

# Grain Sedimentation Time in a Gaseous Protoplanet

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**Abstract** Segregation times for solid grains inside a gaseous protoplanet have been calculated for three different initial grain sizes by polytropic method. The result is found to be in good agreement with results obtained by other authors with more rigorous treatment of the problem.

**Keywords** Grain · Sedimentation · Protoplanet · Polytrope

## 1 Introduction

It has long been suggested that the solid body planets in the terrestrial region of the solar system might have formed by segregation of heavy elements in gaseous protoplanets followed by the removal of gaseous envelope. Two mechanisms, available in the literature, for formation of giant gaseous protoplanets are (1) core accretion and (2) disk instability in solar nebula (Pollack et al. 1996; Boss 1998, 2000, 2003). In the core accretion model, planetesimals collide and accrete to form a core which grows and gains mass by attracting the surrounding gas forming Jupiter mass objects (Pollack et al. 1996; Hubickyj et al. 2005). In the alternative scenario gas giants are formed due to local gravitational instability in the solar nebula having solar composition of elements and no core at all. This theory, once in vogue and then quickly forgotten, has been revived and reformulated by several authors (e.g. Boss 1997, 1998; Cha and Nayakshin 2010; Boley et al. 2010). Recent models of Jupiter indicate that Jupiter has probably a core of several Earth masses (e.g. Saumon and Guillot 2004). Thus core formation in protoplanets is possibly a reality. Boss (1997, 1998) suggests that solid cores in protoplanets can form by sedimentation of dust grains to

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the center of the protoplanets. Grain sedimentation has been investigated earlier by McCrea and Williams (1965) and recently by Boss (1997, 1998) and Helled et al. (2008). In calculating the sedimentation time McCrea and Williams (1965) assumed a uniform density model of a protoplanet, while Boss (1998) and Helled et al. (2008) investigated the problem in some great detail. In all cases the sedimentation time was found to be short on astronomical time scale. In their calculation Boss (1998) assumed the protoplanet to be in radiative equilibrium while Helled et al. (2008) found the gas blob to be fully convective with a thin outer radiative zone. This is consistent with Bodenheimer et al. (1980) and Wuchterl et al. (2000). In this communication we investigate the problem by simple polytropic model. Instead of going into detailed physics involved in the problem we simply calculate the time taken by a dust grain in falling from the surface to the centre of a protoplanet by determining its polytropic structure assuming that the density distribution inside the protoplanet is given by a polytropic law, and intend to see whether in such cases a core can form in a protoplanet before the central temperature of the protoplanet becomes, due to initial contraction, high enough to vaporize the dust grains. The time scale required for a protoplanet to reach such a high temperature is of the order of a few times  $10^5$  years (see Helled et al. 2008).

## 2 Model Used

We consider a spherical globe of gas of mass  $M = 2 \times 10^{30}$  g and radius  $R = 3 \times 10^{12}$  cm. The globe is, in reality, a protoplanet of solar composition in quasi-static equilibrium with no core in which ideal gas law holds. The protoplanet is assumed to behave as a polytrope of index  $n$  meaning that the density distribution inside is given by

$$P = K\rho^{1+\frac{1}{n}}, \quad (1)$$

where  $P$  is the pressure,  $\rho$  is the density, and  $K$  is the polytropic constant.

As for grain growth we assume that the grains falling from the surface towards the centre accrete all other grains with which they come into contact and grow, and that rate of growth of a grain is given by (Baines and Williams 1965)

$$\frac{dr_g}{dt} = -\frac{\lambda\rho}{4\rho_g} \frac{dr}{dt}, \quad (2)$$

where  $r_g$  is the grain radius at any time,  $\rho_g$  is the density of the grain material, assumed constant,  $r$  is the distance from the centre and  $\lambda$  is the proportion by weight of the grain in the gas. For the initial grain radius we consider three different values, namely  $r_0 = 10^{-1}$  cm,  $r_0 = 10^{-2}$  cm and  $r_0 = 10^{-3}$  cm.

