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Acceleration waves in thermoelastic complex media with temperature-dependent phase fields

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Abstract We analyze homothermal acceleration waves in complex materials (those with active microstructure) in the presence of internal constraints that link the temperature to a manifold-valued phase-field describing a generic material microstructure at a certain spatial scale. Such a constraint leads to hyperbolic heat conduction even in the absence of macroscopic strain; we show how it influences the way acceleration waves propagate. The scheme describes a thermoelastic behavior that is compatible with dependence of the free energy on temperature gradient (a dependence otherwise forbidden by the second law of thermodynamics in the traditional non-isothermal description of simple bodies). We eventually provide examples in which the general treatment that we develop applies.

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1 Introduction

Acceleration waves are moving discontinuities for the acceleration. Pertinent analyses are a classical topic in the mechanics of bodies described in the traditional setting of continuum mechanics (see, e.g., [9, 10, 58–61, 63]) even accounting for the presence of internal constraints [24, 27, 50]. In this last case, under certain restrictions, acceleration waves do not propagate in a material which has three or more internal constraints, as shown by Scott in 1975 [50].

Less frequently discussed is the behavior of acceleration waves in complex bodies, those with microstructure that is active in a way it implies interactions not strictly representable in terms of the standard stress. We record the analyses in [38] and [43] for the general model-building framework of the mechanics of complex materials, [49] in the special cases of beams described by curves with directors, [48] for nematic liquid crystals in non-isothermal setting, [15] and [55] for porous media.

In the present paper, we extend the previous analyses of acceleration waves in general complex bodies accounting for the presence of internal constraints that link microstructural descriptors (phase fields) $\hat{\nu}$, taking values $\nu := \tilde{\nu}(x, t)$ over a

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finite-dimensional manifold $\mathcal{M}$, to temperature θ, namely a relation of the type $\nu := \gamma(\theta(x,t))$, with γ a $\mathcal{M}$-valued differentiable function. If direct external bulk actions on the material microstructure are absent, even when we neglect macroscopic strain, the specific internal constraint indicated by γ implies a hyperbolic type description of heat conduction, as shown in [34]: under the conditions just listed, the microstructural inner power becomes an extra heat flux perturbing the standard Fourier description (for the influence of this extra heat flux on stability see [54] and [13], while the propagation of ordinary heat waves is discussed in [37] and [39] at least under specific conditions).

Thinking of acceleration waves, we thus consider a moving discontinuity surface, so the definition of conditions specifying even a generic internal constraint require to add an interface condition that we provide.

What emerges is that the structure of wave amplitude evolution equation does not change due to the presence of microstructural interactions and the related constraint between ν and θ, while its factors strongly depend on those interactions and the constraint considered.

2 Notations and preliminary notions

2.1 Linear spaces and their dual, duality pairing, second and third-rank tensors

Let X be a m-dimensional real linear space with a basis $\{\mathbf{e}_i\}$, $i = 1, \ddots, m$, and corresponding dual X^* endowed with a basis $\{\mathbf{e}^i\}$. Every $\mathbf{e}^i$ is defined to be such that $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_{\ j}$, where $\delta^i_{\ j}$ is the Kronecker delta and the interposed dot indicates from now on the duality pairing, namely the value taken by the linear form $\mathbf{e}^i$ over the vector $\mathbf{e}_j$ (precisely, $\mathbf{e}^i \cdot \mathbf{e}_j = \mathbf{e}^i(\mathbf{e}_j) \in \mathbb{R}$). The duality relation is such that $(X^*)^* = X$.

For $a \in X$ and $b \in X^*$, a second-rank tensor from X into itself is given by the dyad $a \otimes b$, defined to be such that for every $h \in X$, $(a \otimes b)h = (b \cdot h)a$. Dyadic products $\mathbf{e}_i \otimes \mathbf{e}_j$, $\mathbf{e}_i \otimes \mathbf{e}^j$, $\mathbf{e}^i \otimes \mathbf{e}_j$, $\mathbf{e}^i \otimes \mathbf{e}^j$ constitute bases of different second-rank tensor spaces. Specifically, with A a linear map from X onto itself, with respect to a basis $\{\mathbf{e}_i\}$ in X, we express A as $\mathsf{A} = \mathsf{A}^i_{\ j}\mathbf{e}_i \otimes \mathbf{e}^j$, where we adopt the usual summation over repeated indices, as in the rest of this paper.

For A and B two tensors from X onto itself, we indicate by $\mathsf{A} \cdot \mathsf{B}$ the scalar given by $\mathsf{A}^i_{\ j}\mathsf{B}^j_{\ i}$, and by AB the second rank tensor given by $\mathsf{AB} = \mathsf{A}^i_{\ k}\mathsf{B}^k_{\ j}\mathbf{e}_i \otimes \mathbf{e}^j$. Analogous notations hold for linear maps from X onto its dual X^* and vice versa, or from X^* onto itself, with appropriate lowering or raising component indexes by means of pertinent metrics.

Let Y be another m-dimensional real linear space. Take a linear operator G from X to Y; in short $G \in \mathrm{Hom}(X, Y)$. Two linear operators are associated with G: the transpose $G^\mathsf{T} \in \mathrm{Hom}(Y, X)$ and the adjoint $G^* \in \mathrm{Hom}(Y^*, X^*)$. If g and $\tilde{g}$ are metrics in X and Y respectively, we have $G^\mathsf{T} = g^{-1}G^*\tilde{g}$.

If $G \in \mathrm{Hom}(X, X)$ and the metric refers to orthonormal frames, G^* and G^T coincide. Also if G has components G^{ij} or G_{ij}, transpose and adjoint operators coincide. Finally, if $G \in \mathrm{Hom}(X, X)$ is non-singular, meaning $\det G \neq 0$, its inverse G^{-1} exists and belongs also to $\mathrm{Hom}(X, X)$.

Every second-rank tensor $G \in \mathrm{Hom}(X, X)$ admits the decomposition $G = \mathrm{sym}G + \mathrm{skw}G$, with $\mathrm{sym}G = \frac{1}{2}(G + G^\mathsf{T})$ and $\mathrm{skew}G = \frac{1}{2}(G - G^\mathsf{T})$.

For K a third-rank tensor, e.g. $K = K^i_{\ hj}\mathbf{e}_i \otimes \mathbf{e}^h \otimes \mathbf{e}^j$, we indicate with $\overset{*}{K}$ the *major* adjoint involving components i and j, a linear operator with components $K_{jh}^{\ \ i}$, while with ${}^*K$ the minor-left adjoint, a linear operator with components $K_{h\ j}^{\ i}$; eventually $K^{\underline{*}}$ indicates minor-right adjoint, which involves the last two components, namely h and j. In analogous way, K^T, ${}^tK$, and K^t indicate major, minor-left and minor-right transpositions. Left and right minor adjoint or transpose operations hold also for fourth-rank tensors, for which they involve the first or the last two components of the tensor considered.

Let $\mathfrak{A}$ be a tensor with rank higher than two. We adopt the following notations: $\mathfrak{A}^{\overset{t}{*}}$ is the adjoint between first and last component; $\mathfrak{A}^{\underline{*}}$ involves second and fourth component ($rank > 3$); $\mathfrak{A}^{\bar{*}}$ second and fifth ($rank > 4$); $\mathfrak{A}^{\check{*}}$ second and sixth (rank>5); $\mathfrak{A}^{\hat{*}}$ third and fourth ($rank > 3$); $\mathfrak{A}^{\acute{*}}$ fourth and seventh ($rank > 6$); $\mathfrak{A}^{\grave{*}}$ fifth and sixth ($rank > 5$); $\mathfrak{A}^{\breve{*}}$ third and fifth; $\mathfrak{A}^{\tilde{*}}$ fourth and sixth. The combination of these symbols, e.g. $\mathfrak{A}^{\bar{*}\bar{*}\bar{*}}$, should be read by considering their action from left to right.

If $a \in X$, we indicate by $a^\flat$ the counterpart of a into X^*, namely $a^\flat = ga$, in components $a_i^\flat = g_{ij}a^j$, where $g = g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$ is the metric in X. If $c \in X^*$, with $c^\sharp$ we indicate the vector with components $c^{\sharp\, i} = g^{ij}\mathbf{e}_j$, where g^{ij} is the ij-th component of g^{-1}.

2.2 Piecewise continuous maps, jumps, integral identities

Take a simply connected bounded open domain $\Omega \in \mathbb{R}^n$ with almost smooth boundary (its closure is indicated by $\bar{\Omega}$), and consider the space-time tube $\mathsf{T} := \{(x,t) \in \Omega \times (t_1,t_2)\}$. A closed subset Υ of $\bar{\mathsf{T}} := \overline{\Omega \times (t_1,t_2)}$ is called a singular surface in T if there exist a number of pairwise disjoint regions with almost smooth boundary $\mathsf{T}^p \subset \mathsf{T}$, $p = 1,\dots,q$, such that $\mathsf{T} \setminus \Upsilon = \cup_{p=1}^q \mathsf{T}^p$ and $\Upsilon = \Upsilon \cap (\cup_{p=1}^q \partial\mathsf{T}^p)$. Let $B(x,r) \in \mathbb{R}^n$ a ball with radius r, centered at $x \in \Upsilon$. Such x is called a regular point of Υ if there exists $r > 0$ such that $B(x,r) \cap \Upsilon$ is a smooth hypersurface, so locally (meaning in a neighborhood of x) Υ admits an implicit representation as a level set $\varphi(x,t) = 0$, with φ a smooth function. Write Υ^r for the set of regular point of Υ; it needs not to be connected.

A singular surface Υ can be oriented by exploiting the orientations of all $\partial\mathsf{T}^p$. The orientation has spatial and temporal components. Υ is considered to be an evolving spatial surface if it is a smooth hypersurface with spatial orientation m different from zero. This implies that the tangent space is never space-like only. Thus, an evolving singular surface on T is a closed subset Υ of $\bar{\mathsf{T}}$ which is a singular surface and is such that Υ^r is an evolving spatial surface. Specifically, we will indicate by Σ the section $\Sigma = \Sigma(t) := \{(x,t) \in \mathsf{T} | \varphi(x,t) = 0\}$. Σ is oriented by the spatial normal $m^\sharp := \frac{\nabla\varphi(x,t)}{|\nabla\varphi(x,t)|}$ and has intrinsic velocity with amplitude $U = -\frac{\dot{\varphi}(x,t)}{|\nabla\varphi(x,t)|}$. The normal above is a vector. It can be also viewed as a covector when we define it by $m := \frac{D\varphi(x,t)}{|D\varphi(x,t)|}$, with D the derivative with respect to x. Precisely, with g the metric in space, we have in terms of components $(D\varphi)_i = g_{ij}(\nabla\varphi)^j$. $f : \mathsf{T} \longrightarrow \mathbb{R}$ is said to be piecewise continuous (piecewise continuously differentiable), with singular surface Υ if there are nonempty subsets of T, namely T^+ and T^-, such that $\Upsilon \subset \mathsf{T}^+ \cap \mathsf{T}^-$ and f admits on T^+ and T^- continuous (continuously differentiable) extensions; they are not unique but their values at Υ are uniquely determined by the limits

$$f^\pm := \lim_{\alpha\downarrow 0} f(x \pm \alpha m, t), \qquad f^\pm := \lim_{\alpha\downarrow 0} f(x, t \mp \alpha).$$

If f is a piecewise continuously differentiable function, in short we write $f \in PC^1(\mathsf{T})$. We define the jump of f as the difference $[f] := f^+ - f^-$ and call average of f the sum $\langle f\rangle := \frac{1}{2}(f^+ + f^-)$. If f_1 and f_2 are two functions with the properties of f and the same discontinuity surface, we have $[f_1 f_2] = \langle f_1\rangle[f_2] + [f_1]\langle f_2\rangle$.

