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Connecting beams and continua: variational basis and mathematical analysis

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Abstract We present a new variational principle for linking models of beams and deformable solids, providing also its mathematical analysis. Despite the apparent differences between the two types of governing equations, it will be shown that the equilibrium of systems combining beams and solids can be obtained from a joint constrained variational principle and that the resulting boundary-value problem is well posed.

Keywords Variational principles · Beams · Elasticity · Well-posedness

1 Introduction

The problems of beams and deformable solids refer both to the mechanical response of bodies when subjected to external actions, including forces, torques, and imposed displacements. However, from the mathematical viewpoint, these two problems are intrinsically very different. Even when restricted to small strains, the kinematics of these two types of bodies are disparate: whereas the former is described by a displacement field on an open set of two or three-dimensional Euclidean space, the latter depends on the displacement and the rotation on an interval of the real line. The equilibrium equations of a deformable solid, moreover, are *partial* differential equations, in contrast with the *ordinary* differential equations that describe the equilibrium of forces and momenta in a beam.

Despite the apparent differences between the mathematical description of the mechanics of beams and deformable solids, there are deep relations between them. After all, beams are nothing but a special class of solids whose equations can be obtained from the equations of solid mechanics by exploiting some asymptotic behavior or by constraining the class of admissible kinematics (see, for example, [6, 7] for a description of these two avenues for model reduction).

One specific aspect that is of both theoretical and practical interest is the combination of the equations of beams and solids within a single mechanical system or structure. From the theoretical point of view, the interest lies in the formulation of links between these two types of equations and the well-posedness of the resulting boundary-value problems. From the practical side, joint beam/solid equations lead to numerical methods that can efficiently represent the behavior of (beam) structures with subsets studied as three-dimensional solids.

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In a related article, we have presented novel formulations of coupled beam/solid mechanics that lead to numerical methods, both in the linear and nonlinear regimes [11]. These formulations, based on new variational principles, can be easily discretized using, for example, finite elements, and replace commonly employed ad hoc links between beams and solids (e.g., [5, 12, 13]). The latter, often based on constraints on the discrete solution, lack a variational basis and thus neither their well-posedness nor their stability can be ascertained.

In this article, we study boundary-value problems of linked, deformable, beams and solids in the context of linearized elasticity, as defined by a constrained variational principle. The main goal is to prove the well-posedness of problems with beams and solids involving the minimum set of boundary conditions, effectively proving that the linking terms provide the right stability to the equations, precluding rigid body motions of the system. The boundary-value problems that will be studied have the structure of saddle-point optimization problems in Hilbert spaces (e.g., [2]) and standard analysis techniques can be used to study their stability and well-posedness. The problem in consideration is actually a paradigmatic example of a larger class of problems of linearized elasticity in which the Dirichlet boundary conditions are weakly imposed.

In Sect. 2 we summarize the equations that govern deformable solids and beams in the context of small strain kinematics, highlighting the variational statement of these two problems and their essential mathematical properties. Section 3 formulates the simplest problem consisting of a beam and a solid that share an interface with the minimum set of Dirichlet boundary conditions. A joint variational principle, where the kinematic compatibility is introduced with Lagrange multipliers, is presented as well. The well-posedness of the resulting boundary-value problem is analyzed in Sect. 4. Section 5 provides an illustration of the variational principle proposed in Sect. 3, using it to find the optimal distribution of the tractions on a surface of a body when only its resultant and moment are known. The article concludes with a summary of the main results in Sect. 6.

2 Problem statement

This article analyses boundary value problems of joint continuum solids and beams whose solutions correspond to the mechanical equilibrium of both types of bodies, as well as certain compatibility relations in their shared interfaces. Before formulating the global problem, the governing equations of elasticity and beams are briefly reviewed, and their main mathematical properties are identified.

The choice of boundary conditions in these problems is crucial. To show that the constraints that are later introduced effectively link beams and solids, we will present the pure traction problem of an elastic solid and a mixed traction-displacement problem of a beam. Later, we will prove that these two bodies, when appropriately connected, result in a stable structure.

2.1 The Neumann problem of small strain, elastic solids

We start by describing the continuum solid, and we restrict our presentation to an elastic one that occupies a bounded open set $\mathcal{B} \subset \mathbb{R}^3$ with volume $|\mathcal{B}|$. The boundary of the solid is denoted $\partial\mathcal{B}$ and we identify a subset $\Sigma \subsetneq \partial\mathcal{B}$ that will later be linked to a beam.

In classical elasticity, the unknown is the displacement $\boldsymbol{u} \in \mathcal{U} := \left[H^1(\mathcal{B})\right]^3$, where H^1 is the Hilbert space of vector fields with (Lebesgue) square-integrable components and square-integrable (weak) first derivatives. The stored energy of the deformable body is given by a scalar function $W = \hat{W}(\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u} := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$ is the infinitesimal strain tensor and ∇ is the gradient operator. More specifically, for linear isotropic materials this function takes the form $\hat{W}(\boldsymbol{\varepsilon}) = \mu \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \frac{\lambda}{2}\mathrm{tr}[\boldsymbol{\varepsilon}]^2$ where λ, μ are the two Lamé constants, the dot product refers to the complete index contraction, and $\mathrm{tr}[\cdot]$ is the trace operator.

Considering that the body might be subject to body forces $\boldsymbol{f} \in [L^2(\mathcal{B})]^3$ and surface tractions $\boldsymbol{t}$ on $\partial\mathcal{B}\backslash\Sigma$, the total potential energy of the body is

$$\Pi_{\mathcal{B}}(\boldsymbol{u}) := \frac{1}{2} a_{\mathcal{B}}(\boldsymbol{u}, \boldsymbol{u}) - f_{\mathcal{B}}(\boldsymbol{u}), \tag{1}$$

with

$$a_{\mathcal{B}}(\boldsymbol{u},\boldsymbol{v}) := \int_{\mathcal{B}} (2\mu\nabla^s\boldsymbol{u}\cdot\nabla^s\boldsymbol{v} + \lambda(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{v}))\,\mathrm{d}V \tag{2a}$$

$$f_{\mathcal{B}}(\boldsymbol{u}) := \int_{\mathcal{B}} \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}V + \int_{\partial\mathcal{B}\setminus\Sigma} \boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}A, \tag{2b}$$

for all $\boldsymbol{u},\boldsymbol{v}\in\mathcal{U}$. We note, in passing, that the potential energy (1) might not have any minimiser in $\mathcal{U}$ unless the forces are statically equilibrated, i.e., they must satisfy

$$\begin{aligned}\int_{\mathcal{B}} \boldsymbol{f}\,\mathrm{d}V + \int_{\partial\mathcal{B}\setminus\Sigma} \boldsymbol{t}\,\mathrm{d}S = \boldsymbol{0},\\ \int_{\mathcal{B}} \boldsymbol{x}\times\boldsymbol{f}\,\mathrm{d}V + \int_{\partial\mathcal{B}\setminus\Sigma} \boldsymbol{x}\times\boldsymbol{t}\,\mathrm{d}S = \boldsymbol{0},\end{aligned} \tag{3}$$

where $\boldsymbol{x}$ denotes the position vector of points on the body or its boundary. In this case, a solution exists and is unique modulo infinitesimal rigid body motions [8].

