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Analytic contact solutions of the Boussinesq and Cattaneo problems for an ellipsoidal power-law indenter

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Abstract Based on Galin's theorem for the indentation of an elastic half-space by a polynomial punch and Barber's theorem for the determination of the contact area in elastic normal contact problems, an exact contact solution is developed for the frictionless indentation of an elastic half-space by an ellipsoidal power-law punch. Within the Cattaneo–Mindlin-approximation of tangential contacts, the tangential contact problem with friction can be reduced to the frictionless normal contact via the Ciavarella–Jäger principle, based on which also the tangential contact solution for the ellipsoidal power-law indenter is given. All known solutions for special cases (axisymmetric contact, elliptical Hertzian contact) are exactly recovered. A comparison of the pressure distribution obtained analytically for the 4-th power ellipsoidal indenter with a corresponding numerical solution based on the boundary element method shows no noticeable differences. The proposed solution procedure can also be used exactly for an arbitrary finite superposition of ellipsoidal power-law profiles, and, in an approximate sense, is applicable for arbitrary monotonous ellipsoidal profiles.

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1 Introduction

Contact mechanics as a branch of natural sciences started with the work of Hertz [1] on the elastic normal contact of parabolically ellipsoidal bodies. As any smooth curved surface in the vicinity of the contact "point" can be approximated as a second order polynomial in cartesian coordinates x and y, that has been and still is a fairly general solution for the elastic normal contact problem.

Nonetheless, ending the Taylor series of the contacting surfaces at the second order remains an approximation, and ellipsoidal contact profiles, which do not fall into the Hertzian description, can be easily thought of. On the other hand, while for the limiting cases of axisymmetric and plane contacts general solutions to the respective elastic normal contact problems were published by Schubert [2], a corresponding general solution for ellipsoidal contact partners is, yet, absent, because it is not as easy as in these limiting cases to make ad hoc statements about the geometrical structure of the contact area.

However, some special cases have been solved in the literature in exact or, at least, approximate, analytic fashion: Vorob'ev [3] gave an exact contact solution for the indentation by an elliptical cone; Argatov

[4] published an asymptotic solution for a superposition of the Hertzian profile with a 4-th order perturbance, and, very recently, Popov [5], based on Fabrikant's [6] approximation for the pressure distribution under a rigid flat punch of arbitrary planform and the approximate procedure of Barber & Billings [7] for general single normal contact problems of smooth indenters, gave an approximate general solution for profiles slightly deviating from axial symmetry, which also can be applied for the (slightly eccentric) elliptical case.

In the present manuscript, a general exact elastic normal contact solution will be presented for ellipsoidal contact profiles in the form of a power-law (while the elliptical eccentricity of different horizontal cross-sections of the indenters remains constant). As, under some simplifying assumptions, the solution to the tangential contact problem can be reduced to the normal contact solution, this will also provide an analytic (but approximate) solution for the respective contacts under tangential loading.

The remainder of the present manuscript is structured as follows: First, the contact problem of interest will be stated in physically rigorous fashion. After that, the analytic solutions to the normal and tangential contact problems will be deduced and illustrated by some examples. Finally, several conclusive remarks finish the manuscript.

2 Problem statement

Let us consider single contacts of linearly elastic, homogeneous, isotropic bodies with the shear moduli G_1 and G_2 and Poisson's ratios ν_1 and ν_2 under normal and tangential loading. To ensure elastic decoupling of the normal and tangential contact problems, the materials are assumed to be elastically similar,

$$\frac{1-2\nu_1}{G_1} - \frac{1-2\nu_2}{G_2} = 0. \tag{1}$$

The contacting bodies shall obey the restrictions of the half-space approximation. Moreover, effects of surface roughness, adhesion, or surface tension are neglected and friction is considered within the framework of a local Amontons-Coulomb law with a constant coefficient of friction μ. Within this set of assumptions, the contact is equivalent to the one between an elastic half-space with the shear modulus and Poisson's ratio

$$G := \frac{G_1 G_2}{G_1 + G_2}, \quad \nu := \frac{\nu_2 G_1 + \nu_1 G_2}{G_1 + G_2}, \tag{2}$$

and a rigid indenter. We can also introduce the effective elastic modulus

$$E^* := \left[\frac{1-\nu_1}{2G_1} + \frac{1-\nu_2}{2G_2}\right]^{-1} = \frac{2G}{1-\nu}. \tag{3}$$

