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On the entropy inequality and its exploitation in continuum physics

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Abstract The paper deals with the statement and the application of the entropy principle, through the Clausius–Duhem inequality, in continuum physics. The conceptual role is taken from the Coleman–Noll paper of 1963 thus leading to the physical admissibility of constitutive equations. The statement is generalized by letting the rate of entropy production be a constitutive property per se. This generalization proves essential in connection with the modelling of hysteretic phenomena. As to the application, the view of the Coleman–Noll procedure is maintained but a representation formula is shown to generalize the consequences of the entropy principle; as an example the modelling of heat conduction is investigated. Furthermore, while applying the entropy principle to magnetic materials, it is shown an interesting connection between the balance of angular momentum and the thermodynamic restrictions. Also, the modelling through rate-type equations shows the need of Lagrangian fields to obey objectivity and that of the entropy production as a constitutive function to account for the difference between loading and unloading processes.

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1 Introduction

Continuum physics approaches are based on two essential tools namely the balance equations and the constitutive functions. Once these tools are settled, practical methods can be applied for solving mathematical problems involving mainly differential equations, boundary-value formulations, and integral equations. To fix ideas here we restrict attention to single-phase continua though the basic concepts apply also to mixtures and continua with internal structure.

The balance equations are established through axioms that are commonly based on global properties. As to the balance of linear momentum, angular momentum, and energy the pertinent equation is assumed as the analogue of the balance for systems of particles (see, e.g., [1], ch. I). Classical thermodynamics provides the analogous reference for the first and second law of continuous bodies; the balance of energy is based on both thermodynamic and mechanical analogues.

In local form, the second law leads to the entropy or Clausius–Duhem (CD) inequality. As we show in the next section, though we are rightly accustomed to the term inequality, the second law too is expressed by an equality.

While the balance of linear momentum and energy involve pertinent external supplies, upon accounting for the balance of energy the second law resulted (and results) in a supply-free inequality, for appropriate constitutive functions. The question about the conceptual role of the CD inequality was answered by Coleman and Noll in 1963 [2] by stating that the CD inequality has to be valid for any set of functions which are compatible with the balance equations. Hence if the balance equations are valid, the CD inequality is a test for the admissibility of the constitutive functions. Though with different methods, the paper by Coleman and Noll initiated much research into the restrictions imposed by thermodynamics on materials models.

The purpose of this paper is threefold. First, to revisit the formulations of the second law (or entropy principle) subsequent to the original one of Coleman and Noll. Secondly, to exhibit the methods of exploitation of the second law, through the CD inequality. Thirdly, to point out recent methods of exploitation and to apply them to the modelling of dissipative processes via rate equations. In particular, attention is addressed to hysteretic models and rate-type equations for heat conductors.

Notation. We let Ω be the time-dependent region occupied by the continuous body in the three-dimensional space. The position vector of a point in Ω is denoted by $\mathbf{x}$. Hence $\rho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ are the mass density and the velocity fields at $\mathbf{x}$, at time $t \in \mathbb{R}$. The symbol ∇ denotes the gradient, with respect to $\mathbf{x}$, while $\nabla\cdot$ is the divergence operator. For any pair of vectors $\mathbf{u}, \mathbf{w}$, or tensors $\mathbf{A}, \mathbf{B}$, the notations $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{A} \cdot \mathbf{B}$ denote the inner product. Cartesian coordinates are used and then, in the suffix notation, $\mathbf{u} \cdot \mathbf{w} = u_i w_i$, $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$, the summation over repeated indices being understood. Also, sym$\mathbf{A}$ and skw$\mathbf{A}$ denote the symmetric and skew-symmetric parts of $\mathbf{A}$ while Sym is the space of symmetric tensors. A superposed dot denotes the total time derivative and hence, for any function $f(\mathbf{x}, t)$ on $\Omega \times \mathbb{R}$ we have $\dot{f} = \partial_t f + (\mathbf{v} \cdot \nabla) f$. The symbol $\mathbf{L}$ denotes the velocity gradient, $L_{ij} = \partial_{x_j} v_i$, while $\mathbf{D} = \mathrm{sym}\mathbf{L}$ and $\mathbf{W} = \mathrm{skw}\mathbf{L}$. Further, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ is the specific body force, and $\otimes$ denotes the dyadic product.

Let R be the region occupied by the body in a reference configuration. Any point in R is associated with the position vector $\mathbf{X}$ relative to a chosen origin. The motion of the body is a C^2 function $\chi(\mathbf{X}, t) : \mathrm{R} \times \mathbb{R} \to \Omega = \chi(\mathrm{R}, t)$. The gradient, with respect to $\mathbf{X}$, of χ is the deformation gradient $\mathbf{F}$, $F_{iK} = \partial_{X_K} \chi_i$.

2 Balance equations

Let $\mathrm{P} \subset \mathrm{R}$ be any sub-region of R and denote by $\mathcal{P}_t$ the image through the motion in that $\mathcal{P}_t = \chi(\mathrm{P}, t)$; we can say that $\mathcal{P}_t$ convects with the body.

Let $\Phi(\mathbf{x}, t)$ be a density, per unit volume. If Φ is differentiable then

$$\frac{d}{dt} \int_{\mathcal{P}_t} \Phi \, dv = \int_{\mathcal{P}_t} [\partial_t \Phi + \nabla \cdot (\Phi \mathbf{v})] dv. \tag{1}$$

Let ρ be the mass density and hence

$$m(\mathcal{P}_t) = \int_{\mathcal{P}_t} \rho dv$$

is the mass of the convecting region $\mathcal{P}_t$. The conservation of mass is expressed by

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho dv = 0.$$

Applying Eq. (1) and using the continuity of the integrand and the arbitrariness of $\mathcal{P}_t$ we find that

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{2}$$

Consequently, for any specific density φ we have

$$\frac{d}{dt} \int_{\mathcal{P}_t} \rho \varphi dv = \int_{\mathcal{P}_t} \rho \dot{\varphi} dv, \tag{3}$$

this result being referred to as Reynolds' transport theorem.

The global form of a balance law is stated by assuming that the rate of a volume integral consists of volume and surface integrals. If $\rho\beta$ is the supply, per unit volume, and λ the supply per unit area then the balance law is given the form

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\varphi\, dv = \int_{\mathcal{P}_t} \rho\beta\, dv + \int_{\partial\mathcal{P}_t} \lambda\, da. \tag{4}$$

If $\varphi = \mathbf{v}$ this is the balance of linear momentum and $\beta = \mathbf{b}$ is the body force while $\lambda = \mathbf{t}$ is the tension. Cauchy's theorem implies the existence of the stress tensor so that $\mathbf{t}(\mathbf{x}, \mathbf{n}, t) = \mathbf{T}(\mathbf{x}, t)\mathbf{n}$. Hence we find the local form

$$\rho\dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho\mathbf{b}. \tag{5}$$

By the balance of angular momentum it follows that $\mathbf{T} = \mathbf{T}^T$; for materials with couple stress tensors, the Cauchy stress $\mathbf{T}$ is no longer symmetric.

A purely mechanical balance of energy is not consistent except for particular cases (e.g. incompressible fluids). A general form is then considered with an energy density ε, a supply ρr and a surface power density h,

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho(\tfrac{1}{2}\mathbf{v}^2 + \varepsilon)dv = \int_{\mathcal{P}_t} \rho(\mathbf{b}\cdot\mathbf{v} + r)dv + \int_{\partial\mathcal{P}_t} (\mathbf{t}\cdot\mathbf{v} + h)da.$$

A Cauchy-like proof shows that there is a vector field, say $-\mathbf{q}$, such that

$$h = -\mathbf{q}\cdot\mathbf{n}.$$

Using Eqs. (3) and (5) we then find the local form

$$\rho\dot{\varepsilon} = \mathbf{T}\cdot\mathbf{D} - \nabla\cdot\mathbf{q} + \rho r; \tag{6}$$

the consistency with the classical first law of thermodynamics allows us to view ε as the specific internal energy, r the specific heat supply, and $\mathbf{q}$ the heat flux vector.