To set up the analysis framework for the study of three-dimensional solids, we first recall the norm on the space $\mathcal{U}$ which has the standard form

$$\|\boldsymbol{u}\|_{\mathcal{U}} := \left(\|\boldsymbol{u}\|^2_{[L^2(\mathcal{B})]^3} + \ell^2\|\nabla\boldsymbol{u}\|^2_{[L^2(\mathcal{B})]^3}\right)^{1/2}, \tag{4}$$

where ℓ is a characteristic length of the solid, for example its diameter. The bilinear form (2a) verifies the following continuity and stability bounds

$$|a_{\mathcal{B}}(\boldsymbol{u},\boldsymbol{v})| \le C_{\mathcal{B}}\,\|\boldsymbol{u}\|_{\mathcal{U}}\,\|\boldsymbol{v}\|_{\mathcal{U}}, \tag{5a}$$

$$a_{\mathcal{B}}(\boldsymbol{u},\boldsymbol{u}) + \|\boldsymbol{u}\|^2_{[L^2(\mathcal{B})]^3} \ge \alpha_{\mathcal{B}}\,\|\boldsymbol{u}\|^2_{\mathcal{U}}, \tag{5b}$$

for some positive constants $C_{\mathcal{B}}, \alpha_{\mathcal{B}}$, and all $\boldsymbol{u},\boldsymbol{v}\in\mathcal{U}$. It bears emphasis that, due to the lack of Dirichlet boundary conditions on the boundary of the body, the bilinear form $a_{\mathcal{B}}(\cdot,\cdot)$ is not coercive in $\mathcal{U}$. Rather, and based on Korn's second inequality [9], only the weaker statement (5b) can be made. Also, the linear form $f_{\mathcal{B}}$ is assumed to be continuous, i.e.,

$$f_{\mathcal{B}}(\boldsymbol{u}) \le c_{\mathcal{B}}\,\|\boldsymbol{u}\|_{\mathcal{U}}, \tag{6}$$

with $c_{\mathcal{B}} > 0$ for all $\boldsymbol{u}\in\mathcal{U}$. Under the previous conditions, the Neumann problem is well-posed in the quotient space $\mathcal{U}/\mathrm{ker}[\nabla^s]$ (see, e.g. [4]).

2.2 Beam mechanics

We describe next the equations that govern the so-called Timoshenko beam elasto-statics. This is a classical problem of mechanics (see, e.g., [1, 14, 14]) and we present it here in a succinct fashion that is enough for the goals of this article.

A cantilever beam of length L is now studied. Its curve of centroids is described by a known smooth curve $\boldsymbol{r} : [0,L] \to \mathbb{R}^3$, with a cross-section attached to each point of the curve and oriented according to a known smooth rotation field $\boldsymbol{\Lambda} : [0,L] \to SO(3)$, the latter referring to the set of proper orthogonal tensors. The points on $\boldsymbol{r}$ and sections $\boldsymbol{\Lambda}$ are parameterized by the arclength $s\in[0,L]$ and we choose $s=0$ and $s=L$ to correspond, respectively, to the clamped section and free tip.

Let $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ be a Cartesian basis. Then $\boldsymbol{\Lambda}(s)$ maps $\boldsymbol{e}_3$ to the unit tangent vector to the curve of centroids at the point $\boldsymbol{r}(s)$, and $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ to the directions of the principal axis of the cross-section at the same point. The displacement of the centroids will be given by the vector field $\boldsymbol{w}\in\mathcal{W} := [H^1_0(0,L)]^3$ and the incremental rotation vector of the cross sections as $\boldsymbol{\theta}\in\mathcal{R} := [H^1_0(0,L)]^3$. Following our previous notation, $[H^1_0(0,L)]^3$ refers to the Hilbert space of vectors fields on $(0, L)$ with vanishing trace at $s=0$.

Shear deformable, three-dimensional beams employ two deformation measures, namely,

$$\begin{aligned}\boldsymbol{\Gamma} = \hat{\boldsymbol{\Gamma}}(\boldsymbol{w},\boldsymbol{\theta}) &:= \boldsymbol{\Lambda}^T(\boldsymbol{w}' - \boldsymbol{\theta}\times\boldsymbol{r}'),\\ \boldsymbol{\Omega} = \hat{\boldsymbol{\Omega}}(\boldsymbol{\theta}) &:= \boldsymbol{\Lambda}^T\boldsymbol{\theta}',\end{aligned} \tag{7}$$

where the prime symbol denotes the derivative with respect to the arc-length. The strain $\boldsymbol{\Gamma}$ holds the shear and axial deformations, whereas the vector $\boldsymbol{\Omega}$ contains the two bending curvatures and the torsion deformation.

The simplest *section* constitutive law for a beam of a linear elastic and isotropic material with Young's and shear moduli E, G, respectively, is based on a quadratic stored energy function per unit length. It has the form

$$U(\boldsymbol{\Gamma},\boldsymbol{\Omega}) := \frac{1}{2}\boldsymbol{\Gamma}\cdot\boldsymbol{C}_{\Gamma}\boldsymbol{\Gamma} + \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{C}_{\Omega}\boldsymbol{\Omega} \tag{8}$$

with section stiffness $\boldsymbol{C}_{\Gamma} = \mathrm{diag}[GA_1, GA_2, EA]$ and $\boldsymbol{C}_{\Omega} = \mathrm{diag}[EI_1, EI_2, GI_t]$, where A is the cross section area, A_1, A_2 are the (shear) reduced sections areas in the two principal directions, I_1, I_2 are the two principal moments of inertia, and I_t is the torsional inertia. When the beam is under distributed loads and moments, denoted respectively as $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$, and subject to a concentrated load $\bar{\boldsymbol{P}}$ and torque $\bar{\boldsymbol{Q}}$ at the tip, its total potential energy can be expressed as

$$\Pi_b(\boldsymbol{w},\boldsymbol{\theta}) := \frac{1}{2} a_b(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{w},\boldsymbol{\theta}) - f_b(\boldsymbol{w},\boldsymbol{\theta}), \tag{9}$$

with $(\boldsymbol{w},\boldsymbol{\theta}) \in \mathcal{W}\times\mathcal{R}$ and

$$\begin{aligned} a_b(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{t},\boldsymbol{\beta}) &:= \int_0^L \left(\hat{\boldsymbol{\Gamma}}(\boldsymbol{w},\boldsymbol{\theta})\cdot \boldsymbol{C}_\Gamma \hat{\boldsymbol{\Gamma}}(\boldsymbol{t},\boldsymbol{\beta}) + \hat{\boldsymbol{\Omega}}(\boldsymbol{\theta})\cdot \boldsymbol{C}_\Omega \hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})\right) \mathrm{d}S, \\ f_b(\boldsymbol{t},\boldsymbol{\beta}) &:= \int_0^L (\bar{\boldsymbol{n}}\cdot\boldsymbol{t} + \bar{\boldsymbol{m}}\cdot\boldsymbol{\beta})\,\mathrm{d}S + \bar{\boldsymbol{P}}\cdot\boldsymbol{u}(L) + \bar{\boldsymbol{Q}}\cdot\boldsymbol{\theta}(L), \end{aligned} \tag{10}$$

for all $(\boldsymbol{t},\boldsymbol{\beta}) \in \mathcal{W}\times\mathcal{R}$.