Lemma 1 (*J. Hadamard*) *If f is a real-valued continuous and piecewise continuously differentiable function with evolving discontinuity surface for the* 4-*gradient* (*meaning the one with components ∇f and $\dot{f}$*) *with section Σ at t, there exists a continuous real-valued function* c *on Σ such that*

$$[\nabla f] = \mathrm{c}m \qquad \text{and} \qquad [\dot{f}] = -U\mathrm{c} \tag{1}$$

at every t.

Conversely, if $f \in PC^1$, Σ is connected, and (1) holds with some function c *on Σ, $[f] = 0$ everywhere.*

The following identities hold for $f \in PC^1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f\,\mathrm{d}x = \int_\Omega \dot{f}\,\mathrm{d}x - \int_{\Omega\cap\Sigma} U[f]\,\mathrm{d}\mathcal{H}^{n-1}(x), \tag{2}$$

$$\int_{\partial\Omega} f\,m^\sharp\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_\Omega \nabla f\,\mathrm{d}x + \int_{\Omega\cap\Sigma} [f]\,m^\sharp\,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3}$$

What has been discussed above holds also for $f : \mathsf{T} \longrightarrow \mathfrak{L}$, with $\mathfrak{L}$ a linear space over the field of real numbers, provided necessary adaptations (pertinent proofs are available in treatises like, for example, [52]).

3 Geometry of motions and observers

3.1 Deformations and phase fields

Write $\mathbb{R}^3$ and $\tilde{\mathbb{R}}^3$ for two isomorphic copies of the three-dimensional real space, the isomorphism $\imath : \mathbb{R}^3 \longrightarrow \tilde{\mathbb{R}}^3$ simply being the *identification*. We select in $\mathbb{R}^3$ a bounded regularly open set $\mathcal{B}$ with a piecewise smooth boundary, oriented by the outward unit normal n to within a finite number of corners and edges; we consider $\mathcal{B}$ as a (macroscopic) reference configuration. One-to-one continuous and piecewise continuously differentiable orientation-preserving maps

$$(x,t) \longmapsto y := \tilde{y}(x,t) \in \tilde{\mathbb{R}}^3, \tag{4}$$

with $x \in \mathcal{B}$ and t in a time interval $[0,\bar{t}]$, select in $\tilde{\mathbb{R}}^3$ configurations that we consider deformed with respect to $\mathcal{B}$. We indicate by a superposed dot the velocity $\dot{y} := \frac{\mathrm{d}\tilde{y}(x,t)}{\mathrm{d}t}$ in the so-called Lagrangian representation, while F stands for the spatial derivative $D\tilde{y}(x,t)$.

The orientation-preserving assumption implies $\det F > 0$.

A continuous and piecewise continuously differentiable field

$$(x,t) \longmapsto \nu := \tilde{\nu}(x,t) \tag{5}$$

represents at continuous scale essential geometric features of the microstructure at a certain spatial scale λ. The nature of ν depends on specific circumstances. A general model-building setting, which encompass known cases and is flexible as to offer new modeling perspectives, is to consider ν as an element of a differentiable manifold $\mathcal{M}$ with finite dimension (see [2, 3, 30, 32, 33, 40, 51]). Here, we avoid the case in which $\mathcal{M}$ coincides with an interval of the real line; reasons will be explained below.

A superposed dot indicates also for ν its time rate of ν, namely $\dot{\nu} := \frac{\mathrm{d}\tilde{\nu}(x,t)}{\mathrm{d}t} \in T_\nu\mathcal{M}$, where $T_\nu\mathcal{M}$ is the tangent space to $\mathcal{M}$ at ν. Notice that $\dot{\nu}$ can be defined intrinsically, meaning independently of the choice of a specific coordinate atlas over $\mathcal{M}$. In addition, we indicate by N the spatial derivative $D\tilde{\nu}(x,t)$.

With Σ a moving surface within $\mathcal{B}$, we will consider $\tilde{y}$ and $\tilde{\nu}$ to be continuous across Σ, while those other maps taken to be discontinuous at Σ will be taken as endowed with bounded discontinuities; the specific circumstances will be clearly stated below.

3.2 Observers in the physical space and the way their changes influence the perception of microstructures

Observers are frames of reference in all spaces that are chosen to describe the morphology of a body and its motion. To what we refer to here the list of pertinent spaces includes $\mathbb{R}^3$, $\tilde{\mathbb{R}}^3$, the manifold $\mathcal{M}$, and the time scale, say the interval $[0,\bar{t}]$ or the entire real line $\mathbb{R}$.

We consider changes of observers that leave invariant the reference space $\mathbb{R}^3$ and the time scale, while they are determined by a rigid-body motion into the physical space $\tilde{\mathbb{R}}^3$. Specifically, if $\mathcal{O}$ and $\mathcal{O}'$ are two different observers in $\tilde{\mathbb{R}}^3$, we consider that a point $y \in \tilde{\mathcal{B}} = \tilde{y}(\mathcal{B},t) \in \tilde{\mathbb{R}}^3$ as recorded by $\mathcal{O}$ is viewed by $\mathcal{O}'$ to be $y' := \mathsf{w}(t) + Q(t)(y - y_0)$, where w and Q depend smoothly on time t only and take values into $\tilde{\mathbb{R}}^3$ and $SO(3)$, respectively. When $\dot{y}$ is for $\mathcal{O}$ the velocity pertaining to the material element mapped onto y by a deformation, the other observer, namely $\mathcal{O}'$, reads a velocity $\dot{y}' = \dot{\mathsf{w}} + \dot{Q}(y - y_0) + Q\dot{y}$. To allow a comparison between the two velocities, we pull-back $\dot{y}'$ into the frame defining $\mathcal{O}$ by means of $Q^\top$, the translation being irrelevant because we consider here $\dot{y}$ as a free vector. By defining $\dot{y}^\diamond := Q^\top \dot{y}'$, we get the standard formula

$$\dot{y}^\diamond := \mathsf{c}(t) + \mathsf{q} \times (y - y_0) + \dot{y}, \tag{6}$$

where c and q are translational and rotational relative velocities between the two observers, respectively; precisely, $\mathsf{c} = Q^\top \dot{w}$ and q is the characteristic vector of the skew-symmetric second-rank tensor $Q^\top \dot{Q}$.

Rotation and translation of frames in the physical space $\tilde{\mathbb{R}}^3$ may alter, in principle, the perception of microstructures whose features—at least those considered to be essential in the specific modeling—are represented by ν. To take into account the circumstance, according to [32] (see also refinements in [35]), we presume the existence of a (possibly empty) family of differentiable homeomorphisms

$$\{\phi : SO(3) \times \mathbb{R} \longrightarrow \mathrm{Diff}(\mathcal{M},\mathcal{M})\}$$

where $\mathrm{Diff}(\mathcal{M},\mathcal{M})$ is the group of diffeomorphisms mapping $\mathcal{M}$ onto itself; ϕ takes values $\phi(Q_\epsilon) \in \mathrm{Diff}(\mathcal{M},\mathcal{M})$, with $\epsilon \in \mathbb{R}$ and Q_0 = identity. Specifically, ϕ transfers over $\mathcal{M}$ a possible discrepancy on the observation of microstructure between two observers rotating relatively one another. Thus, the counterpart of $\dot{y}^{\diamond}$ for $\dot{v}$ is

$$\dot{v}^{\diamond} = \dot{v} + \mathcal{A}(v)\mathsf{q}, \tag{7}$$

where, when the set $\{\phi\}$ is not empty, with $v_{\phi(Q)}$ the value of v after the action of $\phi(Q) \in \mathrm{Diff}(\mathcal{M}, \mathcal{M})$, the linear operator $\mathcal{A}(v) \in \mathrm{Hom}\left(\tilde{\mathbb{R}}^k, T_v\mathcal{M}\right)$ is given by

$$\mathcal{A}(v) = \frac{\mathrm{d}v_{\phi(Q_\epsilon)}}{\mathrm{d}Q_\epsilon}\frac{\mathrm{d}Q_\epsilon}{\mathrm{d}q_\epsilon}\Big|_{\epsilon=0}, \tag{8}$$

after identification of ϵ with t. Of course, the explicit expression of $v_{\phi(Q_\epsilon)}$ depends on the tensor nature of v and $\phi\left(Q_\epsilon\right)$); also, the choice of ϕ characterizes the specific change of observer considered and depends on the microstructure. When $\{\phi\}$ is empty, v is declared to be not observable, so per se an internal variable associated only with the production of entropy, not with true balanced interactions – we exclude such a circumstance here.

In writing (8) we take into account that every $Q \in SO(3)$ is such that $Q = \exp(\mathsf{eq}) = \exp(\mathsf{q}\times)$, with e the third-rank Ricci's indicator. Specific expressions of $\mathcal{A}$ can be found in [32, 33, 35], together with further details of foundational character.

The map $(x,t) \longmapsto \mathcal{A}(\tilde{v}(x,t))$ is taken to be continuous across Σ.

4 Balance of standard and microstructural actions in the presence of evolving interfaces

Balance equations are independent of constitutive structure. Also, they may or not be considered as first principles because they can be derived from invariance requirements imposed to entities through which the interactions between body parts and the external environment are defined.

Here we summarize an approach followed in [30] and [32] (with refinements in [35]), and adapt the derivation of inertial terms to the presence of a moving unstructured surface, as an acceleration wave is. The approach rests on a requirement of external power invariance under changes of observers described by the rules (6) and (7). At a conceptual level, we presume that every mechanism determining (in principle) independent changes of body morphology has its own interactions defined by the power performed to allow such changes. We postulate only the external power to a generic part $\mathfrak{b} \subseteq \mathcal{B}$ (a subset of $\mathcal{B}$ taken to be regularly open, with non-vanishing volume, and a piecewise smooth boundary oriented by n, as is for $\mathcal{B}$).

We *do not* postulate the inner power as it is necessary when one starts writing down the principle of virtual power as basic source. Resorting to it as a primary origin of the balance equations means postulating a priori the weak form of the balance equations themselves, which implies that the representation of macroscopic and microstructural actions is assumed, not derived, and so is for the microstructural self-action, which, at variance, emerges naturally from the procedure we follow, together with the representation of contact actions. So, aiming at an approach reducing assumptions with respect to postulating them in strong or weak form, we will not follow the path based on the virtual power principle stating the equality between external and internal power; the latter will emerge as a consequence, rather than a postulate.

At every t we will select $\mathfrak{b}$ to be *across* Σ, namely to be such that $\partial\mathfrak{b} \cap \Sigma$ is a piecewise smooth curve.