Let η be the specific entropy density. The balance of entropy too can be assumed in the form (4) with

$$\varphi = \eta, \quad \beta = \beta_\eta, \quad \lambda = \lambda_\eta.$$

Again a Cauchy-like theorem shows that there is a vector field, say $\mathbf{j}$, such that

$$\lambda_\eta = -\mathbf{j}\cdot\mathbf{n},$$

$\mathbf{j}$ being viewed as the entropy flux. We then obtain the balance of entropy in the local form as

$$\rho\dot{\eta} = \rho\beta_\eta - \nabla\cdot\mathbf{j}.$$

Simple models in classical thermodynamics (see, e.g., [3], ch. 9; [4], ch. 6) indicate that the rate of entropy is related to the rate of heat divided by the absolute temperature θ. Further, since entropy is not conserved then the balance of entropy is subject to the general requirement (e.g. [4], §6.5)

entropy change = entropy transfer + entropy production.

This suggests that

$$\beta_\eta = \frac{r}{\theta} + \gamma, \tag{7}$$

where γ is the specific (rate of) production of entropy. Hence the local form of the balance of entropy can be written as

$$\rho\dot{\eta} = \frac{\rho r}{\theta} - \nabla\cdot\mathbf{j} + \rho\gamma. \tag{8}$$

The assumption

$$\gamma \geq 0$$

then results in the relation

$$\rho\dot{\eta} - \frac{\rho r}{\theta} + \nabla\cdot\mathbf{j} \geq 0; \tag{9}$$

let us call it the *entropy inequality*. The definition (7) denotes that both r/θ and γ happen as volume densities; r/θ arises from an external source [12], while γ is an internal production.

3 Entropy principle and entropy inequality

The non-negative character of the entropy production has been taken by Coleman and Noll as a criterion for the selection of admissible constitutive equations. To make this point precise let a thermodynamic process be the set of fields entering the model of the continuum (through balance and constitutive equations). Then we state the following

Entropy principle *A thermodynamic process is admissible if the inequality*

$$\gamma \geq 0$$

is valid.

It is natural to ask about the presumed argument, by Coleman and Noll, underlying the statement of the entropy principle. Also, we need operative methods for the exploitation of the entropy principle as a procedure for the selection of admissible constitutive equations. Before answering these questions we point out that different expressions have been established of the entropy production while preserving the entropy principle.

Consistent with the view that the rate of entropy is related to the rate of heat divided by the absolute temperature, Coleman and Noll [2] and Truesdell and Toupin ([5], §257), assume $\mathbf{j} = \mathbf{q}/\theta$. Hence the entropy inequality (9) becomes

$$\rho\dot{\eta} - \frac{\rho r}{\theta} + \nabla \cdot \frac{\mathbf{q}}{\theta} \geq 0. \tag{10}$$

The requirement (10) is given in [5], §258, and is regarded as a *postulate of irreversibility*. Accordingly, sometimes (e.g. [6]) equation (9) is called the Clausius–Duhem–Truesdell–Toupin inequality; the name Clausius–Duhem inequality would be more appropriate for the special case $r = 0$. Really the name Clausius–Duhem inequality was given to (10) by Truesdell himself [7], Eq. (28.5), for the case $r = 0$. Following a wide literature, we keep denoting (9), and the particular case (10), as the entropy or Clausius–Duhem inequality.

In 1967 Müller [8] generalized the entropy inequality by letting $\mathbf{j}$ as given by an unknown constitutive function and hence avoiding the assumption $\mathbf{j} = \mathbf{q}/\theta$. In this connection it is standard to let

$$\mathbf{j} = \frac{\mathbf{q}}{\theta} + \mathbf{k},$$

$\mathbf{k}$ being viewed as the extra-entropy flux. Accordingly the expression of γ is given by

$$\rho\theta\gamma = -\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T} \cdot \mathbf{D} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta + \theta\nabla \cdot \mathbf{k}. \tag{11}$$

Some years later, Green and Naghdi [9] while keeping $\mathbf{j} = \mathbf{q}/\theta$ as in the original CD inequality, observed that Eq. (8) is an equation

$$\rho\dot{\eta} = \rho(\frac{r}{\theta} + \gamma) - \nabla \cdot \frac{\mathbf{q}}{\theta}, \tag{12}$$

and furthermore that the entropy production γ, as well as $\psi, \eta, \mathbf{T}, \mathbf{q}$, is specified by a constitutive equation (while $\mathbf{k} = \mathbf{0}$ in Ref. [9]). If $\mathbf{j} = \mathbf{q}/\theta + \mathbf{k}$ then (12) is replaced by

$$\rho\dot{\eta} = \rho(\frac{r}{\theta} + \gamma) - \nabla \cdot \frac{\mathbf{q}}{\theta} - \nabla \cdot \mathbf{k};$$

a priori, the sign of $\nabla \cdot \mathbf{k}$ is undetermined whereas γ is required to be non-negative. If $r = 0$ then Eq. (12) can be written in the form

$$\rho\dot{\eta} = \rho\gamma - \nabla \cdot (\frac{\mathbf{q}}{\theta} + \mathbf{k}),$$

still with $\gamma \geq 0$ and sign of $\nabla \cdot \mathbf{k}$ a priori undetermined. Letting both $\mathbf{j}$ and γ be given by constitutive equations is a generalization of the entropy principle by Coleman and Noll [2] and, as shown by a number of examples, allows for a consistent framework of behaviours in matter.

To give a possible motivation underlying the statement of the entropy principle we observe that the balance of linear momentum and energy express the equilibrium between the pertinent fields and the supplies $(\mathbf{b}, r)$. The balance of mass instead is a constraint on the fields ρ and $\mathbf{v}$. Rather, once we substitute for r from (6) into (9), the CD inequality (11), possibly with $\mathbf{k} = \mathbf{0}$, involves constitutive equations but not supplies. It is then a constraint on the constitutive functions subject to the condition $\gamma \geq 0$. If $r = 0$ then (12), with $\gamma \geq 0$, is a constraint.

3.1 Remarks on the rate of entropy production

It is worth reporting the objection, by an anonymous reviewer, to viewing the (rate of) entropy production as independent of other conventional constitutive equations. The entropy production γ is just an abbreviation for the left-hand-side (LHS) of e.g. Eq. (9). Indeed, in examining some of the standard books in continuum mechanics this is made explicitly clear. For example, in Ch. 5 of [10], γ is not even defined; Liu just uses directly the inequality displayed in Eq. (9). Or one can examine § 4.6 of [11] or §§ 4.3 and 4.11 of [12] where it is explicitly noted that γ *is defined by* the LHS of Eq. (9).

Answering the objection, we say that for many models the entropy production γ is in fact given by the other constitutive functions. For instance, in Fourier-like models of heat conduction the CD inequality reduces to

$$-\mathbf{q} \cdot \nabla\theta = \rho\theta^2\gamma$$

and hence $\gamma = -(\mathbf{q} \cdot \nabla\theta)/\rho\theta^2$ is determined by the function $\mathbf{q}$, provided $\gamma \geq 0$. Instead there are cases where γ is independent of the other constitutive functions. In Sect. 7.2 we show that this happens for the modelling of magnetic hysteresis where, e.g., $\gamma = \zeta|\dot{H}|$ is non-negative and independent of the other functions, see Eqs. (35) and (36). Further examples are given, e.g., in [13, 14].

4 Exploitation methods of the entropy inequality

In light of the entropy principle, and the relation (9) for $\rho\theta\gamma$, the CD inequality can be written in the form

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T} \cdot \mathbf{D} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta + \theta\nabla \cdot \mathbf{k} = \rho\theta\gamma \geq 0. \tag{13}$$

The entropy inequality and the entropy principle hold for any single-phase, non-polar continuum; things change according to the nature of the continuum as to the balance equations and hence for the analogue of (11). To fix ideas we let (13) be the form of the CD inequality; later on we will consider deformable magnetic continua.