The indenter shall be under a constant normal load (Boussinesq's [8] problem) and a subsequently applied increasing tangential load (Cattaneo's [9] problem). The indenter's horizontal cross sections shall be ellipses with constant eccentricity ε. Introducing cartesian coordinates $\{x,y,z\}$—z being the contact normal direction, and x and y pointing along the ellipses' half-axes—we can write that condition as follows:

$$\frac{x^2}{A(z)^2} + \frac{y^2}{B(z)^2} = 1, \quad A(z) > B(z), \quad \varepsilon := \sqrt{1 - \frac{B(z)^2}{A(z)^2}} = \text{const.} \tag{4}$$

Note that the choice $A(z) > B(z)$ does not constitute any loss of generality, as the x- and y-directions can always be chosen accordingly. Hence, the indenter profile, i.e., the gap between the contacting bodies at the moment of first contact, reads

$$z = f(x,y) = h\left\{\left(1-\varepsilon^2\right)x^2 + y^2\right\}, \tag{5}$$

with some arbitrary monotonous function h. We will restrict our attention to power functions for h, i.e.,

$$f(x,y) = C_n\left[\left(1-\varepsilon^2\right)x^2 + y^2\right]^n, \quad C_n > 0, \quad n \in \mathbb{N}^+. \tag{6}$$

The case $n = 1$ corresponds to the elliptical Hertzian contact problem.

3 Normal contact solution

According to Galin's theorem [10, p. 120], an indenter with the profile (6) will produce a pressure distribution

$$p(x,y)=S_n\left(x^2,y^2\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2},\quad a>b,$$
$$e:=\sqrt{1-\frac{b^2}{a^2}},\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1, \tag{7}$$

where S_n (x^2, y^2) is a polynomial of the order n in x^2 and y^2. Note that Galin's theorem is formulated specifically in that direction (although it is sometimes cited the other way around): *If* the shape of the indenter is a polynomial in x and y, *then* the pressure distribution will be of the given form (uneven terms in x and y have been disregarded above due to symmetry). On the other hand, for general elliptical punches, whose shape isn't a polynomial, the following procedure would not be exact, but a Rayleigh–Ritz type approximation, because the contact region might deviate from the elliptical shape.

Moreover, the indenter profile is smooth and therefore the contact pressure must be bound at the edge of contact. Therefore, the pressure distribution will take the form

$$p(x,y)=S_{n-1}\left(x^2,y^2\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2},\quad a>b,$$
$$e:=\sqrt{1-\frac{b^2}{a^2}},\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1. \tag{8}$$

Hence, for the exact solution to the normal contact problem, we only require to determine the polynomial S_{n-1} and the half-axes a and b of the contact ellipse, as functions of the indentation depth δ (or the total normal force F_N).

3.1 The normal force integral

Based on Betti's reciprocal theorem, the elastic normal displacements in the contact area, $\delta-f(x,y)$, and the pressure distribution are connected via the integral relation [11, p. 52]

$$\iint\limits_{\Omega}\left[\delta-f(x,y)\right]p^*(x,y)\mathrm{d}x\mathrm{d}y=\iint\limits_{\Omega}p(x,y)\mathrm{d}x\mathrm{d}y, \tag{9}$$

where Ω is the (elliptical) contact area, and $p^*(x,y)$ is the pressure distribution, which produces a unit normal displacement of Ω [12, p. 64],

$$p^*(x,y):=\frac{E^*}{2b\mathrm{K}(e)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2},\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}<1, \tag{10}$$

with the complete elliptic integral of the first kind, K. The right-hand side of Eq. (9) is simply the total normal force. Hence, switching to polar coordinates $\{r,\varphi\}$, and using the indenter profile (6), Eq. (9) reads