The grains moving through the gas experience resistive force. In the model considered, the appropriate resistive force is either Epstein drag or Stokes' drag depending on whether the mean free path of the particles is greater or less than the radius of the grain. In our model, for a brief initial period, Epstein drag may be applicable to some grains. Because the radius of such grains may remain smaller than the mean free path for some time. However, the grains are growing quickly. As is evident from Fig. 2, the grains are found to grow soon to sizes where the grain radii are larger than the mean free path of the particles. Over most of the time Stokes' drag is thus operative. Ignoring that initial brief period of time, we assume that the correct resistance to the motion of the falling grains is given by Stokes' drag, being given by

$$F_{\text{res}} = 6\pi\eta r_g \frac{dr}{dt}, \quad (3)$$

where  $\eta$  is the coefficient of viscosity.

### 3 Structure of the Protoplanet and the Growth of the Grains

If the protoplanet is a polytrope of index  $n$ , then its structure can be determined by the Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \quad (4)$$

where  $\xi = \xi_1 \frac{r}{R}$ ,  $\xi_1$  being the first zero of  $\theta$ ,  $r$  the central distance and  $\theta$  is related to the thermodynamic variables  $P$ ,  $\rho$  and  $T$  through

$$\theta = \left( \frac{\rho}{\rho_c} \right)^{\frac{1}{n}} = \left( \frac{P}{P_c} \right)^{\frac{1}{1+n}} = \frac{T}{T_c}. \quad (5)$$

Here  $\rho_c$ ,  $P_c$  and  $T_c$  are the central values and are given by

$$\rho_c = a_n \frac{3M}{4\pi R^3}, \quad P_c = b_n \frac{GM^2}{R^4} \quad \text{and} \quad T_c = c_n \frac{GM}{R},$$

where  $G$  is the universal gravitational constant and  $a_n$ ,  $b_n$  and  $c_n$  are numerical constants having different values for different  $n$  and are available in the table (e.g. Menzel et al. 1963).

It is obvious that the structure depends on the  $n$ . For initial protoplanets  $n$  cannot be high. We consider three different values  $n = 0, 0.5, 1$ . Inserting for  $a_n$ ,  $b_n$  and  $c_n$  and the values of the other parameters, we find  $\rho_c$ ,  $P_c$  and  $T_c$ . These together with  $\xi_1$  for different  $n$  are given in Table 1.

To determine  $\theta$  at any distance  $x$  from the centre, we replace  $\xi$  in Eq. (4) by  $x\xi_1$ . The Eq. (4) then reduces to

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d\theta}{dx} \right) = -\xi_1^2 \theta^n, \quad (6)$$

where  $\theta = 1$ , and  $\frac{d\theta}{dx} = 0$  at  $x = 0$ .

Mass distribution inside the protoplanet is given by  $dM(r) = 4\pi r^2 \rho dr$ , which with the help of Eq. (5) and on nondimensionalisation with  $M(r) = q(x)M$  and  $r = xR$  gives

**Table 1** Some important quantities for the polytrope for some values of the polytropic index  $n$

$n$	$\xi_1$	$\rho_c$ (g cm $^{-3}$ )	$P_c$ (dynes cm $^{-2}$ )	$T_c$ (°K)
0	2.4494	$1.7683 \times 10^{-8}$	393.4657	161.25
0.5	2.7528	$3.2469 \times 10^{-8}$	630.1199	141.1422
1	3.14159	$5.8178 \times 10^{-8}$	1294.4523	161.8219

$$q(x) = \frac{4\pi R^3 \rho_c}{M} \int_0^x x^2 \theta^n dx, \quad (7)$$

where  $q(x) = 1$  at  $x = 1$  and  $q(x) = 0$  at  $x = 0$ .

The structure is thus determined by Eqs. (5–7). The distribution is shown in Fig. 1.

### 3.1 Growth of Grains

Using  $\rho = \frac{\mu P}{\mathfrak{R} T}$ , the equation of state for an ideal gas, in Eq. (2) and nondimensionalising with  $r_g = r_0 R_g$ ,  $t = 10^7 \tau$  and other nondimensional variables, we get

$$\frac{dR_g}{dt} = -\frac{\lambda \mu R}{4 \rho_g r_0 \mathfrak{R}} \frac{P_c p}{T_c} \frac{dx}{\theta}, \quad (8)$$