For technical reasons it is often convenient to work within the Lagrangian description. In this connection we multiply Eq. (13) by $J = \det \mathbf{F} > 0$ and write

$$-J\rho(\dot{\psi} + \eta\dot{\theta}) + J\mathbf{T} \cdot \mathbf{D} - \frac{1}{\theta}J\mathbf{q} \cdot \nabla\theta + \theta\nabla \cdot \mathbf{k} = J\rho\theta\gamma \geq 0. \tag{14}$$

Let

$$\mathbf{T}_{RR} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}, \qquad \mathbf{q}_R = J\mathbf{F}^{-1}\mathbf{q}, \qquad \mathbf{k}_R = J\mathbf{F}^{-1}\mathbf{k}$$

denote the second Piola stress, the referential heat flux and the referential extra-entropy flux. Notice that, for any function $f(\mathbf{x}, t)$,

$$\nabla_R f := \partial_{\mathbf{X}} f = \nabla f \mathbf{F}$$

while [15, 16]

$$J\mathbf{q} \cdot \nabla\theta = \mathbf{q}_R \cdot \nabla_R\theta, \qquad J\nabla \cdot \mathbf{k} = \nabla_R \cdot \mathbf{k}_R.$$

Further,

$$J\mathbf{T} \cdot \mathbf{D} = \mathbf{T}_{RR} \cdot \dot{\mathbf{E}},$$

where $\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{1})/2$. By (14), since $J\rho = \rho_R$ is the referential density then we can write the referential entropy inequality in the form

$$-(\dot{\psi}_R + \eta_R\dot{\theta}) + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\theta + \theta\nabla_R \cdot \mathbf{k}_R = \rho_R\theta\gamma \geq 0, \tag{15}$$

where $\psi_R = \rho_R\psi, \eta_R = \rho_R\eta$.

Granted the balance equations, the entropy principle, through the CD inequality, becomes a criterion of admissibility of the assumed constitutive functions. To derive the restrictions on the constitutive equations two procedures are used.

4.1 The Coleman–Noll procedure

To exploit the requirement of the entropy principle it is essential to envisage the freedom allowed by the balance equations for the admissible thermodynamic processes. In dynamic problems $\mathbf{b}$ and r are usually regarded as assigned a priori. Here we take the view that conceptually the functions $\mathbf{b}(\mathbf{x}, t)$ and $r(\mathbf{x}, t)$ can be selected arbitrarily. Hence for any value of the right-hand sides of

$$\mathbf{b} = \dot{\mathbf{v}} - \frac{1}{\rho}\nabla \cdot \mathbf{T}, \qquad r = \dot{\varepsilon} - \frac{1}{\rho}\mathbf{T} \cdot \mathbf{D} + \frac{1}{\rho\theta}\nabla \cdot \mathbf{q} \tag{16}$$

the balance equations hold because $\mathbf{b}$ and r are just taken to be given by (16). Analogous claims hold for the space and time derivatives. Hence Eq. (16) provide the body force $\mathbf{b}$ and the heat supply r, and possibly their derivatives, that must be applied to support the process. Based on this view, we find that, given any spatial point $\mathbf{x}$ and any time t, it is possible to find a process such that $\dot{\theta}, \ddot{\theta}, \nabla\dot{\theta}$ and $\mathbf{L}, \dot{\mathbf{L}}, \ddot{\mathbf{L}}, \nabla\mathbf{L}$ or $\dot{\mathbf{F}}, \ddot{\mathbf{F}}$ have arbitrarily prescribed values at that point and time (see, e.g., [16], §3.4).

Let Σ be a set of variables and assume the entropy inequality takes the form

$$f(\Sigma)\alpha + p(\Sigma, \xi) \geq 0. \tag{17}$$

If α can take arbitrary (positive or negative) values, independent of Σ and ξ, then (17) implies $f = 0$ and $p \geq 0$. Likewise, if vectors are involved then

$$\mathbf{f} \cdot \boldsymbol{\alpha} + p \geq 0$$

implies $\mathbf{f} = \mathbf{0}$. If tensors are concerned in the form

$$\mathcal{F} \cdot \mathbf{A} + p \geq 0$$

than the arbitrariness of $\mathbf{A}$ implies $\mathcal{F} = \mathbf{0}$. If $\mathbf{A} \in \mathrm{Sym}$ (or Skw) then $\mathcal{F} \in \mathrm{Skw}$ (or Sym).

If (17) is changed to

$$f(\Sigma)\alpha + p(\Sigma,\alpha) \geq 0, \qquad p(\Sigma,\alpha) = o(|\alpha|),$$

then dividing by $|\alpha|$ we have

$$\text{sgn}\,(f(\Sigma)\frac{\alpha}{|\alpha|} + \frac{p(\Sigma,\alpha)}{|\alpha|}) = \text{sgn} f(\Sigma)\frac{\alpha}{|\alpha|} = \text{sgn} f(\Sigma)\alpha.$$

Hence it follows that $f(\Sigma) = 0$. The analogue holds for $\mathbf{f}$ and $\mathcal{F}$.

4.2 The Liu procedure

In a different procedure established by Liu [17] the entropy inequality is exploited for supply-free bodies. Hence all of the balance equations are constraints and the CD (or entropy) inequality is written in the (Eulerian) form

$$\begin{aligned}
&\partial_t(\rho\eta) + \nabla\cdot(\rho\eta\mathbf{v}) + \mathbf{j}) - \Lambda_\rho[\partial_t\rho + \nabla\cdot(\rho\mathbf{v})] \\
&\quad - \boldsymbol{\Lambda}_\mathbf{v}\cdot[\partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \mathbf{T})] \\
&\quad - \Lambda_\varepsilon[\partial_t(\rho(\varepsilon + \tfrac{1}{2}\mathbf{v}^2)) \\
&\quad + \nabla\cdot(\rho(\varepsilon + \tfrac{1}{2}\mathbf{v}^2)\mathbf{v} - \mathbf{v}\mathbf{T} + \mathbf{q}))] \geq 0,
\end{aligned}$$

$\Lambda_\rho, \boldsymbol{\Lambda}_\mathbf{v}, \Lambda_\varepsilon$ being the Lagrange multipliers, viewed as additional constitutive functions of the chosen independent functions. The linearity and arbitrariness of $\partial_t\rho, \partial_t\theta, \partial_t\mathbf{v}$ imply that

$$\begin{aligned}
&\partial_\rho(\rho\eta) - \Lambda_\rho - \Lambda_\varepsilon[\tfrac{1}{2}\mathbf{v}^2 + \partial_\rho(\rho\varepsilon)] = 0, \\
&\partial_\theta\eta - \Lambda_\varepsilon\partial_\theta\varepsilon = 0, \quad \Lambda_\mathbf{v} + \Lambda_\varepsilon\mathbf{v} = \mathbf{0},
\end{aligned}$$

whence we obtain formally the Lagrange multipliers $\Lambda_\rho, \Lambda_\varepsilon, \boldsymbol{\Lambda}_\mathbf{v}$. Further restrictions are derived via the reduced inequality and the appropriate constitutive assumptions.

The conceptual difference of the two methods is given by the use of the supplies $\mathbf{b}, r$. In the Coleman–Noll procedure we make use of the conceivable arbitrariness of the functions $\mathbf{b}(\mathbf{x},t), r(\mathbf{x},t)$ whereas in Liu's procedure $\mathbf{b} = \mathbf{0}, r = 0$. Henceforth we apply the Coleman–Noll procedure and show how further methods are applicable to investigate the compatibility with the entropy principle.

5 Exploitation through a representation formula

From the mathematical side, the entropy principle requires that we determine the constitutive functions that satisfy the balance equations and inequality (13). About the inequality, and still into line with Coleman–Noll's procedure, we need an improvement when the entropy inequality has the form

$$\mathcal{F}(\Sigma,\mathbf{A})\cdot\dot{\mathbf{A}} + p(\Sigma,\mathbf{A}) = \gamma(\Sigma,\mathbf{A})$$

for the unknown constitutive tensor (or vector) function $\dot{\mathbf{A}}$. We then look for the solution $\dot{\mathbf{A}}$ to

$$\mathcal{F}(\Sigma,\mathbf{A})\cdot\dot{\mathbf{A}} = g(\Sigma,\mathbf{A}), \qquad g = \gamma - p.$$

5.1 Representation formula

Let $\mathbf{N}$ be a tensor, of any order n, subject to $\mathbf{N}\cdot\mathbf{N} = 1$. Any tensor $\mathbf{Z}$, of the same order, can be decomposed in the form

$$\mathbf{Z} = (\mathbf{Z}\cdot\mathbf{N})\mathbf{N} + \mathbf{Z}_\perp, \qquad \mathbf{Z}_\perp\cdot\mathbf{N} = 0.$$

Assume $\mathbf{Z}\cdot\mathbf{N}$ is known, say $\mathbf{Z}\cdot\mathbf{N} = g$. If $\mathbf{Z}_\perp$ is unknown we can represent it in the form

$$\mathbf{Z}_\perp = (\mathbf{I} - \mathbf{N}\otimes\mathbf{N})\mathbf{G} = \mathbf{G} - (\mathbf{G}\cdot\mathbf{N})\mathbf{N}$$

where $\mathbf{G}$ is an arbitrary tensor of order n and $\mathbf{I}$ is the identity $2n$-order tensor.