$$F_N=\frac{E^*}{2b\mathrm{K}(e)}\int_0^{2\pi}\int_0^{R(\varphi)}\left[\delta-C_n r^{2n}\left(1-\varepsilon^2\cos^2\varphi\right)^n\right]\frac{R(\varphi)}{\sqrt{R(\varphi)^2-r^2}}\,r\mathrm{d}r\mathrm{d}\varphi, \tag{11}$$

with the contour of the contact ellipse in polar coordinates,

$$R(\varphi):=b\left(1-e^2\cos^2\varphi\right)^{-1/2}. \tag{12}$$

Evaluating the normal force integral (11), we arrive at

$$F_N=\frac{\pi E^*}{\mathrm{K}(e)\sqrt{1-e^2}}\left[\delta b-C_n\frac{\kappa(2n)}{2n+1}b^{2n+1}\frac{P_n\left(e^2,\varepsilon^2\right)}{\left(1-e^2\right)^n}\right],$$
$$\kappa(m):=\sqrt{\pi}\,\frac{\Gamma(m/2+1)}{\Gamma((m+1)/2)}, \tag{13}$$

with the gamma function Γ, and P_n (e^2, ε^2) being a polynomial of the order n in e^2 and ε^2, defined as

$$P_n\left(e^2,\varepsilon^2\right):=\frac{\left(1-e^2\right)^{n+1/2}}{2\pi}\int_0^{2\pi}\frac{\left(1-\varepsilon^2\cos^2\varphi\right)^n\mathrm{d}\varphi}{\left(1-e^2\cos^2\varphi\right)^{n+1}}. \tag{14}$$

The first of these polynomials are given by

$$\begin{aligned}
P_0\left(e^2,\varepsilon^2\right)&=1,\\
P_1\left(e^2,\varepsilon^2\right)&=1-\frac{1}{2}\left(e^2+\varepsilon^2\right),\\
P_2\left(e^2,\varepsilon^2\right)&=1-\left(e^2+\varepsilon^2\right)+\frac{3}{8}\left(e^4+\varepsilon^4\right)+\frac{1}{4}e^2\varepsilon^2,\\
P_3\left(e^2,\varepsilon^2\right)&=1-\frac{3}{2}\left(e^2+\varepsilon^2\right)+\frac{9}{8}\left(e^4+\varepsilon^4\right)+\frac{3}{4}e^2\varepsilon^2\\
&\quad-\frac{5}{16}\left(e^6+\varepsilon^6\right)-\frac{3}{16}e^2\varepsilon^2\left(e^2+\varepsilon^2\right).
\end{aligned} \tag{15}$$

3.2 Maximizing the Normal Force Integral

According to Barber's theorem [13] (for circular contact areas this idea was first published by Shield [14]), the correct contact area, characterized by the smaller half-axis b and the eccentricity e, will maximize the expression (13) for a given indentation depth. Thus, maximizing first with respect to b, we obtain

$$\frac{\partial F_N}{\partial b}=0 \quad \Rightarrow \quad \delta = C_n\kappa(2n)b^{2n}\frac{P_n\left(e^2,\varepsilon^2\right)}{\left(1-e^2\right)^n}. \tag{16}$$

Hence,

$$F_N=\frac{\pi E^*}{\mathrm{K}(e)}\frac{2n}{2n+1}\delta^{\frac{2n+1}{2n}}\left[C_n\kappa(2n)P_n\left(e^2,\varepsilon^2\right)\right]^{-\frac{1}{2n}}, \tag{17}$$

which in the axisymmetric case, $\varepsilon=e=0$, of course, coincides with the known axisymmetric solution [15, p. 27]. Maximizing the force in Eq. (17) with respect to e results in

$$\frac{\partial F_N}{\partial e}=0 \quad \Rightarrow \quad 1-\frac{\mathrm{E}(e)}{\mathrm{K}(e)\left(1-e^2\right)}=\frac{e}{2nP_n\left(e^2,\varepsilon^2\right)}\frac{\partial P_n}{\partial e}, \tag{18}$$

with the complete elliptic integral of the second kind, E.