To set up the functional setting for the beam problem, we recall the norms on the space of displacements and rotations which are

$$\begin{aligned} \|\boldsymbol{w}\|_{\mathcal{W}} &:= \left(\|\boldsymbol{w}\|^2_{[L^2(0,L)]^3} + L^2\|\boldsymbol{w}'\|^2_{[L^2(0,L)]^3}\right)^{1/2}, \\ \|\boldsymbol{\theta}\|_{\mathcal{R}} &:= \left(\|\boldsymbol{\theta}\|^2_{[L^2(0,L)]^3} + L^2\|\boldsymbol{\theta}'\|^2_{[L^2(0,L)]^3}\right)^{1/2}. \end{aligned} \tag{11}$$

Also, the product space $\mathcal{W}\times\mathcal{R}$, the natural setting for the beam problem, has the product norm

$$\|(\boldsymbol{w},\boldsymbol{\theta})\|_{\mathcal{W}\times\mathcal{R}} = \left(\|\boldsymbol{w}\|^2_{\mathcal{W}} + L^2\|\boldsymbol{\theta}\|^2_{\mathcal{R}}\right)^{1/2}. \tag{12}$$

The bilinear form (10) verifies the continuity and stability bounds

$$|a_b(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{t},\boldsymbol{\beta})| \le C_b\ \|(\boldsymbol{w},\boldsymbol{\theta})\|_{\mathcal{W}\times\mathcal{R}}\ \|(\boldsymbol{t},\boldsymbol{\beta})\|_{\mathcal{W}\times\mathcal{R}}\ , \tag{13a}$$

$$a_b(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{w},\boldsymbol{\theta}) \ge \alpha_b\ \|(\boldsymbol{w},\boldsymbol{\theta})\|^2_{\mathcal{W}\times\mathcal{R}}, \tag{13b}$$

for some constants $C_b, \alpha_b > 0$ and all $(\boldsymbol{w},\boldsymbol{\theta}), (\boldsymbol{t},\boldsymbol{\beta}) \in \mathcal{W}\times\mathcal{R}$. In contrast with the bilinear form of the solid, and precisely due to the boundary conditions on the beam, the bilinear form $a_b(\cdot,\cdot)$ is coercive in $\mathcal{W}\times\mathcal{R}$ (see A). The linear form f_b will be assumed to be continuous as well, i.e., there exists a constant $c_b > 0$ such that for all $(\boldsymbol{t},\boldsymbol{\eta}) \in \mathcal{W}\times\mathcal{R}$

$$f_b(\boldsymbol{t},\boldsymbol{\eta}) \le c_b\ \|(\boldsymbol{t},\boldsymbol{\eta})\|_{\mathcal{W}\times\mathcal{R}}. \tag{14}$$

3 Joint formulation of solids and beams

We consider now the formulation of a problem in which a beam and a three-dimensional solid, connected at some plane interface, deform to reach equilibrium under the action of external forces. Two issues need to be discussed. First, the minimal *compatibility conditions* that can be used to link the kinematics of the beam and the solid on their shared interface. Second, the *stability* and *well-posedness* of the global problem under the smallest set of Dirichlet boundary conditions.

The first issue will be addressed in this section, and follows our previous work [11]. The second issue is studied in Sect. 4. To analyse both of them, we consider the simplest case, an elastic solid as the one described in Sect. 2.1, devoid of Dirichlet boundary conditions, attached through a surface Σ to the tip of a cantilever beam, of the type defined in Sect. 2.2. The number of Dirichlet boundary conditions for the *global* problem is thus six, and it remains to be proven that, when the right links are employed, the former suffice to ensure the stability of the problem. Other, apparently more complex situations (with more beams or solids), are essentially equivalent to this one.

3.1 Link formulation

We define next two constraints relating the displacement and rotation vector of the beam at the free end, denoted respectively as $\boldsymbol{w}_*$ and $\boldsymbol{\theta}_*$, with the displacement field $\boldsymbol{u}$ of the body on the connected surface Σ. To describe these constraints let us define $\boldsymbol{x}_G$ to be the barycenter of the surface Σ and $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{x}_G$ denote the relative vector of an arbitrary point on the surface Σ from the barycenter.

Using this notation, we introduce a first constraint imposing that the tip displacement on the beam is equal to the average displacement of the body on Σ, or equivalently, that the zero moment of the solid and beam displacements on the interface surface be the same, that is

$$
\begin{aligned}
\mathbf{0} &= \int_\Sigma (\boldsymbol{u} - \boldsymbol{w}_* - \boldsymbol{\theta}_* \times \boldsymbol{\xi})\,\mathrm{d} \\
A &= \int_\Sigma (\boldsymbol{u} - \boldsymbol{w}_*)\,\mathrm{d}A = \int_\Sigma \boldsymbol{u}\,\mathrm{d}A - |\Sigma|\,\boldsymbol{w}_*.
\end{aligned}
\tag{15}
$$

The second constraint we will employ imposes that the first moment of the displacement of the beam and solid on the interface surface should be the same, i.e.,

$$
\begin{aligned}
\mathbf{0} &= \int_\Sigma \boldsymbol{\xi} \times (\boldsymbol{u} - \boldsymbol{w}_* - \boldsymbol{\theta}_* \times \boldsymbol{\xi})\,\mathrm{d} \\
A &= \int_\Sigma (\boldsymbol{\xi} \times \boldsymbol{u} - \boldsymbol{\xi} \times (\boldsymbol{\theta}_* \times \boldsymbol{\xi}))\,\mathrm{d} \\
A &= \int_\Sigma \boldsymbol{\xi} \times \boldsymbol{u}\,\mathrm{d}A - |\Sigma|\,\boldsymbol{J}\boldsymbol{\theta}_*,
\end{aligned}
\tag{16}
$$

where the tensor $\boldsymbol{J}$ is the tensor of average (surface) inertia

$$
\boldsymbol{J} := \frac{1}{|\Sigma|} \int_\Sigma (|\xi|^2 \boldsymbol{I} - \boldsymbol{\xi} \otimes \boldsymbol{\xi})\,\mathrm{d}A,
\tag{17}
$$

and $\otimes$ denotes the dyadic product between vectors in $\mathbb{R}^3$.