Let $\mathbf{Z}$ be a vector subject to

$$\mathbf{Z}\cdot\mathcal{F} = g$$

and select

$$\mathbf{N} = \frac{\mathcal{F}}{|\mathcal{F}|}.$$

Consequently

$$\mathbf{Z}\cdot\mathbf{N} = \frac{g}{|\mathcal{F}|}$$

and then

$$\mathbf{Z} = \frac{g}{|\mathcal{F}|^2}\mathcal{F} + (\mathbf{I} - \frac{\mathcal{F}\otimes\mathcal{F}}{|\mathcal{F}|^2})\mathbf{G}. \tag{18}$$

If Γ is the set of variables then g and $\mathbf{G}$ are functions of Γ.

5.2 Application to the modelling of heat conduction

We consider a thermoelastic body and follow a Lagrangian description. We have in mind that heat conduction is described by a rate equation and hence we let $\theta, \mathbf{E}, \mathbf{q}_R, \nabla_R\theta$ be the set of variables for the constitutive functions

$\psi_R, \eta_R, \gamma, \dot{\mathbf{q}}_R$.

Computation of $\dot{\psi}_R$ and substitution in Eq. (15) result in

$$- (\partial_\theta\psi_R + \eta)\dot{\theta} + (\mathbf{T}_{RR} - \partial_{\mathbf{E}}\psi_R) \cdot \dot{\mathbf{E}} - \partial_{\mathbf{q}_R}\psi_R \cdot \dot{\mathbf{q}}_R$$
$$- \partial_{\nabla_R\theta}\psi_R \cdot \nabla_R\dot{\theta} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\theta + \theta\nabla_R \cdot \mathbf{k}_R = \rho_R\theta\gamma.$$

By applying repeatidly the argument associated with (17) we conclude that the factors of $\dot{\theta}, \dot{\mathbf{E}}, \nabla_R\dot{\theta}$ should vanish, which implies that

$$\eta_R = -\partial_\theta\psi_R, \qquad \mathbf{T}_{RR} = \partial_{\mathbf{E}}\psi_R, \qquad \partial_{\nabla_R\theta}\psi_R = \mathbf{0}.$$

Likewise we find that $\mathbf{k}_R$ is independent of $\mathbf{E}, \mathbf{q}_R, \nabla_R\theta$ and hence

$$\nabla_R \cdot \mathbf{k}_R = \partial_\theta\mathbf{k}_R \cdot \nabla_R\theta.$$

This term might be associated with $\mathbf{q}_R \cdot \nabla_R\theta$; yet the isotropy of the body implies the vanishing of a vector $\mathbf{k}_R$ produced by the scalar θ. Thus we let $\mathbf{k}_R = \mathbf{0}$. Hence we are left with

$$-\partial_{\mathbf{q}_R}\psi_R \cdot \dot{\mathbf{q}}_R - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\theta = \rho_R\theta\gamma \geq 0. \tag{19}$$

Equation (19), along with the balance equations (2), (5), and (6), gives a system of differential equations for the unknowns $\theta, \chi, \mathbf{q}_R$ and $\rho = \rho_R/J, J = [\det(2\mathbf{E} + \mathbf{1})]^{1/2}$, with ρ_R a given constant. The form of (19) depends on the functions $\psi_R(\theta, \mathbf{E}, \mathbf{q}_R)$ and $\gamma(\theta, \mathbf{E}, \mathbf{q}_R, \nabla_R\theta)$. Using the representation formula (18) we now derive explicit forms of $\dot{\mathbf{q}}_R$.

Assume $\partial_{\mathbf{q}_R}\psi_R \neq \mathbf{0}$ and let $\mathbf{N} = \partial_{\mathbf{q}_R}\psi_R/|\partial_{\mathbf{q}_R}\psi_R|$. Using the representation formula (18), applied to vectors, and Eq. (19) we find

$$\dot{\mathbf{q}}_R = -\frac{\mathbf{q}_R \cdot \nabla_R\theta + \rho_R\theta^2\gamma}{\theta|\partial_{\mathbf{q}_R}\psi_R|^2}\partial_{\mathbf{q}_R}\psi_R + \Big(\mathbf{1} - \frac{\partial_{\mathbf{q}_R}\psi_R}{|\partial_{\mathbf{q}_R}\psi_R|} \otimes \frac{\partial_{\mathbf{q}_R}\psi_R}{|\partial_{\mathbf{q}_R}\psi_R|}\Big)\mathbf{g}, \tag{20}$$

$\mathbf{g}$ being any vector function of $\theta, \mathbf{E}, \mathbf{q}_R, \nabla_R\theta$. For definiteness, if $\mathbf{g} = \mathbf{J}_R\nabla_R\theta$, where $\mathbf{J}_R$ is a second-order tensor possibly dependent on $\theta, \mathbf{E}, \mathbf{q}_R$, then we have

$$\dot{\mathbf{q}}_R = -\mathbf{K}_R(\theta, \mathbf{q}_R)\nabla_R\theta - \rho_R\theta\gamma\frac{\partial_{\mathbf{q}_R}\psi_R}{|\partial_{\mathbf{q}_R}\psi_R|},$$

where

$$\mathbf{K}_R = -\frac{\partial_{\mathbf{q}_R}\psi_R \otimes \mathbf{q}_R}{\theta|\partial_{\mathbf{q}_R}\psi_R|^2} + \mathbf{J}_R - \frac{\partial_{\mathbf{q}_R}\psi_R \otimes (\partial_{\mathbf{q}_R}\psi_R\mathbf{J}_R)}{|\partial_{\mathbf{q}_R}\psi_R|^2}.$$

We now go back to Eq. (20) and consider the particular case $\mathbf{g} = \mathbf{0}$. Hence we have

$$\dot{\mathbf{q}}_R = -\frac{\mathbf{q}_R \cdot \nabla_R\theta + \rho_R\theta^2\gamma}{\theta|\partial_{\mathbf{q}_R}\psi_R|^2}\partial_{\mathbf{q}_R}\psi_R.$$

Since $\gamma > 0$ it follows that, in stationary conditions,

$$\mathbf{q}_R \cdot \nabla_R\theta = -\rho_R\theta^2\gamma \leq 0;$$

this is a necessary condition, subject only to $\partial_{\mathbf{q}_R}\psi_R \neq \mathbf{0}$, and provides the classical result of heat conduction inequality.

An interesting particular case of (20) follows by letting $\partial_{\mathbf{q}_R}\psi_R = \alpha\mathbf{q}_R$ so that

$$\dot{\mathbf{q}}_R = -\frac{\mathbf{q}_R \cdot \nabla_R\theta}{\theta\alpha\mathbf{q}_R^2}\mathbf{q}_R - \frac{\rho_R\theta\gamma}{\alpha\mathbf{q}_R^2}\mathbf{q}_R - \frac{\mathbf{q}_R \cdot \mathbf{g}}{\mathbf{q}_R^2}\mathbf{q}_R + \mathbf{g}.$$

Hence selecting $\mathbf{g} = -\nabla_R\theta/\theta\alpha$ we have

$$\dot{\mathbf{q}}_R = -\frac{\rho_R\theta\gamma}{\alpha\mathbf{q}_R^2}\mathbf{q}_R - \frac{1}{\alpha\theta}\nabla_R\theta. \tag{21}$$

Equation (21) has the standard form of the Maxwell-Cattaneo equation (see, e.g., [18, 19]) with relaxation time τ and conductivity κ given by

$$\tau = \frac{\alpha\mathbf{q}_R^2}{\rho_R\theta\gamma}, \qquad \kappa = \frac{\mathbf{q}_R^2}{\rho_R\theta^2\gamma}.$$

If, rather, the conductivity κ, e.g. as a function of θ, is given then we determine the entropy production,

$$\gamma = \frac{\mathbf{q}_R^2}{\rho_R\theta^2\kappa},$$

as a function of the variables $\mathbf{q}_R$ and θ. The positive valuedness of the entropy production γ implies the positive valuedness of the heat conductivity κ.