We consider bulk and contact actions of first-neighbor type for both standard and microstructural classes. Thus, we define the *external power* $\mathcal{P}^{ext}_{\mathfrak{b}}$ over a generic part $\mathfrak{b}$ of $\mathcal{B}$ as a functional depending on $\dot{y}$ and $\dot{v}$ with values

$$\mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y},\dot{v}) := \int_{\mathfrak{b}} (b^{\ddagger}\cdot\dot{y} + \beta^{\ddagger}\cdot\dot{v})\,\mathrm{d}x + \int_{\partial\mathfrak{b}} (\mathsf{t}_\partial\cdot\dot{y} + \tau_\partial\cdot\dot{v})\,\mathrm{d}\mathcal{H}^2(x) + \int_{\mathfrak{b}\cap\Sigma} (b^{in}_\Sigma\cdot\langle\dot{y}\rangle + \beta^{in}_\Sigma\cdot\langle\dot{v}\rangle)\,\mathrm{d}\mathcal{H}^2(x). \tag{9}$$

In the previous expression $b^{\ddagger}$ represents standard bulk forces and is the sum $b^{\ddagger} = b^{in} + b$ of inertial (b^{in}) and non-inertial (b) contributions (the latter is objective while the former is not so); t_∂ is the standard traction, which depends on x, t, and the boundary $\partial\mathfrak{b}$ of $\mathfrak{b}$, a circumstance indicated by the subscript ∂. Analogous meaning have $\beta^{\ddagger}$ and τ_∂, the latter a microstructural contact interaction (at x and t both actions are elements of $T^*_{\tilde{v}(x,t)}\mathcal{M}$, the cotangent space of $\mathcal{M}$ at v, namely a dual space to the tangent space of $\mathcal{M}$ at v); also, even $\beta^{\ddagger}$ is taken to be the sum $\beta^{in} + \beta$ into inertial (β^{in}) and non-inertial (β) components, the former associated with a possible (relative) inertia with respect to the macroscopic motion, the latter a possible non-inertial external bulk action directly over the microstructure (example is the consequence of an electric field on a microstructure given by polarization). Finally, b_{Σ}^{in} and β_{Σ}^{in} are surface inertial actions pertaining to Σ. The laws (6) and (7) allows us to test the same actions over two different sets of time rates. So, the invoked invariance request reduces to impose that

$$\mathcal{P}_{\mathfrak{b}}^{ext}(\dot{y},\dot{\nu}) = \mathcal{P}_{\mathfrak{b}}^{ext}(\dot{y}^{\diamond},\dot{\nu}^{\diamond}) \tag{10}$$

for *any choice* of c, q, and the part $\mathfrak{b}$ considered. The axiom implies that the integral balances

$$\int_{\mathfrak{b}} b^{\ddagger}\,\mathrm{d}x + \int_{\partial\mathfrak{b}} \mathsf{t}_{\partial}\,\mathrm{d}\mathcal{H}^2(x) + \int_{\mathfrak{b}\cap\Sigma} b_{\Sigma}^{in}\,\mathrm{d}\mathcal{H}^2(x) = 0 \tag{11}$$

and

$$\int_{\mathfrak{b}} ((y-y_0)\times b^{\ddagger} + \mathcal{A}^*\beta^{\ddagger})\,\mathrm{d}x + \int_{\partial\mathfrak{b}} ((y-y_0)\times \mathsf{t}_{\partial} + \mathcal{A}^*\tau_{\partial})\,\mathrm{d}\mathcal{H}^2(x) + \int_{\mathfrak{b}\cap\Sigma} ((y-y_0)\times b_{\Sigma}^{in} + \mathcal{A}^*\beta_{\Sigma}^{in})\,\mathrm{d}\mathcal{H}^2(x) = 0 \tag{12}$$

hold for any choice of $\mathfrak{b}$.

Equation (11) is the standard integral balance of forces while (12) is a non-standard balance of couples. The last equation does not imply by the way that the microstructural actions $\beta^{\ddagger}$ and τ_{∂} are couples: they are not necessarily so per se, rather their projections through $\mathcal{A}^*$ over the physical space are couples.

Standard assumptions apply to Eq. (11): boundedness of $|b^{\ddagger}|$ implies that in the regions where $\mathsf{t}_{\partial}(\cdot,t)$ is continuous, that is outside Σ, and for every t, we find a map $\tilde{\mathsf{t}}$ such that $\mathsf{t}_{\partial}(x,t) = \tilde{\mathsf{t}}(x,t,n) = -\tilde{\mathsf{t}}(x,t,-n)$, where n is the outward unit normal to $\partial\mathfrak{b}$ (considered as a covector), and also $\tilde{\mathsf{t}}$ is linear with respect to n, meaning there is a second-rank tensor field P such that $\tilde{\mathsf{t}}(x,t,n) = P(x,t)n$, with P the standard first Piola-Kirchhoff stress.

Since $\mathcal{B}$ is bounded, we can choose y_0 is a way such that the boundedness of $|b^{\ddagger}|$ assures the one of $|(y-y_0)\times b^{\ddagger}|$ and $|(y-y_0)\times \mathsf{t}_{\partial}|$ (in the analysis of Eq. (11) the boundedness of $|b^{\ddagger}|$ grants the one of $|\mathsf{t}_{\partial}|$). Also, if $|\mathcal{A}^*\beta|$ is also bounded, in the regions where $\tau_{\partial}(\cdot,t)$ is continuous, that is outside Σ, and for every t, we find a map $\tilde{\tau}$ such that $\tau_{\partial}(x,t) = \tilde{\tau}(x,t,n)$ and

$$\mathcal{A}^*\tilde{\tau}(x,t,n) = -\mathcal{A}^*\tilde{\tau}(x,t,-n);$$

also $\tilde{\tau}$ is linear with respect to n, namely there exists a second-rank tensor mapping n into the cotangent space of $\mathcal{M}$ at ν, namely $\tilde{\tau}(x,t,n) = S(x,t)n \in T_{\nu}^*\mathcal{M}$. We call S a microstress (for further details see [32, 35]; notice also that to prove the existence of S we do not need to embed $\mathcal{M}$ into a linear space, as required in [7]; the embedding is always available because $\mathcal{M}$ is with finite dimension but is not unique; to grant an intrinsic theory we need to avoid the embedding, as we do here in accord with [32, 35]).

When $P(\cdot,t)$ is a PC^1 map while $b^{\ddagger}$ and b_{Σ}^{in} are continuous over their domains, use of the Gauss theorem (3) and the arbitrariness of $\mathfrak{b}$ imply from Eq. (11)

$$b^{\ddagger} + \mathrm{Div}P = 0 \tag{13}$$

pointwise in the bulk and

$$[P]m + b_{\Sigma}^{in} = 0 \tag{14}$$

over Σ. Also, when $S(\cdot,t)$ is a PC^1 map while $\beta^{\ddagger}$ and β_{Σ}^{in} are continuous over their domains, by exploiting the local Eqs. (13) and (14), the arbitrariness of $\mathfrak{b}$, and using Gauss theorem, from Eq. (12) we get that there is an element z of $T_{\nu}^*\mathcal{M}$, defined to within an arbitrary element of $\ker\mathcal{A}^*$, such that

$$\beta^{\ddagger} - z + \mathrm{Div}S = 0 \tag{15}$$

and

$$\mathrm{skew}(PF^*) = \frac{1}{2}\mathsf{e}(\mathcal{A}^*z + (D\mathcal{A}^*)^t S), \tag{16}$$

in the bulk, while over Σ

$$\mathcal{A}^*([S]m + \beta_{\Sigma}^{in}) = 0,$$

which means that there is $\zeta := \tilde{\zeta}(x,t) \in T_{\tilde{\nu}(x,t)}^*\mathcal{M}$ such that

$$[S]m + \beta_{\Sigma}^{in} = \zeta \qquad \text{with} \qquad \mathcal{A}^*\zeta = 0, \tag{17}$$

(compare analyses in [8, 19, 30, 32]).

The validity of pointwise balances implies the identity

$$\mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y},\dot{\nu}) = \int_{\mathfrak{b}} (P\cdot\dot{F} + z\cdot\dot{\nu} + S\cdot\dot{N})\,\mathrm{d}x + \int_{\mathfrak{b}\cap\Sigma} (\langle P\rangle m\cdot[\dot{y}] + \langle S\rangle m\cdot[\dot{\nu}] + \zeta\cdot\langle\dot{\nu}\rangle)\,\mathrm{d}\mathcal{H}^2(x). \tag{18}$$

The right-hand-side integrals is what we call *internal power* and indicate in short with the symbol $\mathcal{P}^{int}_{\mathfrak{b}}(\dot{y},\dot{\nu})$.

The inertial components of $b^{\ddagger}$, namely b^{in}, and the surface inertial action require to be characterized. A reasonable and rather standard way we can follow is to adapt a prescription that the power developed by inertial terms over a generic body part plus the pertinent kinetic energy vanishes for every part of $\mathcal{B}$.

A suggestion by Capriz [3] is to consider the kinetic energy density of a complex body described in the multi-field setting sketched above as the sum

$$\frac{1}{2}\rho|\dot{y}|^2 + \mathfrak{k}(\nu,\mu),$$

where ρ it the referential mass density, taken here to be constant, μ an element of $T^*_\nu\mathcal{M}$, and $\mathfrak{k}$ a twice differentiable non-negative function over the cotangent bundle of $\mathcal{M}$, namely the disjoint union $T^*\mathcal{M} := \bigsqcup_{\nu\in\mathcal{M}} T^*_\nu\mathcal{M}$, such that $\frac{\partial^2\mathfrak{k}}{\partial\mu\partial\dot{\nu}^\flat}\cdot\dot{\nu}^\flat\otimes\dot{\nu}^\flat > 0$ and $\mathfrak{k}(\nu,0)=0$. Then, the relation between kinetic energy and the set of inertial actions is attributed to the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}}(\frac{1}{2}\rho|\dot{y}|^2 + \mathfrak{k}(\nu,\mu))\,\mathrm{d}x = -\int_{\mathcal{B}}(b^{in}\cdot\dot{y} + \beta^{\ddagger}\cdot\dot{\nu})\,\mathrm{d}x - \int_{\mathcal{B}\cap\Sigma}(b^{in}_\Sigma\cdot\langle\dot{y}\rangle + \beta^{in}_\Sigma\cdot\langle\dot{y}\rangle)\,\mathrm{d}\mathcal{H}^2(x), \tag{19}$$

which is presumed to hold for any choice of the velocity fields involved. The assumptions on $\mathfrak{k}$ allow us to consider a function $\chi : T\mathcal{M} \longmapsto \mathbb{R}$, with values $\chi(\nu,\dot{\nu})$, where $T\mathcal{M}$ it the tangent bundle of $\mathcal{M}$, namely the disjoint union $T\mathcal{M} := \bigsqcup_{\nu\in\mathcal{M}} T_\nu\mathcal{M}$, that is convex with respect to $\dot{\nu}$ and such that

$$\mathfrak{k}(\nu,\mu) = \frac{\partial\chi(\nu,\dot{\nu})}{\partial\dot{\nu}}\cdot\dot{\nu} - \chi(\nu,\dot{\nu}), \tag{20}$$

with

$$\mu := \frac{\partial\chi(\nu,\dot{\nu})}{\partial\dot{\nu}} \in T^*_\nu\mathcal{M},$$

so that when $\mathfrak{k}$ is quadratic with respect to its second entry, μ coincides with $\dot{\nu}^\flat$. In other words, $\mathfrak{k}$ is the Legendre transform of χ with respect to $\dot{\nu}$.

By taking into account the relation (20), arbitrariness of the time rates involved in the balance (19) and use of the transport theorem (2) imply that (19) is compatible with the identifications

$$b^{in} = -\rho\ddot{y}^\flat, \quad \beta^{in} = \frac{\partial\chi}{\partial\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\chi}{\partial\dot{\nu}}, b^{in}_\Sigma = \rho U[\dot{y}^\flat], \quad \beta^{in}_\Sigma = U\left[\frac{\partial\chi}{\partial\dot{\nu}}\right], \tag{21}$$

to within powerless terms. The possibility of powerless terms allows one to define inertial frames of reference with respect to the gross motion as those for which powerless terms vanish in the expression of b^{in} and b^{in}_Σ.

This analysis is open to discussion: when $\mathfrak{k}$ is quadratic and we compute the overall kinetic energy over a rigid-body motion characterized by $\dot{y} = \mathsf{q}\times(y-y_0)$ and $\dot{\nu} = \mathcal{A}\mathsf{q}$, what results is that the previous choice for the kinetic energy density would imply an augmented inertial tensor due to a microstructure. When the field $\tilde{\nu}$ is used to represent at a low-dimensional setting the features of a physical body filling a non-vanishing volume in higher dimensional space, as it occurs in the case of one-dimensional curves with directors modeling rods [11], the augmented inertia tensor accounts for the three-dimensional physical body. Otherwise, when $\tilde{\nu}$ describes internal degrees of freedom, as for example in the case of polarization, we could presumably consider a kinetic effect that is only relative to the macroscopic rigid-body motion, in order to obtain the inertial tensor of the real body. We do not enter further the issue. Rather, here we neglect microstructural inertia for the sake of simplicity.