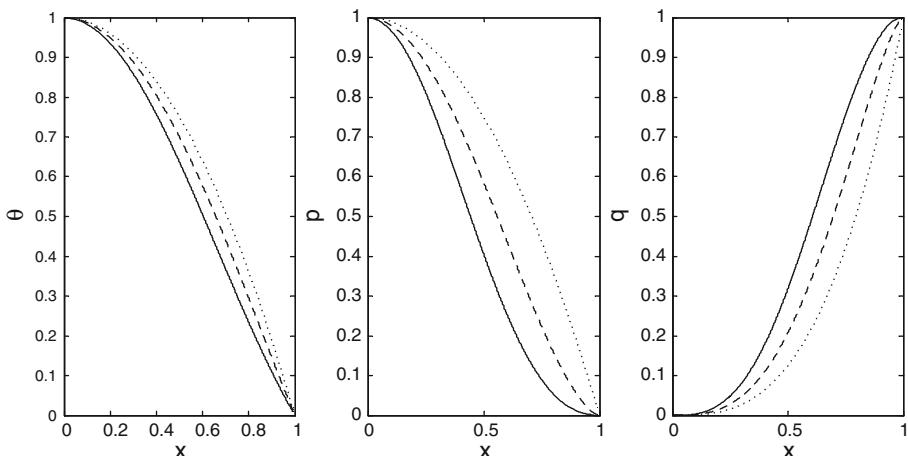
$\mathfrak{R}$  being the gas constant.

Equation (8), on integration, gives

$$R_g = 1 + \frac{\lambda \mu R}{4 \rho_g r_0 \mathfrak{R} T_c} \int_x^1 \frac{p}{\theta} dx, \quad (9)$$

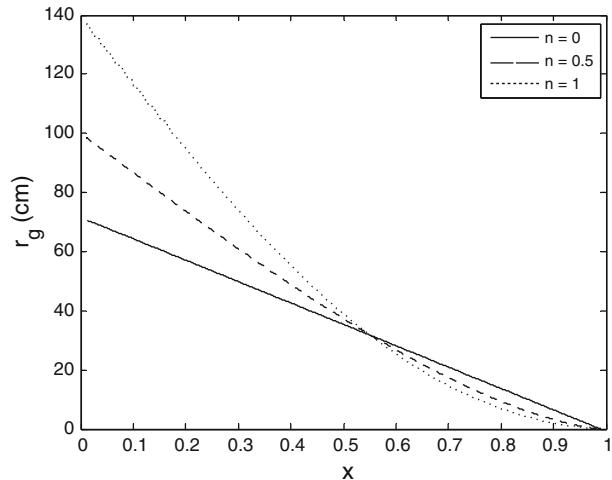
since  $R_g = 1$  at  $x = 1$ .

Taking  $\lambda = 4 \times 10^{-3}$  (Helled et al. 2008),  $\mu = 2.3$  (Dullemond and Dominik 2004),  $\rho_g = 2.8 \text{ gm cm}^{-3}$  (appropriate for silicon grain) and substituting for  $\mathfrak{R}$ ,  $P_c$  and  $T_c$ , we have evaluated the integral in Eq. (9) by Simpson's one third rule for  $r_0 = 10^{-1} \text{ cm}$ ,  $r_0 = 10^{-2} \text{ cm}$  and  $r_0 = 10^{-3} \text{ cm}$  for each value of  $n$ . The result for  $r_0 = 10^{-2} \text{ cm}$  is shown in Fig. 2. The curves for other values of  $r_0$  look similar and have not been included.



**Fig. 1** The distribution of  $\theta$ ,  $p(P/P_c)$ ,  $q$  for  $n = 0$  (dotted curves),  $n = 0.5$  (dashed curves) and  $n = 1$  (solid curves)

**Fig. 2** Growth of grain radius for different  $n$  with  $r_0 = 10^{-2}$  cm



#### 4 Calculation of Sedimentation Time

Let a dust grain move from rest at the surface of a protoplanet toward its centre through the ambient gas. If  $m_g$  be the mass of the grain at time  $t$  at distance  $r$  from the centre, then the equation of motion of the grain is given by

$$\frac{d}{dt} \left( m_g \frac{dr}{dt} \right) = -\frac{GM(r)m_g}{r^2} - 6\pi\eta r_g \frac{dr}{dt}. \quad (10)$$

In general, any body moving in a resisting medium reaches a velocity close to its terminal velocity very quickly and proceeds to travel at such velocity. Therefore, ignoring the acceleration term, Eq. (10) can be written as

$$\frac{dm_g}{dt} \frac{dr}{dt} = -\frac{GM(r)m_g}{r^2} - 6\pi\eta r_g \frac{dr}{dt}. \quad (11)$$