6 Entropy inequality and selection of variables

It is customary to assume that a thermodynamic process satisfies the balance equations and then constitutive restrictions are determined so that the entropy inequality holds. There are cases though where the entropy inequality can be employed to select appropriate independent variables. This feature is exemplified through the modelling of electromagnetic continua.

For simplicity consider a deformable magnetic body. Let $\mathbf{m}$ and $\mathbf{M}$ be the magnetization per unit mass and unit volume. The balance of angular momentum leads to ([16], §2.16.1)

$$\mathbf{T} + \mu_0 \mathbf{H} \otimes \mathbf{M} \in \mathrm{Sym}. \tag{22}$$

Owing to the balance of energy,

$$\rho \dot{\varepsilon} = \mu_0 \rho \mathbf{H} \cdot \dot{\mathbf{m}} + \mathbf{T} \cdot \mathbf{L} + \rho r - \nabla \cdot \mathbf{q}, \tag{23}$$

we obtain the entropy inequality in the form

$$-\rho(\dot{\psi} + \eta \dot{\theta}) + \mathbf{T} \cdot \mathbf{L} + \mu_0 \rho \mathbf{H} \cdot \dot{\mathbf{m}} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta = \rho \theta \gamma.$$

Since

$$(-\dot{\psi} + \mu_0 \mathbf{H} \cdot \dot{\mathbf{m}}) = (-\psi + \mu_0 \mathbf{H} \cdot \mathbf{m})^{\cdot} - \mu_0 \mathbf{m} \cdot \dot{\mathbf{H}}$$

we let

$$\phi = \psi - \mu_0 \mathbf{H} \cdot \mathbf{m}$$

and write the entropy inequality in the form

$$-\rho(\dot{\phi} + \eta \dot{\theta}) - \mu_0 \mathbf{M} \cdot \dot{\mathbf{H}} + \mathbf{T} \cdot \mathbf{L} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta = \rho \theta \gamma. \tag{24}$$

If $\mathbf{F}, \mathbf{H}$ and e.g. $\theta, \nabla\theta$ are the variables then we have

$$\dot{\phi} = \partial_{\mathbf{F}} \phi \cdot \dot{\mathbf{F}} + \cdots = \partial_{\mathbf{F}} \phi \cdot (\mathbf{L}\mathbf{F}) + \cdots = (\partial_{\mathbf{F}} \phi \mathbf{F}^T) \cdot \mathbf{L} + \cdots$$

the dots denoting the terms in $\dot{\theta}, \dot{\mathbf{H}}, (\nabla\theta)^{\cdot}$. Upon substitution of $\dot{\phi}$ we can write Eq. (24) in the form

$$\begin{aligned} &- \rho(\partial_\theta \phi + \eta)\dot{\theta} + (\mathbf{T} - \rho \partial_{\mathbf{F}} \phi \mathbf{F}^T) \cdot \mathbf{L} \\ &\quad - (\rho \partial_{\mathbf{H}} \phi + \mu_0 \mathbf{M}) \cdot \dot{\mathbf{H}} \\ &\quad - \rho \partial_{\nabla\theta} \phi \cdot (\nabla\theta)^{\cdot} - \frac{1}{\theta} \mathbf{q} \cdot \nabla\theta = \rho\theta\gamma. \end{aligned}$$

The linearity an arbitrariness of $(\nabla\theta)^{\cdot}, \dot{\theta}, \mathbf{L}, \dot{\mathbf{H}}$ imply that

$$\begin{aligned} &\partial_{\nabla\theta} \phi = \mathbf{0}, \qquad \eta = -\partial_\theta \phi, \\ &\mathbf{T} = \rho \partial_{\mathbf{F}} \phi \mathbf{F}^T, \qquad \mu_0 \mathbf{M} = -\rho \partial_{\mathbf{H}} \phi, \\ &\mathbf{q} \cdot \nabla\theta = -\rho\theta^2\gamma \le 0. \end{aligned} \tag{25}$$

We now investigate the compatibility of the symmetry requirement (22) with the thermodynamic restrictions (25). The function ϕ might depend separately on $\mathbf{F}$ (e.g. through $|\mathbf{F}|$ or $|\mathbf{E}|$) and $\mathbf{H}$ (through $|\mathbf{H}|$). Hence we notice that

$$\begin{aligned} \partial_{\mathbf{F}} |\mathbf{F}| &= \partial_{\mathbf{F}} (\mathbf{F} \cdot \mathbf{F})^{1/2} = \mathbf{F}/|\mathbf{F}|, \\ \partial_{\mathbf{F}} |\mathbf{E}| &= \partial_{\mathbf{F}} (\mathbf{E} \cdot \mathbf{E})^{1/2} = (\mathbf{E} \cdot \mathbf{E})^{-1/2} (\partial_{\mathbf{F}} \mathbf{E}) \mathbf{E}, \\ &\qquad \partial_{F_{iK}} \tfrac{1}{2}(F_{lP} F_{lQ} - \delta_{PQ}) = \tfrac{1}{2}(F_{iQ} \delta_{KP} + F_{iP} \delta_{KQ}), \\ \partial_{\mathbf{H}} |\mathbf{H}| &= \mathbf{H}/|\mathbf{H}|. \end{aligned}$$

Consequently it follows

$$\partial_{\mathbf{F}} \phi \mathbf{F}^T = \partial_{|\mathbf{F}|} \phi \, \mathbf{F}\mathbf{F}^T \in \mathrm{Sym},$$

or

$$\partial_{\mathbf{F}} \phi \mathbf{F}^T = \partial_{|\mathbf{E}|} \phi \frac{1}{|\mathbf{E}|} \mathbf{F}\mathbf{E}\mathbf{F}^T \in \mathrm{Sym},$$

and

$$\partial_{\mathbf{H}} \phi = \frac{1}{|\mathbf{H}|} \partial_{|\mathbf{H}|} \phi \, \mathbf{H}.$$

Hence

$$\begin{aligned} &\mathbf{T} = \rho \partial_{\mathbf{F}} \phi \mathbf{F}^T \in \mathrm{Sym}, \\ &\qquad \mathbf{H} \otimes \mathbf{M} = -\frac{1}{|\mathbf{H}|} \partial_{|\mathbf{H}|} \phi \, \mathbf{H} \otimes \mathbf{H} \in \mathrm{Sym}. \end{aligned}$$

Thus the symmetry requirement (22) holds in the special case where $\mathbf{T} \in \mathrm{Sym}$ and $\mathbf{H} \otimes \mathbf{M} \in \mathrm{Sym}$.

More realistic models should allow for a cross dependence on $\mathbf{F}$ and $\mathbf{H}$ so that a magnetomechanical coupling [29] is obtained. Both $\mathbf{F}^T\mathbf{H}$ (i.e. $(\mathbf{F}^T\mathbf{H})_P = F_{iP} H_i$) and $\mathbf{F}^{-1}\mathbf{H}$ are invariant under Euclidean transformations (SO(3) invariant). Based on SO(3) invariance we might look for a dependence on $\mathbf{F}^T\mathbf{H}$ or $\mathbf{F}^{-1}\mathbf{H}$ possibly multiplied by any power of the scalar invariant $J = \det \mathbf{F}$. First we check the SO(3) invariance of $J^n \mathbf{F}^T \mathbf{H}$ and $J^n \mathbf{F}^{-1} \mathbf{H}$, with $n \in \mathbb{N}$.

