathcal{P}_L(0)$ is the set of pair partitions of $\{1, 2, \dots, L\}$.

We show

Lemma 10

$$|\mathcal{P}_{2L}(0)| = (2L - 1)!! = \frac{2L!}{L! 2^L}$$

and

$$|\mathcal{P}_{2L+1}(0)| = 0.$$

Proof The claim is true for $L = 1$. Suppose $|\mathcal{P}_{2L}(0)| = (2L - 1)!!$. Then to build $\{\pi_1, \dots, \pi_{L+1}\}$ we can match the number $2L + 2$ with any of the other $2L - 1$ numbers. Thus, by induction hypothesis, we have $(2L - 1)!!$ choices to build pair partitions from the remaining $2L$ unmatched elements. $\square$

Proposition 11 Let $L, r \in \mathbb{N}$.

If $L - r$ is even, then

$$p_L(r) := |\mathcal{P}_L(r)| = \binom{L}{r} (L - r - 1)!!$$

and if $L - r$ is odd, then $p_L(r) = 0$.

Proof If $\{\pi_1, \dots, \pi_\ell\} \in \mathcal{P}_L(r)$ then $2\ell - r = 2(\ell - r) + r = L$, so $L - r$ is even. This proves the second assertion of the Proposition. Let $L - r$ be even. Then we have

$\binom{L}{r}$ choices for the sets π_i with $|\pi_i| = 1$. There remain $L - r$ elements to build pair partitions from. By Lemma 10 this can be done in $(L - r - 1)!!$ ways. $\square$

Proof (Proposition 8): We prove the final assertion of the proposition, the other assertions are easier to prove. There are $\frac{Q+r}{2}$ different indices to choose from a total of M possible choices. This gives $\frac{M!}{\left(M - \frac{Q+r}{2}\right)!}$ possibilities. To distribute them on the Q indices with r single and $\frac{Q-r}{2}$ double occurrences gives

$$\binom{Q}{r} (Q - r - 1)!! = \frac{Q!}{(Q - r)! r!} \frac{(Q - r)!}{\left(\frac{Q-r}{2}\right)! 2^{\frac{Q-r}{2}}}.$$

Multiplying gives the assertion. $\square$

Now, we consider a disjoint partition $\{1, 2, \dots, 2L\}$ into two sets K and M with $k = |K|$ and $m = |M|$, and look for all pair partitions $\Pi = (\pi_1, \dots, \pi_L)$ such that exactly r of the π_i are mixed, i.e. $\pi_i \cap K \neq \emptyset$ and $\pi_i \cap M \neq \emptyset$.

Proposition 12 *The number $p_{K,M}(r)$ of pair partitions $\Pi = (\pi_1, \dots, \pi_L)$ of $\{1, 2, \dots, 2L\}$ with exactly r mixed π_i is given by*

$$p_{K,M}(r) = \frac{k!}{(k-r)!} \frac{m!}{(m-r)!} \frac{1}{r!} (k-r-1)!! (m-r-1)!!$$

if both $k-r$ and $m-r$ are even and non-negative, and

$$p_{K,M}(r) = 0 \quad \text{otherwise.}$$

Proof There are $\frac{k!}{(k-r)! r!}$ choices for elements of K in the mixed π_i and $\frac{m!}{(m-r)!}$ ways to fill them with elements from M . The choices for the $(k-r)$ pure π_i from K and $(m-r)$ pure π_i from M are given by Lemma 10. $\square$

Corollary 13 *If both k and m are even, then*

$$(k+m-1)!! = \sum_{\rho=0}^{\frac{k}{2} \wedge \frac{m}{2}} \frac{k! m!}{(k-2\rho)! (m-2\rho)! 2^\rho!} (k-2\rho)!! (m-2\rho)!!$$

If both k and m are odd, then

$$(k+m-1)!! = \sum_{\rho=0}^{\frac{k-1}{2} \wedge \frac{m-1}{2}} \frac{k! m!}{(k-2\rho+1)! (m-2\rho+1)! (2\rho+1)!} (k-2\rho+1)!! (m-2\rho+1)!!$$

This Corollary follows by summing $p_{k,m}(\rho)$ over all possible ρ . We end this appendix with the proof of Lemma 6.

Proof (Lemma 6): By the Theorem of Isserlis [8], we have

$$\mathbb{E} \left(Z_1^{2K} Z_2^{2L} \right) = \sum_{\Pi=(\pi_1, \dots, \pi_{K+L}) \in \mathcal{P}_{2K+2L}(0)} \prod_{j=1}^{K+L} \mathbb{E}_{\pi_j},$$

where

$$\mathbb{E}_{(\kappa, \lambda)} = \begin{cases} \mathbb{E}(Z_1^2) = \sigma_1^2, & \text{if } \kappa, \lambda \leq 2K \\ \mathbb{E}(Z_2^2) = \sigma_2^2, & \text{if } \kappa, \lambda > 2K \\ \mathbb{E}(Z_1 Z_2) = \bar{\sigma} & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} &= \sum_{r=0}^{K \wedge L} \sum_{\Pi \in \mathcal{P}_{2K, 2L}(2r)} (\sigma_1^2)^{K-r} \bar{\sigma}^{2r} (\sigma_2^2)^{L-r} \\ &= \sum_{r=0}^{K \wedge L} \binom{2K}{2K-2r} \binom{2L}{2L-2r} (2r)! (2K-2r-1)!! (2L-2r-1)!! \\ &\quad \cdot (\sigma_1^2)^{K-r} \bar{\sigma}^{2r} (\sigma_2^2)^{L-r} \end{aligned}$$

by Proposition 12. Equation (23) can be proved analogously. $\square$

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