3.2 Global problem statement

In this joint problem, the equilibrium of the structure consisting of the clamped beam, the deformable body and the connecting link is obtained from the stationarity condition of a Lagrangian (see Fig. 1). To define the latter, consider first the space of Lagrange multipliers

$$
\mathcal{Q} := \mathbb{R}^3 \times \mathbb{R}^3
\tag{18}
$$

with norm

$$
\|(\lambda, \boldsymbol{\mu})\|_{\mathcal{Q}} := \left(\frac{1}{L^2} \|\lambda\|_2^2 + \|\boldsymbol{\mu}\|_2^2 \right)^{1/2}.
\tag{19}
$$

Since the global problem involves two types of bodies, we start by defining one last product space $\mathcal{V} := \mathcal{U} \times \mathcal{W} \times \mathcal{R}$ with norm

$$
\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})\|_{\mathcal{V}} := \left(\|\boldsymbol{u}\|_{\mathcal{U}}^2 + \|\boldsymbol{w}\|_{\mathcal{W}}^2 + L^2 \|\boldsymbol{\theta}\|_{\mathcal{R}}^2 \right)^{1/2},
\tag{20}
$$

for all $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) \in \mathcal{V}$. On this space, we can define the bilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and the linear form $f : \mathcal{V} \to \mathbb{R}$ by

$$
\begin{aligned}
&a(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{t}, \boldsymbol{\eta}) := a_{\mathcal{B}}(\boldsymbol{u}, \boldsymbol{v}) + a_b(\boldsymbol{w}, \boldsymbol{\theta}; \boldsymbol{t}, \boldsymbol{\eta}), \\
&f(\boldsymbol{v}, \boldsymbol{t}, \boldsymbol{\eta}) := f_{\mathcal{B}}(\boldsymbol{v}) + f_b(\boldsymbol{t}, \boldsymbol{\eta}).
\end{aligned}
\tag{21}
$$

The joint equilibrium of the solid and beam will be obtained as the saddle point of the Lagrangian $L : \mathcal{V} \times \mathcal{Q}$ defined as

$$
\begin{aligned}
L(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}, \lambda, \boldsymbol{\mu}) := {}& \frac{1}{2} a(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) - f(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) \\
&+ \langle \lambda, \boldsymbol{\xi} \times \boldsymbol{u} - \boldsymbol{J}\boldsymbol{\theta}_* \rangle_\Sigma + \langle \boldsymbol{\mu}, \boldsymbol{u} - \boldsymbol{w}_* \rangle_\Sigma.
\end{aligned}
\tag{22}
$$

where the notation $\langle \cdot, \cdot \rangle_\Sigma$ denotes the L_2 product on the surface Σ. The optimality conditions of the Lagrangian give the mixed variational problem: find $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}, \lambda, \boldsymbol{\mu}) \in \mathcal{V} \times \mathcal{Q}$ such that

$$
\begin{aligned}
&a(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{t}, \boldsymbol{\beta}) + b(\lambda, \boldsymbol{\mu}; \boldsymbol{v}, \boldsymbol{t}, \boldsymbol{\beta}) = f(\boldsymbol{v}, \boldsymbol{t}, \boldsymbol{\beta}), \\
&b(\boldsymbol{\gamma}, \boldsymbol{\nu}; \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) = 0,
\end{aligned}
\tag{23}
$$

for all $(\boldsymbol{v}, \boldsymbol{t}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu})$ in $\mathcal{V} \times \mathcal{Q}$, with

$$
b(\boldsymbol{\gamma}, \boldsymbol{\nu}; \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) = \langle \boldsymbol{\gamma}, \boldsymbol{\xi} \times \boldsymbol{u} - \boldsymbol{J}\boldsymbol{\theta}_* \rangle_\Sigma + \langle \boldsymbol{\nu}, \boldsymbol{u} - \boldsymbol{w}_* \rangle_\Sigma.
\tag{24}
$$

The solvability of problem (23) requires the careful consideration of the properties of both bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, as well as the spaces on which they are defined.

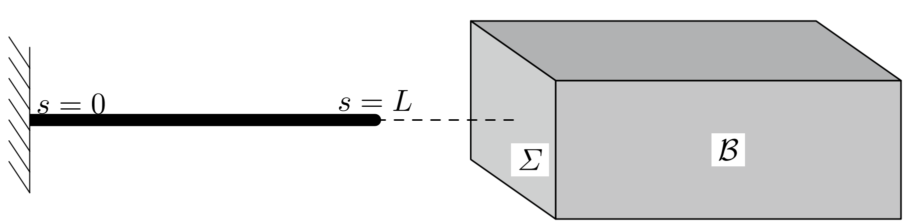

Fig. 1 Schematic of the linked problem. A body $\mathcal{B}$ that is not supported is connected through Σ, a subset of its boundary, with a cantilever beam

4 Analysis

Mixed variational problems such as the one described in Eq. (23) have been extensively studied in the literature [3, 10]. Their well-posedness pivots on two conditions: the ellipticity of the bilinear form $a(\cdot,\cdot)$ on a certain set $\mathcal{K} \subset \mathcal{V}$ defined below, and the inf-sup condition of the bilinear form $b(\cdot,\cdot)$.

Before stating the main result we note that, based on Eqs. (5) and (13), the global bilinear form $a(\cdot,\cdot)$ verifies the following bounds

$$\begin{aligned} &|a(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{v},\boldsymbol{t},\boldsymbol{\eta})| \le C\, \|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})\|_{\mathcal{V}}\, \|(\boldsymbol{v},\boldsymbol{t},\boldsymbol{\eta})\|_{\mathcal{V}},\\ &a(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}) + \|\boldsymbol{u}\|^2_{[L^2(\mathcal{B})]^3} \ge \alpha\, \|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})\|^2_{\mathcal{V}}, \end{aligned} \tag{25}$$

for some $C,\alpha > 0$ and all $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}),(\boldsymbol{v},\boldsymbol{t},\boldsymbol{\eta}) \in \mathcal{V}$. Likewise, and due to Eqs. (6) and (14) the global linear form $f(\cdot)$ is continuous, that is,

$$f(\boldsymbol{v},\boldsymbol{t},\boldsymbol{\eta}) \le c\, \|(\boldsymbol{v},\boldsymbol{t},\boldsymbol{\eta})\|_{\mathcal{V}}, \tag{26}$$

for some $c > 0$ and all $(\boldsymbol{v},\boldsymbol{t},\boldsymbol{\eta}) \in \mathcal{V}$. We note, again, that the bilinear form $a(\cdot;\cdot)$ is not coercive in $\mathcal{V}$ as a result of the lack of coercivity of the bilinear form in the problem of the deformable solid.