Under the Euclidean transformation

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \qquad \det \mathbf{Q} = 1,$$

we have [15, 16]

$$\mathbf{F}^* = \mathbf{QF}, \quad \mathbf{H}^* = \mathbf{QH},$$
$$(J^n\mathbf{F}^T\mathbf{H})^* = (\det \mathbf{F}^*)^n\mathbf{F}^{T*}\mathbf{H}^* = (\det \mathbf{F})^n(\mathbf{QF})^T\mathbf{QH}$$
$$= J^n\mathbf{F}^T\mathbf{Q}^T\mathbf{QH} = J^n\mathbf{F}^T\mathbf{H}$$

and the like for $J^n\mathbf{F}^{-1}\mathbf{H}$.

Now let

$$\mathbf{A} = J^n\mathbf{F}^T\mathbf{H}$$

and assume $\phi = \phi(\theta, \mathbf{E}, \mathbf{A})$. We find that

$$T_{ij} = \rho\partial_{F_{iK}}\phi F_{jK} = \rho\partial_{E_{PQ}}\phi\tfrac{1}{2}(F_{iQ}F_{jP} + F_{iP}F_{jQ}) + \rho\partial_{A_P}\phi(nJ^n\delta_{ij}A_P + J^nH_iF_{jP}),$$
$$\mu_0H_iM_j = -\rho H_i\partial_{H_j}\phi = -\rho H_i\partial_{A_P}\phi J^nF_{jP}.$$

Though $\mathbf{T}$ and $\mathbf{H}\otimes\mathbf{M}$ are non-symmetric we have

$$T_{ij} + \mu_0H_iM_j = \rho\partial_{E_{PQ}}\phi\tfrac{1}{2}(F_{iQ}F_{jP} + F_{iP}F_{jQ}) + \rho nJ^nA_P\partial_{A_P}\phi\,\delta_{ij}$$

and then $\mathbf{T} + \mu_0\mathbf{H}\otimes\mathbf{M} \in \mathrm{Sym}$. Hence the dependence on $J^n\mathbf{F}^T\mathbf{H}$ makes the symmetry requirement (22) valid for any $n \in \mathbb{N}$. It is worth remarking that this conclusion is consistent with the literature where the field $\mathbf{A} = \mathbf{F}^T\mathbf{H}$ ($n = 0$) is adopted as the Lagrangian form of the magnetic field [20–23].

Now, $J^n\mathbf{F}^{-1}\mathbf{H}$ is also objective and hence we may look for the possible function $\phi(\theta, \mathbf{E}, \boldsymbol{\mathcal{A}})$, with

$$\boldsymbol{\mathcal{A}} = J^n\mathbf{F}^{-1}\mathbf{H}.$$

Since

$$\partial_{F_{iK}}F^{-1}_{Pm} = -F^{-1}_{Pi}F^{-1}_{Km}$$

then we find

$$T_{ij} = \rho\partial_{F_{iK}}\phi F_{jK} = \rho\partial_{E_{PQ}}\phi\tfrac{1}{2}(F_{iQ}F_{jP} + F_{iP}F_{jQ}) + \rho\partial_{\mathcal{A}_P}\phi(nJ^nF^{-1}_{Ki}F^{-1}_{Pm}H_m - J^nF^{-1}_{Pi}F^{-1}_{Km}H_m)F_{jK}$$
$$\mu_0H_iM_j = -J^n\rho H_i\partial_{\mathcal{A}_P}\phi F^{-1}_{Pj}.$$

Consequently

$$T_{ij} + \mu_0H_iM_j = \rho\partial_{E_{PQ}}\phi\tfrac{1}{2}(F_{iQ}F_{jP} + F_{iP}F_{jQ}) + \rho\partial_{\mathcal{A}_P}\phi\, nJ^nF^{-1}_{Pm}H_m\delta_{ij} - \rho\partial_{\mathcal{A}_P}\phi J^n(F^{-1}_{Pi}H_j + F^{-1}_{Pj}H_i).$$

Hence $\mathbf{T} + \mu_0\mathbf{H}\otimes\mathbf{M} \in \mathrm{Sym}$ and also the dependence on $\boldsymbol{\mathcal{A}}$ is objective and satisfies the requirement (22). Nevertheless the dependence on $\boldsymbol{\mathcal{A}}$ seems to be uncommon in the literature.

7 Dissipation through the entropy production or the dissipation potential

Dissipative phenomena are associated with a positive entropy production. Indeed, relative to the CD inequality, dissipative phenomena are characterized by the reduced inequality. Here we show how different approaches to the analysis of the reduced inequality lead to qualitatively different models of continua.

For definiteness we consider magnetic solids subject to the fields $\mathbf{H}, \mathbf{M}, \mathbf{B}$. For simplicity it is assumed that the material is thermally and electrically non-conducting. The possible electric field $\mathbf{E}$ is then viewed as a vector parameter.

7.1 Relation to some reversible-irreversible decompositions

We review Ref. [24] and start with the balance of energy in the form

$$\rho\dot{u} = \mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}} + \rho r, \tag{26}$$

where u is the internal energy density. The mechanical power $\mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}}$ means that we assume $\mathbf{T}\in\mathrm{Sym}$ and $\mathbf{D}$ is replaced with the time derivative of the infinitesimal strain $\boldsymbol{\varepsilon}$. The magnetic power $\mathbf{H}\cdot\dot{\mathbf{B}}$ is consistent with the expression $\mu_0\mathbf{H}\cdot\dot{\mathbf{M}}$ in that ρ is assumed to be constant and hence (26) is consistent with (23) by letting $\rho\varepsilon = \rho u - \mu_0|\mathbf{H}|^2/2$. In light of (26) the CD inequality reads

$$-(\dot{\Psi} + S\dot{\theta}) + \mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}} \geq 0. \tag{27}$$

We assume the additive decomposition of $\mathbf{B}$ and $\boldsymbol{\varepsilon}$ in the form

$$\mathbf{B} = \mathbf{B}^{\mathrm{rev}} + \mathbf{B}^{\mathrm{irr}}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{rev}} + \boldsymbol{\varepsilon}^{\mathrm{irr}},$$

the irreversible parts are considered to describe dissipative effects, here magnetostriction or hysteresis. A possible internal variable ξ is here ignored. Let S be the entropy density, $\Psi = \rho u - \theta S$ the free energy density. We let $\Psi, S, \mathbf{T}, \mathbf{H}$ be functions of the variables

$\theta, \mathbf{B}, \mathbf{B}^{\rm irr}, \boldsymbol{\varepsilon}^{\rm rev}, \theta$.

Inequality (27) becomes

$$-(\partial_\theta \Psi + S)\dot\theta + (\mathbf{H} - \partial_{\mathbf{B}}\Psi)\cdot\dot{\mathbf{B}} + (\mathbf{T} - \partial_{\boldsymbol{\varepsilon}}\Psi)\cdot\dot{\boldsymbol{\varepsilon}} - \partial_{\mathbf{B}^{\rm irr}}\Psi\cdot\dot{\mathbf{B}}^{\rm irr} + \mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}}^{\rm irr} \ge 0.$$

The arbitrariness of $\dot\theta, \dot{\mathbf{B}}$, and $\dot{\boldsymbol{\varepsilon}}^{\rm rev}$ implies

$$S = -\partial_\theta\Psi, \qquad \mathbf{H} = \partial_{\mathbf{B}}\Psi, \qquad \mathbf{T} = \partial_{\boldsymbol{\varepsilon}}\Psi,$$

and hence

$$-\partial_{\mathbf{B}^{\rm irr}}\Psi\cdot\dot{\mathbf{B}}^{\rm irr} + \mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}}^{\rm irr} \ge 0. \tag{28}$$

For technical convenience the free energy Ψ is then assumed to be a function with appropriate dependences, $\Psi = \Psi(\theta, \mathbf{B} - \mathbf{B}^{\rm irr}, \boldsymbol{\varepsilon}^{\rm rev})$. Hence it follows

$$\partial_{\mathbf{B}^{\rm irr}}\Psi = -\partial_{\mathbf{B}}\Psi = -\mathbf{H}$$

and the reduced inequality (28) can be written in the form

$$\mathbf{H}\cdot\dot{\mathbf{B}}^{\rm irr} + \mathbf{T}\cdot\dot{\boldsymbol{\varepsilon}}^{\rm irr} \ge 0. \tag{29}$$