Note that the trivial solution of Eq. (18), $e=0$, corresponds to a local minimum of the normal force. As one might expect, Eq. (18) does not depend on the indentation depth, i.e., the contact area eccentricity is constant during the indentation process. Eqs. (16), (17) and (18), together with the definition (14) exactly solve the normal contact problem. For $n=1$, it can be shown easily that the known solution for the elliptical Hertzian contact [11, pp. 33 ff.] is recovered.

The normal contact stiffness is given by

$$k_z:=\frac{\partial F_N}{\partial \delta}=\frac{\pi E^*}{\mathrm{K}(e)}\frac{b}{\sqrt{1-e^2}}=\frac{\pi E^* a}{\mathrm{K}(e)}. \tag{19}$$

As the indentation can be understood as a series of incremental flat punch indentations (see below), this, unsurprisingly, exactly coincides with the normal stiffness of the contact between a rigid elliptical flat punch (with larger half-axis a and eccentricity e) and an elastic half-space.

4 The pressure distribution

As was first pointed out by Mossakovskii [16] and later Jäger [17] for the axisymmetric case, the indentation process can be understood as a series of incremental indentations $\mathrm{d}\delta$ by rigid flat punches, whose planforms are given by the contact regions at different indentation depths. During the indentation procedure the indentation depth changes from $\tilde{\delta}=0$ to $\tilde{\delta}=\delta$ (note that the contact area eccentricity remains constant, as was shown above). Accordingly, the pressure distribution at the end of the indentation is given by the integral of pressure distributions resulting from the incremental flat punch indentations.

Hence, due to Eq. (10) (the variable quantities during the indentation process are characterized by the tilde),

$$p(x,y)=\frac{E^*}{2\mathrm{K}(e)}\int\limits_{\delta_c(x,y)}^{\delta}\frac{1}{\tilde{b}}\left(1-\frac{x^2}{\tilde{a}^2}-\frac{y^2}{\tilde{b}^2}\right)^{-1/2}\mathrm{d}\tilde{\delta}, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1, \tag{20}$$

where $\delta_c\,(x,y)$ is the indentation depth, for which the point $\{x,y\}$ first comes into contact. Switching once again to polar coordinates, and changing the parametrization of the integral, we obtain

$$p(r,\varphi)=\frac{E^*}{2\mathrm{K}(e)}\int\limits_{r}^{R(\varphi)}\frac{1}{\tilde{b}}\frac{\tilde{R}(\varphi)}{\sqrt{\tilde{R}(\varphi)^2-r^2}}\frac{\mathrm{d}\tilde{\delta}}{\mathrm{d}\tilde{b}}\frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tilde{R}(\varphi)}\mathrm{d}\tilde{R}(\varphi). \tag{21}$$

Using Eqs. (12) and (16), Eq. (21) simplifies to

$$p(r,\varphi)=\frac{nE^*\delta}{b\mathrm{K}(e)}\int\limits_{\rho}^{1}\frac{u^{2n-1}\mathrm{d}u}{\sqrt{u^2-\rho^2}}, \quad \rho^2:=\frac{r^2}{R(\varphi)^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}, \tag{22}$$

which, except for the constant factor before the integral, is the same as in the axisymmetric case. In other words, the pressure distribution for the power-law ellipsoidal indenter is given by the axisymmetric distribution, scaled to the elliptical contact area. A similar effect was recently discovered by Popov [5] in the framework of an approximate analytical solution for power-law indenters with an arbitrary planform, which is slightly deviating from the axisymmetric one.