As a preliminary property, let us verify that the bilinear form $b(\cdot,\cdot)$ is continuous in the test space.

Lemma 1 *The bilinear form $b(\cdot,\cdot) : \mathcal{V}\times\mathcal{Q} \to \mathbb{R}$ is continuous, i.e, there exists a constant $C > 0$ such that*

$$|b(\boldsymbol{\gamma},\boldsymbol{v};\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})| \le C\|(\boldsymbol{\gamma},\boldsymbol{v})\|_{\mathcal{Q}}\, \|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})\|_{\mathcal{V}}. \tag{27}$$

Proof We first show a preliminary result that we will need to prove the continuity of $b(\cdot,\cdot)$. Since $\boldsymbol{\theta},\boldsymbol{w} \in H^1_0(0,L)$, and $H^1_0(0,L) \hookrightarrow C^0[0,L]$, we have that

$$|\boldsymbol{\theta}_*| \le \int_0^L |\boldsymbol{\theta}'(s)|\, \mathrm{d}s \le L^{1/2}\, |\boldsymbol{\theta}|_{[H^1_0(0,L)]^3} \le \|\boldsymbol{\theta}\|_{[H^1_0(0,L)]^3} = \|\boldsymbol{\theta}\|_{\mathcal{R}}, \tag{28}$$

and, similarly,

$$\begin{aligned} |\boldsymbol{w}_*| &\le \int_0^L |\boldsymbol{w}'(s)|\, \mathrm{d}s \le L^{1/2}\, |\boldsymbol{w}|_{[H^1_0(0,L)]^3}\\ &\le \|\boldsymbol{w}\|_{[H^1_0(0,L)]^3} = \|\boldsymbol{w}\|_{\mathcal{W}}. \end{aligned} \tag{29}$$

Next, by the trace theorem, we have that

$$\begin{aligned} &\int_\Sigma \boldsymbol{\gamma}\cdot\boldsymbol{\xi}\times\boldsymbol{u}\, \mathrm{d}A \le C\, \|\boldsymbol{\gamma}\|_2 \left(\int_\Sigma |\boldsymbol{u}|^2\, \mathrm{d}A\right)^{1/2}\\ &\le C\, \|\boldsymbol{\gamma}\|_2\, \|\boldsymbol{u}\|_{[H^1(\mathcal{B})]^3} = C\, \|\boldsymbol{\gamma}\|_2\, \|\boldsymbol{u}\|_{\mathcal{U}}. \end{aligned} \tag{30}$$

The continuity of $b(\cdot,\cdot)$ then follows from the previous result, the application of the Cauchy-Schwartz inequality and the bounds for $\boldsymbol{\theta}_*$ and $\boldsymbol{w}_*$:

$$\begin{aligned} b(\boldsymbol{\gamma},\boldsymbol{v};\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}) &= \langle\boldsymbol{\gamma},\boldsymbol{\xi}\times\boldsymbol{u} - \boldsymbol{J}\boldsymbol{\theta}_*\rangle_\Sigma + \langle\boldsymbol{v},\boldsymbol{u}-\boldsymbol{w}_*\rangle_\Sigma\\ &\le |\langle\boldsymbol{\gamma},\boldsymbol{\xi}\times\boldsymbol{u}\rangle_\Sigma| + |\langle\boldsymbol{\gamma},\boldsymbol{J}\boldsymbol{\theta}_*\rangle_\Sigma| + |\langle\boldsymbol{v},\boldsymbol{u}\rangle_\Sigma| + |\langle\boldsymbol{v},\boldsymbol{w}_*\rangle_\Sigma|\\ &\le C\big(\|\boldsymbol{\gamma}\|_2\, \|\boldsymbol{u}\|_{\mathcal{U}} + \|\boldsymbol{\gamma}\|_2\, \|\boldsymbol{\theta}\|_{\mathcal{R}} + \|\boldsymbol{v}\|_2\, \|\boldsymbol{u}\|_{\mathcal{U}} + \|\boldsymbol{v}\|_2\, \|\boldsymbol{w}\|_{\mathcal{W}}\big)\\ &\le C\|(\boldsymbol{\gamma},\boldsymbol{v})\|_{\mathcal{Q}}\, \big(\|\boldsymbol{u}\|_{\mathcal{U}} + \|\boldsymbol{\theta}\|_{\mathcal{R}} + \|\boldsymbol{w}\|_{\mathcal{W}}\big)\\ &\le C\, \|(\boldsymbol{\gamma},\boldsymbol{v})\|_{\mathcal{Q}}\, \|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})\|_{\mathcal{V}}, \end{aligned} \tag{31}$$

□

The set $\mathcal{K} \subset \mathcal{V}$ consists of all the functions where the bilinear form $b(\cdot,\cdot)$ vanishes, i.e.,

$$\mathcal{K} = \{(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}) \in \mathcal{V},\ b(\boldsymbol{\gamma},\boldsymbol{v};\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}) = 0 \quad \text{for all } (\boldsymbol{\gamma},\boldsymbol{v}) \in \mathcal{Q}\}. \tag{32}$$

From the definition of the bilinear form $b(\cdot;\cdot)$ it follows that the elements in $\mathcal{K}$ are ones that satisfy the constraints (15) and (16).

The well-posedness of the saddle point problem is the result of two theorems that we state and prove next.

Theorem 2 *The bilinear form $a(\cdot;\cdot)$ is $\mathcal{V}$-elliptic on $\mathcal{K}$.*

Proof Let the function $|||\cdot||| : \mathcal{V} \to \mathbb{R}$ be defined as

$$|||(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})||| = a(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}), \tag{33}$$

for all $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}) \in \mathcal{V}$. We prove first that this function is positive definite on $\mathcal{K}$. For $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta}) \in \mathcal{K}$, $|||(\boldsymbol{u},\boldsymbol{w},\boldsymbol{\theta})||| = 0$ if and only if

$$0 = a_{\mathcal{B}}(\boldsymbol{u},\boldsymbol{u}) + a_b(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{w},\boldsymbol{\theta}). \tag{34}$$

The bilinear forms $a_{\mathcal{B}}(\cdot,\cdot)$ and $a_b(\cdot,\cdot)$ are positive semidefinite and positive definite, respectively. Hence, $(\boldsymbol{w},\boldsymbol{\theta})$ must be equal to $(\boldsymbol{0},\boldsymbol{0})$ and $\boldsymbol{u}$ must be an infinitesimal rigid body motion. The only rigid body deformation in $\mathcal{K}$ is

$$\boldsymbol{u} = \boldsymbol{w}_* + \boldsymbol{\theta}_*\times\boldsymbol{\xi}, \tag{35}$$

with $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{x}_G$, as before, and $\boldsymbol{x}_G$ being the position of the barycenter of Σ. But, since $\boldsymbol{w} \equiv \boldsymbol{0}$ and $\boldsymbol{\theta} \equiv \boldsymbol{0}$, the function $\boldsymbol{u}$ must also be identically zero.