To satisfy (29), appeal is made to a function $\varphi(\mathbf{H}, \mathbf{T})$ and the rates $\dot{\mathbf{B}}^{\rm irr}, \dot{\boldsymbol{\varepsilon}}^{\rm irr}$ are required to be given by

$$\dot{\mathbf{B}}^{\rm irr} = \partial_{\mathbf{H}}\varphi, \qquad \dot{\boldsymbol{\varepsilon}}^{\rm irr} = \partial_{\mathbf{T}}\varphi,$$

while

$$\mathbf{H}\cdot\partial_{\mathbf{H}}\varphi + \mathbf{T}\cdot\partial_{\mathbf{T}}\varphi \ge 0.$$

Granted these conditions, φ is called a pseudopotential [25]. If, further, $\varphi = \hat\varphi(|\mathbf{H}|, |\mathbf{T}|)$ then the inequality reads

$$\partial_{|\mathbf{H}|}\varphi|\mathbf{H}| + \partial_{|\mathbf{T}|}\varphi|\mathbf{T}| \ge 0$$

and holds if $\hat\varphi$ is convex, relative to $|\mathbf{H}|$ and $|\mathbf{T}|$, with $\hat\varphi(0,0) = 0$. More generally we might have

$$\varphi = f(\mathbf{H}\cdot\mathbf{AH}, \mathbf{T}\cdot\mathsf{C}\mathbf{T}),$$

where $\mathbf{A}$ (C) is a second-order (fourth-order) positive definite tensor. In that case we have

$$f_1\mathbf{H}\cdot\mathbf{AH} + f_2\mathbf{T}\cdot\mathsf{C}\mathbf{T} \ge 0$$

if only the derivatives f_1, f_2, with respect to the first and second variable, are non-negative.

Thus the model is characterized by two potentials, Ψ and φ. Yet, as a general remark on the use of pseudopotentials, we observe that the functions Ψ and φ depend on different variables,

$$\theta, \mathbf{B}, \mathbf{B}^{\rm irr}, \boldsymbol{\varepsilon}^{\rm rev} \quad \text{and} \quad \theta, \mathbf{H}, \mathbf{T},$$

respectively.

A similar approach is developed in [26] where the variables are

$$\mathbf{C}, \mathbf{C}_v, \mathbf{B}, \mathbf{B}_v,$$

where $\mathbf{C}$ is the Cauchy-Green tensor, $\mathbf{B}$ is the Lagrangian induction field; the additive decomposition is assumed for $\mathbf{B}$,

$$\mathbf{B} = \mathbf{B}_e + \mathbf{B}_v,$$

$\mathbf{B}_e, \mathbf{B}_v$ being viewed as elastic and viscous parts. Instead, the deformation gradient $\mathbf{F}$ is factorized as

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_v,$$

and $\mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e$, $\mathbf{C}_v = \mathbf{F}_v^T\mathbf{F}_v$. A free energy density Ω is considered and the CD inequality is written in the form

$$\tfrac{1}{2}\mathbf{T}_{RR}\cdot\dot{\mathbf{C}} + (\mathbf{H} - \partial_{\mathbf{B}}\Omega)\cdot\dot{\mathbf{B}} - \partial_{\mathbf{C}_v}\Omega\cdot\dot{\mathbf{C}}_v - \partial_{\mathbf{B}_v}\Omega\cdot\dot{\mathbf{B}}_v \ge 0,$$

where $\mathbf{T}_{RR}$ is the second Piola stress to within a pressure term due to the possible incompressibility constraint. It follows

$$\mathbf{H} = \partial_{\mathbf{B}}\Omega, \qquad \partial_{\mathbf{C}_v}\Omega\cdot\dot{\mathbf{C}}_v + \partial_{\mathbf{B}_v}\Omega\cdot\dot{\mathbf{B}}_v \le 0.$$

Next, for simplicity of the model it is assumed that the separate inequalities

$$\partial_{\mathbf{C}_v}\Omega\cdot\dot{\mathbf{C}}_v \le 0, \qquad \partial_{\mathbf{B}_v}\Omega\cdot\dot{\mathbf{B}}_v \le 0 \tag{30}$$

are satisfied while

$$\Omega(\mathbf{C}, \mathbf{C}_v, \mathbf{B}, \mathbf{B}_v) = \Omega_e(\mathbf{C}, \mathbf{B}) + \Omega_v(\mathbf{C}, \mathbf{C}_v, \mathbf{B}, \mathbf{B}_v).$$

The second requirement of (30) is satisfied by letting

$$\dot{\mathbf{B}}_v = -\kappa\partial_{\mathbf{B}_v}\Omega, \qquad \kappa > 0. \tag{31}$$

It is worth mentioning that this type of solution is adopted also by appealing to relaxation toward equilibrium [27]. Indeed, if e.g. Ω_v depends on $\mathbf{B}_v$ in the form

$$\Omega_v = q_v|\mathbf{B} - \mathbf{B}_v|^2$$

then

$\dot{\mathbf{B}}_v = -2\kappa q_v(\mathbf{B}_v - \mathbf{B});$

if $\mathbf{B}$ is constant then $\mathbf{B}_v - \mathbf{B}$ approaches zero as t approaches infinity.

Differently from the approach of [24], in [26] the set of variables is kept unchanged when establishing the equation for the rate $\dot{\mathbf{B}}_v$.

In both approaches the set of variables does not involve the time derivative $\dot{\mathbf{B}}$, or $\dot{\mathbf{B}}^{\rm irr}$, and hence hysteretic models would seem out of reach. Now, in [28] the dissipation potential is assumed to be a function of $\dot{\mathbf{B}}^{\rm irr}$ whereas the constitutive functions are not dependent on $\dot{\mathbf{B}}^{\rm irr}$. Next, by Legendre transformations, a constitutive equation is established in the form

$$\dot{\mathbf{B}}^{\rm irr} = F(|\mathbf{H}-\mathbf{G}|^n/D)\frac{\mathbf{H}-\mathbf{G}}{|\mathbf{H}-\mathbf{G}|}$$

and experimental data are simulated through

$$\begin{aligned}\dot{B} &= \mu\dot{H} + F(|H-G|^n/D)\frac{H-G}{|H-G|},\\ \dot{G} &= \nu[\dot{B}^{\rm irr} - (G-K)|\dot{B}^{\rm irr}|/Z] + \dot{B}^{\rm irr}/\mu,\\ \dot{K} &= \dot{B}^{\rm irr}/\mu,\end{aligned}$$

where μ is the permeability and ν, Z are parameters. This system follows by suitable assumptions on the dissipation potential, the dependence on an additional internal variable, ξ, and Legendre transformations. Things would be different by maintaining the same variables for all of the constitutive functions.

7.2 Hysteretic models through the entropy production

Consider the CD inequality (24) and, for formal simplicity, neglect heat conductivity so that

$$-\rho(\dot{\phi} + \eta\dot{\theta}) - \mu_0 \mathbf{M}\cdot\dot{\mathbf{H}} + \mathbf{T}\cdot\mathbf{L} = \rho\theta\gamma. \tag{32}$$

Hysteretic models necessarily involve the rates of the magnetic fields $\mathbf{H}, \mathbf{B}, \mathbf{M}$. Since the time rates $\dot{\mathbf{H}}, \dot{\mathbf{B}}, \dot{\mathbf{M}}$ are not objective then we consider an objective analogue of $\mathbf{H}$, namely

$$\mathfrak{H} = \mathbf{F}^T\mathbf{H},$$

which is Euclidian invariant. Now,

$$\dot{\mathbf{H}} = \mathbf{F}^{-T}\dot{\mathfrak{H}} - \mathbf{L}\mathbf{H}$$

and

$$\mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot\mathbf{D} + \mathbf{T}_{RR}\cdot\dot{\mathbf{E}}.$$

Substitution in (32) results in

$$\begin{aligned}&-\rho(\dot{\phi}+\eta\dot{\theta}) - \mu_0\mathfrak{M}\cdot\dot{\mathfrak{H}}\\ &\quad + \mathcal{T}_{RR}\cdot\dot{\mathbf{E}} + (\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M})\cdot\mathbf{W} = \rho\theta\gamma,\end{aligned} \tag{33}$$

where

$$\mathfrak{M} = J\mathbf{F}^{-1}\mathbf{M}, \qquad \mathcal{T}_{RR} = \mathbf{T}_{RR} + \mu_0\mathbf{C}^{-1}\mathfrak{H}\otimes\mathfrak{M}.$$