The remaining integral in Eq. (22) can be evaluated in closed analytical fashion as

$$\int\limits_{\rho}^{1}\frac{u^{2n-1}\mathrm{d}u}{\sqrt{u^2-\rho^2}}=\frac{\rho^{2n-1}}{2}\left[\mathrm{B}\left(1;\frac{1-2n}{2},\frac{1}{2}\right)-\mathrm{B}\left(\rho^2;\frac{1-2n}{2},\frac{1}{2}\right)\right], \tag{23}$$

with the incomplete beta function B expressed in terms of the hypergeometric function ${}_2\mathrm{F}_1$,

$$\mathrm{B}(z;a,b) := \frac{z^a}{a}\,{}_2\mathrm{F}_1(a, 1-b;a+1;z). \tag{24}$$

Alternatively, the following recursion scheme can be used,

$$\begin{aligned}
&\int_\rho^1 \frac{u\mathrm{d}u}{\sqrt{u^2-\rho^2}} = \sqrt{1-\rho^2},\\
&\int_\rho^1 \frac{u^{2n-1}\mathrm{d}u}{\sqrt{u^2-\rho^2}} = \frac{\sqrt{1-\rho^2}}{2n-1} + \rho^2\frac{2n-2}{2n-1}\int_\rho^1 \frac{u^{2n-3}\mathrm{d}u}{\sqrt{u^2-\rho^2}},
\end{aligned} \tag{25}$$

from which it is also clearer that the pressure distribution can always be written in the form of Eq. (8).

Similarly, i.e., based on the appropriate superposition of incremental flat punch indentations, the elastic normal displacements outside the contact area can be determined from the respective flat punch solution [11, p. 22]. However, the resulting integral cannot be evaluated in any "useful" way and shall therefore not be detailed here.

5 The subsurface stress state

As will be shown below, the complete subsurface stress state $\sigma_{ij}(x, y, z)$ can be given in terms of simple one-dimensional integrals of the subsurface stresses in an elliptical Hertzian contact, which are known analytically. For that purpose, we first recall again that, according to Eq. (18), the eccentricity of the contact ellipse is a function only of the eccentricity of the indenter profile, and does not depend on the contact configuration. Hence, similarly to Eq. (20) for the contact pressure distribution, we can write the subsurface stress state as a superposition of states arising from the incremental indentations by elliptical flat punches with increasing semiaxes (but with constant eccentricity),

$$\sigma_{ij}(x, y, z;b;e) = \int_0^b \sigma_{ij}^*\left(x, y, z;\tilde{b};e\right)\frac{\mathrm{d}\tilde{\delta}}{\mathrm{d}\tilde{b}}\,\mathrm{d}\tilde{b}. \tag{26}$$

Here, $\sigma_{ij}^*\left(x, y, z;\tilde{b};e\right)$ is the subsurface stress state arising from the unit indentation by an elliptical flat punch, i.e., due to the contact pressure distribution (10). Comparing that to the pressure distribution under an elliptical Hertzian contact (R_1 being the radius of curvature along the x-axis) [11, pp. 33 f.],

$$\begin{aligned}
p^{\mathrm{H}}\left(x, y;\tilde{b};e\right) = \frac{E^*}{2R_1}\frac{e^2}{\left(1-e^2\right)[\mathrm{K}(e)-\mathrm{E}(e)]}\\
\left[\tilde{b}^2-\left(1-e^2\right)x^2-y^2\right]^{1/2}, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,
\end{aligned} \tag{27}$$

we immediately recognize that

$$\begin{aligned}
p^*\left(x, y;\tilde{b};e\right) &= \frac{2R_1}{e^2\mathrm{K}(e)}\left(1-e^2\right)[\mathrm{K}(e)-\mathrm{E}(e)]\frac{\partial p^{\mathrm{H}}\left(x, y;\tilde{b};e\right)}{\partial\left(\tilde{b}^2\right)}\\
&= :R_1\mathrm{D}(e)\frac{\partial p^{\mathrm{H}}\left(x, y;\tilde{b};e\right)}{\partial\left(\tilde{b}^2\right)}.
\end{aligned} \tag{28}$$