4.1 A spin-free version of the bulk inner power density

Define $L := \dot{F}F^{-1}$. It is an element of $\mathrm{Hom}(\tilde{\mathbb{R}}^3,\tilde{\mathbb{R}}^3)$. As such it decomposes as $L = \mathsf{D} - \mathsf{Y}$, where $\mathsf{D} := \mathrm{sym}L$ and $\mathsf{Y} := -\mathrm{skew}L$ is the spin tensor. Associated with Y is the spin vector

$$\mathsf{r} := \frac{1}{2}\mathsf{e}\mathsf{Y} = \frac{1}{2}\mathrm{Curl}\dot{y}.$$

We thus define $\mathring{v} := \dot{v} - \mathcal{A}\mathsf{r}$.

Write in short w for the inner power bulk density, namely $w := P \cdot \dot{F} + z \cdot \dot{v} + S \cdot D\dot{v}$. By exploiting the local balance of couples (16), we thus have

$$
\begin{aligned}
w &= PF^* \cdot \dot{F}F^{-1} + z \cdot \dot{v} + S \cdot \dot{N} \\
&= \mathrm{sym}(PF^*) \cdot \mathsf{D} - \mathrm{skew}(PF^*) \cdot \mathsf{er} + z \cdot \dot{v} + S \cdot \dot{N} \\
&= \mathrm{sym}(PF^*) \cdot \mathsf{D} - (\mathcal{A}^* z + (D\mathcal{A}^*)^t S) \cdot \mathsf{r} + z \cdot \dot{v} + S \cdot \dot{N} \\
&= \mathrm{sym}(PF^*) \cdot \mathsf{D} + z \cdot \mathring{v} + S \cdot D\mathring{v} - \mathcal{A}^* S \cdot D\mathsf{r}.
\end{aligned}
$$

5 Energies and entropy

5.1 Internal energy and the first law of thermodynamics

We write e for the internal energy density, a piecewise differentiable function of space and time with discontinuity set Σ at every instant. It satisfies the first law of thermodynamics, namely the energy balance, given by

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} e \,\mathrm{d}x - \mathcal{P}^{ext}_{\mathfrak{b}}(\dot{y}, \dot{v}) - \int_{\mathfrak{b}} r \,\mathrm{d}x + \int_{\partial\mathfrak{b}} \tilde{q}(x,t) \cdot n \,\mathrm{d}\mathcal{H}^{k-1}(x) = 0, \tag{22}
$$

or, thanks to the identity (18), as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} e \,\mathrm{d}x - \mathcal{P}^{int}_{\mathfrak{b}}(\dot{y}, \dot{v}) - \int_{\mathfrak{b}} r \,\mathrm{d}x + \int_{\partial\mathfrak{b}} q \cdot n \,\mathrm{d}\mathcal{H}^{k-1}(x) = 0, \tag{23}
$$

where $q := \tilde{q}(x,t)$ is the heat flux across the boundary $\partial\mathfrak{b}$ oriented almost everywhere by the normal n, a piecewise differentiable function of space and time with discontinuity set still coinciding with Σ at every instant; r is a heat source, a continuous function of space and time. By using the transport Eq. (2), the arbitrariness of $\mathfrak{b}$ implies the local energy balance

$$
\dot{e} = P \cdot \dot{F} + z \cdot \dot{v} + S \cdot \dot{N} - \mathrm{Div} q + r \tag{24}
$$

in the bulk and

$$
\begin{aligned}
U[e] &= \langle P \rangle m \cdot [\dot{y}] + \langle S \rangle m \cdot [\dot{v}] + \zeta \cdot \langle \dot{v} \rangle + [q] \cdot m \\
&= U\langle P \rangle \cdot [F] + U\langle S \rangle \cdot [N] - \zeta \cdot \langle \dot{v} \rangle + [q] \cdot m
\end{aligned} \tag{25}
$$

across Σ, where the last identity follows frm Hadamard's lemma.

5.2 Entropy inequality

With η the entropy density, a piecewise differentiable function of space and time with discontinuity set coinciding with Σ at every instant, as usual we write the second law of thermodynamics as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}} \eta \,\mathrm{d}x \geq -\int_{\partial\mathfrak{b}} h \cdot n \,\mathrm{d}\mathcal{H}^{k-1}(x) + \int_{\mathfrak{b}} s \,\mathrm{d}x, \tag{26}
$$

where s is the entropy source that depends on space and time, while h is the entropy flux, a piecewise differentiable function with discontinuity set Σ at every instant. We assume standard relations given by

$$
h = \frac{q}{\theta}, \qquad s = \frac{r}{\theta}. \tag{27}
$$

Relation $(27)_1$ can eb extended to $h = \frac{q}{\theta} + \varpi$, with ϖ an extra entropy flux (such a relation has been introduced first in [41]; here, a natural attribution of ϖ is to microstructural events, namely to fluctuations; however, for the sake of simplicity, we restrict ourselves to the classical relation $(27)_1$). Thus, by exploiting arbitrariness of $\mathfrak{b}$ and the identity (2), we get

$$
\dot{\eta} \geq -\mathrm{Div}\frac{q}{\theta} + \frac{r}{\theta} \tag{28}
$$

pointwise in the bulk and

$$
-U[\eta] \geq \left[\frac{q}{\theta}\right] \cdot m \tag{29}
$$

over Σ. By expanding the right-hand side term and using the surface balance (25), we can rewrite the inequality (29) as

$$
\begin{aligned}
&-U\langle\theta\rangle[\eta] \geq -U([e] - \langle P \rangle \cdot [F] - \langle S \rangle \cdot [N]) \\
&- \zeta \cdot \langle \dot{v} \rangle + [\theta]\left\langle \frac{q}{\theta} \right\rangle \cdot m.
\end{aligned} \tag{30}
$$

5.3 Free energy and the Clausius–Duhem inequality

Helmoltz's free energy density ψ is usually defined as $\psi := e - \theta\eta$. By combining Eq. (24) with (28) in the common way, we obtain an appropriate local version of the Clausius-Duhem inequality:

$$\dot{\psi} + \eta\dot{\theta} - P \cdot \dot{F} - z \cdot \dot{v} - S \cdot \dot{N} - \frac{1}{\theta} q \cdot D\theta \leq 0, \qquad (31)$$

while from the surface inequality (30) we get

$$\begin{aligned} U([\psi] - [\theta]\langle\eta\rangle) \geq\; & U(\langle P\rangle \cdot [F] + \langle S\rangle \cdot [N]) \\ & - \zeta \cdot \langle \dot{v}\rangle + [\theta]\left\langle \frac{q}{\theta}\right\rangle \cdot m. \end{aligned} \qquad (32)$$

6 Internal constraints in the multi-field description of complex media

A generic material element is said to be internally constrained when it may assume only states in a defined strict subset of the state space. A body is said to undergo an internal constraint when some or all its elements are constrained. Examples of internal constraints are incompressibility and inextensibility along a prescribed direction. A systematic treatment of kinematic internal constraints in simple bodies is in the 1964 treatise by Truesdell and Noll [58, Sect. 30]; the principle of determinism is modified by the prescription that purely kinematic internal constraints foresee workless reactive stress. The analysis of the non-isothermal case dates back the 1970 work by Green, Naghdi, and Trapp [22]. A general view on internal constraints was proposed in 1973 by Gurtin and Podio-Guidugli [23]. Further analyses dealt with theoretical refinements and applications, above all to the deduction of beam, plate, and shell theories from three-dimensional standard elasticity of simple bodies (see, e.g., [27–29, 42, 46, 62]). Within the available literature, pertinent to the analysis developed here is the way Capriz and Podio-Guidugli discussed internal constraints in the multi-field setting concerning the continuum mechanics of complex bodies [6] (see also [2] and, for applications, [16] and [21]), an approach also developed by Giovine considering partially constrained microstructures [14, 17–20] and Capriz and Giovine in the case of sparse phases [5].

We come back here to internal constraints in the mechanics of complex bodies, but before entering details, a few remarks deserve to be mentioned in order to clarify the stage. First is the awareness that internal constraints generically are ideal conditions not strictly realized in nature, at least within what falls under the realm of continuum mechanics in classical space-time. Physical appearance seems to encompass weakened forms of internal constraints (for example, incompressibility is not under any load in imagination but it appears ideal under certain classes of loads). As regards this aspect, in 1978 Rostamian showed that in the elastostatics of nonlinear and linear hyperelastic materials, at least for the cases considered, including incompressibility, "slight relaxation of the internal constraint does not greatly affect the solutions of the boundary value problems" [47, p. 638]. Also, we notice that the view just described on internal constraints is local. There is at least one non-local approach [1] to internal constraints but here we maintain the local view. We continue to adopt the principle of determinism:

Axiom 1. ***Principle of determinism***

1. Every state functional $\mathcal{F}$ (free energy, entropy, stress, etc.) admits additive decomposition into active $\overset{a}{\mathcal{F}}$ and reactive $\overset{r}{\mathcal{F}}$ components.
2. Only the active components are connected by suitable constitutive rules to the independent thermokinetic variables.
3. Although restricted by the constraint explicit specification and a requirement of objectivity, the reactive terms maintain undetermined aspects. They depend on the specific process under analysis; they are fields in the orthogonal complement of the respective active functionals, orthogonality defined in some suitable way.

Definition 1 Internal constraints are said to be *perfect* when the pertinent reactive entities satisfy identically the identity

$$\dot{\overset{r}{\psi}} + \overset{r}{\eta}\dot{\theta} - \overset{r}{P} \cdot \dot{F} - \overset{r}{z} \cdot \dot{\nu} - \overset{r}{S} \cdot D\dot{\nu} - \frac{1}{\theta}\overset{r}{q} \cdot D\theta = 0 \tag{33}$$

in the bulk and for every process, and

$$\begin{aligned} &U([\overset{r}{\psi}] - [\theta]\left\langle \overset{r}{\eta} \right\rangle) = -\left\langle \overset{r}{P} \right\rangle m \cdot [\dot{y}] \\ &- (\left\langle \overset{r}{S} \right\rangle m + \overset{r}{\zeta}) \cdot \langle \dot{\nu} \rangle + [\theta]\left\langle \frac{\overset{r}{q}}{\theta} \right\rangle \cdot m \end{aligned} \tag{34}$$

over Σ for every process allowed by the constraint.

The bulk condition can be substituted by

$$\begin{aligned} &\dot{\overset{r}{\psi}} + \overset{r}{\eta}\dot{\theta} - \mathrm{sym}(\overset{r}{P}F^*) \cdot \mathsf{D} - \overset{r}{z} \cdot \mathring{\nu} - \overset{r}{S} \cdot D\mathring{\nu} \\ &+ (\mathcal{A}^*\overset{r}{S}) \cdot D\mathsf{r} - \frac{1}{\theta}\overset{r}{q} \cdot D\theta = 0, \end{aligned}$$

when we consider the inner power density involving $\mathring{\nu}$.

Imperfect internal constraints could be defined in some way, their treatment requiring to be at least defined. We do not enter this possible topic and maintain focus on constraints satisfying the definition above.

7 Temperature dependent internal constraint

Among several possible choices of internal constraints, here we consider ν as a function of the temperature, namely we presume the validity of a constraint

$$\nu = \gamma(\theta), \tag{35}$$

where γ is $\mathcal{M}$-valued, continuous and piecewise continuously differentiable. We also set

$$\nu' := \frac{\mathrm{d}\gamma(\theta)}{\mathrm{d}\theta} \tag{36}$$

and consider $|\nu'| \neq 0$.