Since we have in mind hysteretic effects in deformable ferromagnets, the minimal set of variables is $\theta, \mathbf{F}, \mathfrak{H}, \mathfrak{M}, \dot{\mathfrak{H}}$ or possibly with $\dot{\mathfrak{H}}$ replaced by $\dot{\mathfrak{M}}$. The free energy ϕ is Euclidean invariant and hence we let

$$\phi = \phi(\theta, \mathbf{E}, \mathfrak{H}, \mathfrak{M}, \dot{\mathfrak{H}}).$$

Computation of $\dot{\phi}$ and substitution in (33) yields

$$\begin{aligned}&-\rho(\partial_\theta\phi + \eta)\dot{\theta} + (\mathcal{T}_{RR} - \rho\partial_{\mathbf{E}}\phi)\cdot\dot{\mathbf{E}}\\ &\quad - (\mu_0\mathfrak{M} + \rho\partial_{\mathfrak{H}}\phi)\cdot\dot{\mathfrak{H}} - \rho\partial_{\mathfrak{M}}\phi\cdot\dot{\mathfrak{M}}\\ &\quad - \rho\partial_{\dot{\mathfrak{H}}}\phi\cdot\ddot{\mathfrak{H}} + (\mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M})\cdot\mathbf{W} = \rho\theta\gamma.\end{aligned}$$

The linearity and arbitrariness of $\dot{\theta}, \ddot{\mathfrak{H}}, \mathbf{W}$ imply

$$\eta = -\partial_\theta\phi, \qquad \partial_{\dot{\mathfrak{H}}}\phi = \mathbf{0}, \qquad \mathbf{T}+\mu_0\mathbf{H}\otimes\mathbf{M} \in \mathrm{Sym}.$$

If $\dot{\mathbf{E}}$ is independent of $\dot{\mathfrak{H}}$ and $\dot{\mathfrak{M}}$ then it follows

$$\mathcal{T}_{RR} = \rho\partial_{\mathbf{E}}\phi.$$

Consequently we obtain the reduced inequality

$$-(\mu_0\mathfrak{M} + \rho\partial_{\mathfrak{H}}\phi)\cdot\dot{\mathfrak{H}} - \rho\partial_{\mathfrak{M}}\phi\cdot\dot{\mathfrak{M}} = \rho\theta\gamma \geq 0 \tag{34}$$

where $\gamma(\theta, \mathbf{F}, \mathfrak{H}, \mathfrak{M}, \dot{\mathfrak{H}})$.

Inequality (34) is the condition for the occurrence of dissipative processes. To describe hysteretic effects we need a distinction between increasing and decreasing of the source field. We may consider the magnetic field as the source field and then we model the entropy production in the form

$$\rho\theta\gamma = \zeta(\theta, \mathbf{F}, \mathfrak{H}, \mathfrak{M})|\dot{\mathfrak{H}}|, \qquad \zeta \geq 0. \tag{35}$$

Since $\gamma \geq 0$ then (34) is satisfied and we can view

$$-(\mu_0\mathfrak{M} + \rho\partial_{\mathfrak{H}}\phi)\cdot\dot{\mathfrak{H}} - \rho\partial_{\mathfrak{M}}\phi\cdot\dot{\mathfrak{M}} = \zeta|\dot{\mathfrak{H}}| \tag{36}$$

as the rate equation governing the hysteretic processes.

For simplicity, and also with reference to experimental setups [29, 30], we restrict attention to one-dimensional models. Let $\mathbf{e}$ be a fixed direction and let

$$\mathbf{F} = \mathrm{diag}(F_{11}, F_{22}, F_{33}), \qquad \mathbf{H} = H\mathbf{e}, \qquad \mathbf{M} = M\mathbf{e},$$
$$\mathfrak{H} = F_{11}H\mathbf{e}, \qquad \mathfrak{M} = F_{22}F_{33}M\mathbf{e}.$$

We observe that $\dot{H}$ and $\dot{M}$ are Euclidean invariants. This is so because

$$H^* = (\mathbf{H} \cdot \mathbf{e})^* = (\mathbf{QH}) \cdot (\mathbf{Qe}) = \mathbf{H} \cdot \mathbf{Q}^T\mathbf{Qe} = \mathbf{H} \cdot \mathbf{e} = H,$$

and the like for M. Hence H, M are scalar Euclidean invariants and so are $\dot{H}, \dot{M}$.

In a linear approximation, relative to $H, M, \dot{F}$ we can write

$$\dot{\mathfrak{H}} = \dot{F}_{11}H + F_{11}\dot{H} \simeq \dot{H}, \qquad \dot{\mathfrak{M}} \simeq \dot{M}\mathbf{e}.$$

Hence Eq. (36) is approximated by

$$-(\mu_0 M + \rho\partial_{\mathfrak{H}}\phi)\dot{H} - \rho\partial_{\mathfrak{M}}\phi\dot{M} = \zeta|\dot{H}|. \tag{37}$$

Subject to $\dot{H} \neq 0$, it is

$$\frac{\dot{M}}{\dot{H}} = \frac{dM}{dH},$$

that is the differential magnetic susceptibility. Thus dividing (37) by $\dot{H}$ we have

$$\frac{dM}{dH} = -\frac{\mu_0\mathfrak{M} + \rho\partial_{\mathfrak{H}}\phi}{\rho\partial_{\mathfrak{M}}\phi} - \frac{\zeta}{\rho\partial_{\mathfrak{M}}\phi}\mathrm{sgn}\,\dot{H}. \tag{38}$$

The occurrence of $\mathrm{sgn}\,\dot{H}$ produces different dependences of M in terms of H, in hysteretic systems, depending on whether H increases or decreases. Furthermore Eq. (38) is determined by the free energy ϕ and the entropy production γ through ζ. It is worth remarking that the crucial term $\mathrm{sgn}\,\dot{H}$ arises because of the constitutive property of the entropy production. The free energy ϕ cannot depend on $\dot{\mathfrak{H}}$ because ϕ occurs through the time derivative $\dot{\phi}$; instead γ occurs through the value possibly dependent on $\dot{\mathfrak{H}}$. Examples of hysteresis cycles determined by equations like (38) are given in [16], ch. 15.

8 Conclusions

The paper deals with the application of the entropy principle, through the Clausius–Duhem inequality, for the analysis of constitutive properties in continuum physics. Though the application is meant in the sense of Coleman–Noll procedure [2], some generalizations are established. First, the use of the representation formula (18) allows us to solve equations of the form (19) which are customary when dealing with vectors and tensors.

Secondly, the Clausius–Duhem inequality may lead to the suitable selection of variables as is the case of the restrictions for magnetic solids,

$$\mathbf{T} = \rho\partial_{\mathbf{F}}\phi\mathbf{F}^T, \qquad \mu_0\mathbf{M} = \rho\partial_{\mathbf{H}}\phi,$$

subject to the balance equation

$$\mathbf{T} + \mu_0\mathbf{H} \otimes \mathbf{M} \in \mathrm{Sym}. \tag{39}$$

The Lagrangian variable $\mathfrak{H} = \mathbf{F}^T\mathbf{H}$ proves to satisfy (39) and in addition is invariant under Euclidean transformations.

Thirdly, letting the entropy production γ be a constitutive function per se allows the modelling of hysteresis. As a remarkable example, in ferromagnetism we let the constitutive functions depend also on $\dot{\mathfrak{H}} \simeq \dot{\mathbf{H}}$ and the dependence of γ on $\dot{\mathfrak{H}}$ allows us to select a non-negative function $\zeta|\dot{\mathbf{H}}|$ which, upon division by $\dot{H}$, eventually distinguishes the susceptibility dM/dH in magnetizing and demagnetizing processes. In relation to other approaches in the literature, the analogue dependence on $\dot{\mathbf{B}}^{\mathrm{irr}}$ is obtained by using a Legendre transformation thus establishing the model with two different sets of variables.

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Declarations

Conflict of interest The author declares that they have no conflict of interest.

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