Due to linearity, this superposition by differentiation will also be correct for the full subsurface stress state. Hence,

$$\sigma_{ij}(x, y, z;b;e) = R_1\mathrm{D}(e)\int_0^b \frac{\partial\sigma_{ij}^{\mathrm{H}}\left(x, y, z;\tilde{b};e\right)}{\partial\left(\tilde{b}^2\right)}\frac{\mathrm{d}\tilde{\delta}}{\mathrm{d}\tilde{b}}\,\mathrm{d}\tilde{b}, \tag{29}$$

which, after introducing the coordinate transform

$$\begin{aligned}
&\tilde{B}: = \tilde{b}^2, \quad B: = b^2, \quad \hat{\sigma}_{ij}\left(x, y, z;\tilde{B};e\right): = \sigma_{ij}\left(x, y, z;\tilde{b};e\right),\\
&\tilde{\delta}\left(\tilde{b}\right) = :g\left(\tilde{B}\right) = C_n\kappa(2n)\tilde{B}^n\frac{P_n\left(e^2, \varepsilon^2\right)}{\left(1-e^2\right)^n}
\end{aligned} \tag{30}$$

and partial integration results in

$$\hat{\sigma}_{ij}(x, y, z;B;e) = R_1\mathrm{D}(e)\left[\hat{\sigma}_{ij}^{\mathrm{H}}\left(x, y, z;\tilde{B};e\right)g'(B) - \int_0^B \hat{\sigma}_{ij}^{\mathrm{H}}\left(x, y, z;\tilde{B};e\right)g''\left(\tilde{B}\right)\mathrm{d}\tilde{B}\right]. \tag{31}$$

Thus, the problem of determining the subsurface stresses has been reduced to the evaluation of an elementary integral of the subsurface stresses $\hat{\sigma}_{ij}^{\mathrm{H}}\left(x, y, z;\tilde{B};e\right)$ under an elliptical Hertzian contact, which were given by Sackfield & Hills [18] and recently, using Carlson elliptical integrals, Greenwood [19].

6 Example: 4-th power ellipsoidal indenter

As was mentioned, for $n=1$, the long-known Hertzian solution is easily recovered from the general solution shown above. Therefore, as an example, in the following, the case $n=2$, i.e., the 4-th power ellipsoidal indenter, is discussed in little more detail.

The relation between the indenter and contact eccentricities follows from Eq. (18):

$$1-\frac{\mathrm{E}(e)}{\mathrm{K}(e)\left(1-e^2\right)}=\frac{3e^4+e^2\left(\varepsilon^2-4\right)}{8-8\left(e^2+\varepsilon^2\right)+3\left(e^4+\varepsilon^4\right)+2e^2\varepsilon^2}, \tag{32}$$

with the asymptotic solution

$$e\approx\sqrt{\frac{6}{5}}\,\varepsilon. \tag{33}$$

For the pressure distribution one obtains from Eq. (22):

$$p(x,y)=\frac{2E^*\delta}{3b\mathrm{K}(e)}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\left(1+\frac{2x^2}{a^2}+\frac{2y^2}{b^2}\right),\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1. \tag{34}$$

In Fig. 1 the normalized pressure distribution $pb_0/(E^*\delta)$ with $b_0=[3\delta/(8C_2)]^{1/4}$ is shown in a contour line diagram as a function of the normalized cartesian coordinates x/b_0 and y/b_0 for a contact profile eccentricity of $\varepsilon=0.5$. The same results were also obtained with a numerical solution of the contact problem, based on the boundary element method [20].

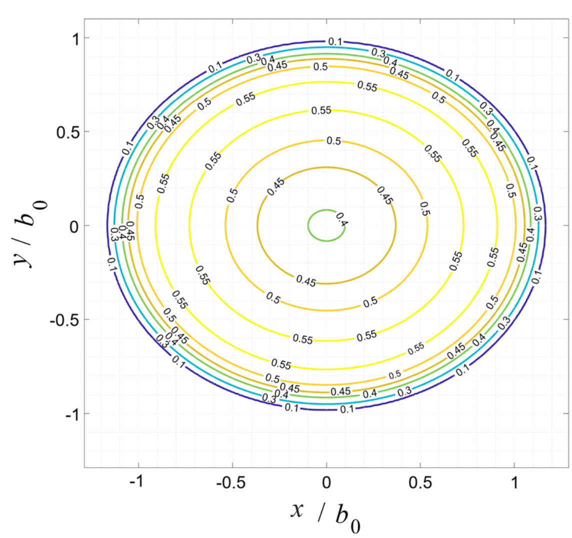

Fig. 1 Contour line diagram of the normalized contact pressure $pb_0/(E^*\delta)$ as a function of the normalized cartesian coordinates x and y for the indentation of an elastic half-space by a 4-th power ellipsoidal indenter with $\varepsilon=0.5$