To prove next that $|||(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})||| \geq \alpha \, \|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})\|_{\mathcal{V}}$ for some constant $\alpha > 0$, and any $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) \in \mathcal{K}$, suppose that it is not true. Then there is a sequence $\{(\boldsymbol{u}_i, \boldsymbol{w}_i, \boldsymbol{\theta}_i)\} \in \mathcal{K}$ with

$$\|(\boldsymbol{u}_i, \boldsymbol{w}_i, \boldsymbol{\theta}_i)\|_{\mathcal{V}} = 1, \qquad \text{and} \qquad \lim_{i \to \infty} |||(\boldsymbol{u}_i, \boldsymbol{w}_i, \boldsymbol{\theta}_i)||| = 0. \tag{36}$$

Since $1 = \|(\boldsymbol{u}_i, \boldsymbol{w}_i, \boldsymbol{\theta}_i)\|_{\mathcal{V}} \geq \|\boldsymbol{u}_i\|_{[H^1(\mathcal{B})]^3}$, the sequence $\{\boldsymbol{u}_i\}$ is bounded in $[H^1(\mathcal{B})]^3$ and, by Rellich's theorem, there is a subsequence $\{\boldsymbol{u}_{i_j}\}$ that converges in $[L^2(\mathcal{B})]^3$ to a function $\bar{\boldsymbol{u}}$. But, noting that $\lim_{j \to \infty} |||(\boldsymbol{u}_{i_j}, \boldsymbol{w}_{i_j}, \boldsymbol{\theta}_{i_j})||| = 0$, this must be a Cauchy sequence in the norm

$$(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) \mapsto \left(\|\boldsymbol{u}\|^2_{[L^2(\mathcal{B})]^3} + |||(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})|||^2 \right)^{1/2}. \tag{37}$$

But this norm is equivalent to $\|\cdot\|_{\mathcal{V}}$ due to Korn's second inequality and the ellipticity of $a_b(\cdot, \cdot)$. Hence, the sequence is Cauchy with respect to $\|\cdot\|_{\mathcal{V}}$ and since $\mathcal{V}$ is a Hilbert space, it converges to $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{\theta}}) \in \mathcal{K}$. The two norms being equivalent proves that

$$0 = \lim_{j \to \infty} |||(\boldsymbol{u}_{i_j}, \boldsymbol{w}_{i_j}, \boldsymbol{\theta}_{i_j})||| = |||(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{\theta}})|||. \tag{38}$$

Above we showed that $|||\cdot|||$ is positive definite in $\mathcal{K}$, hence $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{\theta}}) = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ but

$$0 = \|(\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{\theta}})\| = \lim_{j \to \infty} |||(\boldsymbol{u}_{i_j}, \boldsymbol{w}_{i_j}, \boldsymbol{\theta}_{i_j})||| = 1. \tag{39}$$

Since this is impossible, we conclude that there exists $\alpha > 0$ such that $|||(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})||| \geq \alpha \, \|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})\|_{\mathcal{V}}$.

□

The second condition required to guarantee the well-posedness of the mixed problem is the *inf-sup* condition on the bilinear form $b(\cdot, \cdot)$.

Theorem 3 *There exists a constant $\beta > 0$ such that for all $(\boldsymbol{\gamma}, \boldsymbol{v}) \in \mathcal{Q}$,*

$$\sup_{(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) \in \mathcal{V}} \frac{b(\boldsymbol{\gamma}, \boldsymbol{v}; \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})}{\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})\|_{\mathcal{V}}} \geq \beta \, \|(\boldsymbol{\gamma}, \boldsymbol{v})\|_{\mathcal{Q}}. \tag{40}$$

Proof To prove this bound, first, we need the following. Let $\boldsymbol{v}, \boldsymbol{\gamma} \in \mathbb{R}^3 \times \mathbb{R}^3$ and, as before, let $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{x}_G$ and $\boldsymbol{x}_G$ denote the position vector of the barycenter of Σ. Then,

$$\begin{aligned} \|\boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{\xi}\|^2_{\mathcal{U}} &\leq \int_{\mathcal{B}} |\boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{\xi}|^2 \, \mathrm{d}V + 2\ell^2 \int_{\mathcal{B}} |\boldsymbol{\gamma}|^2 \, \mathrm{d}V \\ &\leq C \, |\mathcal{B}| \left(|\boldsymbol{v}|^2 + \ell^2 |\boldsymbol{\gamma}|^2 \right) \\ &\leq C \, \|(\boldsymbol{\gamma}, \boldsymbol{v})\|^2_{\mathcal{Q}} \, . \end{aligned} \tag{41}$$

Next, we find an upper bound for the inf-sup quotient by selecting the triplet $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})$ to be $(\boldsymbol{v} + \boldsymbol{\lambda} \times \boldsymbol{\xi}, \boldsymbol{0}, \boldsymbol{0})$, i.e.,

$$\begin{aligned} \sup_{(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta}) \in \mathcal{V}} \frac{b(\boldsymbol{\gamma}, \boldsymbol{v}; \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})}{\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\theta})\|_{\mathcal{V}}} &\geq \frac{b(\boldsymbol{\gamma}, \boldsymbol{v}; \boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{\xi}, \boldsymbol{0}, \boldsymbol{0})}{\|(\boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{\xi}, \boldsymbol{0}, \boldsymbol{0})\|_{\mathcal{V}}} \\ &= \frac{\langle \boldsymbol{\gamma}, \boldsymbol{\xi} \times (\boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{\xi}) \rangle_{\Sigma} + \langle \boldsymbol{v}, \boldsymbol{v} + \boldsymbol{\lambda} \times \boldsymbol{\xi} \rangle_{\Sigma}}{\|\boldsymbol{v} + \boldsymbol{\gamma} \times \boldsymbol{\xi}\|_{\mathcal{U}}} \\ &\geq C \, \frac{\boldsymbol{\gamma} \cdot \boldsymbol{J} \boldsymbol{\gamma} \, |\Sigma| + |\boldsymbol{v}|^2 \, |\Sigma|}{\|(\boldsymbol{\gamma}, \boldsymbol{v})\|_{\mathcal{Q}}} \\ &\geq \beta \, \|(\boldsymbol{\gamma}, \boldsymbol{v})\|_{\mathcal{Q}}, \end{aligned} \tag{42}$$

where we have employed (41) and the boundedness of $\mathcal{B}$, Σ, and the norm of $\boldsymbol{J}$. □

Theorems 2 and 3 are necessary and sufficient conditions for the well-posedness of problem (23).