Using  $m_g = \frac{4}{3}\pi r_g^3 \rho_g$  and introducing the same dimensionless variables Eq. (11) together with Eqs. (2) and (3) gives on some simplification,

$$c_1 \frac{p}{\theta} \left( \frac{dx}{d\tau} \right)^2 = c_2 \frac{1}{R_g d\tau} + c_3 \frac{q R_g}{x^2}, \quad (12)$$

where  $c_1 = \frac{\lambda\mu R^2}{10^{14} \mathfrak{R}} \frac{P_c}{T_c}$ ,  $c_2 = \frac{6\eta R}{10^7 r_0}$  and  $c_3 = \frac{4GM\rho_g r_0}{3R^2}$ ,  $\mu$  being the mean molecular weight.

Solving Eq. (12) for  $\frac{dx}{d\tau}$ , we have

$$\frac{dx}{d\tau} = \frac{\frac{c_2}{R_g} \pm \sqrt{\left( \frac{c_2^2}{R_g^2} + 4c_1 c_3 \frac{pq R_g}{\theta x^2} \right)}}{2c_1 \frac{p}{\theta}}. \quad (13)$$

Since  $c_1$ ,  $c_2$  and  $c_3$  are positive and  $\frac{dx}{d\tau}$  is always negative, we must take the negative sign. This gives the time of fall as

$$\tau = - \int_{x=1}^0 F(x, p, q, \theta, R_g) dx, \quad (14)$$

where

$$F(x, p, q, \theta, R_g) = \frac{2c_1 \frac{p}{\theta}}{-\frac{c_2}{R_g} + \sqrt{\left(\frac{c_2^2}{R_g^2} + 4c_1 c_3 \frac{pqR_g}{\theta x^2}\right)}}.$$

It is evident that the integral in Eq. (14) cannot be evaluated analytically. Resort has to be taken to numerical technique. In order to do this, although we know  $p$ ,  $q$ ,  $\theta$  and  $R_g$  at distance  $x$ , we also need to know  $c_1$ ,  $c_2$  and  $c_3$ .  $c$ 's have been calculated for different  $r_0$  and  $n$  by taking  $\lambda = 4 \times 10^{-3}$ ,  $\mu = 2.3$ ,  $\rho_g = 2.8 \text{ g cm}^{-3}$ , as before, and  $\eta = 5 \times 10^{-5} \text{ g cm}^{-1} \text{ s}^{-1}$  together with known values of other parameters. It may be mentioned that for protoplanets of solar composition of temperature of about  $100^\circ\text{K}$  the adopted value of  $\eta$  is a reasonable estimate, as can be seen from the definition of coefficient of viscosity. Calculated values of  $c_1$ ,  $c_2$  and  $c_3$  are shown in Table 2.

Using these values of  $c_1$ ,  $c_2$  and  $c_3$ , we have evaluated the integral in Eq. (14) by Simpson's one third rule for different values of  $n$  for each  $r_0$ . To avoid the singularity, integration has been performed between limits  $x = .99$  to  $x = .01$ , required initial values have been obtained by series solution near the singular point. Results of our calculation are shown below in Table 3.

## 5 Discussion

We have calculated the grain sedimentation time inside a gaseous protoplanet. We have not considered detailed physics involved in the problem. We simply assume that a protoplanet is an isolated gaseous object which behaves as a polytrope of some index  $n$ , and that the

**Table 2** Values of  $c_1$ ,  $c_2$  and  $c_3$  for different values of  $r_0$  and  $n$

$n$	$r_0 = 10^{-1} \text{ cm}$			$r_0 = 10^{-2} \text{ cm}$			$r_0 = 10^{-3} \text{ cm}$		
	$c_1$	$c_2$	$c_3$	$c_1$	$c_2$	$c_3$	$c_1$	$c_2$	$c_3$
0	24.30	$9.00 \times 10^2$	$5.54 \times 10^{-3}$	24.30	$9.00 \times 10^3$	$5.54 \times 10^{-4}$	24.30	$9.00 \times 10^4$	$5.54 \times 10^{-5}$
0.5	44.46	$9.00 \times 10^2$	$5.54 \times 10^{-3}$	44.46	$9.00 \times 10^3$	$5.54 \times 10^{-4}$	44.46	$9.00 \times 10^4$	$5.54 \times 10^{-5}$
1	79.66	$9.00 \times 10^2$	$5.54 \times 10^{-3}$	79.66	$9.00 \times 10^3$	$5.54 \times 10^{-4}$	79.66	$9.00 \times 10^4$	$5.54 \times 10^{-5}$