As the limit for $n\to\infty$ corresponds to the contact with a rigid elliptical flat punch, for which obviously $e=\varepsilon$, one might expect that the difference between the eccentricities of the contact ellipse and the indenter profile decreases for increasing n. That is the case, as can be seen in Fig. 2, where the difference between e and ε is shown as a function of ε for different values of the exponent n. The thin lines correspond to the respective asymptotic solutions for small eccentricities

$$e\approx\sqrt{\frac{2n+2}{2n+1}}\,\varepsilon. \tag{35}$$

The latter relation seems to be generally correct, although no attempt has been made to rigorously prove it for arbitrary n, because it is sufficiently simple to just use the exact solution by solving Eq. (18).

7 Tangential contact solution

In elastic single contacts under tangential loading, the contact area generally consists of an inner stick region and an outer area of local slip. The statement of the tangential contact solution comprises the relations between the tangential force, the extent of the stick zone, and the macroscopic relative tangential displacement (i.e., between two distant points of the

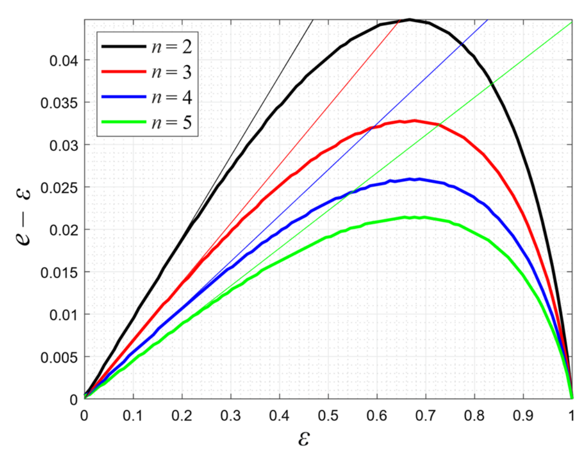

Fig. 2 Difference between the contact area and profile eccentricities as a function of the profile eccentricity for different exponents of the profile power-law. The thin lines correspond to the asymptotic solution (35)

contacting bodies), as well as the determination of the frictional shear tractions.

Within the framework of assumptions stated in Sect. 2, the tangential contact problem can be reduced to the normal contact problem via the principle of Jäger [21] and Ciavarella [22, 23]. While that principle is exactly correct for plane contact problems of elastically similar materials, in the general three-dimensional case—as the lateral displacements due to the tangential tractions are neglected and therefore the isotropy of the friction law is slightly violated in the region of local slip—it is associated with a small error (unless the materials do not exhibit the Poisson effect), even for elastically similar materials, which, however, is usually small [24].

With the standard Ciavarella–Jäger procedure, we obtain for the shear stress distribution

$$\tau_x(r,\varphi) = \mu\left[p(r,\varphi;\delta) - p\left(r,\varphi;\delta_1\right)\right]. \tag{36}$$

The (fictious) indentation depth δ_1—i.e., the indentation depth that induces a contact area which coincides with the stick area in the actual tangentially loaded contact—can be determined from the total tangential force,

$$F_x = \mu\left[F_N(\delta) - F_N\left(\delta_1\right)\right] = \mu F_N\left[1 - \left(\frac{\delta_1}{\delta}\right)^{\frac{2n+1}{2n}}\right]. \tag{37}$$

or the macroscopic tangential displacement for loading along the larger half-axis,

$$u_x^{(0)} = \frac{\mu}{1-\nu}\frac{f_x(e)}{\mathrm{K}(e)}\left(\delta - \delta_1\right),$$
$$f_x(e) := \mathrm{K}(e) + \frac{\nu}{e^2}\left[\left(1-e^2\right)\mathrm{K}(e) - \mathrm{E}(e)\right]. \tag{38}$$