5 Application: bodies under concentrated forces and moments

In mechanics, it is natural to consider the response of structures under point loads and concentrated moments. This is in contrast with the mechanics of two- or three-dimensional continuum bodies since the latter can only be subject to surface tractions and body forces. Remarkably, the constrained formulation introduced in Sect. 3 can be used as a convenient mechanism to effectively impose concentrated loads on continua, one that naturally introduces the most natural surface tractions on the boundary of the body that are statically equivalent to the imposed loads.

To illustrate this application consider an arbitrary body $\mathcal{B} \subset \mathbb{R}^3$ and three disjoint sets $\partial_D \mathcal{B}, \partial_N \mathcal{B}, \Sigma \subset \partial \mathcal{B}$. On the Dirichlet boundary $\partial_D \mathcal{B}$ the displacement of the body is assumed to be zero; the Neumann boundary $\partial_N \mathcal{B}$ is free of tractions; finally, the boundary Σ, assumed to be plane for simplicity, is subject to an unknown vector field of tractions $\boldsymbol{t}$ that satisfy

$$\boldsymbol{F} = \int_{\Sigma} \boldsymbol{t}\, \mathrm{d}A, \quad \boldsymbol{M}_G = \int_{\Sigma} \boldsymbol{\xi} \times \boldsymbol{t}\, \mathrm{d}A, \tag{43}$$

where $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{x}_G$ and $\boldsymbol{x}_G$ is the position vector of the barycenter of Σ.

The vectors $\boldsymbol{F}, \boldsymbol{M}_G \in \mathbb{R}^3$ are the resultant force and moment on the surface Σ. When the body $\mathcal{B}$ is a prismatic bar or flat the theories of beams and shells posit stress distributions in the cross section following from the theories proposed by Navier, Bernoulli, Timoshenko, Coulomb, Mindlin, among other. In contrast, here we will show that we can recover Eq. (43), as well as the equilibrium equations on the solid in $\mathcal{B}$ and the Neumann boundary from the stationarity of the constrained functional (22).

More specifically, let $\boldsymbol{w}_*, \boldsymbol{\theta}_* \in \mathbb{R}^3$ and consider the problem of finding the displacement $\boldsymbol{u}$ in the space

$$\mathcal{U} = \left\{ \boldsymbol{u} \in [H^1(\mathcal{B})]^3,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial_u \mathcal{B} \right\} \tag{44}$$

and the Lagrange multipliers $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^3$ that make stationary the functional

$$L(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{1}{2} a_{\mathcal{B}}(\boldsymbol{u}, \boldsymbol{u}) - f_{\mathcal{B}}(\boldsymbol{u}) - \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \times \boldsymbol{u} - \boldsymbol{J}\boldsymbol{\theta}_* \rangle_{\Sigma} - \langle \boldsymbol{\mu}, \boldsymbol{u} - \boldsymbol{w}_* \rangle_{\Sigma}, \tag{45}$$

with $a_{\mathcal{B}}(\cdot,\cdot), f_{\mathcal{B}}(\cdot)$ being the bilinear and linear forms, respectively, of the elastic body under body forces $\boldsymbol{f}$ defined in Sect. 2.1. This problem falls in the category studied in Sects. 3 and 4, but removes the beam which is unnecessary for the example.

The Euler–Lagrange equations of the Lagrangian (45) are

$$a_{\mathcal{B}}(\boldsymbol{u}, \boldsymbol{v}) - \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \times \boldsymbol{v} \rangle_{\Sigma} - \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Sigma} = f_{\mathcal{B}}(\boldsymbol{v}), \tag{46a}$$

$$\langle \boldsymbol{\gamma}, \boldsymbol{\xi} \times \boldsymbol{u} \rangle_{\Sigma} + \langle \boldsymbol{\nu}, \boldsymbol{u} \rangle_{\Sigma} = |\Sigma|\, \boldsymbol{\gamma} \cdot \boldsymbol{J}\boldsymbol{\theta}_* + |\Sigma|\, \boldsymbol{\nu} \cdot \boldsymbol{w}_*, \tag{46b}$$

and the solution consists of the triplet $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{U} \times \mathbb{R}^3 \times \mathbb{R}^3$ that verifies (46) for all $(\boldsymbol{v}, \boldsymbol{\gamma}, \boldsymbol{\nu}) \in \mathcal{U} \times \mathbb{R}^3 \times \mathbb{R}^3$.

Defining the stress tensor

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = 2\mu\, \boldsymbol{\varepsilon} + \lambda\, \mathrm{tr}\boldsymbol{\varepsilon}\, \boldsymbol{I}, \tag{47}$$

and the surface tractions $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$, integrating by parts the bilinear form $a_{\mathcal{B}}(\cdot,\cdot)$ in Eq. (46a) and using the arbitrariness of the test function $\boldsymbol{v} \in \mathcal{U}$ we get that, weakly,

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \mathcal{B}, \tag{48a}$$

$$\boldsymbol{t} = \boldsymbol{0} \qquad \text{on } \partial_t \mathcal{B}, \tag{48b}$$

$$\boldsymbol{t} = \boldsymbol{\lambda} \times \boldsymbol{\xi} + \boldsymbol{\mu} \quad \text{on } \Sigma. \tag{48c}$$

The first of these equations expresses the equilibrium of forces in the interior of the body; the second one reveals that the tractions on the Neumann boundary are zero; the third one shows that the tractions on the surface Σ depend on the Lagrange multipliers. Integrating this last equation over all the surface Σ, and using the property that $\boldsymbol{x}_G$ is the barycenter of Σ, we can obtain

$$\int_{\Sigma} \boldsymbol{t}\, \mathrm{d}A = |\Sigma|\, \boldsymbol{\mu}. \tag{49}$$

Also, taking the cross product of $\boldsymbol{\xi}$ of both sides of Eq. (48c) and integrating again over the boundary Σ we get

$$\int_{\Sigma} \boldsymbol{\xi} \times \boldsymbol{t}\, \mathrm{d}A = |\Sigma|\, \boldsymbol{J}\boldsymbol{\lambda}. \tag{50}$$

Combining Eqs. (49) and (50) with (43) we find that the Lagrange multipliers have the value

$$\boldsymbol{\lambda} = |\Sigma|^{-1} \boldsymbol{J}^{-1} \boldsymbol{M}_G, \quad \boldsymbol{\mu} = |\Sigma|^{-1} \boldsymbol{F}. \tag{51}$$

Alternatively, we can use these last two expressions to identify in Eq. (48c) that the tractions on Σ can be split as

$$\boldsymbol{t} = \boldsymbol{t}_M + \boldsymbol{t}_F. \tag{52}$$

The first contribution, $\boldsymbol{t}_M = (\boldsymbol{J}^{-1}\boldsymbol{M}_G) \times \boldsymbol{\xi}$, are the tractions due to the concentrated moment. The second part, $\boldsymbol{t}_F = |\Sigma|^{-1}\boldsymbol{F}$, are the (uniform) tractions due to the concentrated load. Remarkably, this example shows that the constrained formulation that corresponds to the optimality points of the Lagrangian (45) yields the displacement of the body in equilibrium under the body forces $\boldsymbol{f}$ and a traction field whose *pointwise* value is found in closed form. Naturally, these tractions verify Eq. (43).