As already recalled, for macroscopically rigid bodies endowed with microstructure not undergoing external bulk actions, even in the acceptance of Fourier's law, such an internal constraint implies finite speed propagation of temperature disturbances [34] (see also [38, 39]). This is the reason justifying our attention over it.

When we consider the constraint (35), so that $\dot{\nu} = \nu'\dot{\theta}$, Eq. (33) becomes

$$\begin{aligned} &\dot{\overset{r}{\psi}} + \overset{r}{\eta}\dot{\theta} - \overset{r}{P} \cdot \dot{F} - \overset{r}{z} \cdot \nu'\dot{\theta} - \left(\overset{r}{S}{}^*\nu'\right) \\ &\cdot D\dot{\theta} - \left(\overset{r}{S} \cdot D\nu'\right)\dot{\theta} - \frac{1}{\theta}\overset{r}{q} \cdot D\theta = 0. \end{aligned} \tag{37}$$

The arbitrariness of the fields involved implies

$$\overset{r}{P} = 0, \tag{38}$$

$$-\overset{r}{\eta} + \overset{r}{z} \cdot \nu' + \overset{r}{S} \cdot D\nu' = 0, \tag{39}$$

$$\overset{r}{S}{}^*\nu' = 0, \tag{40}$$

$$\dot{\overset{r}{\psi}} = 0, \tag{41}$$

$$\overset{r}{q} = 0, \tag{42}$$

We assume that $\overset{a}{\psi}$, $\overset{a}{\eta}$, $\overset{a}{P}$, $\overset{a}{z}$, and $\overset{a}{S}$, all depend on (F, ν, N, θ), a list of variables defining the state at a point.

The internal constraint (35) implies that the state representation reduces essentially to the list $(F, \theta, D\theta)$; so, e.g.,

$$\overset{a}{\psi} = \tilde{\psi}(F, \nu, N, \theta) = \bar{\psi}(F, \theta, D\theta).$$

The Clausius-Duhem inequality thus reads

$$\begin{aligned} &\frac{\partial\overset{a}{\psi}}{\partial F} \cdot \dot{F} + \frac{\partial\overset{a}{\psi}}{\partial\theta}\dot{\theta} + \frac{\partial\overset{a}{\psi}}{\partial D\theta} \cdot D\dot{\theta} + \overset{a}{\eta}\dot{\theta} - \overset{a}{P} \cdot \dot{F} - \overset{a}{z} \cdot \nu'\dot{\theta} \\ &\qquad -(\overset{a}{S}{}^*\nu') \cdot D\dot{\theta} - (\overset{a}{S} \cdot D\nu')\dot{\theta} - \frac{1}{\theta}\overset{a}{q} \cdot D\theta \leq 0. \end{aligned} \tag{43}$$

Once again, arbitrariness of the rate fields involved, namely $\dot{F}, \dot{\theta}, D\dot{\theta}$, implies

$$-\frac{\partial\overset{a}{\psi}}{\partial F} + \overset{a}{P} = 0, \tag{44}$$

$$-\frac{\partial\overset{a}{\psi}}{\partial\theta} - \overset{a}{\eta} + \overset{a}{z} \cdot \nu' + \overset{a}{S} \cdot D\nu' = 0, \tag{45}$$

$$-\frac{\partial \overset{a}{\psi}}{\partial D\theta} + \overset{a}{S^*} \nu' = 0, \tag{46}$$

$$\frac{1}{\theta} \overset{a}{q} \cdot D\theta \geq 0. \tag{47}$$

The last inequality is compatible with Fourier's law

$$q = -\kappa \nabla \theta, \tag{48}$$

which implies

$$[\dot{q}] = \frac{1}{U} \kappa [\ddot{\theta}] m^\sharp \tag{49}$$

across Σ.

By summing up active and reactive components, we get

$$P = \frac{\partial \overset{a}{\psi}}{\partial F} \tag{50}$$

$$S^* \nu' = \frac{\partial \overset{a}{\psi}}{\partial D\theta}, \tag{51}$$

$$\eta = -\frac{\partial \overset{a}{\psi}}{\partial \theta} + \mathrm{Div}\left(\frac{\partial \overset{a}{\psi}}{\partial D\theta}\right) + \nu' \cdot \beta. \tag{52}$$

From the identity (51) and taking into account (41) we get

$$S = \frac{\nu'^\flat}{|\nu'|^2} \otimes \frac{\partial \psi}{\partial D\theta}, \tag{53}$$

where $\nu'^\flat$ is the covector associated with ν', namely $\nu'^\flat = g_{\mathcal{M}} \nu'$ with $g_{\mathcal{M}}$ the metric over $\mathcal{M}$. Thus, by exploiting the relations (52) and (53), and taking into account that we do not consider here microstructural inertia, so $\beta^\ddagger = \beta$, from the local balance (15) we get

$$z = \left(\mathrm{Div}\left(\frac{\partial \psi}{\partial D\theta}\right) + \nu' \cdot \beta + \left(\frac{\nu'^\flat}{|\nu'|^2} \otimes \frac{\partial \psi}{\partial D\theta}\right) \cdot D\nu'\right) \frac{\nu'^\flat}{|\nu'|^2}. \tag{54}$$

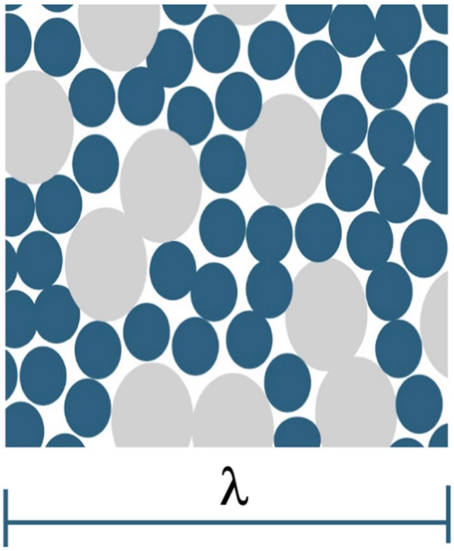

Fig. 1 Representative volume element of a mixture made of two families of grains. The gray ones are those highly temperature sensitive with respect to the others

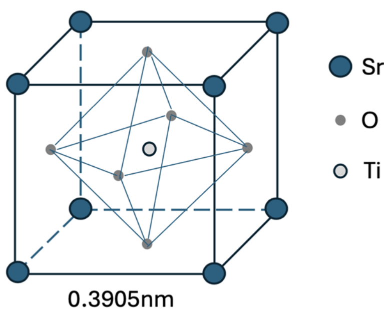

Fig. 2 Strontium titanate structure in an electrically neutral state

8 Examples of complex bodies to which the special internal constraint considered may apply

8.1 Two families of grains

Consider a mixture of two densely packed families of grains and take one of them to be highly temperature sensitive with respect to the other. At every x we attribute properties of a (statistically) representative volume element (RVE) characterized by a scale λ. Figure 1 describes such an RVE.

The (macroscopic) deformation gradient F is here an average over the RVE. Then, we may consider ν as a perturbation to F given by a (micro)deformation gradient $\mathfrak{G}$, due to the influence of temperature variations on thermally sensitive grains; so, $\nu = \mathfrak{G}$ should depend on θ. The resulting scheme is the one of a micromorphic continuum (see, e.g., [12]) with an internal constraint. In [19] P. Giovine discussed details of a similar case in which the two families of grains are replaced by a spongy body with voids filled by a gas. In both cases, specific constitutive choices depend on the type of RVE considered (about problems concerning the choice of an RVE, see [26] and [57]).

8.2 Strontium titanate and other ceramics

Strontium titanate ceramics, formally the class $Ba_{1-x}Sr_xTiO_3$, experience polarization as a consequence of heating. A primary effect is simply due to temperature variations at constant volume; another effect (the secondary one) depends on temperature-induced strain; the tertiary effect is determined by non-uniform heating [25]. Figure 2 schematically depicts the Strontium titanate structure in an electrically neutral state.

Also, piezoelectric materials such as quartz, lead titanate ($PbTiO_3$), Barium titanate ($BaTiO_3$), lead zirconate titanate experience such an effect and are used in cooling devices and energy harvesters [44]. In a reverse way, the application of an external electric field determines a change of temperature through the induced polarization; this effect is evident in ferroelectric perovskite ceramics, for example based on Barium titanate [53].

Thus, to represent the situation, we choose ν to be the polarization $\mathbf{p}$, so that $\mathcal{M}$ is a ball $B_{p^{max}}$ in $\mathbb{R}^3$ with radius the maximum polarization intensity p^{max} that the material may sustain.

To consider even the electrocaloric reverse effect, we account also for the presence of an electric field $\mathfrak{E}$ filling $\tilde{\mathbb{R}}^3$, where we detect the deformed configurations; so $\mathfrak{E}$ depends on y and t and satisfies Maxwell's equations in the vacuum, that is the space free of other bodies. We do not look at the space outside $\mathcal{B}$ while focus attention on $\mathcal{B}$ and its interaction with the environment.

To account explicitly for effects due to $\mathfrak{E}$, we consider the commonly adopted additive decomposition of the interactions occurring into the external power (inertia apart) into (electro)mechanical and electrical components indicated by superscripts m and e, respectively. So, we have

$$b^{\ddagger} = b^{\ddagger m} + b^e, \quad \beta = \beta^m + \beta^e, \quad \mathrm{t}_\partial = \mathrm{t}_\partial^m + \mathrm{t}_\partial^e, \quad \tau_\partial = \tau_\partial^m + \tau_\partial^e. \tag{55}$$

We thus identify the electric components by equation the negative of their power to the time rate of the electric energy in the matter, here indicated by $\mathcal{D}(\mathcal{B})$. We thus write

$$\frac{\mathrm{d}\mathcal{D}(\mathcal{B})}{\mathrm{d}t} = -\int_{\mathcal{B}} (b^e \cdot \dot{y} + \beta^e \cdot \dot{\mathbf{p}})\,\mathrm{d}x - \int_{\partial\mathcal{B}} (\mathrm{t}_\partial^e \cdot \dot{y} + \tau_\partial^e \cdot \dot{\mathbf{p}})\,\mathrm{d}\mathcal{H}^2(x), \tag{56}$$

where, by definition,

$$\mathcal{D}(\mathcal{B}) = -\frac{1}{2}\int_{\mathfrak{b}} \rho\mathfrak{E} \cdot \dot{\mathbf{p}}\,\mathrm{d}x, \tag{57}$$

and the Eq. (56) is presumed to hold for every choice of compactly supported time rates involved. A result by H. F. Tiersten [56], here rewritten in Lagrangian representation, reads

$$\begin{aligned}\frac{\mathrm{d}\mathcal{D}(\mathcal{B})}{\mathrm{d}t} &= -\int_{\mathfrak{b}} (\rho (D_y\mathfrak{E})\mathbf{p} \cdot \dot{y} + \rho\mathfrak{E} \cdot \dot{\mathbf{p}})\,\mathrm{d}x \\ &- \int_{\partial\mathcal{B}} \frac{1}{2}(\det F) p_n^2 F^{-*} n \cdot \dot{y}\,\mathrm{d}\mathcal{H}^2(x),\end{aligned} \tag{58}$$

where D_y indicates the derivative with respect to y and $p_n := \mathbf{p} \cdot n$. The arbitrariness of the rate fields involved implies the identifications

$$b^e = \rho(D_y\mathfrak{E})\mathbf{p}, \quad \beta^e = \rho\mathfrak{E}, \quad \tau_\partial^e = 0, \quad \mathrm{t}_\partial^e = \frac{1}{2}(\det F) p_n^2 F^{-*} n. \tag{59}$$

The last identity implies that, with $\mathbf{f}$ a continuous distribution of forces over a part of the body boundary or over a sub-body boundary, we have $Pn = \mathbf{f} - \frac{1}{2}(\det F) p_n^2 F^{-*} n$ at all points of the boundary considered where the normal n is well-defined.