**Table 3** Grain sedimentation time

$n$	$r_0 = 10^{-1} \text{ cm}$		$r_0 = 10^{-2} \text{ cm}$		$r_0 = 10^{-3} \text{ cm}$	
	time $t$ (in years)		time $t$ (in years)		time $t$ (in years)	
0	$8.58 \times 10^1$		$8.90 \times 10^2$		$3.22 \times 10^4$	
0.5	$1.42 \times 10^2$		$1.52 \times 10^3$		$3.24 \times 10^4$	
1	$3.24 \times 10^2$		$3.91 \times 10^3$		$5.02 \times 10^4$	

grains falling under gravity inside the protoplanet grow by cold welding. Considering three different initial grain sizes  $r_0 = 10^{-1}$  cm,  $r_0 = 10^{-2}$  cm and  $r_0 = 10^{-3}$  cm, we have determined the falling time of a grain for each of  $n = 0, 0.5$ , and  $1$ . It should be noted that  $n$  measures the degree of central condensation, and as such,  $n$  can not be large for young protoplanet. The considered values of  $n$  are thus reasonable. It is found that the sedimentation times do not differ much for differing  $n$  and  $r_0$ . In all cases time is well within the initial contraction time  $\sim 10^5$  years of a protoplanet (e.g. Boss 1998; Helled et al. 2008; Donnison and Williams 1974). The calculated time is also found to be in general agreement with Boss (1998) and Helled et al. (2008). The best agreement with Boss is obtained for  $r_0 = 10^{-2}$ ,  $n = 1$  and  $n = 0.5$  giving a sedimentation time  $\sim 10^3$  years the same as that estimated by Boss (1998) while the best agreement with Helled et al. (2008) comes from  $r_0 = 10^{-3}$  cm for all  $n$ , the segregation time being  $\sim 10^4$  years. Besides  $r_0$  and  $n$ , the segregation time is also dependent on the parameters  $\lambda$ ,  $\mu$  and  $\eta$ . Although we have used particular values for these parameters the effect of their possible variation on segregation time has also been tested. The results are always found to be in general agreement with that given in Table 3. The polytropic investigation of gravitational settling of grains in protoplanets thus demonstrates that segregation of heavy elements in protoplanets occur in a reasonably short time scale lending support to different numerical codes. It is suggestive that the gaseous planets can form from giant gaseous protoplanets possibly by contraction and sedimentation of solid grains. For terrestrial planets, gaseous envelopes of the protoplanets must be removed leaving the solid cores behind where the core formation time must have to be less than time scale required for removing the gas. Two mechanisms for removing the gas may be the solar wind and solar tidal effect. While an early enhanced solar wind can remove the gas from all protoplanets out to the asteroidal belt presumably in a long time (Williams and Handbury 1974) the tidal dispersal time for the envelopes, as obtained by Donnison and Williams (1975) is very short, so short that the protoplanets would be disrupted long before any solid core can form inside. In fact, due to the strong tidal effect of the Sun no protoplanet having the assumed density can at all be formed in the terrestrial region. If the protoplanets migrate inward from outside the Roche limit with heavy cores inside, then the tidal disruptions of the gaseous envelopes may lead to the formation of terrestrial type planets. In that case the segregation time scale must have to be shorter than the migration time scale. The protoplanets then would have enough time to form cores before they are disrupted by the tidal force of the Sun. It may be mentioned that the time scale for planetary migration is fairly long (e.g. Armitage 2007) in comparison with our estimated time scale for segregation. We thus conclude that the segregation time scale obtained by the polytropic method is quite realistic and also in agreement with other recently published results of different numerical codes.

In our calculation the effect of radiative heating and the contraction of the gas near the centre due to the increased pressure have not been considered. Moreover, we have used Stocks' drag as the resistive force. However, initially for some grains, Epstein drag may be applicable. It is, therefore, instructive to calculate the sedimentation time using both Epstein drag and Stocks' drag with a proper boundary and including the effects of radiative heating and contraction of the central part, as suggested by Helled et al. (2008) and see how the result compares with our estimated time.

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