As advanced in the introduction and illustrated with this example, the linked formulation of Sect. 3 provides the right variational formulation for *loading* *devices* that apply onto the three-dimensional elastic body concentrated forces and torques.

6 Summary

We have presented the small strain form of a variational principle that governs the collective equilibria of linked beams and deformable solids. This is a remarkable principle in that it combines the mechanical response of two types of bodies with very different kinematic descriptions.

The variational principle rests on two compatibility conditions that link, in the weakest possible way, the kinematics of beams and solids on their common interface. These conditions express that the zero-th and first-order moments of the displacement fields of the beam and solid on the interface surface are equal, following previous work of one of the authors [11].

The optimality conditions of this variational principle give rise to a saddle point problem whose well-posedness is proven. In addition to the mathematical consequences of such a result, it evinces that it can be the basis of convergent numerical discretizations for structural models combining beams and deformable solids. This result has important practical implications for the analysis of large structures that are modeled with beams, shells, as well as continua and whose link is typically achieved with ad hoc connections lacking any mechanical foundation.

We close by noting that the well-posedness of the problem does not rely on the elastic response of either the solid or the beam. Rather, only some (weak) coercivity conditions of the bilinear forms of the solid and the beam are required for the proof. Hence, the result obtained can be, in principle, extended to inelastic structures in which the same stability estimates hold, even if just incrementally.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

A Coercivity of the Timoshenko beam

For completeness, we present a proof of the coercivity relation (13b). If $(\boldsymbol{w}, \boldsymbol{\theta}) \in \mathcal{W} \times \mathcal{R}$, then

$$\begin{aligned} a(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{w},\boldsymbol{\theta}) &= \int_0^L \left[(\boldsymbol{w}'-\boldsymbol{\theta}\times\boldsymbol{r}')\cdot\boldsymbol{C}_\Gamma(\boldsymbol{w}'-\boldsymbol{\theta}\times\boldsymbol{r}')+\boldsymbol{\theta}'\cdot\boldsymbol{C}_\Omega\boldsymbol{\theta}'\right]\,\mathrm{d}S \\ &\geq C_1\int_0^L \left[|\boldsymbol{w}'-\boldsymbol{\theta}\times\boldsymbol{r}'|^2+L^2\,|\boldsymbol{\theta}'|^2\right]\,\mathrm{d}S \\ &\geq C_1\int_0^L \left[\left(1-\frac{1}{\epsilon^2}\right)|\boldsymbol{w}'|^2+(1-\epsilon^2)\,|\boldsymbol{\theta}\times\boldsymbol{r}'|^2+L^2\,|\boldsymbol{\theta}'|^2\right], \end{aligned} \tag{53}$$

for some constant C_1 depending on the section stiffness $\boldsymbol{C}_\Gamma, \boldsymbol{C}_\Omega$ and $\boldsymbol{\Lambda}$, and any $\epsilon > 0$. Then, using Poincaré's inequality, it follows that

$$a(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{w},\boldsymbol{\theta}) \geq C_1\,C_P\left(1-\frac{1}{\epsilon^2}\right)\frac{1}{L^2}\|\boldsymbol{w}\|^2_{\mathcal{W}} + C_1\,C_r\,(1-\epsilon^2) \int_0^L |\boldsymbol{\theta}|^2\,\mathrm{d}S \quad + C_1\,C_P\int_0^L \left(|\boldsymbol{\theta}|^2+L^2|\boldsymbol{\theta}'|^2\right)\,\mathrm{d}S, \tag{54}$$

where $C_P > 0$ is Poincaré's constant and $C_r = |\boldsymbol{r}'|^2_{[L^\infty(0,L)]^3}$. Collecting all the terms on $|\boldsymbol{\theta}|$ we obtain

$$a(\boldsymbol{w},\boldsymbol{\theta};\boldsymbol{w},\boldsymbol{\theta}) \geq C_1\,C_P\left(1-\frac{1}{\epsilon^2}\right)\frac{1}{L^2}\|\boldsymbol{w}\|^2_{\mathcal{W}} + C_1(C_r(1-\epsilon^2)+C_P) \int_0^L |\boldsymbol{\theta}|^2\,\mathrm{d}S \quad + C_1\,C_P\int_0^L L^2\,|\boldsymbol{\theta}'|^2\,\mathrm{d}S. \tag{55}$$

By selecting ϵ that verifies

$$1 < \epsilon^2 < 1 + \frac{C_P}{C_r}, \tag{56}$$

the constants multiplying $\|\boldsymbol{w}\|_{\mathcal{W}}$ and the L_2-norm of $\boldsymbol{\theta}$ are both positive and (13b) follows.

References

1. Antman SS (1973) The theory of rods. In: Handbuch der physik. Springer, Berlin, pp. 641–703
2. Boffi D, Brezzi F, Fortin M (2013) Mixed finite element methods and applications, vol 44. Springer Series in Computational Mathematics. Springer, Berlin
3. Brezzi F, Fortin M (1991) Mixed and hybrid finite element methods. Springer, Berlin
4. Ciarlet PG (1988) Mathematical elasticity. North-Holland
5. Gmür TC, Kauten RH (1993) Three-dimensional solid-to-beam transition elements for structural dynamics analysis. Int J Numer Methods Eng 36(9):1429–1444
6. Green AE, Naghdi PM, Wenner ML (1974) On the theory of rods I. Derivations from the three dimensional equations. Proc R Soc Lond Ser A 337:451–483
7. Green AE, Naghdi PM, Wenner ML (1974) On the theory of rods II. Developments by direct approach. Proc R Soc Lond Ser A 337:485–507
8. Gurtin ME (1965) The linear theory of elasticity. In: Flügge S, Truesdell C (eds) Encyclopedia of physics. Springer, Berlin
9. Marsden JE, Hughes TJR (1983) Mathematical foundations of elasticity. Prentice-Hall, Englewood Cliffs
10. Roberts JE, Thomas JM (1989) Mixed and hybrid methods. In: Ciarlet PG, Lions JL (eds) Handbook of numerical analysis. North-Holland, Amsterdam, pp 523–639
11. Romero I (2018) Coupling nonlinear beams and continua: variational principles and finite element approximations. Int J Numer Methods Eng 114:1192–1212. https://doi.org/10.1002/nme.5782
12. Surana KS (1980) Transition finite elements for axisymmetric stress analysis. Int J Numer Methods Eng 15:809–832
13. Surana KS (1980) Transition finite elements for three-dimensional stress analysis. Int J Numer Methods Eng 15:991–1020
14. Timoshenko S, Gere JM (1972) Mechanics of materials, 1st edn. Van Nostrand Reinhold, New York

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