With these premises, the internal constraint can be written as

$$\mathbf{p} = f(\theta)\mathfrak{l}, \tag{60}$$

where f is a differentiable function and $\mathfrak{l}$ a unit vector, which allows one to describe polarization switching. According to previous assumptions, we take $f'(\theta) \neq 0$. When $\mathbf{p}$ reduces to the total remnant polarization, an appropriate expression for $f(\theta)$ seems to be the polynomial $\mathfrak{a}_1 + \mathfrak{a}_2\theta + \mathfrak{a}_3\theta^2$ (values of the factors $\mathfrak{a}_i$, $i = 1, 2, 3$, are determined by experiments on a piezoelectric described in [45]).

In the presence of the special internal constraint (60), we have to consider active and reactive components of free energy, entropy, and mechanical actions. By following the previous procedure, we thus get

$$\overset{r\,m}{P} = 0, \quad \overset{r}{q} = 0, \quad \overset{r\,m}{S} = 0, \quad \overset{r}{\eta} = f'\overset{r}{z} \cdot \mathfrak{l}, \quad \overset{r}{z} \parallel \mathfrak{l}, \tag{61}$$

where the last relation indicates that the reactive component of the self-action z is undetermined and proportional to $\mathfrak{l}$.

9 Homothermal acceleration waves when $\nu = \gamma(\theta)$

At first we assume absence of non inertial bulk forces: $b^{\ddagger} = b^{in}$. Also, we presume $[\beta] = 0$.

We also set $\zeta = 0$ as a constitutive choice.

We consider homothermal acceleration waves. They are defined by the following conditions:

$$[y] = 0, [\dot{y}] = 0, [\theta] = 0, [\dot{\theta}] = 0,$$
$$[D\theta] = 0, [\nu] = 0, [\dot{\nu}] = 0.$$

From Eqs. (13) and $(21)_1$, we have

$$-U[\dot{P}] \otimes m = \rho U^2 [\ddot{y}^{\flat}]. \tag{62}$$

With the assumed constitutive structures, by taking into account the assumed homothermal state, namely $[\dot{\theta}] = 0$, the interfacial balance (14) becomes

$$\left(\frac{\overset{*}{\partial P}}{\partial F}(m \otimes m^{\sharp})\right)[\ddot{y}] + \frac{\partial P}{\partial D\theta}(m \otimes m)[\ddot{\theta}] = \rho U^2 [\ddot{y}^{\flat}]; \tag{63}$$

it includes two unknowns, namely $[\ddot{y}]$ and $[\ddot{\theta}]$; so, we need an additional condition for one of them; it is provided by the microstructural balance.

We presume that the self-action z is continuous across the wavefront, so that the interfacial balance (17) reads

$$\left(\frac{\overset{*}{\partial S}}{\partial F}(m \otimes m^{\sharp})\right)[\ddot{y}^{\flat}] + \frac{\partial S}{\partial D\theta}(m \otimes m)[\ddot{\theta}] = 0. \tag{64}$$

Multiplying by ν', we get

$$\begin{aligned}[\ddot{\theta}] = -\left(\nu' \cdot \left(\frac{\partial S}{\partial D\theta}(m \otimes m)\right)\right)^{-1} \\ \left(\left(\nu' \frac{\overset{*}{\partial S}}{\partial F}(m \otimes m^{\sharp})\right) \cdot [\ddot{y}^{\flat}]\right),\end{aligned} \tag{65}$$

Substitution into Eq. (63) implies

$$\begin{aligned}&\left(\left(\frac{\partial P}{\partial D\theta}(m \otimes m)\right)\left(\nu' \cdot \frac{\partial S}{\partial D\theta}(m \otimes m)^{-1}\right)\right. \\ &\left.\left(\nu' \frac{\overset{*}{\partial S}}{\partial F}(m \otimes m) + \rho\hat{\mathtt{I}}U^2 - \frac{\overset{*}{\partial P}}{\partial F}(m \otimes m^{\sharp})\right)\right)[\ddot{y}] = 0,\end{aligned} \tag{66}$$

with $[\ddot{y}^{\flat}]_i = (\hat{\mathtt{I}})_{ij}[\ddot{y}]^j$, where $\hat{\mathtt{I}}$ is a spatial metric (in orthonormal frames it is the identity tensor).

Also, from the pointwise balance of forces (13), we also get $[\mathrm{Div}\dot{P}] = \rho[\ddot{y}^{\flat}]$, so, since in the case treated here $P = \bar{P}(F, \theta, D\theta)$, we compute

$$\begin{aligned}\rho[\ddot{y}^{\flat}] = \frac{\partial P}{\partial F}[D\dot{F}^*] + \left[\mathrm{Div}\left(\left(\frac{\overset{*}{\partial P}}{\partial F}\right)^{\underline{*}}\right)\right] \\ (\dot{F}^*)^+ - \left[\mathrm{Div}\left(\left(\frac{\overset{*}{\partial P}}{\partial F}\right)^{\underline{*}}\right)\right][\dot{F}^*] \\ + \left(\mathrm{Div}\left(\left(\frac{\overset{*}{\partial P}}{\partial F}\right)^{\underline{*}}\right)\right)^+ [\dot{F}^*] + \frac{\partial P}{\partial \theta}[D\dot{\theta}] \\ + \left[\mathrm{Div}\left(\frac{\partial P}{\partial \theta}\right)\right]\dot{\theta} + \frac{\partial P}{\partial D\theta}[D(D\dot{\theta})] \\ + \left[\mathrm{Div}\left(\frac{\partial P}{\partial D\theta}\right)^*\right](D\dot{\theta})^+ \\ - \left[\mathrm{Div}\left(\frac{\partial P}{\partial D\theta}\right)^*\right][D\dot{\theta}] + \left(\mathrm{Div}\left(\frac{\partial P}{\partial D\theta}\right)^*\right)^+ [D\dot{\theta}]\end{aligned} \tag{67}$$

By using Hadamard's lemma, since $[\dot{\theta}] = 0$, we thus obtain

$$\begin{aligned}
&\left(\rho \hat{\mathbb{I}} - \frac{1}{U^2}\left(\overset{*}{\frac{\partial P}{\partial F}}\right)(m\otimes m)\right)[\ddot{y}] - \left(\frac{1}{U^2}\frac{\partial P}{\partial D\theta}(m\otimes m)\right)[\ddot{\theta}] \\
&= -\left(Div\left(\left(\overset{*}{\frac{\partial P}{\partial F}}\right)^{\overset{*}{-}}\right)\right)^{+}\left(\frac{m}{U}\otimes[\dot{y}]\right) - \frac{\partial P}{\partial F}D\left(\frac{m}{U}\otimes[\dot{y}]\right) - \frac{\partial P}{\partial F}\left(\frac{m}{U}\otimes D[\dot{y}]\right) \\
&- \left(Div\left(\frac{\partial P}{\partial D\theta}\right)^{*}\right)\left(\frac{m}{U}[\dot{\theta}]\right) - \frac{\partial P}{\partial D\theta}D\left(\frac{m}{U}[\dot{\theta}]\right) - \frac{\partial P}{\partial D\theta}\left(\frac{m}{U}\otimes D[\dot{\theta}]\right) \\
&+ \frac{1}{U^2}\left\{\left(\frac{\partial^2 P}{\partial F\partial F}\right)^{***}((\dot{F}^*)^{+}\otimes m\otimes m)\right. \\
&\quad \left. + \left(\overset{*}{\frac{\partial^2 P}{\partial\theta\partial F}}\right)(\dot{\theta}m\otimes m) + \left(\frac{\partial^2 P}{\partial D\theta\partial F}\right)^{\overset{**}{-}}((D\dot{\theta}^{+})\otimes m\otimes m)\right\}[\dot{y}] \\
&+ \frac{1}{U^2}\left\{-U\frac{\partial P}{\partial\theta}m + \left(\frac{\partial^2 P}{\partial D\theta\partial D\theta}\right)^{\hat{\overset{**}{-}}}((D\dot{\theta}^{+})\otimes m\otimes m)\right\}[\dot{\theta}] \\
&+ \frac{1}{U^3}\left(\overset{*}{\frac{\partial^2 P}{\partial F\partial F}}\right)^{\overset{*}{-}}(m\otimes m\otimes m\otimes[\dot{y}]\otimes[\dot{y}]) \\
&+ \frac{1}{U^3}\left(\frac{\partial P}{\partial D\theta\partial F}\right)^{\overset{*}{-}}(m\otimes m\otimes m\otimes[\dot{y}][\dot{\theta}]) + \frac{1}{U^3}\left(\frac{\partial P}{\partial D\theta\partial D\theta}\right)(m\otimes m\otimes m[\dot{\theta}]^2).
\end{aligned} \tag{68}$$

The jump $[\ddot{\theta}]$ enters a list of unknowns. So, for $[\ddot{\theta}]$ we exploit Eq. (67), while for $[\ddot{\theta}]$ we refer to the entropy inequality and obtain

$$\begin{aligned}
&\theta U^2\left[\mathrm{Div}\left(\dot{\overline{\frac{\partial \overset{a}{\psi}}{\partial D\theta}}}\right)\right] - \theta U^2\left[\dot{\overline{\frac{\partial \overset{a}{\psi}}{\partial\theta}}}\right] + \theta U^2[(\nu''\,\dot{\theta})\cdot\beta] \\
&+ \theta U^2[\nu'\cdot\dot{\beta}] = -U[\dot{q}]\cdot m.
\end{aligned} \tag{69}$$

We define $\bar{\eta} := -\frac{\partial\overset{a}{\psi}}{\partial\theta}$ and $\mathfrak{f} := \frac{\partial\overset{a}{\psi}}{\partial D\theta}$. Then, we compute

$$\begin{aligned}
[\dot{\eta}] = [\dot{\bar{\eta}}] + [\mathrm{Di\dot{v}}\,\mathfrak{f}] &= \left[\frac{\partial\bar{\eta}}{\partial F}\cdot\dot{F}^* + \frac{\partial\bar{\eta}}{\partial\theta}\,\dot{\theta} + \frac{\partial\bar{\eta}}{\partial D\theta}\cdot\dot{D\theta}\right] \\
&+ \left[\left(\frac{\partial^2\mathfrak{f}}{\partial F\partial F}\dot{F}^* + \frac{\partial^2\mathfrak{f}}{\partial F\partial\theta}\,\dot{\theta} + \frac{\partial^2\mathfrak{f}}{\partial F\partial D\theta}\,D\dot{\theta}\right)\cdot({}^*(DF))\right] \\
&+ \left[\frac{\partial\mathfrak{f}}{\partial F}\cdot({}^*(D\dot{F}))\right] + \left[\left(\frac{\partial^2\mathfrak{f}}{\partial\theta\partial F}\dot{F}^* + \frac{\partial^2\mathfrak{f}}{\partial\theta\partial\theta}\,\dot{\theta}\right.\right. \\
&+ \left.\left.\frac{\partial^2\mathfrak{f}}{\partial\theta\partial D\theta}\,D\dot{\theta}\right)\cdot D\theta\right] + \left[\frac{\partial\mathfrak{f}}{\partial\theta}\cdot D\dot{\theta}\right] \\
&+ \left[\left(\frac{\partial^2\,\mathfrak{f}}{\partial D\theta\partial F}\dot{F}^* + \frac{\partial^2\mathfrak{f}}{\partial D\theta\partial\theta}\,\dot{\theta} + \frac{\partial^2\mathfrak{f}}{\partial D\theta\partial D\theta}\,D\dot{\theta}\right)\cdot D(D\theta)\right] \\
&+ \left[\frac{\partial\mathfrak{f}}{\partial D\theta}\cdot D(D\dot{\theta})\right].
\end{aligned} \tag{70}$$