Hence, the corresponding tangential contact stiffness is

$$k_x := \frac{\partial F_x}{\partial u_x^{(0)}} = \frac{\partial F_x}{\partial \delta_1}\frac{\partial \delta_1}{\partial u_x^{(0)}} = k_z\left(\frac{\delta_1}{\delta}\right)^{\frac{1}{2n}}\frac{(1-\nu)\mathrm{K}(e)}{f_x(e)} = \frac{2\pi G a_1}{f_x(e)}, \tag{39}$$

where a_1 is the larger half-axis of the sticking ellipse. As the final form of Eq. (39) does not contain any explicit dependence on n, it, of course, coincides with Mindlin's [25] respective result for the Hertzian case. Similarly, for tangential loading along the smaller half-axis,

$$u_y^{(0)} = \frac{\mu}{1-\nu}\frac{f_y(e)}{\mathrm{K}(e)}\left(\delta - \delta_1\right),$$
$$f_y(e) := \mathrm{K}(e) - \frac{\nu}{e^2}[\mathrm{K}(e) - \mathrm{E}(e)], \quad k_y = \frac{2\pi G a_1}{f_y(e)}, \tag{40}$$

while Eqs. (36) and (37) remain unchanged (using y instead of x).

While the maximum values of the relative tangential displacement at the onset of gross slip,

$$u_{x,\max}^{(0)} = \frac{\mu}{1-\nu}\frac{f_x(e)}{\mathrm{K}(e)}\delta, \quad u_{y,\max}^{(0)} = \frac{\mu}{1-\nu}\frac{f_y(e)}{\mathrm{K}(e)}\delta, \tag{41}$$

for a given indentation depth, do not exhibit explicit dependencies on the profile geometry, they, in contrast to the axisymmetric case, do implicitly depend on the indenter via the contact area eccentricity e (except for materials without Poisson effect, i.e., with $\nu = 0$). This, however, is only noticeable for large values of Poisson's ratio and large contact area eccentricities. Also, bearing in mind, that the error of the Ciavarella–Jäger procedure for three-dimensional contacts itself is of the order of ν, this result might be "academic".

8 Discussion and conclusions

Based on Galin's theorem for the indentation of an elastic half-space by a polynomial punch and Barber's theorem for the determination of the contact area in elastic normal contact problems, an exact contact solution has been developed for the frictionless indentation of an elastic half-space by an ellipsoidal power-law punch. Within the Cattaneo–Mindlin-approximation of tangential contacts, the tangential contact problem with friction can be reduced to the frictionless normal contact via the Ciavarella–Jäger principle, based on which also the tangential contact solution for the ellipsoidal power-law indenter has been given. The most interesting result is, that the pressure distribution for the power-law ellipsoidal indenter is given by the axisymmetric distribution, scaled to the elliptical contact area.

As is true for many analytical solutions of contact problems, the framework of set assumptions, made to allow for an analytical treatment, may pose restrictions of varying severity for the application of the obtained results to real engineering contacts. With regard to the present manuscript, the strongest model simplifications stem from the assumptions of linear elasticity and of a local Amontons-Coulomb friction law with a constant coefficient of friction. Also, elastic coupling (which severely complicates the rigorous analysis) can be an issue for the contact of dissimilar materials. On the other hand, the advantage of analytic solutions is their exactness, which is not compromised by (spatial) discretization or other numerical procedures; they therefore can serve as benchmark solutions for numerical models, which, in turn, are more flexible with respect to the physical modelling.

Moreover, several extensions of the proposed method are possible. As Galin's theorem is valid for *any* polynomial indenter shape, the solution procedure can be used exactly for an arbitrary finite superposition of ellipsoidal power-law profiles; at least in an approximate sense, it is also applicable for arbitrary monotonous ellipsoidal profiles. Moreover, like in the viscoelastic contact theories of Lee and Radok [26], Ting [27] and Graham [28], the obtained exact elastic normal contact solution can be used to determine the respective solution for linearly viscoelastic materials via the elastic–viscoelastic correspondence principle.

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Declarations

Conflict of interest The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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