By rearranging terms and dividing by θ, we get

$$\begin{aligned}
&\left({}^*\left(\frac{\partial\mathfrak{f}}{\partial F}\right)(m\otimes m)\right)\cdot[\ddot{y}] + \left(\frac{\partial\mathfrak{f}}{\partial D\theta}\cdot(m\otimes m)\right)[\ddot{\theta}] = U^2\frac{\partial\mathfrak{f}}{\partial F}\cdot D\left(\frac{m}{U}\otimes[\dot{y}]\right) \\
&\quad + U^2\frac{\partial\mathfrak{f}}{\partial F}\cdot\left(\frac{m}{U}\otimes D[\dot{y}]\right) + U^2\frac{\partial\mathfrak{f}}{\partial D\theta}\cdot D\left(\frac{m}{U}[\dot{\theta}]\right) \\
&\quad + U^2\frac{\partial\mathfrak{f}}{\partial D\theta}\cdot\left(\frac{m}{U}\otimes D[\dot{\theta}]\right) \\
&\quad + U\left\{\frac{\partial\bar{\eta}}{\partial F}m +^* \left(\frac{\partial^2\mathfrak{f}}{\partial F\partial F}\right)((\dot{F}^*)^+\otimes m\otimes m)\right. \\
&\quad - \frac{1}{U}{}^*\left(\frac{\partial^2\mathfrak{f}}{\partial F\partial F}\right)(({}^*(DF))^+\otimes m) - \frac{1}{U}{}^*\left(\frac{\partial^2\mathfrak{f}}{\partial F\partial\theta}\right)(\dot{\theta}m\otimes m) \\
&\quad - \frac{1}{U}{}^*\left(\frac{\partial^2\mathfrak{f}}{\partial F\partial D\theta}\right)(D\dot{\theta}^+\otimes m\otimes m) +^* \left(\frac{\partial^2\mathfrak{f}}{\partial\theta\partial F}\right)(D\theta\otimes m) \\
&\quad \left. + \left(\overset{*}{-}\left(\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial F}\right)\right)^{\overset{*}{-}}((D(D\theta))^+\otimes m)\right\}\cdot[\dot{y}]
\end{aligned} \tag{71}$$

$$\begin{aligned}
&+ U\left\{\frac{\partial\bar{\eta}}{\partial D\theta}\cdot m + \left(\overset{*}{\frac{\partial^2\mathfrak{f}}{\partial F\partial D\theta}}\right)^{\overset{t}{*}}\cdot(({}^*(DF))^+\otimes m) + \left(\frac{\partial^2\mathfrak{f}}{\partial\theta\partial D\theta}\right)^*\right. \\
&\cdot(D\theta\otimes m) + \frac{\partial\mathfrak{f}}{\partial\theta}\cdot m - \frac{1}{U}\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial F}\cdot((\dot{F}^*)^+\otimes m\otimes m) - \frac{1}{U}\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial\theta} \\
&\cdot(\dot{\theta}m\otimes m) - \frac{1}{U}\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial D\theta}\cdot((D\dot{\theta})^+\otimes m\otimes m) + \left(\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial D\theta}\right)^{\overset{t}{*}} \\
&\left.\cdot((D(D\theta))^+\otimes m) + \frac{K}{\theta U}(m\cdot m)\right\}[\dot{\theta}] - \frac{1}{U}{}^*\left(\frac{\partial^2\mathfrak{f}}{\partial F\partial F}\right)^{\overset{*}{-}} \\
&\cdot(m\otimes m\otimes m\otimes[\dot{y}]\otimes[\dot{y}]) - \left({}^*\left(\frac{\partial^2\mathfrak{f}}{\partial F\partial D\theta}\right)\right) \\
&\cdot(m\otimes m\otimes m\otimes[\dot{y}][\dot{\theta}]) - \frac{1}{U}\overset{*}{-}\left(\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial F}\right) \\
&\cdot(m\otimes m\otimes m\otimes[\dot{y}][\dot{\theta}]) - \frac{1}{U}\left(\frac{\partial^2\mathfrak{f}}{\partial D\theta\partial D\theta}\right)\cdot(m\otimes m\otimes m[\dot{\theta}]^2).
\end{aligned}$$

In Eq. (68), terms multiplying $[\ddot{y}]$ and $[\ddot{\theta}]$ are those that in Eq. (63) multiply $[\ddot{y}]$ and $[\ddot{\theta}]$. Also, as regards Eq. (71), from Eq. (51) we get $S^*\nu' = \frac{\partial\overset{a}{\psi}}{\partial D\theta}$, so that $\mathfrak{f} = \nu' S$ and

$$\left({}^*\left(\frac{\partial\mathfrak{f}}{\partial F}\right)\right) = \left({}^*\left(\frac{\partial}{\partial F}(\nu' S)\right)\right) = \nu'\,\overset{*}{\frac{\partial S}{\partial F}},$$

and

$$\frac{\partial\mathfrak{f}}{\partial D\theta} = \frac{\partial}{\partial D\theta}(\nu' S) = \nu'\,\frac{\partial S}{\partial D\theta}.$$

So, Eq. (64) can be rewritten as

$$\left({}^*\left(\frac{\partial\mathfrak{f}}{\partial F}\right)(m\otimes m)\right)[\ddot{y}] + \frac{\partial\mathfrak{f}}{\partial D\theta}(m\otimes m)[\ddot{\theta}] = 0 \tag{72}$$

Terms multiplying $[\ddot{y}]$ and $\ddot{\theta}$ in Eq. (73) are the same that in Eq. (71) multiply $[\ddot{y}]$ and $[\ddot{\theta}]$.

Define M and R as follows:

$$M = \left(\rho \hat{\mathrm{I}} - \frac{1}{U^2} \left(\overset{*}{\frac{\partial P}{\partial F}} \right) (m \otimes m) \right) [\ddot{y}] - \frac{1}{U^2} \frac{\partial P}{\partial D\theta} (m \otimes m) [\ddot{\theta}],$$

$$R = \left(\overset{*}{\left(\frac{\partial \mathfrak{f}}{\partial F} \right)} (m \otimes m) \right) \cdot [\ddot{y}] + \frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) [\ddot{\theta}].$$

From Eqs. (68) and (71), we get

$$[\ddot{\theta}] = \left(\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) \right)^{-1} R - \left(\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) \right)^{-1} \left(\overset{*}{\left(\frac{\partial \mathfrak{f}}{\partial F} \right)} (m \otimes m) \right) \cdot [\ddot{y}] \tag{73}$$

and

$$- \left(\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) \right)^{-1} R + M = \left(\rho \hat{\mathrm{I}} - \frac{1}{U^2} \left(\overset{*}{\frac{\partial P}{\partial F}} \right) (m \otimes m) + \left(\frac{1}{U^2} \frac{\partial P}{\partial D\theta} (m \otimes m) \right) \left(\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) \right)^{-1} \left(\overset{*}{\left(\frac{\partial \mathfrak{f}}{\partial F} \right)} (m \otimes m) \right) [\ddot{y}] \right) \tag{74}$$

The right-hand side term of Eq. (75) is the same in (66). Thus, we set

$$[\ddot{y}] = \sigma\, r,$$

where σ is the jump amplitude and r the corresponding eigenvector. We multiply Eq. (68) by r^L and substitute $[\ddot{y}]$ with $\sigma\, r^R$, where r^R is the right eigenvector of the same matrix.

Also, by decomposing R and M into

$$R = R1 + R2 + R3 \qquad \text{and} \qquad M = M1 + M2 + M3, \tag{75}$$

where the explicit expressions of the above addenda are in "Appendix", we then get

$$\sigma^{-1} D(r^L \cdot \partial_m \lambda \; r^R \sigma^2) = \left(\frac{1}{U^2} \frac{\partial P}{\partial D\theta} (m \otimes m) \right) \left(\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) \right)^{-1} R1 + M1,$$

where

$$\lambda = \left(\overset{*}{\frac{\partial \bar{P}}{\partial F}} \right) (m \otimes m) - \left(\frac{\partial P}{\partial D\theta} (m \otimes m) \right) \left(\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m) \right)^{-1} \left(\left(\overset{*}{\left(\frac{\partial \mathfrak{f}}{\partial F} \right)} (m \otimes m) \right) \right).$$

By substituting $[\ddot{\theta}]$ with its expression given by Eq. (65) and considering once again $[\ddot{y}] = \sigma\, r$ in the expressions of $R1$ and $M1$, we obtain a standard equation of Bernoulli's form, namely

$$(\mathcal{J} \sigma^2)' = A\, \mathcal{J} \sigma^2 - B\, \mathcal{J} \sigma^3, \tag{76}$$

with coefficients

$$A := r^L \{ (\frac{1}{U^2} \frac{\partial P}{\partial D\theta} (m \otimes m)) (\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m))^{-1} R2 + M2 \} \tag{77}$$

and

$$B := -r^L \{ (\frac{1}{U^2} \frac{\partial P}{\partial D\theta} (m \otimes m)) (\frac{\partial \mathfrak{f}}{\partial D\theta} \cdot (m \otimes m))^{-1} R3 + M3 \} \tag{78}$$

far from being those obtained in the standard analysis of acceleration waves in simple thermoelastic media.

When the wavefront shape is constant so is $\mathcal{J}$. In this case we find a decaying-type solution given by the classical form

$$\sigma(t) = \frac{A}{(\frac{A}{\sigma_0} - B)\, e^{-\frac{A}{2} t} + B}. \tag{79}$$

10 Acceleration waves in a special composite

Consider a composite constituted by wire-type inclusions of a Strontium titanate ceramic into a conducting matrix. Figure 3 depicts the circumstance in a schematic way.

In this case $\nu = \mathbf{p} = f(\theta)\mathfrak{l} = f(\theta)\mathbf{e}_1$, where $\mathbf{e}_1$ is the direction along which the inclusions are aligned, so that $D\nu = D\mathbf{p} = f'(\theta)\mathbf{e}_1 \otimes D\theta$, with $f'(\theta) \neq 0$ by assumption.

In the case of uniform temperature distribution by the formula above we have $D\nu = 0$ because $D\theta = 0$. By allowing large strain we thus consider a free energy $\psi = \tilde{\psi}(F, \nu, \theta) = \tilde{\psi}(F, \theta)$ given explicitly by

$$\psi = \frac{1}{2}\alpha |F|^2 + G \cdot F\theta + \frac{1}{2}\delta\theta^2 + (-\log(\det F) \vee 0),$$

where $G = \mathbf{e}_1 \otimes \mathbf{e}_1$, α and δ are positive constants, and $\vee$ indicates selects the maximum between $-\log(\det F)$ and 0; the last terms avoids the well-known physical incompatibility between convexity of ψ with respect to F and objectivity of the energy itself (see [52, Ch. 11] for further expressions of polyconvex energies).

With these premises, $S = 0$ and also $A = 0$, so that $\sigma = 0$. In other words, a uniform heating does not allow propagation of acceleration waves in the composite specific material considered in Fig. 3.

At variance, in the case of non-uniform heating, since $D\theta \neq 0$, we may choose an expression of the free energy $\psi = \tilde{\psi}(F, \nu, D\nu, \theta) = \tilde{\psi}(F, \theta, D\theta)$ given by

$$\psi = \frac{1}{2}\alpha|F|^2 + G \cdot F\theta + \frac{1}{2}\delta\theta^2 + \mathfrak{c} f'^2(\theta)|F|^2|D\theta|^2 + (-\log(\det F) \vee 0),$$

where $\mathfrak{c}$ is a positive constant. Thus, along the path followed in the previous section we may check that generically $A \neq 0$, so that the propagation of acceleration waves is admissible.

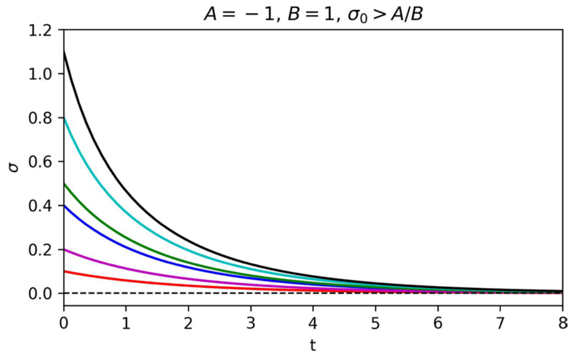

Fig. 4 With $A > -1$, $B = 1$, and $\sigma_0 > \frac{A}{B}$, σ decays asymptotically in time towards 0

11 Additional remarks

Remark 1 A constraint of the type $\nu := \hat{\nu}(F, \theta)$, with F the deformation gradient, implies second-grade thermoelasticity (see [2] for the isothermal case). Thus, proving existence of a microstress which develop internal work on $D\nu$ implies the one of a hyperstress developing internal work on DF. Analyzing acceleration waves in this case is not a straightforward replica of what has been shown in the previous section. Indeed, reproducing the path followed so far, we would meet the occurrence of $[\ddot{y}]$, beyond $[\ddot{\theta}]$ already met here, as an additional unknown. To manage it we could exploit the first law of thermodynamics. The detailed analysis is not tackled here.

Remark 2 When ν is a scalar or a pseudo-scalar taking values in an interval $(0, k)$, $k > 0$, let us say, the procedure based on the external power invariance, as adopted above to derive balance equations, does not work because $\mathcal{A}$ vanishes, since a rigidly rotating observer does change the perception of scalar or pseudo-scalar entities. To avoid the difficulty we need a request of covariance (meaning structure-invariance under diffeomorphism based changes of observers) for the second law of thermodynamics, written in terms of Clausius-Duhem inequality. This covariance principle has been introduced in [31] and analyzed in [36]. Such an approach, also, would eliminate the indeterminacy of z, which is at variance defined by the external power invariance to within an arbitrary element of ker$\mathcal{A}$. Notice also that the presence of an internal constrain like the one discussed here induced implicitly another indeterminacy in the self-action z because we do not deduce naturally a constitutive prescription for $\overset{a}{z}$ alone, as in the absence of internal constraints, where the energetic part of z is fully defined by the derivative $\frac{\partial\psi}{\partial\nu}$.

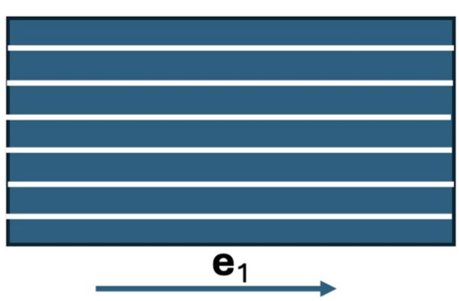

Fig. 3 Composite with Strontium titanate inclusions into a conducting matrix; the distance between neighboring inclusions is magnified

Remark 3 A parametric analysis reveals how A, B, and the initial value σ_0 in (80) influence the decay of σ (see also [43]). In summary, if $A < 0$, $B > 0$, and $\sigma_0 > \frac{A}{B}$, or, vice versa, both A and B are negative and $\sigma_0 > \frac{A}{B}$, the amplitude σ_0 exponentially decays to 0; see Fig. 4. If $A > 0$ and $B\sigma_0 > 0$, σ reaches asymptotically the value $\frac{A}{B}$ (Fig. 5). Finally, σ blows up in a finite time t_b if one of the following conditions occurs:

1. $A > 0$ and $B\sigma_0 < 0$;
2. $A < 0$, $B > 0$, and $\sigma_0 < \frac{A}{B}$;
3. $A < 0$, $B < 0$, and $\sigma_0 > \frac{A}{B}$.

For t_b we have (see also [43])

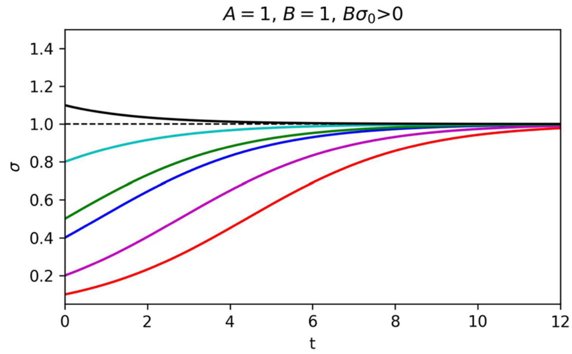

Fig. 5 If $A > 0$ and $B\sigma_0 > 0$, σ reaches asymptotically the value $\frac{A}{B}$

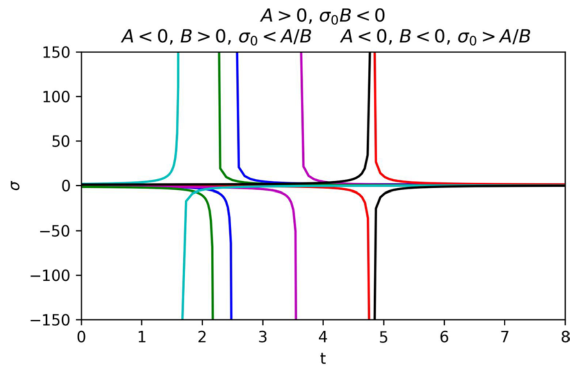

Fig. 6 Blow up behavior under different values of σ_0; the blow up instant corresponds to spikes

$$t_b := \frac{2}{A} \ln\big(1 - \frac{A}{\sigma_0 B}\big)$$

(see Fig. 6).

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Availability of data and materials Not applicable.

Declarations

Ethical approval Not applicable.

Appendix: Explicit expression of the addenda in (76)

$$\begin{aligned} R1 :=& U^2 \frac{\partial \mathfrak{f}}{\partial F} \cdot D\big(\frac{m}{U} \otimes [\ddot{y}]\big) + U^2 \frac{\partial \mathfrak{f}}{\partial F} \cdot \big(\frac{m}{U} \otimes D[\ddot{y}]\big) \\ &+ U^2 \frac{\partial \mathfrak{f}}{\partial D\theta} \cdot D\big(\frac{m}{U}[\ddot{\theta}]\big) + U^2 \frac{\partial \mathfrak{f}}{\partial D\theta} \cdot \big(\frac{m}{U} \otimes D[\ddot{\theta}]\big), \end{aligned}$$

$$\begin{aligned} R2: =& U\Big\{ \frac{\partial \bar{\eta}}{\partial F} m + \Big({}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial F} \Big)\Big) (\dot{F}^{*})^{+} \otimes m \otimes m \\ &- \frac{1}{U} {}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial F} \Big) {}^{*}(DF))^{+} \otimes m) - \frac{1}{U} {}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial \theta} \Big) (\dot{\theta} m \otimes m) \\ &- \frac{1}{U} {}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial D\theta} \Big) (D\dot{\theta}^{+} \otimes m \otimes m) + \Big({}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial \theta \partial F} \Big)\Big) (D\theta \otimes m) \\ &+ \Big({}^{\underline{*}}\Big(\frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial F} \Big)^{\underline{*}}\Big) ((D(D\theta))^{+} \otimes m) \Big\} \cdot [\ddot{y}] \\ &+ U\Big\{ \frac{\partial \bar{\eta}}{\partial D\theta} \cdot m + \overset{*}{\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial D\theta} \Big)}{}^{\overset{t}{*}} \cdot (({}^{*}(DF))^{+} \otimes m) \\ &+ \Big(\frac{\partial^2 \mathfrak{f}}{\partial \theta \partial D\theta} \Big)^{*} \cdot (D\theta \otimes m) + \frac{\partial \mathfrak{f}}{\partial \theta} \cdot m \\ &- \frac{1}{U} \frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial F} \cdot ((\dot{F}^{*})^{+} \otimes m \otimes m) \\ &- \frac{1}{U} \frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial \theta} \cdot (\dot{\theta} m \otimes m) - \frac{1}{U} \frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial D\theta} \cdot ((D\dot{\theta})^{+} \otimes m \otimes m) \\ &+ \Big(\frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial D\theta} \Big)^{\overset{t}{*}} \cdot ((D(D\theta))^{+} \otimes m) \\ &+ \frac{\kappa}{\theta U} (m \cdot m) \}[\ddot{\theta}], \end{aligned}$$

$$\begin{aligned} R3 :=& - \frac{1}{U} {}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial F} \Big)^{\underline{*}} \cdot (m \otimes m \otimes m \otimes [\ddot{y}] \otimes [\ddot{y}]) \\ &- \frac{1}{U} {}^{*}\Big(\frac{\partial^2 \mathfrak{f}}{\partial F \partial D\theta} \Big) \cdot (m \otimes m \otimes m \otimes [\ddot{y}][\ddot{\theta}]) \\ &- \frac{1}{U} {}^{\underline{*}}\Big(\frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial F} \Big) \cdot (m \otimes m \otimes m \otimes [\ddot{y}][\ddot{\theta}]) \\ &- \frac{1}{U} \Big(\frac{\partial^2 \mathfrak{f}}{\partial D\theta \partial D\theta} \Big) \cdot (m \otimes m \otimes m[\ddot{\theta}]^2), \end{aligned}$$

$$\begin{aligned} M1 :=& - \Big(\mathrm{Div} \Big(\Big(\overset{*}{\Big(\frac{\partial P}{\partial F} \Big)} \Big)^{\underline{*}} \Big)^{+} \big(\frac{m}{U} \otimes [\ddot{y}] \big) - \frac{\partial P}{\partial F} D\big(\frac{m}{U} \otimes [\ddot{y}] \big) \\ &- \frac{\partial P}{\partial F} \big(\frac{m}{U} \otimes D[\ddot{y}] \big) - \mathrm{Div} \Big(\frac{\partial P}{\partial D\theta} \Big)^{*} \big(\frac{m}{U} [\ddot{\theta}] \big) \\ &- \frac{\partial P}{\partial D\theta} D\big(\frac{m}{U} [\ddot{\theta}] \big) - \frac{\partial P}{\partial D\theta} \big(\frac{m}{U} \otimes D[\ddot{\theta}] \big), \end{aligned}$$

$$M2:=\frac{1}{U^2}\{\left(\frac{\partial^2 P}{\partial F\partial F}\right)^{\bar{*}\breve{*}\tilde{*}}((\dot{F}^*)^+\otimes m\otimes m)+\left(\frac{\overset{*}{\partial^2 P}}{\partial\theta\partial F}\right)(\dot{\theta}\, m\otimes m)$$

$$+\left(\frac{\partial^2 P}{\partial D\theta\partial F}\right)^{\underline{*}\breve{*}}((D\dot{\theta}^+)\otimes m\otimes\ m)\}[\ddot{y}]$$

$$+\frac{1}{U^2}\left\{-U\frac{\partial P}{\partial\theta}m+\left(\frac{\partial^2 P}{\partial D\theta\partial D\theta}\right)^{\underline{*}\hat{*}}((D\dot{\theta}^+)\otimes\ m\otimes\ \right\}[\ddot{\theta}],$$

$$M3\ :=\frac{1}{U^3}\left(\frac{\overset{*}{\partial^2 P}}{\partial F\partial F}\right)^{\underline{*}}(m\otimes m\otimes m\otimes[\ddot{y}]\otimes[\ddot{y}])$$

$$+\frac{1}{U^3}\left(\frac{\partial P}{\partial D\theta\partial F}\right)^{\underline{*}}(m\otimes\ m\otimes\ m\,[\ddot{y}][\ddot{\theta}])$$

$$+\frac{1}{U^3}\left(\frac{\partial P}{\partial D\theta\partial D\theta}\right)(m\otimes\ m\otimes\ m\,[\ddot{\theta}]^2),$$

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